Ten commented works over the period 2015 - 2025

by Laurent Stolovitch

My initial works study **local analytic foliations in a neighborhood of a singular point in the euclidean complex space**. In particular, we are interested in the the behavior of the flow of a vector field X in a neighborhood of a singular point (i.e. a fixed point of the dynamic) in $\mathbb{C}^n$, $n \geq 2$. We study those, for instance, the associated system of differential equations of which, can be written as

$$\frac{dx_i}{dt} = \lambda_i x_i + f_i(x) \quad i = 1, \ldots, n$$

where λ_i (the eigenvalues of the linear part) are complex numbers, not all zero and where the f_i's are germs of "non-linear" holomorphic functions at the origin ($f_i(0) = 0$ et $Df_i(0) = 0$).

In order to understand the geometry of the foliation and the associated dynamics, one tries to transform, by a change of coordinates preserving the singularity, the differential system into another one, supposed to be "simpler"(there is a formal definition), called **normal form**. In general, such a transformation exists only at formal power series level: there exists a formal change of coordinates which conjuguates X to NF, the later may or not be an analytic vector field. This has to be thought as an infinite dimensional version "Jordan normal form decomposition of matrices".

In order to obtain dynamical or geometrical information on the initial dynamical system from the normal form, one need the transformation to be regular. The more regular it is, the more faithful the information transferred from the normal form are. We mainly concerned with analytic (real or complex) regularity, the finest one. So the problem lead to know under which circumstances there exists an analytic normalizing diffeomorphism (i.e transformation from the original to a normal form). This problem is related to *resonances* (i.e formal obstruction to linearization) and to the so-called *small divisors problems* (which measure how close the system is to resonances).

Our works shed a new light on these problems on both analytic and geometrical aspects. We give here three examples of our former works in order to enlighten our newest ones.

- In "L. Stolovitch, Singular complete integrability. Inst. Hautes Etudes Sci. Publ. Math. 91 (2000), 133-210 (2001)" , I study families of commuting germs of holomorphic vector fields at a common fixed point. I give conditions, that I name "complete integrability", sufficient to transform holomorphically near the fixed point the family to a normal form. One of these conditions is related to the "type of formal normal form" obtained through a formal transformation. Our result not only recovers all similar results known at the time (e.g. Brjuno's work) but also to explain why and how some geometric constraints, such as "symplectic" or "volume preserving" allow to conclude (Vey's theorems). As a corollary, we show that such a family admits a kind of "holomorphic fibration" of invariant analytic varieties : the family of their linear parts at the fixed point defines a finite numbers of *resonant* monomials the level set of which (restricted to a neighborhood of the fixed point) are all left invariant under the dynamical system defined by the family of normal forms. This has been extended to to families with possibly vanishing linear part in our Annals's article [St05a]. The existence of germs of invariant analytic subset passing throught the origin is proved in [Sto94], ideas of which has been used in [Sto15] and among others.
- The natural question that follows is the persistence of these invariant analytic sets when perturbing such a holomorphic "completely integrable system". I discovered that the so-called KAM phenomena (in Hamiltonian dynamics : an analytic Hamiltonian perturbation admits a "lot of invariant (manifolds diffeomorphic to) tori" and et the dynamic restricted to such a manifold is conjugated to a constant motion on the torus) exists in fact outside the Hamilltonian framework as soon as we are not looking for invariant tori (L. Stolovitch. A KAM Phenomenon for Singular Holomorphic Vector Fields. Publ. Math. Inst. Hautes ´Etudes Sci. 102 (2005), pp. 99-165). Indeed, it is the level sets of resonant monomials (associated to the unperturbed completely integrable system, as descried above) that serves as natural models for invariant varieties. Our result is essentially that such a "good" perturbed holomorphic vector field admits a lot of analytic invariant sets (in a neighborhood of the origin). The later are biholomorphic to some of these "resonant" level sets. The restriction of the dynamic to one of these invariant sets is holomorphically conjugated to the restriction of a linear vector field to the corresponding resonant level set. The underlining ideas (but not the techniques) are the keys for Publi 2.

