Quelques problèmes de petits diviseurs en mécanique des fluides

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Typical small divisor problem

See the paper by E.Ghys "Resonances et Petits Diviseurs" Héritage de Kolmogorov en Maths, 2007

Reduced model of perturbation theory (celestial mechanics)

$g : (x, y) \in \mathbb{R}/\mathbb{Z}x\mathbb{R}$ (cylinder) $\rightarrow (x + \alpha, y + u(x))$, α irrational

u is C^∞, periodic of period 1, with 0 average.

Q: as $n \rightarrow \infty$, is the sequence $u(x) + u(x + \alpha) + ...u(x + n\alpha)$ bounded ?

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Lemma

$u(x_0) + u(x_0 + \alpha) + ...u(x_0 + n\alpha)$ is bounded as $n \rightarrow \infty$ iff there exits v continuous, periodic of period 1, such that $u(x) = v(x + \alpha) - v(x)$, for all $x \in \mathbb{R}$

Geometry: For $u = 0$ the circles $y = const$ are invariant orbits of g
for $u \neq 0$ the circles $y - v(x) = const$ are invariant orbits of g

Typical small divisor problem - continued

Solve with respect to v periodic (period 1)

$$u(x) = v(x + \alpha) - v(x), \text{ for all } x \in \mathbb{R}, u \text{ known }, \alpha \text{ irrational}$$

Fourier analysis gives a small divisor problem

$$(e^{2i\pi n\alpha} - 1)v_n = u_n, \; n \in \mathbb{Z}$$

Dirichlet: for any α irrational there are infinitely many p/q such that $|\alpha - p/q| < 1/q^2, q > 0$.

Typical small divisor problem - continued

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Dirichlet: for any α irrational there are infinitely many p/q such that $|\alpha - p/q| < 1/q^2, q > 0$.
Diophantine number: $\alpha \notin \mathbb{Q}$ such that there exists $c > 0$ and $r \geq 2$ such that $|\alpha - p/q| > c/q^r$ for any $p/q, q > 0$.
For α diophantine, $|e^{2i\pi n\alpha} - 1| > 4c/|n|^{r-1}$, $|v_n| \leq \frac{|n|^{r-1}}{4c}|u_n|$
Loss of regularity:
if $u \in H^s$, then $v \in H^{s-r+1}$ (linear continuous mapping)

Lyapunov-Schmidt method

$$Lu + R(u,\mu) = 0 \text{ in } \mathcal{X},\ \ u \in \mathcal{Z} \subset_{dense} \mathcal{X} \text{ Hilbert spaces}$$

$$\begin{aligned} R &: \mathcal{Z}x\mathbb{R} \rightarrow \mathcal{X},\ \mathcal{C}^k, R(0,0) = 0, D_u R(0,0) = 0 \\ L &: \mathcal{Z} \rightarrow \mathcal{X} \text{ linear operator,} \end{aligned}$$

0 isolated in the spectrum of L, $ker(L) = E_0 \subset \mathcal{Z}, dimE_0 < \infty$

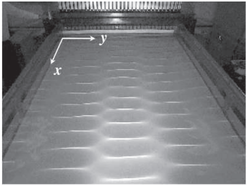
$$\text{pseudo-inverse } \widetilde{L}^{-1} : range(L) = \{kerL^*\}^\perp \rightarrow \{kerL\}^\perp \subset \mathcal{Z}$$

$$u = u_0 + v,\ \ u_0 \in E_0, v \in \{kerL\}^\perp$$

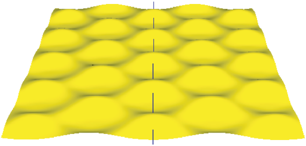
$$v + \widetilde{L}^{-1} QR(u_0 + v, \mu) = 0 \text{ in } \{kerL\}^\perp,\ \ \text{Q:}\perp \text{ proj on } range(L)$$

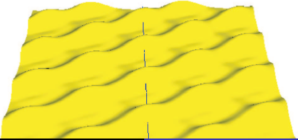
implicit function theorem gives $v = \mathcal{V}(u_0,\mu) = O(|\mu| + |u_0|^2)$
Bifurcation equation in $\{kerL^*\}$: $(\mathbb{I} - Q)R(u_0 + \mathcal{V}(u_0,\mu)) = 0$
Method fails if $\widetilde{L}^{-1}$ unbounded (0 not isolated in spectrum of L)

