FROM ORDERS ON GROUPS TO SPECTRA OF ℓ-GROUPS, AND BACK

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INTRODUCTION

The theory of orderable groups finds its roots in the work of Dedekind and Hölder at the end of 1800 and at the beginning of 1900, respectively.

This theory is often presented as a particular area of the theory of lattice-ordered groups (ℓ-groups).

In fact, the connection between these two fields is deep and well-studied.

EXAMPLE

A group is right orderable if, and only if, it embeds into an ℓ-group.

INTRODUCTION

Orderability of many interesting groups (e.g., braid groups) has recently attracted the interest of people from different areas in mathematics (e.g., low dimensional topology, geometric group theory).

Applications go in both directions.

EXAMPLE

Orderability of the fundamental group of a 3-manifold is related to the existence of certain foliations. (Boyer, Rolfsen, & Wiest, 2005)

Nonetheless, the study of the interplay between *topology* and the *theory of orderable groups* has been almost fully disconnected from the study of the interaction between *orderable groups* and ℓ-*groups*.

INTRODUCTION

Given a group $\langle G, \cdot, ^{-1}, e \rangle$, a *right invariant* (total) order $\leq$ on G is a total order on G such that for every $a, b, t \in G$,

$$a \leq b \Longrightarrow a \cdot t \leq b \cdot t.$$

We call a *right invariant* order on a group G a (total) right order on G.

Given a right order on G, the set of its *non-negative elements* $C \subseteq G$ is a submonoid of G with the properties $C \cup C^{-1} = G$ and $C \cap C^{-1} = \{e\}$, and we call such a submonoid a (total) cone for G.

Conversely, every *cone* C is the positive cone of some right order $\leq_C$ on G, defined via: $a \leq_C b$ if, and only if, $ba^{-1} \in C$.

We identify a right order $\leq$ on G with its cone C, and hence see the set $\mathcal{R}(G)$ of *all possible right orders on* G as a set of subsets of G.

EXAMPLES

EXAMPLE

Finite groups are *not* right orderable. In fact, right orderable groups are necessarily *torsion-free* (if $a^n = e \in G$, then $a = e$).

The notion of a *left invariant* order can be defined analogously.

A right order on G which is also left invariant is a (total) order on G, and orders can be identified with *cones closed under conjugation*.

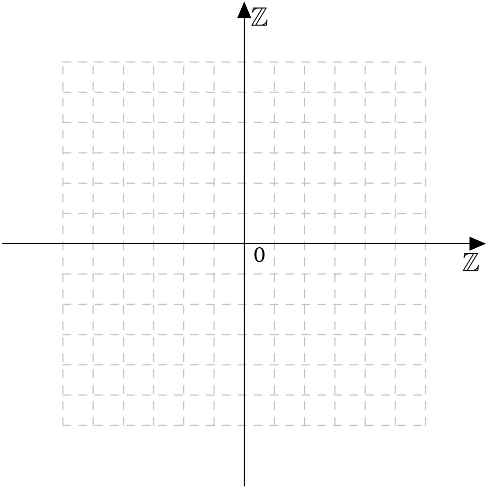
For Abelian groups, *right orders* and *orders* coincide, and *torsion-freeness* is also a *sufficient* condition for orderability.

EXAMPLE

Free groups (Shimbireva, 1947), and *free Abelian groups* are orderable.

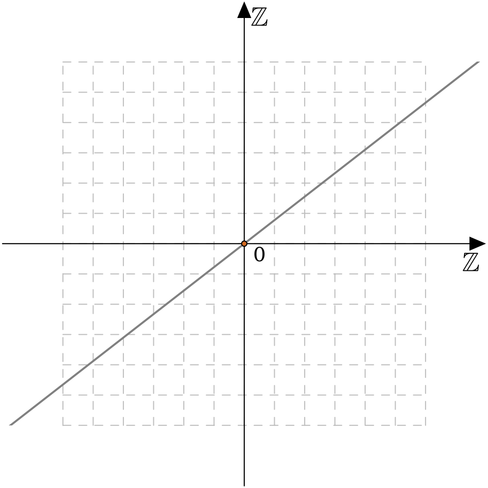
EXAMPLE

The free group $\mathbb{Z}^2$ is orderable.