- In ”E. Lombardi and L. Stolovitch. Normal Forms of Analytic Perturbations of Quasihomogeneous Vector Fields: Rigidity, Invariant Analytic Sets and Exponentially Small Approximation. Ann. Sci. Ec. Norm. Super. (4) 43 (2010), pp. 659-718”, we consider more degenerate singularities of vector fields : holomorphic perturbations of ”quasi-homogeneous” vector fields near a fixed point. We develop a normal form theory as well as a kind of ”Diophantine small divisor” condition relatively to this quasi-homogeneous unperturbed part. Among our results, we obtain one which is the well known ”Siegel-Brjuno theorem” when the initial part is just a linear diagonal vector field : ”if the perturbation is formally conjugate to its unperturbed initial part, then it is also holomorphically conjugated to it as soon as ”the associated small divisors are good” (i.e. not accumulating 0 too fast)”. This work was a source of inspiration my many others (and not only of mine) : for holomorphic normal form theory of vector field with a nilpotent linear part at the origin, in any dimension [StV16] ; in Publi 5 (used in Publi 4) ; in CR Geometry (Kolar-Kossovskiy), in fast-slow dynamical systems (see below) ....

For some years, I've been trying to transfer this point of view, these methods and technics to understanding general rigidity problems in Geometry as for instance, Cauchy-Riemann geometry, Poisson structures and also in some PDE's problems. I did important investments in these domains. It turns out that the introduction of our ”new technologies” in these areas led to deep new results along three main directions as well as in Dynamical Systems in return :Several complex variables (SCV)/ Cauchy-Riemann geometry; Dynamical Systems and PDE's.

1.1 Singularities in Cauchy-Riemann Geometry and their interactions with Dynamical Systems

It became clear, since the seminal work of Bishop ('60s), that submanifolds of $\mathbb{C}^n$ that have a singularity of their Cauchy-Riemann (CR) are quite astonishing geometrical and analytical objects. It also became natural to classify them through the action of biholomorphisms that preserve the singular point (say 0). It can be shown that these submanifolds are higher order (≥ 3) perturbations $M : z'' := Q_2(z', \bar{z}') + M_{\geq 3}(z', \bar{z}')$ of some quadric Q_2, where $M_{\geq 3}$ is of order ≥ 3 at the origin in local coordinates $z = (z', z'')$. In dimension $n = 2$, the pioneering and fundamental work of Moser-Webster considered submanifolds with a *minimal* complex dimension at the singularity (the manifold is totally real everywhere but at 0 where it has a 1-dimensional complexe tangent) and classified perturbation of *elliptic* quadrics. One of the main feature of the construction is that the manifold is studied not directly but through an associated (germ of) dynamical system at a fixed point.

•”Real submanifolds of maximum complex tangent space at a CR singular point I”.(with X. Gong), Publi 2

In this work, we sudy subamnifolds with a *maximal* complex dimension at the singularity. This led us to discoveries of situations that does not occur in Moser-Webster case. Among our main contributions is a **holomorphic transformation to a normal form theorem** : we define the notion of *abelian CR singularity* which are shown to be biholomorphically equivalent to a (analytic) normal form near the CR singularity. These geometric problems are very related to a Dynamical Systems considerations. Indeed to obtain our geometric classification, we had to devise and prove a **new theorem on holomorphic normalization of abelian families of biholomorphisms at a common fixed point**. We also have more partial results for submanifolds which are not of abelian type.This is the **very first work in almost 40 years** that overcome the issue of **higher dimension of the degeneracy** (or the dimension $n > 2$). One of the reason, is that, not only the geometric realization is conceptually more difficult but also because of the complexity of the underlying dynamics. It is certainly a starting point for further results such as Publi 1,Publi 8,Publi 10, [StJ21].

