On the minimizing Sard Conjecture in sub-Riemannian geometry

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The Setting

Let M be a smooth connected manifold of dimension n.

Definition

A **sub-Riemannian structure** of rank $m \leq n$ on M is given by a pair (Δ, g) where:

- Δ is a **totally nonholonomic distribution** of rank $m \leq n$ on M which is defined locally by

$$\Delta(x) = \mathrm{Span}\Big\{X^1(x), \cdots, X^m(x)\Big\} \subset T_xM,$$

  where $X^1, \ldots, X^m$ is a family of m linearly independent smooth vector fields satisfying the **Hörmander condition**;

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- g is smooth and for every $x \in M$, g_x is a **scalar product** over $\Delta(x)$.

Chow-Rashevsky's Theorem and SR distance

Theorem (Chow-Rashevsky, 1938)

Let Δ be a totally nonholonomic distribution on M, then every pair of points can be joined by an **horizontal path**, that is, a path $\gamma \in W^{1,2}([0,1]; M)$ such that

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \qquad \text{a.e. } t \in [0,1].$$

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Definition

Given $x, y \in M$, the **sub-Riemannian distance** between x and y is defined by

$$d_{SR}(x,y) := \inf \Big\{ \text{length}^g(\gamma) \,|\, \gamma \text{ hor.}, \gamma(0) = x, \gamma(1) = y \Big\},$$

$$\text{with} \qquad \text{length}^g(\gamma) := \int_0^T |\dot{\gamma}(t)|^g_{\gamma(t)} \, dt.$$

Sub-Riemannian minimizing geodesics

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Let $x, y \in M$ be fixed.

- We call **minimizing horizontal path** between x and y any horizontal path $\gamma : [0,1] \to M$ joining x to y satisfying
$$d_{SR}(x,y) = \text{length}^g(\gamma).$$
- We call **minimizing geodesic** between x and y any horizontal path $\gamma : [0,1] \to M$ joining x to y such that
$$d_{SR}(x,y)^2 = \text{ener}^g(\gamma) := \int_0^1 \left(|\dot{\gamma}(t)|^g_{\gamma(t)} \right)^2 \, dt.$$

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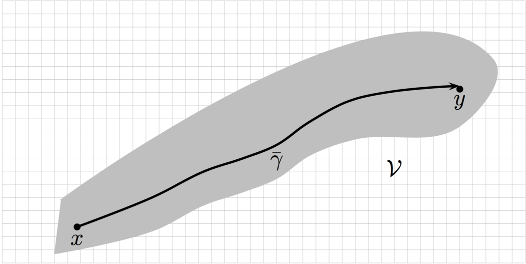
$\rightsquigarrow$ There is a SR Hopf-Rinow Theorem. If (M, d_{SR}) is complete, then we have existence gives of minimizing geodesics.

Study of minimizing geodesics I

Let $x, y \in M$ and $\bar{\gamma}$ be a **minimizing geodesic** between x and y be fixed. The SR structure admits an orthonormal parametrization along $\bar{\gamma}$, which means that there exists a neighborhood $\mathcal{V}$ of $\bar{\gamma}([0,1])$ and an orthonomal family of m vector fields $X^1, \ldots, X^m$ such that

$$\Delta(z) = \mathrm{Span}\left\{X^1(z), \ldots, X^m(z)\right\} \qquad \forall z \in \mathcal{V}.$$



Study of minimizing geodesics II

There exists a control $\bar{u} \in L^2\left([0,1];\mathbb{R}^m\right)$ such that

$$\dot{\bar{\gamma}}(t) = \sum_{i=1}^{m} \bar{u}_i(t)\, X^i\left(\bar{\gamma}(t)\right) \qquad \text{a.e. } t \in [0,1].$$

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Moreover, any control $u \in \mathcal{U} \subset L^2\big([0,1];\mathbb{R}^m\big)$ (u sufficiently close to $\bar{u}$) gives rise to a trajectory γ_u solution of

$$\dot{\gamma}_u = \sum_{i=1}^{m} u^i\, X^i\big(\gamma_u\big) \quad \text{sur } [0,T], \quad \gamma_u(0) = x.$$

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Furthermore, for every horizontal path $\gamma : [0,1] \to \mathcal{V}$ there exists a unique control $u \in L^2\big([0,1];\mathbb{R}^m\big)$ for which the above equation is satisfied.