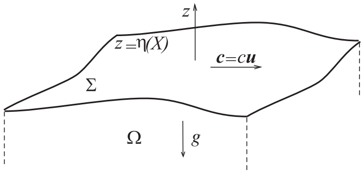
3D-travelling periodic gravity waves



D.Henderson et al JFM 532, 2005

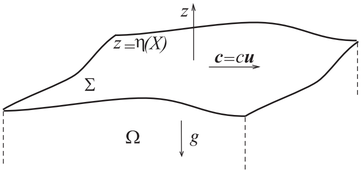






potential flow velocity $\mathbf{U} = (\mathbf{u} + \nabla_X \varphi, \frac{\partial \varphi}{\partial z})$ (moving frame)

$$\Delta\varphi = 0 \quad z < \eta(X), \ \nabla\varphi \to 0 \quad \text{as } z \to -\infty$$



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Boundary conditions on $z = \eta(X)$

$$\nabla \eta \cdot (\mathbf{u} + \nabla_X \varphi) - \frac{\partial \varphi}{\partial z} = 0 \text{ } (\mathbf{U} \text{ orthogonal to the normal of } \Sigma)$$

$$\mathbf{u} \cdot \nabla_X \varphi + \frac{(\nabla \varphi)^2}{2} + \mu \eta = 0 \text{ Bernoulli first integral of Euler equations}$$

$$\mu = gL/c^2$$

Basic solution: (flat free surface) $\varphi = 0, \quad \eta = 0.$

Linearized problem for horizontally periodic waves

$$\Delta\varphi = 0 \quad z < 0,\ \nabla\varphi \to 0 \quad \text{as } z \to -\infty$$
$$\nabla\eta \cdot \mathbf{u} - \frac{\partial\varphi}{\partial z} = 0,\ z = 0,$$
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periodic lattice of wave vectors

$\Gamma = \{K = n_1 K_1 + n_2 K_2; (n_1, n_2) \in \mathbb{Z}^2\}$

$$\eta(X) = \sum_{K\in\Gamma} \eta_K e^{iK\cdot X}, \quad \varphi = \sum_{K\in\Gamma} \varphi_K(z) e^{iK\cdot X}, \ X = (x_1, x_2) \in \mathbb{R}^2$$

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Dispersion relation

$$\mu|K| - (K \cdot \mathbf{u})^2 = 0.$$

$$\mathbf{u} = \mathbf{u}_0 = (1, 0), \ K_1 = (1, \tau_1), \ K_2 = \lambda(1, -\tau_2)$$

$$\mu_c = |K_1|^{-1} = \lambda^2 |K_2|^{-1}$$

Small divisor problem

$$\text{unknown function } U = (\psi, \eta), \quad \psi(X) = \varphi(X, \eta(X)),$$

$$\mathcal{L}_0 U + (\mu - \mu_c)\mathcal{L}_1 U + \mathcal{L}_2(U, \mathbf{u} - \mathbf{u}_0) + \mathcal{N}(U) = 0$$

$$\text{Invariance under horizontal translations } X \quad \mapsto \quad X + \mathbf{h},$$

$$\text{Invariance under symmetry } \ X \quad \mapsto \quad -X$$

$$\mathcal{L}_0 U = \begin{pmatrix} (-\Delta)^{1/2} & -\partial_{x_1} \\ \partial_{x_1} & \mu_c \end{pmatrix} U, \ \text{ker } \mathcal{L}_0 \text{ is 4-dimensional}$$

$$\mathcal{N}(U) : \text{ first order components}$$

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$\mathcal{L}_0$ has an unbounded inverse on $\{\ker \mathcal{L}_0\}^{\perp}$: for Fourier modes, the following factor occuring at the denominator

$$\mu_c|K| - (K \cdot u_0)^2, \; K = n_1 K_1 + n_2 K_2 \in \Gamma \backslash \{\pm K_1, \pm K_2\}$$

may be *very small for large* $|K|$ (no pb with surface tension).