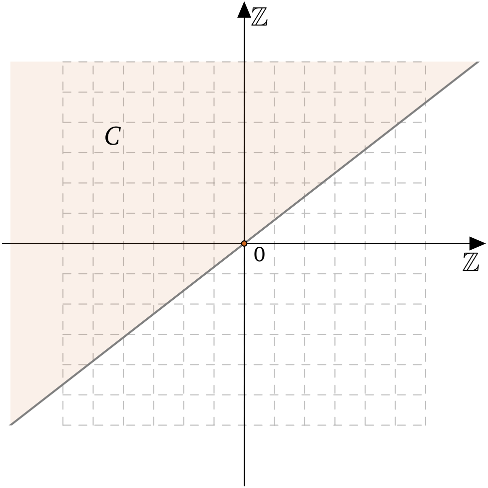
EXAMPLE

Lines through the origin, with irrational slope…



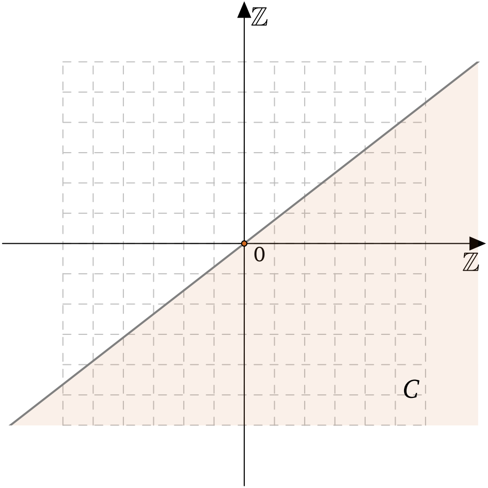
EXAMPLE

… determine *one* (Archimedean) *order*…



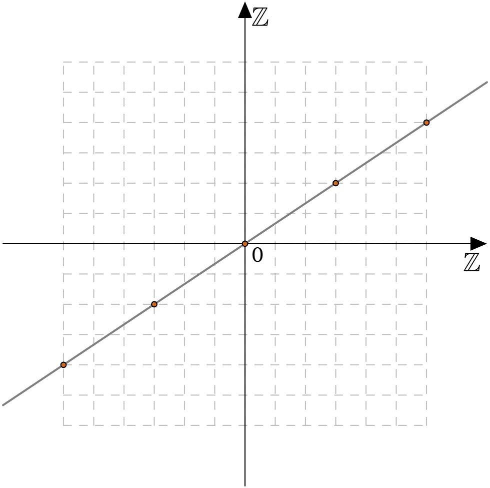
EXAMPLE

… determine *two* (Archimedean) *orders*.



Example

Lines through the origin, with rational slope...



RIGHT PRE-ORDERS

Given a group G, a (total) right pre-order $\preceq$ on G is a (proper) *right invariant pre-order* on G.

Given a right pre-order $\preceq$ on G, the set of its *non-strictly-negative elements* $C \subsetneq G$ is a submonoid of G with the property $C \cup C^{-1} = G$, and we call such a submonoid a (total) pre-cone for G.

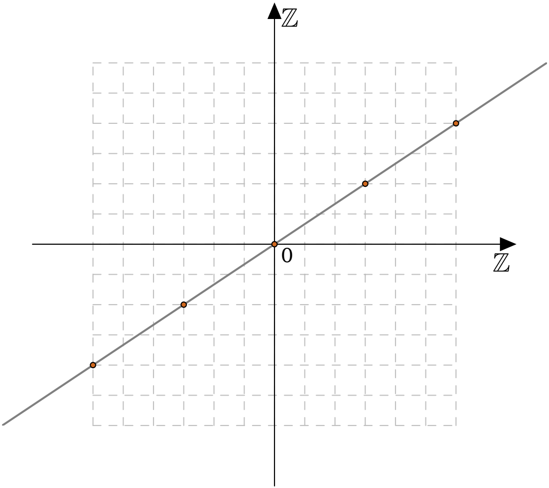
Conversely, every *pre-cone* C is the positive cone of some right pre-order $\preceq_C$ on G, defined via: $a \preceq_C b$ if, and only if, $ba^{-1} \in C$.

We identify a right pre-order $\preceq$ on G with its pre-cone C, and hence see the set $\mathcal{P}(G)$ of *all possible right pre-orders on* G ordered by inclusion as a subposet of the power set of G.

Recall that given a right pre-order C on G, its *poset reflection* Ω_C is a chain. We write $[a]$ for the the equivalence class of $a \in G$ induced by C.