Study of minimizing geodesics III

Consider the **End-Point mapping**

$$E^{x,1} \; : \; L^2\big([0,1];\mathbb{R}^m\big) \longrightarrow M$$

defined by

$$E^{x,1}(u) := \gamma_u(1),$$

and set $C(u) = \|u\|_{L^2}^2$, then $\bar{u}$ is a solution to the following **optimization problem with constraints**:

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(Since the family $X^1, \ldots, X^m$ is orthonormal, we have

$$\mathrm{ener}^g(\gamma_u) = C(u) \qquad \forall u \in \mathcal{U}.)$$

Study of minimizing geodesics IV

By the Lagrange Multipliers Theorem, there are $p \in T_y^*M$ and $\lambda_0 \in \{0,1\}$ with $(\lambda_0, p) \neq (0,0)$ such that

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Case $\lambda_0 = 0$: We have

$$p \cdot D_{\bar{u}} E^{x,1} = 0 \text{ with } p \neq 0,$$

which means that $\bar{u}$ is a critical point of the mapping $E^{x,1}$. The minimizing geodesic $\bar{\gamma}$ is said to be **singular**.

Martinet-like distributions

In $\mathbb{R}^3$, let $\Delta = \text{Vect}\{X^1, X^2\}$ with X^1, X^2 of the form

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = \left(1 + x_1\phi(x)\right)\partial_{x_2} + x_1^2\partial_{x_3},$$

where ϕ is a smooth function and let g be a metric over Δ.

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Theorem (Montgomery, 1991)

There exists $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, the singular horizontal path

$$\gamma(t) = (0, t, 0) \qquad \forall t \in [0, \epsilon],$$

is minimizing (w.r.t. g) among all horizontal paths joining 0 to $(0, \epsilon, 0)$.

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is minimizing (w.r.t. g) among all horizontal paths joining 0 to $(0, \epsilon, 0)$. Moreover, if $\{X^1, X^2\}$ is orthonormal w.r.t. g and $\phi(0) \neq 0$, then γ is not the projection of a normal extremal ($\lambda_0 = 1$).

Summary

Given a complete sub-Riemannian structure (Δ, g) on M and a minimizing geodesic γ from x to y, two cases may happen:

- The geodesic γ is the projection of a normal extremal so it is smooth..
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Questions:

When? How many? How?

The Minimizing Sard Conjecture

Given $x \in M$, we denote by $\mathrm{Abn}^{min}(x)$ (or $\mathrm{Sing}_{\Delta,g}^{x,min}$) the set of points $y \in M$ for which there is a **singular minimizing geodesic** joining x to y, it is a closed subset of M containing x.

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Only few cases are known. In general, we have

Theorem (Agrachev, 2009)

Let M be a smooth connected manifold of dimension n equipped with a complete sub-Riemannian structure (Δ, g). Then for every $x \in M$, the closed set $\mathrm{Abn}^{min}(x)$ has empty interior.

A Partial result

Theorem (R, 2023)

Let M be equipped with a complete SR structure (Δ, g) and $x \in M$ be fixed. If for almost every $y \in M$ all minimizing horizontal paths from x to y have Goh-rank at most 1, then the closed set $\mathrm{Abn}^{min}(x)$ has Lebesgue measure zero in M.

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If M is equipped with a complete SR structure (Δ, g) having minimizing co-rank 1 almost everywhere, then the Minimizing Sard Conjecture holds true.

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If M is equipped with a complete SR structure (Δ, g) of rank $m \geq 2$ where Δ is generic, then the Minimizing Sard Conjecture holds true.

Sketch of proof I

Let (Δ, g) be a SR structure and $x \in M$ be fixed, we define $f_x : M \to \mathbb{R}$ by

$$f_x(y) := d_{SR}(x,y)^2$$

The following properties are equivalent:

- The set $\mathrm{Abn}^{min}(x)$ has Lebesgue measure zero in M.
- For a.e. $y \in M$, $\partial^- f_x(y) \neq \emptyset$.
- The set $\mathrm{Lip}^-(f_x)$ has full Lebesgue measure in M.
- The function f_x is smooth on an open subset of M of full Lebesgue measure.