Asymptotic expansion of 3-dim waves

formal Lyapunov-Schmidt method leads to

$$U = (\psi, \eta) = \sum_{p+q \geq 1} \varepsilon_1^p \varepsilon_2^q U_{pq}, \ \psi \text{ odd}, \ \eta \text{ even in } X$$

$$U_{10} = (-\sin K_1 \cdot X, \frac{1}{\mu_c} \cos K_1 \cdot X), \ \ U_{01} = (-\sin K_2 \cdot X, \frac{\lambda}{\mu_c} \cos K_2 \cdot X)$$

$$\mu - \mu_c = \alpha_1 \varepsilon_1^2 + \alpha_2 \varepsilon_2^2 + O(\varepsilon_1^2 + \varepsilon_2^2)^2$$

$$\mathbf{u} - \mathbf{u}_0 = (\omega_1, \omega_2), \ \omega_1 = -\frac{\omega_2^2}{2} + .., \ \omega_2 = \beta_1 \varepsilon_1^2 + \beta_2 \varepsilon_2^2 + O(\varepsilon_1^2 + \varepsilon_2^2)^2$$

α_j, β_j known analytic functions of τ_1 and τ_2

$U_{pq}, p + q > 1$ needs to invert $\mathcal{L}_0$ on $\{\ker \mathcal{L}_0\}^\perp$, for finite Fourier series. For large $p + q$, U_{pq} may be very large.

$$\{\mathcal{T}_\mathbf{v} U = U(. + \mathbf{v}); \mathbf{v} \in \mathbb{R}^2/\Gamma\} \text{ torus family of solutions}$$

Mathematical results on 3-dim travelling waves

First formal studies

R.Fuchs. U.S.Natl. Bur. Stand. Circ. 521 (1952), 187-200.
L.Sretenskii. Dokl. Akad. Nauk SSSR (N.S.) 89 (1953), 25-28.

Existence results (always with surface tension)

J.Reeder,M. Shinbrot. Nonlinear Anal., T.M.A., 5 (1981), 3, 303-323.
W.Craig, D.Nicholls. SIAM J. Math Anal. 32 (2000) 323-359.
M.Groves, A.Mielke. Proc. Roy. Soc. Edin. A 131 (2001), 83-136.
M.Groves, M.Haragus. J.Nonlinear Sci. (2003) 13, 397-447.

Existence result without surface tension

G.I., P.Plotnikov. Memoirs of A.M.S. vol. 200, No 940, 2009 (128p.) (diamond waves)
G.I., P.Plotnikov. Arch. Rat. Mech. Anal. 200, 3, 789-880, 2011 (asymmetrical waves)

Nash-Moser

Adapt the Nash-Moser theorem. Need to invert the differential at all iterated points of the Newton's method

$$-\mathcal{J}^*\left(\frac{1}{\mathfrak{a}}\mathcal{J}(\phi)\right)+\mathcal{G}_\eta(\phi)=h\in H^s_{odd}(\mathbb{R}^2/\Gamma)$$

$$\mathcal{J}=V\cdot\nabla(\cdot)\ (V \text{ depends on } (\psi,\eta))$$

$\mathcal{G}_\eta$: (first order) Dirichlet-Neumann linear operator

For $(\psi,\eta)=0$, and $\mu=\mu_c$

$$\{\mu_c^{-1}(\partial_x)^2+(-\Delta)^{1/2}\}\phi=h$$

$$\text{Fourier symbol: } -\mu_c^{-1}(K\cdot u_0)^2+|K|$$

Idea of Strategy

Find a diffeomorphism of the torus such that main orders of the diff equ. for ϕ have constant coefficients, leading to

$$\mathfrak{L} = \nu\mathcal{D}^2 + (-\Delta)^{1/2}, \ \mathcal{D} =: \partial_{y_1} + \rho\partial_{y_2}$$

where this operator (diagonal on the Fourier basis) would have a controlled inverse.

The new linear operator to invert would look like

$$\mathfrak{L} + \text{perturbation of lower order}$$

It would then be possible to invert

$$(\mathfrak{L} + \text{perturbation})^{-1} = (\mathbb{I} + \mathfrak{L}^{-1}\text{perturbation})^{-1}\mathfrak{L}^{-1}$$

Unfortunately $(\mathfrak{L}^{-1}\text{perturbation})$ is unbounded

Two problems: i) find the good diffeomorphism;

ii) reduce the new operator to the sum of a diagonal operator with a controllable inverse, plus a nicely smoothing operator.

