Université Nice-Sophia Antipolis
SMEFA302 - ECUE Probabilistic computational methods, 2025-2026
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Final exam (duration : 2h)

Documents, calculators and phones forbidden (except your personnal cheat sheet, which cannot exceed one sheet of paper). Grading will take into account how well you justify your answers. Number of points per questions will be fixed during the grading process. You can write your answers in French or English.

Let $T > 0$, $c \in \mathbb{R}^*$, $x \in \mathbb{R}$ and, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(W_t)_{t\in[0,T]}$ a standard Brownian motion in $\mathbb{R}$. We are interested in the following stochastic differential equation

$$\begin{cases} X_0 &= x\,, \\ dX_t &= dW_t + cX_t dt\,. \end{cases} \tag{0.1}$$

For $N \in \mathbb{N}^*$ and for $0 \le k \le N$, we set $t_k = k\Delta t$ where $\Delta t = T/N$. The Euler scheme with N steps is defined recursively by

$$\begin{cases} \overline{X}_0 = x \\ \forall 0 \le k \le N-1\,,\ \forall t \in [t_k, t_{k+1}]\,,\ \overline{X}_t = \overline{X}_{t_k} + (W_t - W_{t_k}) + c\overline{X}_{t_k}(t - t_k) \end{cases}$$

(1) Show that the solution of the above equation (0.1) is given by

$$\forall t \in [0,T]\,,\ X_t = xe^{ct} + \int_0^t e^{c(t-s)} dW_s\,.$$

(Hint: use integration by parts formula)

Set $Y_t = xe^{ct} + \int_0^t e^{c(t-s)} dW_s$. We have

$$\begin{aligned} Y_t &= e^{ct}\left(x + \int_0^t e^{-cs} dW_s\right) \\ dY_t &= cY_t dt + e^{ct}e^{-ct} dW_t \\ dY_t &= cY_t dt + dW_t \end{aligned}$$

(2) We admit that $(X_T, W_{t_1} - W_0, W_{t_2} - W_{t_1}, \dots, W_{t_N} - W_{t_{N-1}})$ is Gaussian. For $1 \le k \le N$, show that $\mathrm{Cov}(X_T, W_{t_k} - W_{t_{k-1}}) = \frac{e^{cT}}{c}(e^{-ct_{k-1}} - e^{-ct_k})$.

We have $\mathbb{E}(X_T) = xe^{cT}$, so

$$\begin{aligned} &\mathrm{Cov}(X_T, W_{t_k} - W_{t_{k-1}}) \\ &= \mathbb{E}\left(\left(\int_0^{t_{k-1}} e^{c(T-s)} dW_s + \int_{t_{k-1}}^{t_k} e^{c(T-s)} dW_s + \int_{t_k}^T e^{c(T-s)} dW_s\right) \times (W_{t_k} - W_{t_{k-1}})\right) \\ &= 0 + \mathbb{E}\left(\int_{t_{k-1}}^{t_k} e^{c(T-s)} dW_s \times \int_{t_{k-1}}^{t_k} dW_s\right) \\ &= \mathbb{E}\left(\int_{t_{k-1}}^{t_k} e^{c(T-s)} d\langle W\rangle_s\right) \\ &= \left[\frac{e^{c(T-s)}}{-c}\right]_{t_{k-1}}^{t_k} = \frac{e^{cT}}{c}(e^{-t_{k-1}} - e^{-t_k})\,. \end{aligned}$$

(3) Let $\mathcal{F}_N$ the tribe engendered by $(W_{t_1}, \dots, W_{t_N})$. Fact: if (X, Y) is a centered Gaussian vector of dimension 2, $\mathbb{E}(X|Y) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)} Y$.

(a) Show that

$$\mathbb{E}(X_T|\mathcal{F}_N) = xe^{cT} + \sum_{k=1}^{N} \mathbb{E}\left(\int_{t_{k-1}}^{t_k} e^{c(T-s)} dW_s \middle| W_{t_k} - W_{t_{k-1}} \right).$$

For all $1 \leq k \leq N$,

$$(W_s)_{t_{k-1} \leq s \leq t_k} \text{ is independant of } \{W_{t_1} - W_{t_0}, \dots, W_{t_N} - W_{t_{N-1}}\} \setminus \{W_{t_k} - W_{t_{k-1}}\}. \tag{0.2}$$

So

$$\begin{aligned}
\mathbb{E}(X_T|\mathcal{F}_N) &= \mathbb{E}(X_T|W_{t_1} - W_{t_0}, \dots, W_{t_N} - W_{t_{N-1}}) \\
&= xe^{cT} + \mathbb{E}\left(\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} e^{c(T-s)} dW_s \middle| W_{t_1} - W_{t_0}, \dots, W_{t_N} - W_{t_{N-1}} \right) \\
&= xe^{cT} + \sum_{k=1}^{N} \mathbb{E}\left(\int_{t_{k-1}}^{t_k} e^{c(T-s)} dW_s \middle| W_{t_k} - W_{t_{k-1}} \right)
\end{aligned}$$

