Normal form for the non–linear Schrödinger equation

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Nonlinear Schrödinger equation

Consider the Nonlinear Schrödinger equation on the torus $\mathbb{T}^n$.

$$iu_t - \Delta u = F(|u|^2)u \tag{1}$$

where $u := u(t, \varphi)$, $\varphi \in \mathbb{T}^n$,
$F(y)$ is an analytic function, $F(0) = 0$
no dependence on φ.
A good model is

$$iu_t - \Delta u = F(|u|^2)u = |u|^{2q}u \tag{2}$$

with $q \in \mathbb{N}$. I will concentrate on the CUBIC CASE namely $q = 1$

$$iu_t - \Delta u = |u|^2 u$$

The case of the cubic NLS in dimension one is integrable but
- the non-cubic NLS in dim $n \geq 1$
-cubic NLS in dim $n > 1$
should exhibit a very rich dynamics.

Marcel Guardia's lectures: growth of Sobolev norms.

Grebert-Thomann: quintic NLS in dimension 1.

Geng-You-Xu: cubic NLS in dimension $n = 2$ existence of quasi-periodic solutions

Wei Min Wang: analytic NLS existence of quasi-periodic solutions

Procesi-Procesi: cubic NLS existence and stability of quasi-periodic solutions.

Linear equations

This system is highly resonant in $u = 0$. Namely if you linearize the equation at the elliptic fixed point u=0

$$iu_t - \Delta u = F(|u|^2)u0$$

all the bounded solutions are periodic!

$$u(t,\varphi) = \sum_{k \in \mathbb{Z}^n} u_k e^{\mathrm{i}(|k|^2 t + k \cdot \varphi)}$$

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One expects the non-linear dynamics to be much richer!

In particular our aim when we started analizing this problem was to prove existence of quasi-periodic solutions with frequencies close to those of the linear NLS. We wanted to apply a KAM algorithm
It is a standard procedure when applying KAM theory to resonant systems to first apply
one step of Birkhoff Normal Form.
The study of the NLS equation after this step will be the main topic of this lectures!
It is essentially a geometric problem.

Quasi-periodic solutions

Final goal, with C. Procesi

For all $m \in \mathbb{N}$, there exist Cantor families of small quasi–periodic solutions of Equation (1) with m frequencies $\omega_1, \dots, \omega_m$. We prove the existence of an integrable normal form close to the solution.

m is arbitrarily large but finite

Dynamical systems approach

For simplicity concentrate on the cubic NLS on $\mathbb{T}^n$. Passing to the Fourier representation

$$u(t,\varphi) := \sum_{k\in\mathbb{Z}^n} u_k(t)e^{\mathrm{i}(k,\varphi)} \, ,$$

Eq. (1) becomes

$$\mathrm{i}\dot{u}_k(t) = |k|^2 u_k(t) + \sum_{k_1-k_2+k_3=k} u_{k_1}\bar{u}_{k_2}u_{k_3}$$

infinite dimensional Hamiltonian dynamical system:

$$H = \sum_{k\in\mathbb{Z}^n} |k|^2 u_k^+ u_k^- + \sum_{k_i\in\mathbb{Z}^n : k_1+k_3=k_2+k_4} u_{k_1}^+ u_{k_2}^- u_{k_3}^+ u_{k_4}^- \qquad (3)$$

with respect to the complex symplectic form $i\sum_k du_k^+ \wedge du_k^-$. On the real subspace $u_k^+ = u_k$ and $u_k^- = \bar{u}_k$.

If one prefers real notation $u_k = p_k + iq_k$:

$$\sum_{k\in\mathbb{Z}^n} |k|^2(q_k^2 + p_k^2) + o(|p|^2 + |q|^2)$$

a list of harmonic oscillators coupled by a non–linearity usually one writes compactly

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Ignore the order 4 term:

$$|u_k|^2 = \xi_k\,, \qquad u_k(t) = \sqrt{\xi_k}\,e^{\mathrm{i}|k|^2 t}$$

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The system has the constants of motion:

$$L = \sum_{k \in \mathbb{Z}^n} |u_k|^2 , \quad M = \sum_{k \in \mathbb{Z}^n} k |u_k|^2$$

the fact that M is preserved will be crucial to the proof!

Birkhoff Normal Form

$$H = K(p,q) + H^{(4)}(p,q)\,, \quad K(p,q) = \sum_k \lambda_k (p_k^2 + q_k^2)$$

where $H^{(4)}$ is a polynomial of degree 4 and the λ_k (in our case $|k|^2$) are all rational.
With a sympletic change of variables we reduce the Hamiltonian H to

$$H_{Birk} = K(p,q) + H^{(4)}_{res}(p,q) + H^{(6)}$$

where $H^{(6)}$ is small while $H^{(4)}_{res}$ Poisson commutes with K.

Study the dynamics of

$$H_{res} = K(p,q) + H^{(4)}_{res}(p,q),$$

then treat $H^{(6)}$ as a small perturbation.

It turns out that the dynamics of H_{res} is very rich and complex

- Step 1: Prove the existence of invariant subspaces on which H_{res} is integrable (and with a simple and explicit dynamics).
- Step 2: Given a solution from step one, linearize H_{res} at this solution. One gets a quadratic Hamiltonian which we call normal form N.
- Step 3: The structure of N depends on the choice of the invariant subspace. Look for invariant subspaces on which N is as simple as possible.

Functional setting

work on the phase space

$$\ell^{(a,p)} := \{u^{\pm} = \{u_k^{\pm}\}_{k\in\mathbb{Z}^n} \mid |u_0^{\pm}|^2 + \sum_{k\in\mathbb{Z}^n} |u_k^{\pm}|^2 e^{2a|k|}|k|^{2p} := \|u^{\pm}\|_{a,p}^2 < \infty\} \tag{4}$$

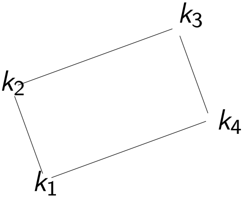
$$a > 0, \; p > n/2 \,, \quad u_k^+ = \bar{u}_k^-$$

$$H = \sum_{k\in\mathbb{Z}^n} |k|^2 u_k^+ u_k^- + \sum_{k_i\in\mathbb{Z}^n : k_1+k_3=k_2+k_4} u_{k_1}^+ u_{k_2}^- u_{k_3}^+ u_{k_4}^- \tag{5}$$

The Hamiltonian is analytic
the vector field generated by $H^{(4)}$ is analytic from $B_R \to \ell^{(a,p)}$.

One step of Birkhoff normal form produces

$$H_{Birk} = \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar u_k + \sum_{\substack{k_i\in\mathbb{Z}^n : k_1+k_3=k_2+k_4 \\ |k_1|^2+|k_3|^2=|k_2|^2+|k_4|^2}} u_{k_1}\bar u_{k_2} u_{k_3}\bar u_{k_4} + H^{(6)} \quad (6)$$



Even if we ignore the term $H^{(6)}$, this equation is still very complicated Note that H_{res} has as constants of motion

$$L = \sum_k |u_k|^2, \quad M = \sum_k k|u_k|^2, \quad K = \sum_k |k|^2|u_k|^2$$

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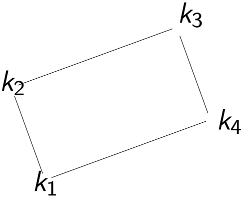

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The simplest rectangles are the degenerate ones $k_1 = k_2 = k_3 = k_4$ and $k_1 = k_2 \neq k_3 = k_4$. This gives a contribution:

$$H_{Res} = \sum_{k \in \mathbb{Z}^n} |k|^2 |u_k|^2 + 2 \sum_{k \neq k} |u_k|^2 |u_h|^2 + \sum_k |u_k|^4 +$$

$$\sum_{\substack{k_i \in \mathbb{Z}^n : k_1 + k_3 = k_2 + k_4 \\ |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2}}^{non-deg} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4}$$