Strategy

1. The diffeomorphism of the torus $Y \in (\mathbb{R}/2\pi\mathbb{Z})^2 \mapsto X(Y)$ allowing to change into constant coefficients the main orders of the diff. equ. for ϕ, satisfies a new equation where two new constants ρ, ν occur (ρ is the rotation number of the velocity vector field on the free surface).
2. Build a formal expansion solution (truncated at order m) of the extended system ($\varepsilon = (\varepsilon_1, \varepsilon_2)$) provided that $\lambda \notin \mathbb{Q}$

$$U_m(Y, \varepsilon), X_m(Y, \varepsilon), \mu_m(\varepsilon), \mathbf{u}_m(\varepsilon), \rho_m(\varepsilon), \nu_m(\varepsilon).$$

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3. Provided that ρ satisfies a diophantine condition, the differential of the extended system reduces to a differential equ. for ϕ with constant main coefficients, with a linear operator of the form

$$\mathfrak{L} + \mathfrak{A}_0 \mathcal{D} + \mathfrak{B}_0 + \mathfrak{L}'_{-1}, \text{ with } \mathfrak{L} = \nu \mathcal{D}^2 + (-\Delta)^{1/2}, \ \mathcal{D} =: \partial_{y_1} + \rho \partial_{y_2}$$

Strategy - continued: Descent method

successive change of variables

starts with $\mathfrak{L} + \mathfrak{A}_0\mathcal{D} + \mathfrak{B}_0 + \mathfrak{L}'_{-1}$

First step leads to

$$\mathfrak{L} + \mathfrak{B}'_0 + \mathfrak{L}''_{-1}$$

Strategy - continued: Descent method

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First step leads to

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Second step introduces a projection Π satisfying

$$\mathfrak{L}^{-1}(\mathbb{I} - \Pi) \text{ and } \mathcal{D}^{-1}\Pi \text{ regularizing operators}$$

and leads to the new operator (triangular form)

$$\Pi(\mathfrak{L} + \mathfrak{V} + \mathfrak{F}_{-1})\Pi + (\mathbb{I} - \Pi)(\mathfrak{L} + \mathfrak{P})$$

with $\mathfrak{V}$ bounded and constant, $\mathfrak{P}$ bounded, $\mathfrak{F}_{-1}$ smoothing.
$\Pi(\mathfrak{L} + \mathfrak{V})^{-1}\Pi$ controllable for suitable (ρ, ν), with the loss of one derivative.

Existence Theorem for 3D periodic travelling waves

Theorem

Choose $l \geq 34$*, m even* ≥ 4, $0 < \delta < 1$*. There is a full measure set* $\mathcal{T} \subset \mathbb{R}^{+2}$ *such that for* $\boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathcal{T}$*, there exists a subset* $\mathcal{E}(\boldsymbol{\tau})$ *of the quadrant* $\{(\varepsilon_1^2, \varepsilon_2^2) \in \mathbb{R}^{+2}\}$ *for which* 0 *is a Lebesgue point, i.e.*

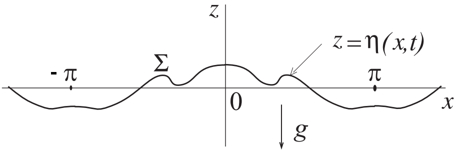
$$(2/\epsilon^2)\mathrm{meas}(\mathcal{E}(\boldsymbol{\tau}) \cap \{\varepsilon_1^2 + \varepsilon_2^2 < \epsilon\}) \to 1 \text{ as } \epsilon \to 0.$$

Moreover, for $\delta < \varepsilon_1/\varepsilon_2 < \delta^{-1}$ *and* $(\varepsilon_1^2, \varepsilon_2^2) \in \mathcal{E}(\boldsymbol{\tau})$*, the nonlinear system has a unique solution* $(U, \mu, \mathbf{u}) \in \mathbb{H}^l_{(S)} \times \mathbb{R} \times \mathbb{S}_1$ *of the form*

$$U = U_{2m} + |\boldsymbol{\varepsilon}|^m \breve{U}(\boldsymbol{\varepsilon}), \;\; \mu = \mu_{2m} + |\boldsymbol{\varepsilon}|^m \breve{\mu}(\boldsymbol{\varepsilon}), \;\; \mathbf{u} = \mathbf{u}_{2m} + |\boldsymbol{\varepsilon}|^m \breve{\mathbf{u}}(\boldsymbol{\varepsilon}),$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$*, and* $(U_{2m}, \mu_{2m}, \mathbf{u}_{2m})$ *is the asymptotic expansion formally computed at order* $|\boldsymbol{\varepsilon}|^{2m}$.