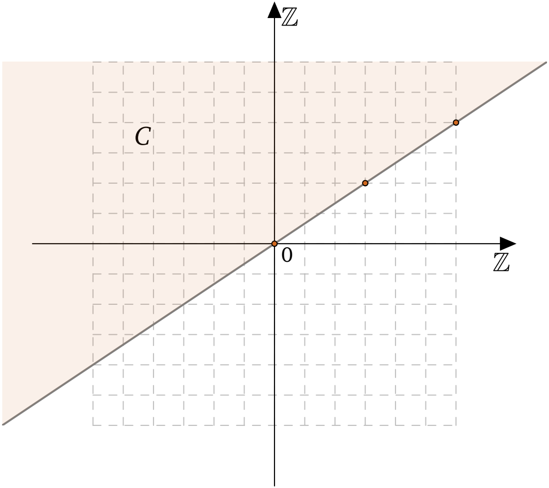
Example

Lines through the origin, with rational slope…



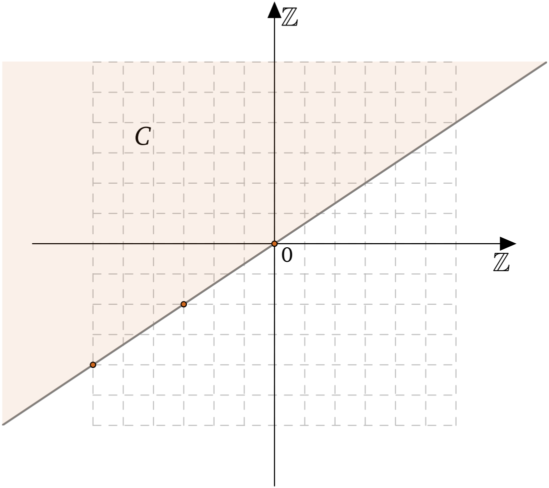
Example

… determine *one* (lexicographic) *order*…



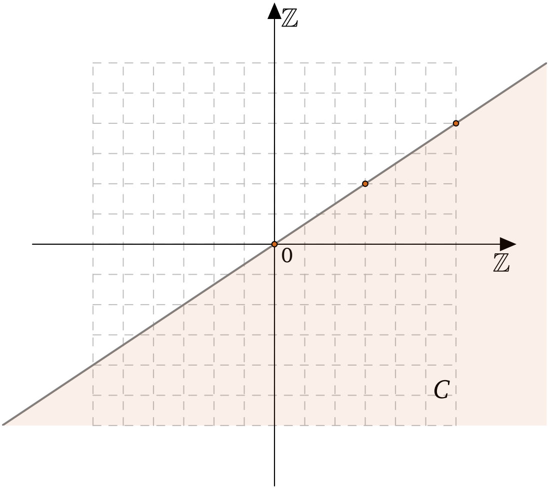
Example

… determine *two* (lexicographic) *orders*…



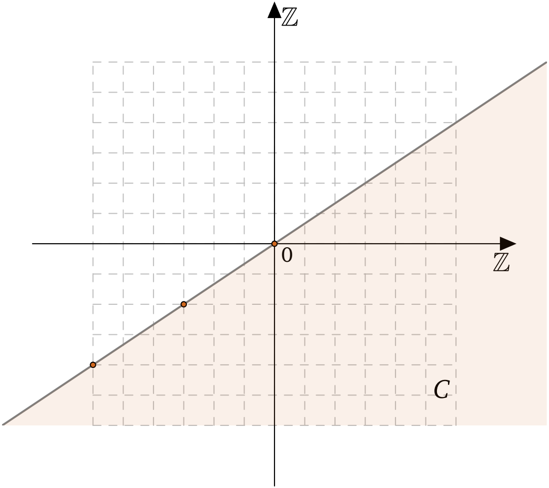
EXAMPLE

… determine *three* (lexicographic) *orders*…



EXAMPLE

... determine *four* (lexicographic) *orders*.



EXAMPLE

The poset $\mathcal{P}(\mathbb{Z}^2)$ is a *root system*, and looks as follows:

NOTE.

It is a general fact that $\mathcal{P}(G)$ is a *root system* with a *minimal layer*, and that every $C \in \mathcal{P}(G)$ extends *at least one* minimal right pre-order.

Moreover, for a *right orderable* group G, the *minimal layer* of $\mathcal{P}(G)$ is made exactly of *all the right orders on G*.

THE TOPOLOGICAL SPACE OF ORDERS

Given a *right orderable* group G, the set $\mathcal{R}(G)$ of right orders on G is non-empty, and it is possible to endow it with a topology τ, namely the *smallest topology* containing the sets

$$\mathbb{R}(a) = \{C \in \mathcal{R}(G) \mid a \in C\}, \quad \text{for } a \in G.$$

For an *orderable group* G, the *subspace topology* can be considered over the subset $\mathcal{O}(G)$ of orders on G.

These spaces were introduced by Adam Sikora in 2004, and were proved to be compact, Hausdorff, and totally disconnected.