(b) Show that

$$\mathbb{E}(X_T|\mathcal{F}_N) = xe^{cT} + \frac{1}{c\Delta t} \sum_{k=1}^{N} e^{c(N-k)\Delta t}(e^{c\Delta t} - 1)(W_{t_k} - W_{t_{k-1}}).$$

$$\begin{aligned}
\mathbb{E}(X_T|\mathcal{F}_N) &= xe^{cT} + \sum_{k=1}^{N} \mathbb{E}\left(\int_{t_{k-1}}^{t_k} e^{c(T-s)} dW_s \middle| W_{t_k} - W_{t_{k-1}} \right) \\
&= xe^{cT} + \sum_{k=1}^{N} \frac{e^{cT}}{c\Delta t}(e^{-c(k-1)\Delta t} - e^{-ck\Delta t})(W_{t_k} - W_{t_{k-1}}) \\
&= xe^{cT} + \frac{1}{c\Delta t} \sum_{k=1}^{N} e^{c(N-k)\Delta t}(e^{c\Delta t} - 1)(W_{t_k} - W_{t_{k-1}})
\end{aligned}$$

(4) Show that (warning: long computation)

$$\mathbb{E}((X_T - \mathbb{E}(X_T|\mathcal{F}_N))^2) = \frac{1}{2c}(e^{2cT} - 1)\left(1 - \frac{2}{c\Delta t} \times \frac{e^{c\Delta t} - 1}{e^{c\Delta t} + 1}\right).$$

$$
\begin{aligned}
\mathbb{E}((X_T - \mathbb{E}(X_T|\mathcal{F}_N))^2) &= \mathbb{E}\left(\left(\sum_{k=1}^{N}\int_{t_{k-1}}^{t_k} e^{c(T-s)}dW_s - \frac{e^{c(N-k)\Delta t}(e^{c\Delta t}-1)}{c\Delta t}(W_{t_k}-W_{t_{k-1}})\right)^2\right) \\
\text{(by Eq. (0.2))} &= \sum_{k=1}^{N}\mathbb{E}\left(\left(\int_{t_{k-1}}^{t_k} e^{c(T-s)}dW_s - \frac{e^{c(N-k)\Delta t}(e^{c\Delta t}-1)}{c\Delta t}(W_{t_k}-W_{t_{k-1}})\right)^2\right) \\
&= \sum_{k=1}^{N}\mathbb{E}\left(\left(\int_{t_{k-1}}^{t_k} e^{c(T-s)}dW_s\right)^2\right) + \mathbb{E}\left(\left(\frac{e^{c(N-k)\Delta t}(e^{c\Delta t}-1)}{c\Delta t}(W_{t_k}-W_{t_{k-1}})\right)^2\right) \\
&\quad -2\frac{e^{c(N-k)\Delta t}(e^{c\Delta t}-1)}{c\Delta t}\mathrm{Cov}\left(\int_{t_{k-1}}^{t_k} e^{c(T-s)}dW_s, W_{t_k}-W_{t_{k-1}}\right) \\
&= \sum_{k=1}^{N}\int_{t_{k-1}}^{t_k} e^{2c(T-s)}ds + \left(\frac{e^{c(N-k)\Delta t}(e^{c\Delta t}-1)}{c\Delta t}\right)^2\Delta t \\
&\quad -2\frac{e^{c(N-k)\Delta t}(e^{c\Delta t}-1)}{c\Delta t}\times\frac{e^{c(N-k)\Delta t}(e^{c\Delta t}-1)}{c} \\
&= \sum_{k=1}^{N}\frac{e^{2c(N-k)\Delta t}(e^{c\Delta t}-1)}{2c} + \left(\frac{e^{c(N-k)\Delta t}(e^{c\Delta t}-1)}{c\Delta t}\right)^2\Delta t \\
&\quad -2\frac{e^{c(N-k)\Delta t}(e^{c\Delta t}-1)}{c\Delta t}\times\frac{e^{c(N-k)\Delta t}(e^{c\Delta t}-1)}{c} \\
&= \sum_{k=1}^{N}\frac{e^{2c(N-k)\Delta t}(e^{2c\Delta t}-1)}{2c} - \frac{e^{2c(N-k)\Delta t}(e^{c\Delta t}-1)^2}{c^2\Delta t} \\
&= \frac{e^{2cN\Delta t}}{2c}\times\left((e^{2c\Delta t}-1) - \frac{2(e^{c\Delta t}-1)^2}{c\Delta t}\right)\times e^{-2c\Delta t}\times\frac{1-e^{-2cN\Delta t}}{1-e^{-2c\Delta t}} \\
&= \frac{e^{2cT}-1}{2c}\left(1-\frac{2(e^{c\Delta t}-1)^2}{c\Delta t(e^{2c\Delta t}-1)}\right) \\
&= \frac{e^{2cT}-1}{2c}\left(1-\frac{2(e^{c\Delta t}-1)}{c\Delta t(e^{c\Delta t}+1)}\right)
\end{aligned}
$$