The Standing Wave problem ("clapotis")



Time scale: $T/2\pi$, Length scale: $\lambda/2\pi$, parameter: $1+\mu = gT^2/2\pi\lambda$

Velocity potential $\varphi(x,z,t)$, 2π - periodic in x and t

$$\Delta\varphi = 0 \qquad -\infty < z < \eta(x,t)$$

Boundary conditions on $z = \eta(x,t)$, 2π - periodic in x and t

$\dfrac{\partial\eta}{\partial t} + \dfrac{\partial\varphi}{\partial x}\dfrac{\partial\eta}{\partial x} - \dfrac{\partial\varphi}{\partial z} = 0$ fluid velocity w.r. to free surface is tangent to Σ

$\dfrac{\partial\varphi}{\partial t} + \dfrac{1}{2}\left\{(\dfrac{\partial\varphi}{\partial x})^2 + (\dfrac{\partial\varphi}{\partial z})^2\right\} + (1+\mu)\eta = 0$ (Bernoulli integral of Euler equ.)

Basic solution: (flat free surface) $\eta = 0, \quad \varphi = 0.$

Linearized problem - complete resonance

look for functions φ and h, $2\pi-$ periodic in x and t, solutions of

$$\Delta\varphi = 0 \qquad -\infty < z < 0$$

$$\frac{\partial\eta}{\partial t} - \frac{\partial\varphi}{\partial z} = 0, \quad \frac{\partial\varphi}{\partial t} + (1+\mu)\eta = 0 \ \text{ on } z = 0$$

$$\eta(x,t) = \sum h_p^{(q)} \cos px \cos qt, \ \varphi(x,z,t) = \sum \varphi_p^{(q)} e^{pz} \cos px \sin qt$$

Dispersion relation: (condition for a non trivial solution)

$$(1+\mu)p - q^2 = 0, \ p, q \in \mathbb{N}$$

For $\mu = 0$: infinite-dimensional kernel

"completely resonant system"

$$\text{Kernel} = span\{\cos q^2 x \cos qt; q \in \mathbb{N}\}.$$

Notice: for any rational value of μ the same phenomenon occurs.
Notice: for irrational values of μ, $(1+\mu)p - q^2$ may be very small.

References

S.Poisson 1818: complete solution of the linearized problem (Laplace 1776 went very close)

J.Boussinesq 1877: first nonlinear study (Lagrangian formulation), expansion up to order ε^2

L.Rayleigh 1915: expansion up to order ε^3

Ya.I.Sekerkh-Zenkovich 1947: exp. up to order ε^4

L.W.Schwartz - A.K.Whitney 1981: Eulerian formulation, Conjecture on an algorithm up to ε^∞, (wrong) conjecture on the set of formal solutions

C.Amick-J.Toland 1987: proof of the validity of formal expansion for the unimodal solution

Infinitely many resonances coupled with a small divisor problem

a conformal map transforms the free surface into $z = 0$, the system takes the form of a scalar second order equation in $w(x, t)$ (new form of η)

$$\mathcal{F}(w, \mu) \stackrel{def}{=} \mathcal{L}_0 w - \mu \mathcal{H} w' + \mathcal{N}(w) = 0$$

$\mathcal{L}_0 w \stackrel{def}{=} \ddot{w} - \mathcal{H} w'$, $\mathcal{H}$:Hilbert transform, $(.)^{\cdot}$ and $(.)'$:time and space deriva

$\mathcal{N}$ second order nonlinear terms

$Ker(\mathcal{L}_0)$ is ∞ dim $Range(\mathcal{L}_0)$ is ∞ codim

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Thm (G.I.2002) Let I be a finite or infinite subset of $\mathbb{N}$

$$\eta = \sum_{p \geq 1} \varepsilon^p \eta_p, \ \ \mu = \varepsilon^2/4, \ \ \eta_1 = \sum_{q \in I} \frac{\pm 1}{q^2} \cos q^2 x \cos qt$$

All orders of the expansion satisfy infinitely many compatibility conditions

Existence result

Adapt Nash-Moser theorem - this introduces a small divisor problem when inverting the differential (diffeomorphism of the torus + change of variable and averaging)

$$\partial_t[\dot{v} - \partial_x(av)] + \mathcal{H}\partial_x\{a\mathcal{H}[\dot{v} - \partial_x(av)]\} - \mathcal{H}\partial_x[(1 + \mu - b)v] = h$$