We consider a generalization of this topology on the set $\mathcal{P}(G)$ of right pre-orders on a group G, by defining a subbasis as:

$$\mathbb{P}(a) = \{C \in \mathcal{P}(G) \mid a \in C \text{ and } a^{-1} \notin C\}, \quad \text{for } a \in G.$$

THE TOPOLOGICAL SPACE OF ORDERS

Already in Sikora's original paper, the topological space of orders was used to prove a result in (computational) algebraic geometry.

EXAMPLE

Given a polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$, every ideal $\mathfrak{i}$ contains a *universal Gröbner basis* of $\mathfrak{i}$. (Schwartz, 1988)

For a comprehensive account of the interaction between topology and the theory of orderable groups, see:

Adam Clay and Dale Rolfsen. *Ordered Groups and Topology*. Vol. 176. American Mathematical Soc., 2016.

EXAMPLES

EXAMPLE

The space $\mathcal{R}(\mathbb{Z}^n) = \mathcal{O}(\mathbb{Z}^n)$, for $n \geq 2$, is the *Cantor space*. (Sikora, 2004)

EXAMPLE

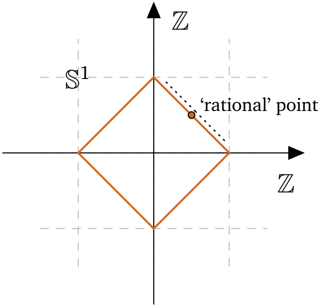
The space $\mathcal{R}(F(n))$ of rank $n \geq 2$ is the *Cantor space*. (McCleary, 1985)

EXAMPLE

The space $\mathcal{O}(F(n))$ of rank $n \geq 2$ is *not known* to be Cantor. (McCleary 1985)

EXAMPLE

The space $\mathcal{P}(\mathbb{Z}^2)$ of (right) pre-orders on the free Abelian group $\mathbb{Z}^2$ is a *non-Hausdorff completion of* $\mathbb{S}^1$ *with the Euclidean topology*.



The space $\mathcal{P}(\mathbb{Z}^n)$ of (right) pre-orders on the free Abelian group $\mathbb{Z}^n$ is the *spectral space* of the *free Abelian* ℓ*-group* $F^{\ell}_{\mathsf{A}}(n)$. (Panti, 1999)

THE SPECTRAL SPACE

The *spectral space*, or *ℓ-spectrum*, was introduced for *Abelian ℓ-groups* by Klaus Keimel in his doctoral dissertation (1971).

His aim was to follow on from the success of scheme theory in algebraic geometry, and introduce sheaf-theoretic methods in the study of ℓ-groups.

The spectral space has been widely studied for Abelian ℓ-groups, and later introduced also for *arbitrary ℓ-groups* (e.g., Conrad & Martinez, 1990).

LATTICE-ORDERED GROUPS

An ℓ-group H is an algebra $\langle H, \cdot, \wedge, \vee, ^{-1}, e \rangle$ where $\langle H, \cdot, ^{-1}, e \rangle$ is a group, $\langle H, \wedge, \vee \rangle$ is a (distributive) lattice, and $\cdot$ distributes over $\wedge, \vee$.

We call an ℓ-group Abelian if its *underlying group* is *Abelian*, and representable if it is a *subdirect product* of *totally ordered groups*.

EXAMPLE

Every Abelian ℓ-group is representable.

The class of (Abelian, representable) ℓ-groups is a variety, that is, it can be defined via *equations*. As such, all their common equational properties are 'encoded' into the *free (Abelian, representable) ℓ-groups*.

NOTE.

From the example above, it follows that the *free Abelian ℓ-group* $F^{\ell}_{\mathsf{A}}(X)$ over X is a *quotient* of the free *representable ℓ-group* $F^{\ell}_{\mathsf{R}}(X)$ over X.

EXAMPLE

Given a chain Ω, the group $\mathrm{Aut}(\Omega)$ of its *order-preserving bijections* can be made into an ℓ-group by defining the *coordinate-wise lattice order*:

$$f \leq g \iff f(a) \leq g(a), \quad \text{for every } a \in \Omega.$$

The ℓ-groups of order-preserving bijections of chains *generate the variety of ℓ-groups*. More precisely, the ℓ-group $\mathrm{Aut}(\mathbb{R})$ *generates the variety*.

THE SPECTRAL SPACE

A convex ℓ-subgroup of an ℓ-group H is a *convex sublattice subgroup* of H.