Note that $L = \sum_k |u_k|^2$ is constant so we may subtract $2L^2$:

$$H_{Res} = \sum_{k \in \mathbb{Z}^n} |k|^2 |u_k|^2 - \sum_k |u_k|^4 + \sum_{\substack{k_i \in \mathbb{Z}^n : k_1 + k_3 = k_2 + k_4 \\ |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2}}^{non-deg} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4}$$

In dimension $n = 1$ there are no non-degenerate rectangles.

$$H_{Res} = \sum_{k \in \mathbb{Z}} |k|^2 |u_k|^2 - \sum_k |u_k|^4$$

$$|u_k|^2 = \xi_k \,, \quad u_k = \sqrt{\xi_k} e^{\mathrm{i}(|k|^2 - 2\xi_k)t}$$

Step1: Choose any finite set $S = \{v_1, \dots, v_m\} \subset \mathbb{Z}$ then

$$U_S := \{u_k = 0 \,, \quad \forall\, k \in \mathbb{Z} \setminus S := S^c\}$$

is an invariant subspace for H_{res}. Solution:
$u_k = \sqrt{\xi_k} e^{\mathrm{i}(|k|^2 - 2\xi_k)t}$ for $k \in S$ and $u_k = 0$ for $k \in S^c$.
We have parameter families of m-tori

$$H_{Res} = \sum_{k\in\mathbb{Z}} |k|^2 |u_k|^2 - \sum_k |u_k|^4$$

Step 2: Consider the solution $u_k = \sqrt{\xi_k} e^{\mathrm{i}(|k|^2 - 2\xi_k)t}$ for $k \in S$ and $u_k = 0$ for $k \in S^c$.

parameter families of invariant tori $\mathbb{T}^m$ Linearize the equation (actually I work on the hamiltonian)

$u_k = \sqrt{\xi_k + y_k} e^{\mathrm{i}x_k}$ for $k \in S$ and $u_k = z_k$ for $k \in S^c$.

$$N = (\omega, y) + \sum_k k^2 |z_k|^2 , \quad \omega_i = |v_i|^2 - 2\xi_{v_i} , \quad S = (v_1, v_2, \dots, v_m)$$

Step 3: N is already as simple as possible! (Kuksin-Pöschel: existence and stability of quasi-periodic solutions)

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Lemma

Given a set $S = \{v_1, \dots, v_m\} \subset \mathbb{Z}^n$ consider the subspace

$$U_S := \{u = \{u_k\}_{k\in\mathbb{Z}^n} : \quad u_k = 0\,,\ \mathrm{if\ k \notin S}\}$$

For generic choices of S the space U_S is invariant for the dynamics of

$$H_{Res} = \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k - \sum_{k\in\mathbb{Z}^n} |u_k|^4 + \sum_{\substack{k_i\in\mathbb{Z}^n : k_1+k_3=k_2+k_4 \\ |k_1|^2+|k_3|^2=|k_2|^2+|k_4|^2}}^{non-deg.} u_{k_1}\bar{u}_{k_2} u_{k_3}\bar{u}_{k_4}$$

the Hamiltonian H_{Res} restricted on U_S is

$$\sum_{k\in S} |k|^2 |u_k|^2 - \sum_{k\in S} |u_k|^4$$

cfr. Marcel Guardia's lectures...

Definition (Completeness:)

$S := \{v_1, \dots, v_m\}$ is complete if:
for all triples $v_{j_1}, v_{j_2}, v_{j_3} \in S$ if

$$k = v_{j_1} - v_{j_2} + v_{j_3} \,, \quad |k|^2 = |v_{j_1}|^2 - |v_{j_2}|^2 + |v_{j_3}|^2$$

then $k \in S$

Lemma

If S is complete then U_S is an invariant subspace for H_{res}. Dim

tangential sites

Definition (Complete and Integrable)

$S := \{v_1, \dots, v_m\}$ is complete and integrable if:
for all triples $v_{j_1} \neq v_{j_2} \neq v_{j_3} \in S$ we have

$$(v_{j_1} - v_{j_2}, v_{j_3} - v_{j_2}) \neq 0$$

Show that this set is non-empty!
If S is complete and integrable then there are no non-degenerate rectangles in S so trivially

$$\sum_{k \in S} |k|^2 |u_k|^2 - \sum_{k \in S} |u_k|^4$$

tangential sites

Choice of the tangential sites Let us now partition

$$\mathbb{Z}^n = S \cup S^c, \quad S := (v_1, \dots, v_m).$$

where:

$S = \{v_1, \dots, v_m\}$ is a finite set

with some constraints.
Start form: S complete and integrable.

The set S is called the *tangential sites* and the set S^c the *normal sites*.

Elliptic/action-angle variables.

Let us now set

$$u_k := z_k \text{ for } k \in S^c\,, \qquad u_{v_i} := \sqrt{\xi_i + y_i}e^{\mathrm{i}x_i} \text{ for } v_i \in S,$$

this puts the tangential sites in action angle variables

$$y := \{y_1, \dots, y_m\}, \quad x := x_1, \dots, x_m$$

the ξ are parameters.

$$\xi \in A_{\varepsilon^2} := \{\xi \, : \, \frac{1}{2}\varepsilon^2 \le \xi_i \le \varepsilon^2\}\,,$$

Let us now set

$$u_k := z_k \text{ for } k \in S^c\,, \qquad u_{v_i} := \sqrt{\xi_i + y_i}\, e^{\mathrm{i} x_i} \text{ for } v_i \in S,$$

For all $r \le \varepsilon/2$ this is a well known analytic and symplectic change of variables in the domain

$$D_{a,p}(s,r) = D(s,r) :=$$

$$\{x, y, z \;:\; |Im(x)| < s\,,\ |y| \le r^2\,,\ \|z\|_{a,p} \le r\} \subset \mathbb{T}^m_s \times \mathbb{C}^m \times \ell^{(a,p)} \times \ell^{(a,p)}.$$

$$||z||^2_{a,p} := |z_0|^2 + \sum_{k \in S^c} |z_k|^2 e^{2a|k|} |k|^{2p}$$

The *normal form Hamiltonian*

substitute

$$u_k := z_k \text{ for } k \in S^c\,, \qquad u_{v_i} := \sqrt{\xi_i + y_i}\, e^{\mathrm{i} x_i} \text{ for } v_i \in S,$$

in

$$H_{res} = \sum_{k \in \mathbb{Z}^n} |k|^2 |u_k|^2 - \sum_{k \in \mathbb{Z}^n} |u_k|^4 + \sum_{\substack{k_i \in \mathbb{Z}^n : k_1 + k_3 = k_2 + k_4 \\ |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2}} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4}$$

We compute the leading term of H_{res}:

$$N := \sum_{1\le i\le m} (|v_i|^2 - 2\xi_i)y_i + \sum_{k\in S^c} |k|^2|z_k|^2 \tag{7}$$

$$+\mathcal{Q}(x,z)$$

set $\omega_i := |v_i|^2 - 2\xi_i.$

Step 3:

We impose some simple constraints:

1. $v_i - v_j \neq 0,\ v_i + v_j \neq 0,\ v_i + v_j - v_l - v_k \neq 0$ and $v_i + v_j - v_l + v_k \neq 0.$
2. $(v_i - v_j, v_k - v_j) \neq 0$.
3. $2(|v_i|^2 - |v_j|^2) + |v_i - v_j|^2 \neq 0$ and $2(|v_i|^2 + |v_j|^2) - |v_i + v_j|^2 \neq 0$.

The *normal form Hamiltonian*

$$\mathcal{Q}(x,z) = 4 \sum^{*}_{\substack{1\le i\neq j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}\, e^{\mathrm{i}(x_i-x_j)} z_h \bar z_k + \tag{8}$$

$$2 \sum^{**}_{\substack{1\le i<j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}\, e^{-\mathrm{i}(x_i+x_j)} z_h z_k + 2 \sum^{**}_{\substack{1\le i<j\le m\\ h,k\in S^c}} \sqrt{\xi_i\xi_j}\, e^{\mathrm{i}(x_i+x_j)} \bar z_h \bar z_k .$$

This is a non-integrable potentially very complicated infinite dimensional quadratic form!