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Thm (G.I.,P.Plotnikov,J.Toland,2003-2005)
Define I a finite set of integers and ε by $\mu = \varepsilon^2/4$, then *there exists a set* $\mathcal{M}_I$ of amplitudes ε, which is *asymptotically of full measure*, where the standing wave exists in a regular function space, with the following asymptotic expansion:

$$\eta = \varepsilon \sum_{q\in I} \frac{\pm 1}{q^2} \cos q^2 x \cos qt + O(\varepsilon^2), \quad \varepsilon \in \mathcal{M}_I$$

$$(1/r) meas\{\mathcal{M}_I \cap [0,r]\} \to 1 \text{ as } r \to 0.$$

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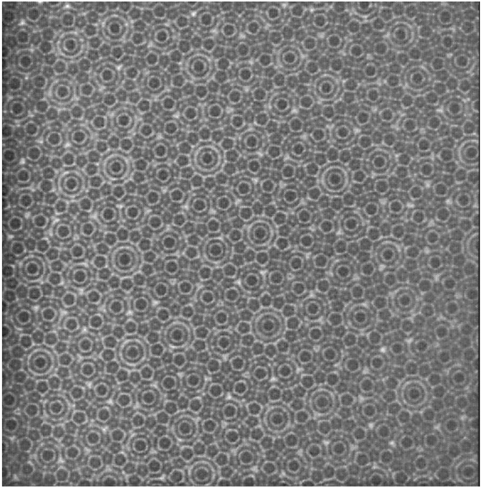
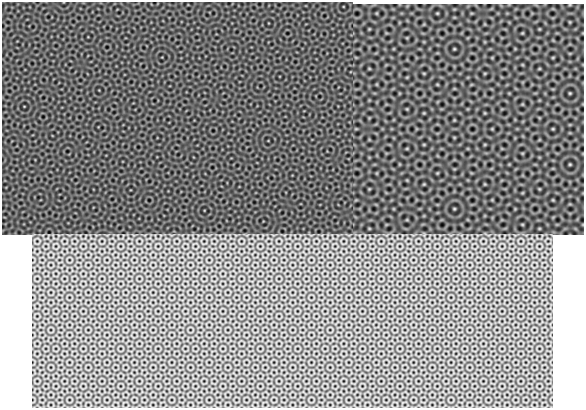
Quasipatterns



Experiment of Faraday type. Kudrolli, Pier, Gollub 1998

Quasipatterns on Swift-Hohenberg PDE



Numerical computation. Rucklidge-Silber 2009
Swift-Hohenberg PDE

Steady Swift-Hohenberg equation in $\mathbb{R}^2$

$$(1+\Delta)^2 U = \mu U - U^3,\ \mathbf{x} \in \mathbb{R}^2 \rightarrow U(\mathbf{x}) \in \mathbb{R}$$

$$e^{i\mathbf{k}.\mathbf{x}} \in Ker\{(1+\Delta)^2 - \mu\}$$

iff Dispersion equation holds:

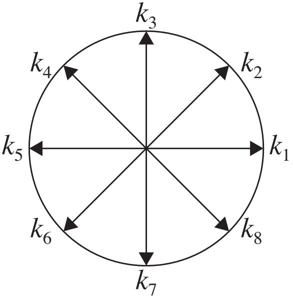
$$(1-|\mathbf{k}|^2)^2 = \mu,\ \mathbf{k} \in \mathbb{R}^2$$

For $\mu = 0$ all wave vectors $\mathbf{k}$ with $|\mathbf{k}| = 1$ are **critical**
We choose to look for solutions quasiperiodic in $\mathbb{R}^2$, invariant under rotations of angle π/q.