(5) Show that, when $\alpha \to 0$,

$$
\frac{e^{\alpha}-1}{\alpha(e^{\alpha}+1)} = \frac{1}{2} - \frac{\alpha^2}{24} + o(\alpha^2)\,.
$$

$$
\begin{aligned}
\frac{e^\alpha - 1}{\alpha(e^\alpha+1)} &= \frac{1}{\alpha}\left(\alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} + o(\alpha^3)\right)\left(2+\alpha+\frac{\alpha^2}{2}+o(\alpha^2)\right)^{-1} \\
&= \left(1+\frac{\alpha}{2}+\frac{\alpha^2}{6}+o(\alpha^2)\right)\times\frac{1}{2}\left(1+\frac{\alpha}{2}+\frac{\alpha^2}{4}+o(\alpha^2)\right)^{-1} \\
&= \left(1+\frac{\alpha}{2}+\frac{\alpha^2}{6}+o(\alpha^2)\right)\times\frac{1}{2}\left(1-\frac{\alpha}{2}-\frac{\alpha^2}{4}+\frac{\alpha^2}{4}+o(\alpha^2)\right) \\
&= \frac{1}{2}\left(1+\frac{\alpha}{2}+\frac{\alpha^2}{6}+o(\alpha^2)\right)\left(1-\frac{\alpha}{2}+o(\alpha^2)\right) \\
&= \frac{1}{2}\left(1-\frac{\alpha^2}{4}+\frac{\alpha^2}{6}+o(\alpha^2)\right) \\
&= \frac{1}{2}\left(1+\frac{\alpha^2}{24}(-6+4)+o(\alpha^2)\right) \\
&= \frac{1}{2}-\frac{\alpha^2}{24}+o(\alpha^2)\,.
\end{aligned}
$$

(6) Deduce from the previous questions that $\mathbb{E}((X_T - \mathbb{E}(X_T|\mathcal{F}_N))^2) = O((\Delta t)^2) = O(1/N^2)$. The quantity $\mathbb{E}((X_T - \mathbb{E}(X_T|\mathcal{F}_N))^2)$ is the smallest mean square error we can get when approximating $\mathbb{E}(X_T)$ with any scheme $\widetilde{X}_T$ based on $W_{t_1} - W_{t_0}, \dots, W_{t_N} - W_{t_{N-1}}$.

$$
\begin{aligned}
\mathbb{E}((X_T - \mathbb{E}(X_T|\mathcal{F}_N))^2) &= \frac{e^{2cT}-1}{2c}\left(1-\frac{2(e^{c\Delta t}-1)}{c\Delta t(e^{c\Delta t}+1)}\right) \\
&= \frac{e^{2cT}-1}{2c}\times\left(\frac{(c\Delta t)^2}{12}+o((c\Delta t)^2\right) \\
&= O((\Delta t)^2)\,.
\end{aligned}
$$

(7) Compare the previous order for with the bound on the mean square error $\mathbb{E}((X_T - \overline{X}_T)^2)$ when $\overline{X}_T$ is the Euler-Maruyama scheme with step-size T/N.

The coefficient $b : y \mapsto cy$ is Lipschitz, grows linearly with y (and does not depend on t). The coefficient in front of dW in Equation (0.1) is constant. So, the course's theorem tells us that

$$
\mathbb{E}((X_T - \overline{X}_T)^2) \leq \frac{C(1+x^2)}{N}
$$

(8) Write a python function that compute an approximation of $\mathbb{E}(X_T)$ (approximation that should use the Euler scheme and the Monte-Carlo method), using N steps for the Euler scheme and M samples for the Monte-Carlo approximation of the $\mathbb{E}$. (This function is thus a function of (N, M).)

(9) What scheme $\widetilde{X}_T$ and result will allow us to get the optimal bound $O((\Delta t)^2)$ for the error $\mathbb{E}((X_T - \widetilde{X}_T)^2)$?

According to the course [insert hypothesis check here ...]: Milstsein's scheme.

Algorithm 1 Question 8

```
import numpy as np
import scipy.stats as sps
T=1
x=1
c=1
def euler(N):
    dt=T/N
    sdt=np.sqrt(dt)
    Z=sps.norm.rvs(0,1,N)
    X=x
    for k in range(N):
      X=X+sdt*Z[k]+c*dt*X
    return(X)
def mc(M,N):
    tab=[N]*M
    tab=list(map(euler,tab))
    return(np.mean(tab))
```