The constraints $\sum^*, \sum^{**}$ mean

that the terms are *resonant with the quadratic part K*, that is:

Definition

- Here $\sum^*$ denotes that (h, k, v_i, v_j) give a rectangle:

$$\{(h, k, v_i, v_j) \,|\, h + v_i = k + v_j,\ \ |h|^2 + |v_i|^2 = |k|^2 + |v_j|^2\}.$$

We say $h \in H_{i,j},\ \ k \in H_{j,i}$.

- $\sum^{**}$ means that (h, v_i, k, v_j) give a rectangle:

$$\{(h, v_i, k, v_j) \,|\, h + k = v_i + v_j,\ \ |h|^2 + |k|^2 = |v_i|^2 + |v_j|^2\}.$$

We say $h, k \in S_{i,j}$

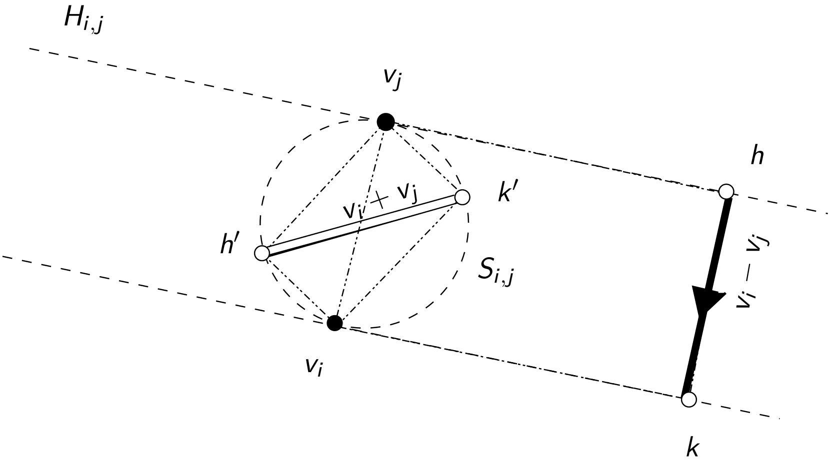

Figure: The plane $H_{i,j}$ and the sphere $S_{i,j}$. The points h, k, v_i, v_j form the vertices of a rectangle. Same for the points h', v_i, k', v_j

Examples: h_1, h_2 belong to one and only one resonant figure:

$$h_2 - h_1 + v_2 - v_1 = 0\,, \quad |h_2|^2 - |h_1|^2 + |v_2|^2 - |v_1|^2 = 0$$

we have the Hamiltonian

$$4\sqrt{\xi_1\xi_2}e^{\mathrm{i}(x_1-x_2)}z_{h_1}\bar{z}_{h_2} + 4\sqrt{\xi_1\xi_2}e^{-\mathrm{i}(x_1-x_2)}\bar{z}_{h_1}z_{h_2}$$

$$h_2 + h_1 - v_2 - v_1 = 0\,, \quad |h_2|^2 + |h_1|^2 - |v_2|^2 - |v_1|^2 = 0$$

we have the Hamiltonian

$$4\sqrt{\xi_1\xi_2}e^{\mathrm{i}(x_1+x_2)}z_{h_1}z_{h_2} + 4\sqrt{\xi_1\xi_2}e^{-\mathrm{i}(x_1+x_2)}\bar{z}_{h_1}\bar{z}_{h_2}$$

a more complicated block:

$$A_{k_1} = \begin{tikzcd} & k_4 \arrow[d, equal, "v_2+v_1"] & \\ k_1 \arrow[r, "v_1-v_3"] & k_2 \arrow[r, "v_2-v_3"] & k_3 \end{tikzcd} , \tag{9}$$

this means that

$$\begin{cases} k_2 - k_1 + v_3 - v_1 = 0 \\ |k_2|^2 - |k_1|^2 + |v_3|^2 - |v_1|^2 = 0 \\ k_3 - k_2 + v_3 - v_2 = 0 \\ |k_3|^2 - |k_2|^2 + |v_3|^2 - |v_2|^2 = 0 \\ k_4 + k_2 - v_2 - v_1 = 0 \\ |k_4|^2 + |k_2|^2 - |v_2|^2 - |v_1|^2 = 0 \end{cases}$$

we have the Hamiltonian:

$$4\sqrt{\xi_1\xi_3}e^{\mathrm{i}(x_1-x_3)}z_{k_1}\bar{z}_{k_2}+4\sqrt{\xi_2\xi_3}e^{\mathrm{i}(x_2-x_3)}z_{k_2}\bar{z}_{k_3}+4\sqrt{\xi_1\xi_2}e^{-\mathrm{i}(x_1+x_2)}z_{k_2}z_{k_4}+ \\ 4\sqrt{\xi_1\xi_3}e^{-\mathrm{i}(x_1-x_3)}\bar{z}_{k_1}z_{k_2}+4\sqrt{\xi_2\xi_3}e^{-\mathrm{i}(x_2-x_3)}\bar{z}_{k_2}z_{k_3}+4\sqrt{\xi_1\xi_2}e^{\mathrm{i}(x_1+x_2)}\bar{z}_{k_2}\bar{z}_{k_4} \tag{10}$$

In dimension $n = 2$ you may strongly simplify by applying arithmetic conditions.

Proposition (Geng-You-Xu, for the cubic NLS)

For $n = 2$, and for every m there exist infinitely many choices of generic tangential sites $S = \{v_1, \dots, v_m\}$ such that, if A is a connect component of the geometric graph, then A is either a vertex or a single edge.

This is the simplest form for the normal form N!

$$(\omega, y) + \sum_k |k|^2 |z_k|^2 + \sum_{i<j} 4\sqrt{\xi_i \xi_i} e^{\mathrm{i}(x_i - x_j)} z_{h_1} \bar{z}_{h_2} + 4\sqrt{\xi_i \xi_j} e^{-\mathrm{i}(x_i - x_j)} \bar{z}_{h_1} z_{h_2} +$$

$$\sum_{i<j} 4\sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_i + x_j)} z_{k_1} z_{k_2} + 4\sqrt{\xi_i \xi_j} e^{-\mathrm{i}(x_i + x_j)} \bar{z}_{k_1} \bar{z}_{k_2}$$

What to do in dimension $n > 2$?

How big can the resonant blocks be?

What to do in dimension $n > 2$?

How big can the resonant blocks be?

GO ASK DADDY!

A first theorem for generic v_i's.

Theorem

- *For generic v_i's the quadratic Hamiltonian $\mathcal{Q}(x, w)$ is an infinite sum of independent (decoupled) terms each depending on a finite number of variables (at most $n+1$ variables z_j together with their conjugates $\bar{z}_j$).*
- *One can exhibit an explicit symplectic change of variables which integrates N, namely makes all the angles disappear from $\mathcal{Q}(x, w)$.*

Theorem

There exists a map

$$S^c \ni k \to L(k) \in \mathbb{Z}^m \,, \quad |L(k)| < 2n$$

such that the analytic symplectic change of variables:

$$z_k = e^{-\mathrm{i}L(k).x} z_k', \;\; y = y' + \sum_{k \in S^c} L(k)|z_k'|^2, \;\; x = x'.$$

$$\Phi : (y', x) \times (z', \bar{z}') \to (y, x) \times (z, \bar{z})$$

from $D(s, r) \to D(s, r/2)$ *such that*

$$N \circ \Phi = (\omega(\xi), y') + \sum_{k \in S^c} (|k|^2 + \sum_i L_i(k)|v_i|^2)|z_k'|^2 + \tilde{\mathcal{Q}}(w') \,, \quad (11)$$

Genericity condition: Resonance polynomials

Definition

Given a list $\mathcal{R} := \{P_1(y), \ldots, P_N(y)\}$ of non–zero polynomials in k vector variables y_i, we say that a list of vectors $S = \{v_1, \ldots, v_m\}, v_i \in \mathbb{C}^n$ is GENERIC relative to $\mathcal{R}$ if, for any list $A = \{u_1, \ldots, u_k\}$ such that $u_i \in S$, $\forall i$, the evaluation of the resonance polynomials at $y_i = u_i$ is non–zero.