A Key Result

Proposition (R-Trélat, 2005)

Let $y \neq x$ in M and $p \in T_y^* M$ be such that

$$\partial^- f_x(y) \neq \emptyset$$

then there is a unique minimizing geodesic $\bar{\gamma}$ between x and y and it is the projection of a normal extremal. In addition, if y is not a critical value of $\exp_{SR} : T_x^* M \to M$ then γ is nonsingular.

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Recall that $p \in T_y^* M$ belongs to the **viscosity subdifferential** $\partial^- f_x(y)$ iff there is a C^1 function $\varphi : M \to \mathbb{R}$ such that

$$p = d_y \varphi, \quad \varphi(y) = f_x(y) \quad \text{and} \quad \varphi \leq f_x \text{ on } M.$$

A Key Result (Sketch of proof)

For every $u \in L^2([0,1];\mathbb{R}^m)$, there holds

$$\varphi\left(E^{x,1}(u)\right) \leq f_x\left(E^{x,1}(u)\right) = d_{SR}\left(x, E^{x,1}(u)\right)^2 \leq C(u)$$

and moreover we have

$$\varphi\left(E^{x,1}(\bar{u})\right) = f_x\left(E^{x,1}(\bar{u})\right) = d_{SR}\left(x,y\right)^2 = C(\bar{u}).$$

As a consequence, we have

$$p \cdot d_{\bar{u}} E^{x,1} = d_{\bar{u}} C,$$

so we are in the $\lambda_0 = 1$ case...

Comments on the proof of our main theorem

- One can construct continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ for which the set of $y \in \mathbb{R}^n$ with $\partial^- f(y) = \emptyset$ is countable.
- Roughly speaking, our assumptions allow to show that f_x is Lipschitz along an hyperplane at each $y \neq x$.
- Our proof follows by a Fubini argument along with the following 1-dimensional dichotomy: Given a continuous function $\varphi : (a, b) \to \mathbb{R}$, for a.e. $x \in (a, b)$, at least one of the following properties is satisfied:
  - (i) φ is differentiable at x.
  - (ii) There is a sequence $\{x_k\}$ converging to x such that $0 \in \partial^- \varphi(x_k)$ for all k.
- We use invoke indeed a more precise version of (ii), which is given by the Denjoy-Young Saks Theorem.

The Denjoy-Young-Saks Theorem

Theorem

Let $f : (a,b) \to \mathbb{R}$ be a function. Then for a.e. $x \in (a,b)$, one of the following assertion holds:

(1) f is differentiable at x,

(2) $D^+f(x) = D^-f(x) = +\infty$, $D_+f(x) = D_-f(x) = -\infty$,

(3) $D^+f(x) = +\infty, D_-f(x) = -\infty$, $D_+f(x) = D^-f(x) \in \mathbb{R}$,

(4) $D^-f(x) = +\infty, D_+f(x) = -\infty$, $D_-f(x) = D^+f(x) \in \mathbb{R}$.

Here, D^+f, D_+f, D^-f, D_-f stand for the Dini derivatives of f defined by $(T_{c,d} := (f(d) - f(c))/(d-c))$

$$D^+f(x) = \limsup_{h \to 0^+} T_{x,x+h}, \quad D_+f(x) = \liminf_{h \to 0^+} T_{x,x+h},$$

$$D_-f(x) = \liminf_{h \to 0^+} T_{x-h,x}, \quad D^-f(x) = \limsup_{h \to 0^+} T_{x-h,x},$$

A Conjecture and an open question

Conjecture (image of the SR exponential mapping)

For every $x \in M$, the set $\exp_x(T_x^* M)$ has full Lebesgue measure in M.

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Given a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ the **limiting subdifferential** of f at $y \in \mathbb{R}^n$ is defined by

$$\partial_L^- f(y) = \left\{ \lim_k p_k \,|\, y_k \to_k y,\ p_k \in \partial^- f(y_k)\, \forall k \right\}.$$

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Open Question

Can we have

$$\partial_L^- f(x) = \emptyset$$

for x in a set of positive Lebesgue measure ?

Thank you for your attention !!