For a convex ℓ-subgroup $\mathfrak{c} \subseteq H$, the quotient $H/\mathfrak{c}$ is a *lattice* with operations $\mathfrak{c}x \wedge \mathfrak{c}y = \mathfrak{c}(x \wedge y)$, and $\mathfrak{c}x \vee \mathfrak{c}y = \mathfrak{c}(x \vee y)$.

We call a convex ℓ-subgroup $\mathfrak{p} \subsetneq H$ prime if for every $x, y \in H$, if $x \wedge y \in \mathfrak{p}$ then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Equivalently, if the quotient $H/\mathfrak{p}$ *is a chain*.

NOTE.

There is a natural ℓ-group homomorphism $\varphi_{\mathfrak{p}} \colon H \to \mathrm{Aut}(H/\mathfrak{p})$, that maps $x \in H$ to the order-preserving bijection $\varphi_{\mathfrak{p}}(x) \colon \mathfrak{p}y \mapsto \mathfrak{p}yx$.

THE SPECTRAL SPACE

The set $\operatorname{Spec} H$ of *prime convex ℓ-subgroups* of H ordered by inclusion is a *root system*. Every element $\mathfrak{p} \in \operatorname{Spec} H$ contains at least one *minimal* element $\mathfrak{m} \in \operatorname{Min} H$ (by a simple application of Zorn's Lemma).

We endow $\operatorname{Spec} H$ with the spectral topology, whose basic open sets are

$$\mathbb{S}(x) = \{\mathfrak{p} \in \operatorname{Spec} H \mid x \notin \mathfrak{p}\}, \quad \text{for } x \in H.$$

These are all the *compact opens*, and the space is a *completely normal generalized spectral space*. In fact, it can be also obtained as the *Stone dual* of the *distributive lattice of principal convex ℓ-subgroups of H*.

We consider $\operatorname{Min} H$ with the subspace topology, whose basic open sets are

$$\mathbb{S}_m(x) = \{\mathfrak{p} \in \operatorname{Min} H \mid x \notin \mathfrak{p}\}, \quad \text{for } x \in H.$$

THE SPECTRAL SPACE

The spectral space of an ℓ-group H provides a tool for employing sheaf-theoretic methods in the study of ℓ-groups.

EXAMPLE

A *representable ℓ-group H* can be embedded into a *Hausdorff sheaf of ℓ-groups* on the space $\operatorname{Min} H$ of minimal prime convex ℓ-subgroups of H. (Darnel, 1995)

THE SPECTRAL SPACE

We write $x \perp y$ to denote $|x| \wedge |y| = e$ for $x, y \in H$, where $|x| = x \vee x^{-1}$ is the absolute value of x. For $T \subseteq H$, set

$$T^{\perp} = \{x \in H \mid x \perp y \text{ for all } y \in T\},$$

and call those subsets *polars*.

We write $\operatorname{Pol} H$ for the *Boolean algebra of polars* of H, and $\operatorname{Pol}_{\mathrm{p}} H$ for its *sublattice of principal polars*, those of the form $\{x\}^{\perp\perp}$ for some $x \in H$.

EXAMPLE

A *representable ℓ-group* H can be embedded into a *Hausdorff sheaf of ℓ-groups* on the *Stone space* associated with the *complete Boolean algebra* $\operatorname{Pol} H$ of polars. (Darnel, 1995)

THE SPECTRAL SPACE

Furthermore, topological properties of the spectrum $\operatorname{Spec} H$ can have important consequences on the structure of the ℓ-group H.

EXAMPLE

The space $\operatorname{Min} H$ is *compact* if, and only if, $\operatorname{Pol}_{\mathrm{p}} H$ is a *Boolean algebra*. In which case, $\operatorname{Min} H$ is the *Stone dual* of $\operatorname{Pol}_{\mathrm{p}} H$. (Conrad & Martinez, 1990)

THE SPECTRAL SPACE

We call an ℓ-group H hyperarchimedean when each of its quotients is Archimedean.

EXAMPLE

The space $\operatorname{Spec} H$ is *Hausdorff* if, and only if, H is *hyperarchimedean*. (Conrad & Martinez, 1990)

An element $u \in H$ is a strong order unit if $u \in H^+$, where H^+ is the *positive cone* of H, and u generates H as a convex ℓ-subgroup.

EXAMPLE

The space $\operatorname{Spec} H$ is *compact* if, and only if, H has a *strong order unit*. (Conrad & Martinez, 1990)

FREE ℓ-GROUPS

For a group G and a variety V of ℓ-groups, there are an ℓ-group $F_{\mathsf{V}}(G)$, and a group homomorphism $\eta\colon G \to F_{\mathsf{V}}(G)$ characterised by the following...