If m is finite this condition is equivalent to requiring that S (considered as a point in $\mathbb{C}^{nm}$) does not belong to the algebraic variety where at least one of the resonance polynomials is zero.

Generiticity condition

In our specific case the required list of *the resonances,* $P_1(y), \dots, P_N(y)$, are non–zero polynomials with integer coefficients depending on $k = 2n$ vector variables $y = (y_1, \dots, y_{2n})$ with $y_i = (y_i^1, \dots, y_i^n)$.

The explicit list of these resonances is generated so to exclude a finite list of undesired graphs. Thus it depends on some non trivial combinatorics of graphs. Example:

$$P_1(y_1, y_2, y_3) = (y_1 - y_2, y_1 - y_3)$$

means that we require

$$(v_i - v_j, v_i - v_k) \neq 0$$

for all $i \neq j \neq k$

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Some remarks

There is no a-priori reason why this change of variables should exist. If one does not impose *good* genericity conditions then this is false.

This change of variables exists for all analytic NLS

$$iu_t - \Delta u = F(|u|^2)u$$

provided that F does not explicitly depends on φ.

We can proceed in the same way also when S is an infinite set.

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The final Theorem and goal for the normal form

We will show that

Theorem

- for generic values of the parameters (outside some algebraic hyper surface) we can find a further symplectic change of coordinates so that
- N is *diagonal* (possibly with some complex terms)
- N is *non degenerate* namely it has distinct eigenvalues..
- there exists a positive measure region of the parameters ξ in which N is elliptic (all real eigenvalues).

$$H_{\mathrm{fin}} = (\omega(\xi), y) + \sum_{k\in S^c} \Omega_k |z_k|^2 + P(\xi, x, y, z, \bar z) \tag{12}$$

$$\omega_i = |v_i|^2 - 2\xi_i$$

$$\Omega_k = |k|^2 + \sum_i L_i(k)|v_i|^2 + \theta_k(\xi)\,, \quad \forall k \in S^c$$

The $L_i(k)$ are integers

$$\theta_k(\xi) \in \{\theta^{(1)}(\xi), \dots, \theta^{(K)}(\xi)\}\,, \quad K := K(n, m), \tag{13}$$

list different analytic homoeogeneous functions of ξ.

Why restrict to the cubic NLS?

In the cubic NLS we are able to prove NON–DEGENERACY conditions:

$$(\omega(\xi),\nu)=0\,,\quad (\omega(\xi),\nu)+\Omega_k(\xi)=0$$

$$(\omega(\xi),\nu)+\Omega_k(\xi)-\Omega_h(\xi)=0,$$

holds true on a proper algebraic hypersurface for all non–trivial choices of $\nu\in\mathbb{Z}^m$ $h,k\in S^c$ (recall that $\mathbb{Z}^n=S\cup S^c$) compatible with momentum conservation. Recall $\omega_i=|v_i|^2-2\xi_i$

$$\Omega_k=|k|^2+\sum_i L_k^{(i)}|v_i|^2+\theta_k(\xi)$$

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$$H = \sum_{k\in\mathbb{Z}^n} |k|^2|u_k|^2 + \sum_{k_i\in\mathbb{Z}^n: k_1+k_3=k_2+k_4} u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4}$$

Do 1 step of Birkhoff normal form:

$$H_{res} = \sum_{k\in\mathbb{Z}^n} |k|^2|u_k|^2 - \sum_{k\in\mathbb{Z}^n} |u_k|^4 + \sum^{Non-deg.}_{\substack{k_i\in\mathbb{Z}^n: k_1+k_3=k_2+k_4\\ |k_1|^2+|k_3|^2=|k_2|^2+|k_4|^2}} u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4} + H^{(6)}$$

Choose a set $S = \{v_1, v_2, \dots, v_m\}$ so that $(v_i - v_j, v_l - v_j) \neq 0$ for all triples $(v_i \neq v_j \neq v_l)$.

Divide $Z^n = S \cup S^c$ and substitute on the board

ANSATZ: the value of $|u_k|^2$ with $k \in S^c$ is much smaller than $|u_v|^2$ with $v \in S$

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Divide $Z^n = S \cup S^c$ and substitute on the board

ANSATZ: the value of $|u_k|^2$ with $k \in S^c$ is much smaller than $|u_v|^2$ with $v \in S$

Show: dynamics on the invariant subspace

Substitute

$$u_k := z_k \text{ for } k \in S^c\,, \qquad u_{v_i} := \sqrt{\xi_i + y_i}e^{\mathrm{i}x_i} \text{ for } v_i \in S,$$

with $|u_{v_i}|^2 \sim \xi_i > 0$, $|y_i| \ll \xi_i$ and $|z_k|^2 \ll |\xi|$

$$H = N + \text{higher order terms}$$

$$N = \sum_{1 \le i \le m} (|v_i|^2 - 2\xi_i) y_i + \sum_{k \in S^c} |k|^2 |z_k|^2 + \mathcal{Q}(\xi; x, z)$$

Write the Hamilton equations

Recall the goal

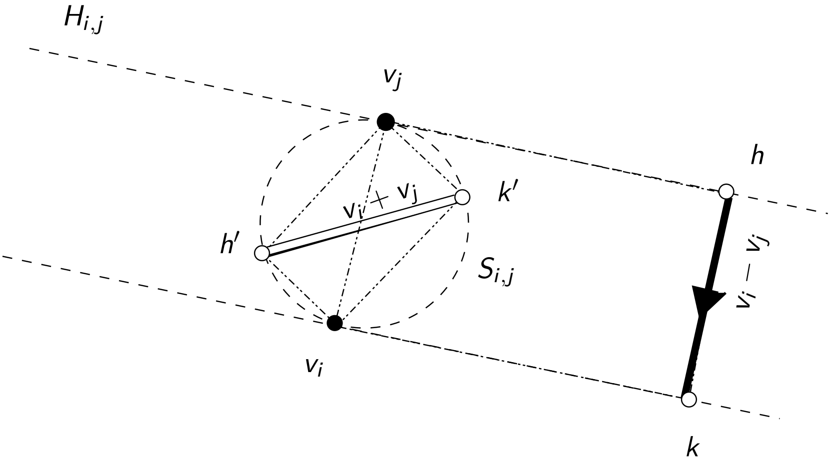
$$N = \sum_{1\le i\le m} (|v_i|^2 - 2\xi_i) y_i + \sum_{k\in S^c} |k|^2 |z_k|^2 +$$

$$4 \sum^{*}_{\substack{1\le i<j\le m\\ h,k\in S^c}} \big(\sqrt{\xi_i\xi_j}\, e^{\mathrm{i}(x_i - x_j)} z_h \bar z_k + \sqrt{\xi_i\xi_j}\, e^{-\mathrm{i}(x_i - x_j)} \bar z_h z_k\big) +$$

$$4 \sum^{**}_{\substack{1\le i<j\le m\\ h,k\in S^c}} \big(\sqrt{\xi_i\xi_j}\, e^{-\mathrm{i}(x_i + x_j)} z_h z_k + \sqrt{\xi_i\xi_j}\, e^{\mathrm{i}(x_i + x_j)} \bar z_h \bar z_k\big).$$

explain the building blocks!

Let us go back to our Hamiltonian and study the blocks of the graph explain the graph!