•”Real submanifolds of maximum complex tangent space at a CR singular point II”.(with X. Gong), Publi 1

In this article, we continue our study of CR singularities. We first classify ”à la Bishop” quadrics having a CR singularity of maximal dimension at the origin. We give also the formal invariants of their *non-degenerate* higher order perturbations and show that they have a **unique formal normal form**. We also show the existence of an analytic submanifold the normal forms of which are all divergent power series at the the origin. This situation cannot appear in dimension 2 (Moser-Webster). Doing so, **we solve a century old Dynamical Systems problem** : the existence of germs of biholomorphism **all the Poincaré-Dulac normal forms** of which **are divergent** at the origin. Although, the existence of *divergent transformation* to a normal form is well known and related to *small divisors*, the problem of *existence of divergent normal forms* was open since Poincaré! since then, another proof in the symplectic case has been devised by R. Krikorian (IHES '22)

• ”Geometry of hyperbolic Cauchy-Riemann singularities and KAM-like theory for holomorphic involutions”.(with Z. Zhao) Publi 10

We show, once more, that the interplay between Dynamical systems and Geometry (here, CR singularity) is crucial. We consider real analytic higher order perturbations of hyperbolic quadrics. In this case, the program of holomorphic transformation to a normal form, initiated by Moser-Webster, fails. Nevertheless, we prove the existence of a large family of complex curves that intersect the submanifold along holomorphic hyperbolas (biholomorphic to z_1z_2 =real constant). This is a consequence of a non-standard KAM like theorem for pairs of holomorphic involutions near a fixed point : there is a "large" compact of real constants such that the involutions have an invariant analytic set biholomorphic to z_1z_2 =real constant and to where they are conjugated to linear maps. The germ of ideas was developed in a different context [Sto05b].

•"Reversible parabolic diffeomorphisms of $(\mathbb{C}^2, 0)$ and exceptional hyperbolic CR-singularities".(with M. Klimes) Publi 8

We classify formally and analytically analytic **exceptional hyperbolic CR singularities**. **There were absolutely no work on that case, even at formal power series level**. *This could be achieved by developing a* **completely new theory of the dynamics of germs of holomorphic parabolic diffeomorphims in dimension 2**. *Up to now, all studies focused on dimension 1 case (except for the existence of "petals"), main achievement being Ecalle-Voronin theory (For germs of vector fields in* $(\mathbb{C}^2, 0)$*, the counter part part theory is due to Martinet-Ramis; It has been extended to higher dimension by the author [Sto96]. It allows to describe (as an infinite dimensional space (of functions)) the space of such biholomorphisms that are holomorphically conjugated in a neighborhood of the fixed point. This can be done by the existence of holomorphic conjugacies to the same (holomorphic) normal form over a finite number of overlapping sectors at the fixed point. We develop a "similar" theory (but with a infinity of "sectorial domains" instead) for some reversible biholomorphisms in dimension 2 and we apply it to classify holomorphically analytic surfaces having such CR singularity. Once more, these "interactions" could allow to obtain these brand new results both in CR geometry as well as in Dynamical Systems.*

1.2 Complex and Cauchy-Riemann Geometry

•"Big demonimators and analytic normal forms. With an appendix of M. Zhitomirskii", Publi 5

This article is our first attempt to transfer concepts and methods from (the local study of) Dynamical Systems to a general geometric framework. This work took its root in our work with Lombardi. In this article, we study the general problem of transformation to *normal form* of an abstract "analytic object" acted on by a group of some (analytic) transformations. We consider analytic perturbations of a given object and we define a notion of normal form of the perturbation wrt to the former. We also build up a sequence of positive numbers that plays the rôle of *small divisors* in Dynamical systems If instead to accumulating the origin, this sequence of numbers tends to infinity with a "sufficient high speed", then we prove the existence of an analytic conjugacy to a normal form. We say that the problem has *Big denominators property* (BD). Applying this theorem to the classical case of conjugating (germs of) analytic vector fields at fixed point, the DB property amount to saying that the linear part of the vector field at the origin is in the *Poincaré domain.* We apply this result to the **normal from problem of holomorphic functions with non-isolated singularity. This is the very first systematic result of the kind**. In the case of isolated singularity, we give a new kind of proof of the conjugacy to a polynomial, this was known from Arnold, Tougeron (late 60's-early 70's)....The results of this article has been used in many other works.