... UNIVERSAL PROPERTY.

For each group homomorphism $p\colon G \to H$ with $H \in \mathsf{V}$, there exists exactly one ℓ-homomorphism $h\colon F_{\mathsf{V}}(G) \to H$ such that the following diagram

$$\begin{tikzcd}
G \arrow[r, "\eta"] \arrow[dr, "p"'] & F_{\mathsf{V}}(G) \arrow[d, dashed, "!h"] \\
 & H
\end{tikzcd}$$

commutes, i.e., $h(\eta(a)) = p(a)$, for each $a \in G$.

We call $F_{\mathsf{V}}(G)$ the V*-free ℓ-group over G*.

It is easy to see that $\eta[G]$ *generates* $F_{\mathsf{V}}(G)$ as a *lattice*.

FROM RIGHT PRE-ORDERS TO ℓ-GROUPS

For a group G and a right pre-order C on G, consider $\pi_C \colon G \to \mathrm{Aut}(\Omega_C)$ sending $a \in G$ to the *order-preserving bijection* $\pi_C(a)([b]) = [ba]$.

The map π_C is a *group homomorphism*, and we write H_C for the ℓ-subgroup of $\mathrm{Aut}(\Omega_C)$ generated by $\pi_C[G]$.

THEOREM (C. & MARRA, 2018)

The (sub)space $\mathcal{P}_{\mathsf{V}}(G)$ of right pre-orders C on G for which $H_C \in \mathsf{V}$ is homeomorphic to the spectral space $\operatorname{Spec} F_{\mathsf{V}}(G)$ of the V-free ℓ-group over G.

The following diagram is central for the construction:

$$\begin{tikzcd}
G \arrow[r, "\eta"] \arrow[dr, "\pi_C"] & F_{\mathsf{V}}(G) \arrow[d, two heads, "h_C"] \\
 & H_C
\end{tikzcd}$$

THE KEY INGREDIENTS

In order to associate a *prime convex* ℓ*-subgroup* $\mathfrak{p} \in \operatorname{Spec} F_{\mathsf{V}}(G)$ to a *right pre-order* $C \in \mathcal{P}_{\mathsf{V}}(G)$, we consider the so-called stabilizer of $[e]$ in H_C:

$$\{h_C(x) \mid x \in F_{\mathsf{V}}(G) \text{ and } h_C(x)([e]) = [e]\}.$$

Its inverse image under h_C^{-1} is a *prime convex* ℓ*-subgroup* of $F_{\mathsf{V}}(G)$.

NOTE.

The set of elements of $a \in G$ whose image $\pi_C(a)$ fixes $[e]$ are exactly the elements of $C \cap C^{-1}$, and are clearly *not enough* to distinguish two different right pre-orders (e.g., right orders).

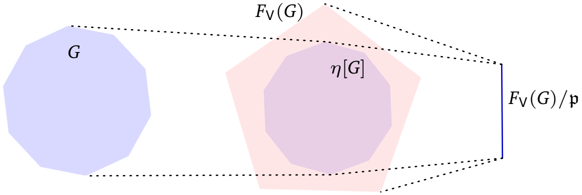
We do have ‘more information’ stored in $F_{\mathsf{V}}(G)$.

If $C \neq D \in \mathcal{P}_{\mathsf{V}}(G)$, consider $a \in C \setminus D$. The element $h_C(\eta(a) \wedge e)$ fixes $[e]$, while $h_D(\eta(a) \wedge e)$ doesn't. (It might still be that $C \cap C^{-1} = D \cap D^{-1}$)

THE KEY INGREDIENTS

For a *prime convex ℓ-subgroup* $\mathfrak{p} \in \operatorname{Spec} F_{\mathsf{V}}(G)$, we get a pre-order on G by:

$$a \preceq_{\mathfrak{p}} b \iff \mathfrak{p}\eta(a) \leq \mathfrak{p}\eta(b),$$

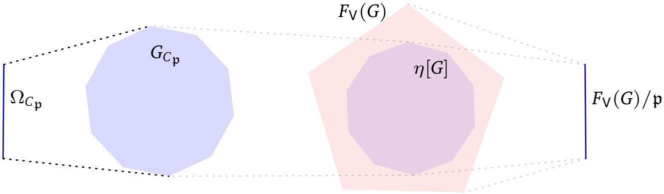


THE KEY INGREDIENTS

For a *prime convex ℓ-subgroup* $\mathfrak{p} \in \operatorname{Spec} F_{\mathsf{V}}(G)$, we get a pre-order on G by:

$$a \preceq_{\mathfrak{p}} b \iff \mathfrak{p}\eta(a) \leq \mathfrak{p}\eta(b),$$

which is a *right pre-order* $C_{\mathfrak{p}} \in \mathcal{P}_{\mathsf{V}}(G)$ such that $\Omega_{C_{\mathfrak{p}}} \cong F_{\mathsf{V}}(G)/\mathfrak{p}$.