Example block:

$$A_{k_1} = \begin{tikzcd} & k_4 \arrow[d, equal, "v_2+v_1"] & \\ k_1 \arrow[r, "v_1-v_3"] & k_2 \arrow[r, "v_2-v_3"] & k_3 \end{tikzcd}, \tag{14}$$

this means that

$$\begin{cases} k_2 - k_1 + v_3 - v_1 = 0 \\ |k_2|^2 - |k_1|^2 + |v_3|^2 - |v_1|^2 = 0 \\ k_3 - k_2 + v_3 - v_2 = 0 \\ |k_3|^2 - |k_2|^2 + |v_3|^2 - |v_2|^2 = 0 \\ k_4 + k_2 - v_2 - v_1 = 0 \\ |k_4|^2 + |k_2|^2 - |v_2|^2 - |v_1|^2 = 0 \end{cases}$$

we have the Hamiltonian:

$$4\sqrt{\xi_1\xi_3}e^{\mathrm{i}(x_1-x_3)}z_{k_1}\bar{z}_{k_2}+4\sqrt{\xi_2\xi_3}e^{\mathrm{i}(x_2-x_3)}z_{k_2}\bar{z}_{k_3}+4\sqrt{\xi_1\xi_2}e^{-\mathrm{i}(x_1+x_2)}z_{k_2}z_{k_4}+$$
$$4\sqrt{\xi_1\xi_3}e^{-\mathrm{i}(x_1-x_3)}\bar{z}_{k_1}z_{k_2}+4\sqrt{\xi_2\xi_3}e^{-\mathrm{i}(x_2-x_3)}\bar{z}_{k_2}z_{k_3}+4\sqrt{\xi_1\xi_2}e^{\mathrm{i}(x_1+x_2)}\bar{z}_{k_2}\bar{z}_{k_4} \tag{15}$$

write the equations in a simpler form

$$\begin{cases} k_2 - k_1 + v_3 - v_1 = 0 \\ |k_2|^2 - |k_1|^2 + |v_3|^2 - |v_1|^2 = 0 \\ k_3 - k_2 + v_3 - v_2 = 0 \\ |k_3|^2 - |k_2|^2 + |v_3|^2 - |v_2|^2 = 0 \\ k_4 + k_2 - v_2 - v_1 = 0 \\ |k_4|^2 + |k_2|^2 - |v_2|^2 - |v_1|^2 = 0 \end{cases}$$

$$\begin{cases} k_2 = k_1 - v_3 + v_1 \\ |k_1 - v_3 + v_1|^2 - |k_1|^2 + |v_3|^2 - |v_1|^2 = 0 \\ k_3 = k_1 + v_1 + v_2 - 2v_3 \\ |k_1 + v_1 + v_2 - 2v_3|^2 - |k_1 - v_3 + v_1|^2 + |v_3|^2 - |v_2|^2 = 0 \\ k_4 = -k_1 + v_3 + v_2 \\ |-k_1 + v_3 + v_2|^2 + |k_1 - v_3 + v_1|^2 - |v_2|^2 - |v_1|^2 = 0 \end{cases}$$

$$\begin{cases} k_2 = k_1 - v_3 + v_1 \\ k_3 = k_1 + v_1 + v_2 - 2v_3 \\ k_4 = -k_1 + v_3 + v_2 \\ |k_1 + v_1 + v_2 - 2v_3|^2 - |k_1 - v_3 + v_1|^2 + |v_3|^2 - |v_2|^2 = 0 \\ |k_1 - v_3 + v_1|^2 - |k_1|^2 + |v_3|^2 - |v_1|^2 = 0 \\ |-k_1 + v_3 + v_2|^2 + |k_1 - v_3 + v_1|^2 - |v_2|^2 - |v_1|^2 = 0 \end{cases}$$

$$\begin{cases} k_2 = k_1 - v_3 + v_1 \\ k_3 = k_1 + v_1 + v_2 - 2v_3 \\ k_4 = -k_1 + v_3 + v_2 \\ 2(k_1, -v_3 + v_1) + |v_3 - v_1|^2 + |v_3|^2 - |v_1|^2 = 0 \\ 2(k_1, v_2 - v_3) + |v_1 + v_2 - 2v_3|^2 - |-v_3 + v_1|^2 + |v_3|^2 - |v_2|^2 = 0 \\ 2|k_1|^2 + 2(k_1, v_1 - 2v_3 - v_2) + |v_3 + v_2|^2 + |v_3 - v_1|^2 - |v_2|^2 - |v_1|^2 = \end{cases}$$

$$\begin{cases} 2(k_1, -v_3 + v_1) + |v_3 - v_1|^2 + |v_3|^2 - |v_1|^2 = 0 \\ 2(k_1, v_2 - v_3) + |v_1 + v_2 - 2v_3|^2 - | - v_3 + v_1|^2 + |v_3|^2 - |v_2|^2 = 0 \\ 2|k_1|^2 + 2(k_1, v_1 - v_2 - 2v_3) + |v_3 + v_2|^2 + |v_1 - v_3|^2 - |v_2|^2 - |v_1|^2 = \end{cases}$$

$$A_{k_1} = \begin{tikzcd} & k_4 & \\ k_1 \arrow[r, "v_1 - v_3"] & k_2 \arrow[u, equal, "v_2 + v_1"'] \arrow[r, "v_2 - v_3"] & k_3 \end{tikzcd} , \qquad (16)$$

corresponds to two linear and one quadratic equation on k_1 the graph appears if this equations have solutions in S^c.

A tree with e edges corresponds to e equations (for $x \in S^c \subset \mathbb{C}^n$):

$$\begin{cases} 2(x, \ell_1(v_1, v_2, \dots, v_m)) = Q_1(v_1, v_2, \dots, v_m) \\ 2(x, \ell_2(v_1, v_2, \dots, v_m)) = Q_2(v_1, v_2, \dots, v_m) \\ \vdots \\ 2(x, \ell_K(v_1, v_2, \dots, v_m)) = Q_K(v_1, v_2, \dots, v_m) \\ 2|x|^2 + 2(x, \ell_{K+1}(v_1, v_2, \dots, v_m)) = Q_{K+1}(v_1, v_2, \dots, v_m) \\ 2|x|^2 + 2(x, \ell_{K+2}(v_1, v_2, \dots, v_m)) = Q_{K+2}(v_1, v_2, \dots, v_m) \\ \vdots \\ 2|x|^2 + 2(x, \ell_e(v_1, v_2, \dots, v_m)) = Q_e(v_1, v_2, \dots, v_m) \end{cases}$$

so if the equations are independent we may have only graphs with at most $n + 1$ vertices.

Unfortunately it is not clear why the equations should be independent!

Suppose we only have linear equations

$$\begin{cases} 2(x, \ell_1(v_1, v_2, \dots, v_m)) = Q_1(v_1, v_2, \dots, v_m) \\ 2(x, \ell_2(v_1, v_2, \dots, v_m)) = Q_2(v_1, v_2, \dots, v_m) \\ \vdots \\ 2(x, \ell_K(v_1, v_2, \dots, v_m)) = Q_K(v_1, v_2, \dots, v_m) \end{cases}$$

If $K < n$ It may be that the equations are resonant because $\sum a_i \ell_i(v_1, \dots, v_m)$ vanishes for some choice of the v_i

choose the v_i's to avoid this resonance Example

$$\begin{cases} 2(x, \ell_1(v_1, v_2, \dots, v_m)) = Q_1(v_1, v_2, \dots, v_m) \\ 2(x, \ell_2(v_1, v_2, \dots, v_m)) = Q_2(v_1, v_2, \dots, v_m) \\ \vdots \\ 2(x, \ell_K(v_1, v_2, \dots, v_m)) = Q_K(v_1, v_2, \dots, v_m) \end{cases}$$

Even when $K > n$ it may be that the equations are resonant because $\sum a_i \ell_i(v_1, \dots, v_m)$ is identically zero.