•"Convergence of the Chern-Moser-Beloshapka normal forms" (with B. Lamel), Publi 4

We define the notion of normal forms for *Levi-nondegenerate* real analytic submanifolds of $\mathbb{C}^n$ of **codimension** $d \geq 1$ under the action of the group of germs of formal biholomorphisms. We give a sufficient condition ensuring that there exists a holomorphic transformation to such a normal form. This gives, in the case of hypersurfaces $(d = 1)$, a new proof of the celebrated Chern-Moser theorem. **It took almost 50 years to overcome the issue of codimension** > 1. To overcome this difficulty, we introduced ideas coming from Dynamical Systems. Our proof uses in an essential way our former works on "Big Denominators"(see above).

•"Equivalence of Cauchy-Riemann manifolds and multisummability theory".(with I. Kossovskiy and B. Lamel), Publi 3

We prove that 2 real analytic hypersurfaces of $\mathbb{C}^2$ that are CR-formally equivalent are also CR-smoothly equivalent. It has recently been found that one may not expect CR-analytic equivalence (Kossovskiy-Shafikov). **Our proof is really unexpected.** Indeed, it rests on *multisummability theory* of formal solution of holomorphic system of ODEs at an irregular singular point (Ramis, Malgrange,...), that is the existence of unique holomorphic solutions in some sectors at the singular point ! **To the best of our knowledge, it is the first time that such method of complex analysis is used in order to solve a problem of geometry.**

•"Equivalence of neighborhoods of embedded compact complex manifolds and higher codimension foliations".(with X. Gong), Publi 6

Let's consider 2 neighborhoods of a complex compact manifolds C embedded into 2 complex manifolds. Assume that these neighborhoods are formally equivalent. Are they automatically holomorphically equivalent ? This is known as Grauert's *Formale Prinzip*. The answer is very related to the curvature of the normal bundle of C in the manifold as it is true in the case of a *negative* normal bundle (Grauert/Hironaka-Rossi). As shown by V.I. Arnold the 70's, the situation is quite different when considering a torus embedded into a surface with a zero self-intersection number. In our work we give a **sufficient condition ensuring that a germ of neighborhood of C of a manifold which is formally equivalent to a neighborhood of the zeroth section of its normal bundle is actually holomorphically equivalent to it.** We also give conditions ensuring that **there exists a holomorphic foliation in a neighborhood of C having C as a leaf.** This extends Ueda's result obtain in the case of a complex curve embedded into a surface as **we do not impose any restriction neither on the dimension nor on the codimension of the C.** These problems can both be formulated as a kind of "linearization problem" for which we need to define appropriate *resonances* and *small divisors*. This article used technics developed in our article [LSt10]. It was the starting point of several other works [GSt25],[GSt24],[StW24],[KSt25].