This follows from the fact that for $x \in F_{\mathsf{V}}(G)$, we have $\mathfrak{p}x = \mathfrak{p}\eta(a)$ for some $a \in G$, and also from the fact that $h_{C_{\mathfrak{p}}}$ 'is' $\varphi_{\mathfrak{p}}$.

THE KEY INGREDIENTS

Finally, the fact that this order-isomorphism lifts to a homeomorphism follows from some basic facts about $F_{\mathsf{V}}(G)$ and its spectrum.

First of all, $\mathbb{S}(x) = \mathbb{S}(|x|)$. Therefore, we focus on $\mathbb{S}(x)$ for $x \in F_{\mathsf{V}}(G)^+$.

Since $F_{\mathsf{V}}(G)$ is generated by $\eta[G]$ as a lattice, it follows that

$$x = \bigwedge_I \bigvee_J (\eta(a_{ij}) \vee e)$$

with finite I, J, and with $a_{ij} \in G$, for every $x \in F_{\mathsf{V}}(G)^+$.

Also, $\mathbb{S}(x \wedge y) = \mathbb{S}(x) \cap \mathbb{S}(y)$ and $\mathbb{S}(x \vee y) = \mathbb{S}(x) \cup \mathbb{S}(y)$, for $x, y \in F_{\mathsf{V}}(G)^+$.

Therefore, we can focus on opens of the form

$$\mathbb{S}(\eta(a) \vee e) = \{\mathfrak{p} \in \operatorname{Spec} F_{\mathsf{V}}(G) \mid (\eta(a) \vee e) \notin \mathfrak{p}\}, \quad \text{for } a \in G.$$

THE KEY INGREDIENTS

Moreover, note that $(\eta(a) \vee e) \notin \mathfrak{p}$ if, and only if,

$$\mathfrak{p}(\eta(a) \vee e) = \mathfrak{p}\eta(a) \vee \mathfrak{p}e > \mathfrak{p}e,$$

that is,

$$\begin{aligned} h_{C_{\mathfrak{p}}}(\eta(a) \vee e)([e]) &= h_{C_{\mathfrak{p}}}(\eta(a))([e]) \vee h_{C_{\mathfrak{p}}}(e)([e]) \\ &= \pi_{C_{\mathfrak{p}}}(a)([e]) \vee \pi_{C_{\mathfrak{p}}}(e)([e]) \\ &= [a] \vee [e] > [e]. \end{aligned}$$

Therefore, the *prime convex ℓ-subgroup* $\mathfrak{p} \in \mathbb{S}(\eta(a) \vee e)$ corresponds to a *right pre-order* $C_{\mathfrak{p}} \in \mathcal{P}_{\mathsf{V}}(G)$ that *properly contains* a, that is, $C_{\mathfrak{p}} \in \mathbb{R}(a)$.

REMARKS

- The result uses only *basic facts* about orderable groups, and ℓ-groups.
- It is fully general, and doesn't make any assumption on G or on η.
- It provides a notion of '*universal object*' for (right) orders on groups.
- It hints at some kind of *correspondence theory*.

  The condition for a right pre-order to correspond to the variety of all (Abelian, representable) ℓ-groups is *elementary*.

RIGHT ORDERABLE GROUPS

Let G be a *right orderable* group.

COROLLARY

Sikora's topological space $\mathcal{R}(G)$ *of right orders on* G *is the minimal layer* $\operatorname{Min} F(G)$ *of the spectrum of the free* ℓ*-group* $F(G)$ *over* G*.*

As a consequence, $\operatorname{Min} F(G)$ is *compact* and hence:

COROLLARY

Sikora's topological space $\mathcal{R}(G)$ *of right orders on* G *is the Stone space of the Boolean algebra of principal polars of the free* ℓ*-group over* G*.*

COROLLARY

The minimal layer $\operatorname{Min} F^{\ell}(n)$ *of the spectrum of the free* ℓ*-group of rank* $n \geq 2$ *is the Cantor space.*

ORDERABLE GROUPS

Let G be an *orderable* group.

COROLLARY

Sikora's topological space $\mathcal{O}(G)$ of orders on G is the minimal layer $\operatorname{Min} F_{\mathsf{R}}(G)$ of the spectrum of the free representable ℓ-group $F_{\mathsf{R}}(G)$ over G.