Then if $\sum a_i Q_1(v_1, v_2, \dots, v_m)$ is not identically zero

choose the v_i's to avoid this resonance Example

$$\begin{cases} 2(x, \ell_1(v_1, v_2, \dots, v_m)) = Q_1(v_1, v_2, \dots, v_m) \\ 2(x, \ell_2(v_1, v_2, \dots, v_m)) = Q_2(v_1, v_2, \dots, v_m) \\ \vdots \\ 2(x, \ell_K(v_1, v_2, \dots, v_m)) = Q_K(v_1, v_2, \dots, v_m) \end{cases}$$

$\sum a_i \ell_i(v_1, \dots, v_m) = 0$, $\sum a_i Q_1(v_1, v_2, \dots, v_m) = 0$ identically

then the resonant equations are unavoidable!

Lemma: the graphs in our construction cannot generate unavoidable resonances made only with vertices of the same colour!

EXPLAIN!

$$\begin{cases} 2(x, \ell_1(v_1, v_2, \dots, v_m)) = Q_1(v_1, v_2, \dots, v_m) \\ 2(x, \ell_2(v_1, v_2, \dots, v_m)) = Q_2(v_1, v_2, \dots, v_m) \\ \vdots \\ 2(x, \ell_K(v_1, v_2, \dots, v_m)) = Q_K(v_1, v_2, \dots, v_m) \end{cases}$$

If $K = n + 1$ then surely the vectors $\ell_1, \dots, \ell_N$ (even if they are formally independent) cannot be independent. If possible: choose the v_i's to avoid this resonance

Then if they produce an unavoidable resonance EXPLAIN!

Lemma: the unique solution is $x = v_i$ (a point in S)

Proposition

Our graphs may have at most $2n+1$ vertices!

$$\begin{cases} 2(x,\ell_1(v_1,v_2,\dots,v_m)) = Q_1(v_1,v_2,\dots,v_m) \\ 2(x,\ell_2(v_1,v_2,\dots,v_m)) = Q_2(v_1,v_2,\dots,v_m) \\ \vdots \\ 2(x,\ell_K(v_1,v_2,\dots,v_m)) = Q_K(v_1,v_2,\dots,v_m) \\ 2|x|^2 + 2(x,\ell_{K+1}(v_1,v_2,\dots,v_m)) = Q_{K+1}(v_1,v_2,\dots,v_m) \\ 2|x|^2 + 2(x,\ell_{K+2}(v_1,v_2,\dots,v_m)) = Q_{K+2}(v_1,v_2,\dots,v_m) \\ \vdots \\ 2|x|^2 + 2(x,\ell_{K+M}(v_1,v_2,\dots,v_m)) = Q_{K+M}(v_1,v_2,\dots,v_m) \end{cases}$$

Lemma1: the linear equations are formally independent
Lemma1: the quadratic equations are formally independent
Lemma2: if either M or K are $\geq n+1$ the only solution is in S

The *normal form Hamiltonian* as matrix

Since $\mathcal{Q}(x, w)$ and N are quadratic Hamiltonians we study their matrix representation:

acting by Poisson bracket on two spaces $V^{0,1}, F^{0,1}$

- $V^{0,1}$ is the space with basis the elements $z_k, \bar z_k, k \in S^c$ (and coefficients trigonometric poly. in x). (compute $(\{\mathcal{Q}, z_k\}, \{\mathcal{Q}, \bar z_k\})$
- $F^{0,1}$ is the subspace of $V^{0,1}$ satisfying conservation of momentum and mass, so it has as basis $e^{i\nu.x} z_k$ with

$$\sum_i \nu_i v_i + k = 0, \ \sum_i \nu_i + 1 = 0, \ k \in S^c$$

and the conjugates $e^{-i\nu.x} \bar z_k$
(compute $(\{\mathcal{Q}, e^{i\nu.x} z_k\}, \{\mathcal{Q}, e^{-i\nu.x} \bar z_k\})$.
Explain

One term

Suppose that $h + v_i = k + v_j$, $|h|^2 + |v_i|^2 = |k|^2 + |v_j|^2$ and $\mathcal{Q}(x, w)$ contains the term

$$4(\sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_i - x_j)} z_h \bar{z}_k + \sqrt{\xi_i \xi_j} e^{-\mathrm{i}(x_i - x_j)} \bar{z}_h z_k)$$

we describe this by two matrices compute:

$$4\mathrm{i} \begin{vmatrix} 0 & \sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_i - x_j)} \\ \sqrt{\xi_i \xi_j} e^{-\mathrm{i}(x_i - x_j)} & 0 \end{vmatrix} , \quad 4\mathrm{i} \begin{vmatrix} 0 & \sqrt{\xi_i \xi_j} \\ \sqrt{\xi_i \xi_j} & 0 \end{vmatrix}$$

respectively in the bases:

$$z_h, z_k \,, \quad \text{resp.} \quad (\nu, +) := e^{\mathrm{i}\nu \cdot x} z_h, \; (\mu, +) := e^{\mathrm{i}\mu \cdot x} z_k \,,$$

$$\sum_i \mu_i v_i + k = 0 \,, \quad \sum_i \nu_i v_i + h = 0 \,, \quad \nu = \mu + e_i - e_j$$

e_i are the basis vectors in $\mathbb{Z}^m$ compute the Poisson bracket.

$$A_{k_1} = \begin{tikzcd} & k_4 & \\ k_1 \arrow[r, "v_1-v_3"] & k_2 \arrow[u, equal, "v_2+v_1"'] \arrow[r, "v_2-v_3"] & k_3 \end{tikzcd} \subset S^c \tag{17}$$

Let now $a = (a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m$ be any vector such that $-\sum a_i v_i = k_1$,
draw a graph in $\mathbb{Z}^m \times \{+, -\}$

$$\mathcal{A}_{(a,+)} = \begin{tikzcd} & (-e_2 - e_3 - a, -) & \\ (a, +) \arrow[r, "e_3-e_1"] & (a + e_3 - e_1, +) \arrow[u, equal, "-e_2-e_1"'] \arrow[r, "e_3-e_2"] & (a - e_1 - e_2 + 2e_3, +) \end{tikzcd} \tag{18}$$

The edges of the graph Λ_S,

We construct a graph Λ_S having as vertices $\mathbb{Z}^m \times \{+,-\})$ we join two basis elements with an edge (with suitable marking) if and only if the corresponding matrix element (of $\{\mathcal{Q},-\}$) is non–zero.

The marking tells us what is the matrix entry.

I impose conditions in order to ensure that the graphs $\mathcal{A}_{(a,+)}$and and A_{k_1} are isomorphic.

$$\begin{tikzcd}
A_{k_1} = & & k_4 \arrow[d, equal, "v_2+v_1"] & \\
& k_1 \arrow[r, "v_1-v_3"] & k_2 \arrow[r, "v_2-v_3"] & k_3
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}_{(a,+)} = & & (-e_2-e_3-a,-) \arrow[d, equal, "-e_2-e_1"] & \\
& (a,+) \arrow[r, "e_3-e_1"] & (a+e_3-e_1,+) \arrow[r, "e_3-e_2"] & (a-e_1-e_2+2e_3,+)
\end{tikzcd}$$

Why do I like to work in this setting:

We have a group structure:

$$(a,+)(b,+) = (a+b,+)\,, \quad (a,-)(b,+) = (a-b,-)\,,$$
$$(a,+)(b,-) = (a+b,-)\,, \quad (a,-)(b,-) = (a-b,+).$$

We use this to traslate to $(0,+)$ our graphs:

$$\mathcal{A}_{(a,+)} = \begin{tikzcd} & (-e_2-e_3-a,-) \arrow[d, dash, "-e_2-e_1"] & \\ (a,+) \arrow[r, "e_3-e_1"] & (a+e_3-e_1,+) \arrow[r, "e_3-e_2"] & (a-e_1-e_2+2e_3,+) \end{tikzcd}$$

goes to

$$\mathcal{A} = \begin{tikzcd} & (-e_2-e_3,-) \arrow[d, dash, "-e_2-e_1"] & \\ (0,+) \arrow[r, "e_3-e_1"] & (e_3-e_1,+) \arrow[r, "e_3-e_2"] & (-e_1-e_2+2e_3,+) \end{tikzcd} \tag{19}$$

By our previous argument the trees

$$A_{k_1} = \begin{tikzcd} & k_4 \arrow[d, equal, "v_2+v_1"] & \\ k_1 \arrow[r, "v_1-v_3"] & k_2 \arrow[r, "v_2-v_3"] & k_3 \end{tikzcd} ,$$

and

$$\mathcal{A} = \begin{tikzcd} & (-e_2-e_3,-) \arrow[d, dash, "-e_2-e_1"] & \\ (0,+) \arrow[r, "e_3-e_1"] & (e_3-e_1,+) \arrow[r, "e_3-e_2"] & (-e_1-e_2+2e_3,+) \end{tikzcd}$$

are isomorphic. To each $k \in S^c$ associate the corresponding vertex $L(k) \in \mathbb{Z}^m$.