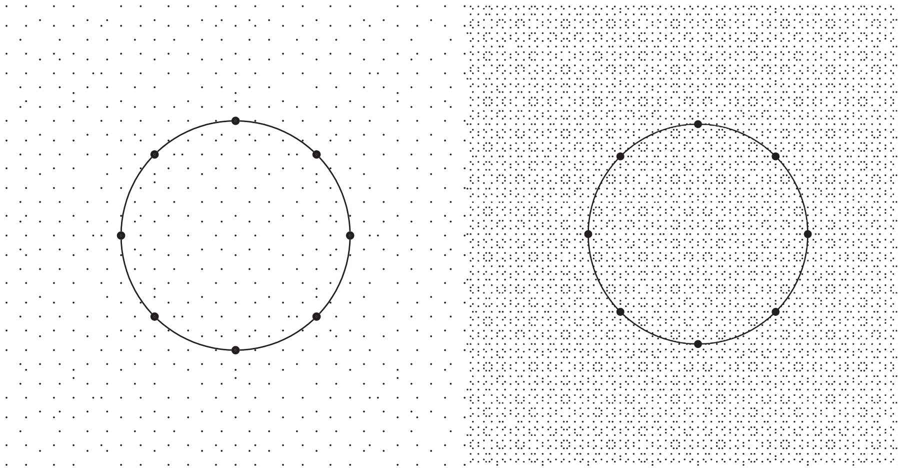
Quasilattices

$$\Gamma = \{\mathbf{k}_m = \sum_{j=1,\dots 2q} m_j \mathbf{k}_j, \quad m \in \mathbb{N}^{2q}, (\mathbf{k}_j, \mathbf{k}_{j+1}) = \pi/q\}$$

For $q = 1, 2, 3$ Γ is a lattice leading to a periodic pattern
For $q \geq 4$ Γ is a quasilattice leading to a quasipattern



Example $q = 4$, the 8 wavevectors which form the basis of the quasilattice



Example with $q = 4$, The truncated quasilattices $\Gamma_9$and Γ_{27}. The small dots mark the combinations of up to 9 or 27 of the 8 basis vectors.

Formal Lyapunov-Schmidt method

$$\mathcal{L}_0 U = \mu U - U^3, \quad \mathcal{L}_0 = (1+\Delta)^2,$$
$$U = \sum_{n\geq 0} \epsilon^{2n+1} U_{2n+1} \text{ invariant under rotations } \mathbf{R}_\pi / q,$$
$$\mu = \sum_{n\geq 1} \epsilon^{2n} \mu_{2n}$$

$$\mathcal{L}_0 U_1 = 0, \; U_1 = \sum_{j=1}^{2q} e^{i\mathbf{k}_j . \mathbf{x}}$$

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$$\mathcal{L}_0 U_3 = \mu_2 U_1 - U_1^3, \; \mu_2 = 3(2q-1) \text{ (compatibility condition)}$$
$$U_3 = \sum_{\mathbf{k}=\mathbf{k}_j+\mathbf{k}_l+\mathbf{k}_r} \alpha_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}}$$

Formal series

Assume U_{2k+1}, μ_{2k} known for $k = 1, ..., n-1$, then U_{2n+1}, μ_{2n} are determined by

$$\mathcal{L}_0 U_{2n+1} = \mu_{2n} U_1 + \sum_{1 \leq k \leq (n-1)} \mu_{2k} U_{2n+1-2k} - \sum_{l+r+s=n-1} U_{2l+1} U_{2r+1} U_{2s+1}$$

Compatibility condition gives μ_{2n}, then we need to invert $\mathcal{L}_0$ in using

$$\mathcal{L}_0^{-1} e^{i\mathbf{k}\cdot\mathbf{x}} = (1 - |\mathbf{k}|^2)^{-2} e^{i\mathbf{k}\cdot\mathbf{x}}, \ \mathbf{k} \neq \mathbf{k}_j, j = 1, ..., 2q$$

Problem: Estimate U_{2n+1}, μ_{2n}

$\Rightarrow$ Small divisor problem

Diophantine estimate

$$N_{\mathbf{k}} = \min\left\{ |\mathbf{m}| = \sum_{j=1,..2q} m_j; \;\; \mathbf{k} = \mathbf{k_m} = \sum_{j=1,...2q} m_j \mathbf{k}_j \right\}$$

$\omega = 2\cos\pi/q,\ \omega$ algebraic integer, $P(\omega) = 0$ of degree $l+1$, coef $\in \mathbb{Z}$

$$\begin{aligned}
|\mathbf{k_m}|^2 - 1 &= p_0 + p_1\omega + ...p_l\omega^l, \; p_j \in \mathbb{Z} \text{ and } |\mathbf{p}| \leq c(q)N_{\mathbf{k}}^2 \\
q &= 4,5,6, \;\; \omega = \sqrt{2}, \; \frac{1+\sqrt{5}}{2}, \; \sqrt{3}, \; \deg(P) = 2 \\
q &= 7, \;\; \omega = 2\cos(\pi/7), \;\; \deg(P) = 3 \\
\deg(P) &= l+1 = \varphi(2q)/2, \; \varphi(2q): \text{ Euler totient function}
\end{aligned}$$

$$|p_0 + p_1\omega + ...p_l\omega^l| \geq C|\mathbf{p}|^{-l}, \;\; \text{for any } p \in \mathbb{Z}^{l+1}\backslash\{0\}$$

$$(|\mathbf{k}|^2 - 1)^2 \geq cN_{\mathbf{k}}^{-4l}, \; \text{if } |\mathbf{k}| \neq 1$$