2. Dynamical Systems

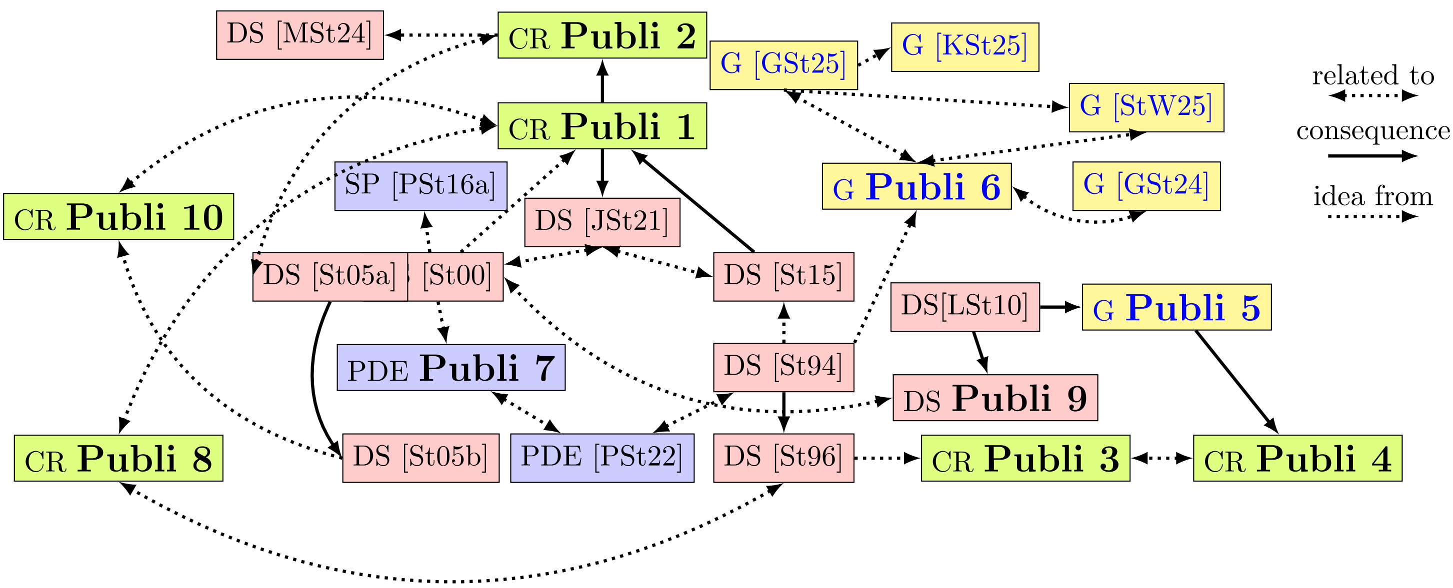
•"Local rigidity of actions of isometries on compact real Riemannian manifolds". (with Z. Zhao) Publi 9

In this article, we consider smooth (resp. analytic) perturbations of isometries of an smooth (resp. analytic) Riemannian manifold M. We prove that, under some conditions, a finitely presented group of such small enough perturbations is smoothly (resp. analytically) conjugate on M to the same group of isometry it is a perturbation of. Our result relies on a "Diophantine-like" condition, relating the actions of the isometry group and the eigenvalues of the Laplace-Beltrami operator. Our results are a **wide generalization Arnold-Herman's theorem about diffeomorphisms of the circle** that are small perturbations of rotations. A version devoted to the analytic case only, with different proofs and other results as well, is contained in [StZ24]. The techniques we developed from the crash should be very crucial in some other related problems such as cocycles over such dynamics

3. PDE's through Dynamical Systems

•"Convergence to normal forms of integrable PDEs".(with D. Bambusi), Publi 7

We consider the problem of transformation to a normal form of an infinite abelian family of (germs of) analytic vector fields at a common fixed point in some (infinite dimensional) Hilbert space. We give a sufficient condition that ensures that such a family can be analytically transformed into a normal form. We apply our result to the normal form problem of completely integrable PDE's such as KdV, NLS (NonLinear Schrödinger) or Toda.**This allows to obtain** *long term behavior* **of solutions that are not otherwise accessible throught classical PDEs analysis.** This work can be seen, to some extend, as an infinite dimensional version of our former work..



The above graph illustrates some of our previous synergetic works (DS=Dynamical systems; CR= Cauchy-Riemann geometry; G=Geometry; SP= Spectral theory)