Theorem

The map

$$S^c \ni k \to L(k) \in \mathbb{Z}^m\,, \quad |L(k)| < 2n$$

is such that the analytic symplectic change of variables:

$$z_k = e^{-\mathrm{i}L(k).x} z'_k,\ y = y' + \sum_{k \in S^c} L(k)|z'_k|^2,\ x = x'.$$

$$\Phi : (y', x) \times (z', \bar{z}') \to (y, x) \times (z, \bar{z})$$

from $D(s,r) \to D(s, r/2)$ *bring you to*

$$N \circ \Phi = (\omega(\xi), y') + \sum_{k \in S^c} (|k|^2 + \sum_i L_i(k)|v_i|^2)|z'_k|^2 + \tilde{\mathcal{Q}}(w')\,, \quad (20)$$

To each graph

$$\begin{tikzcd}
\mathcal{A} = & & (-e_2-e_3,-) \arrow[d, dash, "-e_2-e_1"] & \\
& (0,+) \arrow[r, "e_3-e_1"] & (e_3-e_1,+) \arrow[r, "e_3-e_2"] & (-e_1-e_2+2e_3,+)
\end{tikzcd}$$

associate the equations:

$$\begin{cases} (x,\sum_i a_i v_i) = \frac{1}{2}(|\sum_i a_i v_i|^2 + \sum_i a_i |v_i|^2) \quad \text{for } (a=\sum_i a_i e_i, +) \\ |x|^2 + (x,\sum_i a_i v_i) = -\frac{1}{2}(|\sum_i a_i v_i|^2 + \sum_i a_i |v_i|^2) \quad \text{for } (a,-) \end{cases}. \tag{21}$$

$(a,\pm) \neq (0,+)$ are the vertices of the graph!

Proposition

A combinatorial graph $\mathcal{A}$ contributes to $\mathcal{Q}$ if and only if the equations (21) have solutions in S^c.

Take a graph with only black lines, we have the equations:

$$\begin{cases} (x, \sum_i a_i^{(1)} v_i) = \frac{1}{2}(|\sum_i a_i^{(1)} v_i|^2 + \sum_i a_i^{(1)} |v_i|^2) \\ (x, \sum_i a_i^{(2)} v_i) = \frac{1}{2}(|\sum_i a_i^{(2)} v_i|^2 + \sum_i a_i^{(2)} |v_i|^2) \\ \vdots \\ (x, \sum_i a_i^{(N)} v_i) = \frac{1}{2}(|\sum_i a_i^{(N)} v_i|^2 + \sum_i a_i^{(N)} |v_i|^2) \end{cases}$$

suppose that the left hand side is resonant for all choices of the v_i's (this means that the vertices a of the graph are linearly dependent) then the right hand side cannot cancel identically
we have an avoidable resonance Example
If there are both linear and quadratic equations this is not true!

Compatible and incompatible equations

The main difficulty

is when edges are linearly dependent, we need to exclude certain *resonant graphs* which are always formally compatible but produce no real solutions.

$$\begin{tikzcd}
 & & & e_2-e_3 & \\
-3e_1+e_2 \arrow[r, equal, "-e_1-e_4"] & 2e_1-e_2-e_4 & e_1-e_2 \arrow[l, "e_1-e_4"'] & 0 \arrow[l, "e_1-e_2"'] \arrow[u, "e_2-e_3"'] \arrow[r, equal, "-e_2-e_3"] & -e_2-e_3
\end{tikzcd}.$$

Compatible and incompatible equations

One of the main steps is thus to show that only special combinatorial graphs produce equations which have real solutions for generic values of the tangential sites v_i.

Theorem (Main)

Only graphs which have at most n linearly independent edges appear.

The proof is quite complex

it requires some algebraic geometry and a long combinatorial analysis.

The reduced Hamiltonian

:

$$N = (\omega, y) + \sum_{k \in S^c} (|k|^2 + \sum_{i=1}^{m} L_i(k)|v_i|^2)|z_k|^2 + \sum_{A \in \Gamma_S} \mathcal{Q}_A(\xi, z, \bar{z})$$

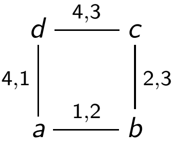
Represent the action as a block diagonal matrix with blocks described by the connected graphs.

one block

$$\begin{vmatrix} \alpha I & 0 \\ 0 & -\alpha I \end{vmatrix} + \begin{vmatrix} Q_A(\xi) & 0 \\ 0 & -Q_A(\xi) \end{vmatrix}$$

to diagonalize I need to compute the eigenvalues

The combinatorial matrices: example



$$\begin{vmatrix} 0 & -2\sqrt{\xi_2\xi_1} & 0 & -2\sqrt{\xi_1\xi_4} \\ -2\sqrt{\xi_2\xi_1} & \xi_2-\xi_1 & -2\sqrt{\xi_2\xi_3} & 0 \\ 0 & -2\sqrt{\xi_2\xi_3} & -\xi_1+\xi_3 & -2\sqrt{\xi_4\xi_3} \\ -2\sqrt{\xi_1\xi_4} & 0 & -2\sqrt{\xi_4\xi_3} & \xi_4-\xi_1 \end{vmatrix}.$$

With characteristic polynomial

$$\begin{gathered}-4\,\xi_1^3\,\xi_2+4\,\xi_1^2\,\xi_2\,\xi_3-4\,\xi_1^3\,\xi_4+8\,\xi_1^2\,\xi_2\,\xi_4+4\,\xi_1^2\,\xi_3\,\xi_4-8\,\xi_1\,\xi_2\,\xi_3\,\xi_4\\ +(\xi_1^3-9\,\xi_1^2\,\xi_2-\xi_1^2\,\xi_3+\xi_1\,\xi_2\,\xi_3-9\,\xi_1^2\,\xi_4+9\,\xi_1\,\xi_2\,\xi_4+\xi_1\,\xi_3\,\xi_4+7\,\xi_2\,\xi_3\,\xi_4)\,t\\ +(3\,\xi_1^2-6\,\xi_1\,\xi_2-2\,\xi_1\,\xi_3-3\,\xi_2\,\xi_3-6\,\xi_1\,\xi_4+\xi_2\,\xi_4-3\,\xi_3\,\xi_4)\,t^2\\ +(3\,\xi_1-\xi_2-\xi_3-\xi_4)\,t^3+t^4\end{gathered}$$

The combinatorial matrices: example

$$a \overset{(1,2)}{=\!=\!=\!=} b \overset{(i,j)}{\rule{3em}{0.4pt}} c \overset{(h,k)}{\rule{3em}{0.4pt}} d \ ,$$

$$\begin{vmatrix} 0 & 2\sqrt{\xi_2\xi_1} & 0 & 0 \\ -2\sqrt{\xi_2\xi_1} & \xi_2+\xi_1 & -2\sqrt{\xi_i\xi_j} & 0 \\ 0 & -2\sqrt{\xi_i\xi_j} & -\xi_i+\xi_j+\xi_2+\xi_1 & -2\sqrt{\xi_k\xi_h} \\ 0 & 0 & -2\sqrt{\xi_h\xi_h} & \xi_k-\xi_h-\xi_i+\xi_j+\xi_2+\xi_1 \end{vmatrix}$$

Non degeneracy: STEP $\boxed{2}$

Non degeneracy

Non degeneracy

Our main goal is to prove that

Theorem

For generic values of the parameters ξ all eigenvalues of all these infinitely many matrices are distinct.