As a consequence, $\operatorname{Min} F_{\mathsf{R}}(G)$ is *compact* and hence:

COROLLARY

Sikora's topological space $\mathcal{O}(G)$ of orders on G is the Stone space of the Boolean algebra of principal polars of the free representable ℓ-group over G.

COROLLARY

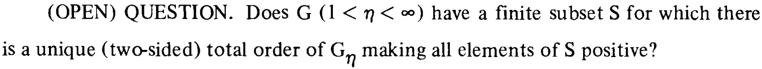
The free representable ℓ-group $F_{\mathsf{R}}(G)$ over G can be ℓ-embedded into a Hausdorff sheaf of ℓ-groups on the space $\mathcal{O}(G)$ of orders on G.

AN OPEN PROBLEM

QUESTION.

Is the *minimal layer* $\operatorname{Min} F^{\ell}_{\mathsf{R}}(n)$ of the spectrum of the *free representable ℓ-group* $F^{\ell}_{\mathsf{R}}(n)$ of rank $n \geq 2$ homeomorphic to the *Cantor space*?

This was first asked by McCleary (1985) in a different form.



(OPEN) QUESTION. Does G $(1 < \eta < \infty)$ have a finite subset S for which there is a unique (two-sided) total order of G_η making all elements of S positive?

CONJECTURE.

The space $\mathcal{O}(F(n))$, for $n \geq 2$, doesn't have any isolated points.

AN OPEN PROBLEM

CONJECTURE.

The space $\operatorname{Min} F^{\ell}_{\mathsf{R}}(n)$, for $n \geq 2$, doesn't have any isolated points.

NOTE.

The space $\operatorname{Max} F^{\ell}_{\mathsf{R}}(n)$ exists, and every element $\mathfrak{p} \in \operatorname{Spec} F^{\ell}_{\mathsf{R}}(n)$ is extended by a *unique element* $\mathfrak{p}^* \in \operatorname{Max} F^{\ell}_{\mathsf{R}}(n)$.

Therefore, the map

$$\begin{array}{rccl} \lambda\colon & \operatorname{Min} F^{\ell}_{\mathsf{R}}(n) & \twoheadrightarrow & \operatorname{Max} F^{\ell}_{\mathsf{R}}(n), \\ & \mathfrak{m} & \mapsto & \mathfrak{m}^* \end{array}$$

is well-defined, *continuous*, and *closed*.

Moreover, $\operatorname{Max} F^{\ell}_{\mathsf{R}}(n) \cong \mathbb{S}^{n-1}$.

AN OPEN PROBLEM

We say that the map λ is irreducible if it sends *proper closed* subsets of $\operatorname{Min} F^{\ell}_{\mathsf{R}}(n)$ to *proper closed* subsets of $\operatorname{Max} F^{\ell}_{\mathsf{R}}(n)$.

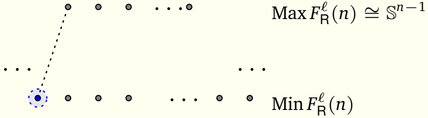
COROLLARY

If the map λ is irreducible, then $\operatorname{Min} F^{\ell}_{\mathsf{R}}(n)$ doesn't have any isolated points.

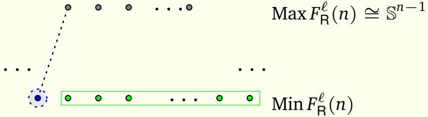
AN OPEN PROBLEM

PROOF...

Suppose that there is an isolated point in $\operatorname{Min} F^{\ell}_{\mathsf{R}}(n)$.



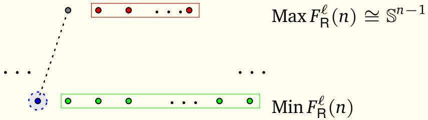
The image of the green points through the map λ must be *proper*.



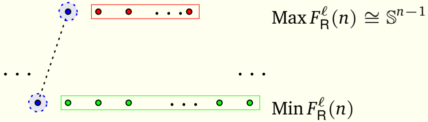
AN OPEN PROBLEM

… PROOF.

Hence, the image of the green points through the map λ is the red part.



This is not possible, since $\operatorname{Max} F^{\ell}_{\mathsf{R}}(n)$ is homeomorphic to $\mathbb{S}^{n-1}$ with the Euclidean topology and hence, it doesn't have isolated points. $\lightning$



Therefore, if λ is irreducible, then $\operatorname{Min} F^{\ell}_{\mathsf{R}}(n)$ doesn't have isolated points. $\square$