Spaces of quasi-periodic functions

$$\mathcal{H}_s = \left\{ U = \sum_{\mathbf{k}\in\Gamma} U_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}; ||U||_s^2 = \sum_{\mathbf{k}\in\Gamma} (1+N_{\mathbf{k}}^2)^s |U_{\mathbf{k}}|^2 < \infty \right\}$$

$$\langle W, V\rangle_s = \sum_{\mathbf{k}\in\Gamma} (1+N_{\mathbf{k}}^2)^s W_{\mathbf{k}} \overline{V_{\mathbf{k}}}$$

Lemma

$\mathcal{H}_s$ is a Banach algebra for $s > q/2$:, $||UV||_s \leq c_s ||U||_s ||V||_s$.
For $s > p + q/2$, we have $\mathcal{H}_s \hookrightarrow \mathcal{C}^p$.

Gevrey estimate

Theorem

Let q be ≥ 4, and choose $s > q/2$, then there exists $K(q,c,s)$, $\gamma(q,s)$ such that the uniquely determined power series $U = \sum_{n\geq 0} \epsilon^{2n+1} U_{2n+1}$, $\mu = \sum_{n\geq 1} \epsilon^{2n} \mu_{2n}$, have coefficients U_{2n+1} (quasi-periodic functions) in $\mathcal{H}_s$ and

$$||U_{2n+1}||_s + |\mu_{2n}| \leq \gamma K^n (n!)^{4l}, \ \textit{for } n \in \mathbb{N}$$

Proof by induction

Borel transform

$$u = \sum_{n\geq 1} u_n \zeta^n, \text{ Gevrey-1 series: } |u_n| \leq c\rho^{-n} n!$$

$$\widehat{u}(\zeta) = \sum_{n\geq 1} \frac{u_n}{n!}\zeta^n \text{ Borel transform of } u, \text{ analytic in } |\zeta| < \rho$$

Convolution prod.: $(\widehat{u} *_G \widehat{v})(\zeta) = \sum_{n\geq 1} \sum_{1\leq k\leq n-1} \frac{u_k v_{n-k}}{n!}\zeta^n, \quad \widehat{u} *_G \widehat{v} = \widehat{(uv)}$

Inverse of Borel transform: Laplace transform (needs $\widehat{u}$ defined for ζ on $\mathbb{R}^+$)

Approximated solution

Define the truncated Laplace transform $\bar{U}, \bar{\mu}$ of the Borel transform of the series $U(\epsilon), \mu(\epsilon)$

Theorem

(G.I., A.Rucklidge 2009) Let $q \geq 4$, $s > q/2$, then there exists K and $C > 0$ such that

$$||(1+\Delta)^2\bar{U}(\epsilon) - \bar{\mu}(\epsilon)\bar{U}(\epsilon)) + [\bar{U}(\epsilon)]^3||_{\mathcal{H}_{s-4}} \leq Ce^{-\frac{K}{\epsilon^{1/8l}}}$$

Existence of quasipatterns

Theorem

(Braaksma, G.I., Stolovitch 2012) For any $q \geq 4, s > q/2$, *there exists* $\mu_0 > 0$, *such that there is a quasipattern solution for* $0 < \mu < \mu_0$ *in* $\mathcal{H}_s$, *invariant under rotations of angle* π/q. *The asymptotic expansion of this bifurcating solution is given by the known formal series.*

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Proof: U_ϵ, μ_ϵ being the approximate solution (truncated at order ϵ^5, the differential (self adjoint in $\mathcal{H}_0$) $(1+\Delta)^2 - \mu_\epsilon \mathbb{I} + 3U_\epsilon^2$ which has quasiperiodic coefficients, has a real spectrum $\geq c\epsilon^2$. Pay attention to the fact that there are not only eigenvalues in the spectrum, and that the perturbation term is not relatively compact with respect to $(1+\Delta)^2$, the spectrum of which fills $\mathbb{R}^+$. A variant of the implicit function theorem is then sufficient to conclude.

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