We can change symplectic coordinates and make the matrix diagonal with distinct eigenvalues (as functions of ξ).

In order to prove this fact we cannot proceed directly but we can prove it as a consequence of a stronger algebraic Theorem.

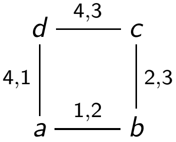
In fact the eigenvalues are solutions of complicated polynomial equations as

$$\text{graph}\quad G := \quad \begin{tikzcd} -e_1 - e_2 \arrow[r, no head, equal] & 0 \arrow[r] & e_1 - e_2 \end{tikzcd}$$

$$\text{matrix}\quad C_G = \begin{pmatrix} -\xi_1 - \xi_2 & 2\sqrt{\xi_1\xi_2} & 0 \\ -2\sqrt{\xi_1\xi_2} & 0 & 2\sqrt{\xi_1\xi_2} \\ 0 & 2\sqrt{\xi_1\xi_2} & \xi_2 - \xi_1 \end{pmatrix}$$

$$\chi_G(t) = \det(tI - C_G) = t^3 + 2\xi_1 t^2 + (\xi_1^2 - \xi_2^2)t - 8\xi_1\xi_2^2 = 0. \quad (22)$$

graph $G :=$



$$-4\,\xi_1^3\,\xi_2+4\,\xi_1^2\,\xi_2\,\xi_3-4\,\xi_1^3\,\xi_4+8\,\xi_1^2\,\xi_2\,\xi_4+4\,\xi_1^2\,\xi_3\,\xi_4-8\,\xi_1\,\xi_2\,\xi_3\,\xi_4$$

$$+(\xi_1^3-9\,\xi_1^2\,\xi_2-\xi_1^2\,\xi_3+\xi_1\,\xi_2\,\xi_3-9\,\xi_1^2\,\xi_4+9\,\xi_1\,\xi_2\,\xi_4+\xi_1\,\xi_3\,\xi_4+7\,\xi_2\,\xi_3\,\xi_4)\,t$$

$$+(3\,\xi_1^2-6\,\xi_1\,\xi_2-2\,\xi_1\,\xi_3-3\,\xi_2\,\xi_3-6\,\xi_1\,\xi_4+\xi_2\,\xi_4-3\,\xi_3\,\xi_4)\,t^2$$

$$+(3\,\xi_1-\xi_2-\xi_3-\xi_4)\,t^3+t^4=0$$

A direct method

one should compute discriminants and resultants, which are polynomials in ξ and show that they are not identically zero.

This can be done by direct computations only for very small cases but in general it is out of question.

So

we prove that all these discriminants and resultants are not identically zero by showing a stronger property of all characteristic polynomials!

MAIN THEOREM of Step $\boxed{2}$

Main Theorem

The characteristic polynomials of the combinatorial matrices are all *irreducible* and *different from each other*.

Implications

1. Eigenvalue separation
2. Validity of the 3 Melnikov conditions.
3. Symplectic coordinates which diagonalize the Hamiltonian.

The proof of this Theorem is very complicated,

the main idea is to apply induction on the size of the graph and on the number of variables ξ_i.

The starting point is the remark that, if we set a variable $\xi_i = 0$ in the matrix associated to a graph G we obtain the matrix of the graph obtained from G by removing all edges where i appears in the marking.

So the characteristic polynomial specializes to the product of characteristic polynomials of the connected components of the obtained graph.

The combinatorial matrices: example

In

$$a \overset{(1,2)}{=\!=\!=\!=} b \overset{(i,j)}{\text{———}} c \overset{(h,k)}{\text{———}} d \,,$$

set $\xi_1 = 0$ get

$$a \qquad\quad b \overset{(i,j)}{\text{———}} c \overset{(h,k)}{\text{———}} d \,,$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \xi_2 & -2\sqrt{\xi_i \xi_j} & 0 \\ 0 & -2\sqrt{\xi_i \xi_j} & -\xi_i + \xi_j + \xi_2 & -2\sqrt{\xi_k \xi_h} \\ 0 & 0 & -2\sqrt{\xi_h \xi_h} & \xi_k - \xi_h - \xi_i + \xi_j + \xi_2 \end{vmatrix}$$

The combinatorial matrices: example

In

$$a \overset{(1,2)}{=\!=\!=\!=} b \overset{(i,j)}{-\!-\!-\!-} c \overset{(h,k)}{-\!-\!-\!-} d \,,$$

set $\xi_i = 0$ get

$$a \overset{(1,2)}{=\!=\!=\!=} b \qquad\quad c \overset{(h,k)}{-\!-\!-\!-} d \,,$$

$$\begin{vmatrix} 0 & 2\sqrt{\xi_2\xi_1} & 0 & 0 \\ -2\sqrt{\xi_2\xi_1} & \xi_2 + \xi_1 & 0 & 0 \\ 0 & 0 & \xi_j + \xi_2 + \xi_1 & -2\sqrt{\xi_k\xi_h} \\ 0 & 0 & -2\sqrt{\xi_h\xi_h} & \xi_k - \xi_h + \xi_j + \xi_2 + \xi_1 \end{vmatrix}$$

Example

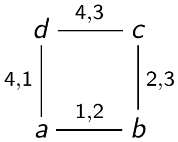
By induction

- in the first case one has that the characteristic polynomial specializes to a product of a linear and a cubic irreducible factor,
- in the second to the product of two irreducible quadratic factors.

This can happen only if we start from an irreducible polynomial.

Example: this argument does not work for

graph $G :=$



whichever variable you set equal zero, you always get a linear and a cubic term.

You need a subtler combinatorial argument. Many special cases thus arise!!

KAM scheme: STEP $\boxed{3}$

KAM scheme

KAM scheme

The new Hamiltonian

$$H_{\rm fin} = (\omega(\xi), y) + \sum_{k \in S^c \setminus \mathcal{H}} \Omega_k |z_k|^2 + P(\xi, x, y, z, \bar z) \tag{23}$$

$$\omega_i = |v_i|^2 - 2\xi_i$$

$$\Omega_k = |k|^2 - \sum_i L_i(k)|v_i|^2 + \theta_k(\xi)\,, \quad \forall k \in S^c$$

The $L_i(k)$ are integers ($|L(k)| < 2n$) and

$$\theta_k(\xi) \in \{\theta^{(1)}(\xi), \dots, \theta^{(K)}(\xi)\}\,, \quad K := K(n,m), \tag{24}$$

list different analytic homoeogeneous functions of ξ. ($\theta_k(\xi)$ are the eigenvalues of the matrices $C_{\mathcal{A}}$ which are a finite list)

The Perturbation P is small in some appropriate norm

Melnikov conditions

$$(\omega(\xi), \nu) = 0\,, \quad (\omega(\xi), \nu) + \Omega_k(\xi) = 0\,, \tag{25}$$
$$(\omega(\xi), \nu) + \Omega_k(\xi) + \sigma\Omega_h(\xi) = 0$$

Theorem

1. *The set of parameter values ξ for which the three Melnikov resonances in (25) occur has zero measure (and for each condition it is algebraic).*

KAM scheme.

1. A *regularity/smallness condition* on the perturbation P namely that $\|X_P\| \ll r^\alpha$.
2. A *regularity condition*, namely $\omega(\xi)$ must be a diffeomorphism and $\Omega_k(\xi) - |k|^2$ must be a bounded Lipschitz function.
3. A *non–degeneracy condition*, that is the three Melnikov resonances 25 hold in a set of measure 0.
4. A *Quasi-Töplitz* condition to control the measure estimates in the second Melnikov condition.