Parametric inference and forcasting for continuously invertible volatilitiy models : application to Egarch models

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Outline of Topics

1. Introduction
   - Motivation
   - EGARCH$(1,1)$
2. Stationarity and Continuous Invertibility
   - Stationarity
   - Invertibility
3. Asymptotics of QMLE
   - Strong Consistency
   - Asymptotic Normality
4. EGARCH$(1,1)$ Case
   - Stationarity
   - Continuous Invertibility for EGARCH$(1,1)$
   - Asymptotics of QMLE
5. Numeric examples
   - Monte Carlo
   - S&P 500

Volatility of finanical time series

- In finance, volatility is a measure for variation of price of a financial instrument over time. It is one of the most important financial quantity.
- Before the mathematical details, lets have a feel of the volatility process.

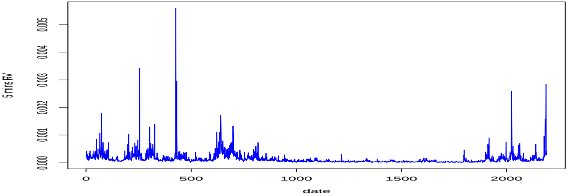

Figure: Realized volatility of the return of S&P 500 index

What should a good volatility capture? (Stylized fact)

- Dependence - Volatility clustering .
- Non-linearity and Asymmetry - Leverage effect.

Some existing models of volatility are ARCH of (Engle 1982), GARCH model of (Bollerslev 1986) and Stochastic Volatility Models (SV). The GARCH model is defined by the following Stochastic Recursive Equation(SRE):

GARCH Model

$$X_t = \sigma_t Z_t \quad \text{with} \quad \sigma_t^2 = \alpha_0 + \sum_{k=1}^{k=p} \alpha_k X_{t-k}^2 + \sum_{k=1}^{k=q} \beta_k \sigma_{t-k}^2$$

- GARCH model fails to capture the Leverage effect.

In view of the default of GARCH, (Nelson 1991) suggest the Exponential GARCH model (EGARCH).

EGARCH$(1,1)$

$$X_t = \sigma_t Z_t \quad \text{with} \quad \log\sigma_t^2 = \alpha_0 + \beta_0 \log\sigma_{t-1}^2 + \gamma_0 Z_{t-1} + \delta_0 |Z_{t-1}| \tag{1}$$

- (Z_t) is iid sequence of random variables not concentrated on two points. $\mathbb{E}Z_0^2 = 1$
- $\theta_0 = (\alpha_0, \beta_0, \gamma_0, \delta_0)$ belong to some compact set Θ.
- EGARCH$(1,1)$ is a ARMA for the log volatilities process.
- EGARCH can model the leverage effect: conditioning on the sign of the noise Z_t, the change of the log-vol $\log(\sigma_{t+1}^2/\sigma_t^2)$ is asymmetric.
- It is widely used by researchers and practitioners. But no general asymptotic result for parameteric estimation.

General Volatility Model

Now we present the our object of this study :

General Volatility Model

$$\begin{aligned} X_t &= \Sigma_t^{1/2} \cdot Z_t \qquad (2) \\ \text{with} \quad (h(\Sigma_k))_{k \le t} &= \psi_t((h(\Sigma_k))_{k \le t-1}, \theta_0) \end{aligned}$$

where h maps the matrix into $\mathbb{R}$ and ψ_t is a $\mathcal{F}_{t-1} = \sigma(Z_k, k \le t-1)$ mesurable function that depended on the ture parameter θ_0.

- The general Volatility contains all the classical models of ARCH type.

QMLE of General Volatility Models

- We want to study the Quasi Maximum Likelihood Estimator (QMLE) for this model. Let $\hat{\Sigma}_t(\theta)$ to be some estimation of the volatility (Σ_t) based on the parameter θ. The QMLE, $\hat{\theta}_n$ is defined by

QMLE

$$\hat{\theta}_n = \text{argmin}_{\theta \in \Theta} \hat{S}_n(\theta)$$

$$\text{where} \quad n\hat{S}_n(\theta) = \sum_{t=1}^{n} s_t = \sum_{t=1}^{n} 2^{-1} \left(X_t^T \hat{\Sigma}_t(\theta)^{-1} X_t + \log(\det(\hat{\Sigma}_t(\theta))) \right)$$

- Since (Z_t) are not observable, we MUST to recover the volatilities (Σ_t) from the observables, i.e. $\hat{\Sigma}_t(\theta)$ only depends on $(X_t)_{t \geq 1}$. Thus the motivation for invertibility of the model.

Stationarity

- Let (E, d) be a polish space. A map $f : E \to E$ is a Lipschitz map if $\Lambda(f) = \sup_{(x,y)\in E^2} d(f(x), f(y))/d(x, y)$ is finite.

The following theorem of (Bougerol 1993) gives the conditions of stationarity.

Theorem 1 (Stationarity of SRE (Bougerol 1993))

Let (Ψ_t) be a stationary ergodic sequence of Lipschitz maps from E to E. Suppose that $\mathbb{E}[\log^+(d(\Psi_0(x), x))] < \infty$ for some $x \in E$, that $\mathbb{E}[\log^+ \Lambda(\Psi_0)] < \infty$ and that for some integer $r \geq 1$,

$$\mathbb{E}[\log \Lambda(\Psi_0^{(r)})] = \mathbb{E}[\log \Lambda(\Psi_0 \circ \cdots \circ \Psi_{-r+1})] < 0.$$

Then the SRE $X_t = \Psi_t(X_{t-1})$ for all $t \in \mathbb{Z}$ is convergent: it admits a unique stationary solution $(Y_t)_{t\in\mathbb{Z}}$ which is ergodic and for any $y \in E$

$$Y_t = \lim_{m\to\infty} \Psi_t \circ \cdots \circ \Psi_{t-m}(y), \quad t \in \mathbb{Z}.$$

The Y_t are measurable with respect to the $\sigma(\Psi_{t-k}, k \geq 0)$ and

$$d(\tilde{Y}_t, Y_t) \xrightarrow{e.a.s.} 0, \quad t \to \infty$$

such that $\tilde{Y}_t = \Psi_t(\tilde{Y}_{t-1})$ for all $t > 0$.

Proposition 1 (Logarithmic moment)

Under the assumptions of Theorem (1) *and* $\mathbb{E}[(\log^+ d(\psi_0(x), x))^2] < \infty$ *the unique stationary solution* (Y_t) *satisfies* $\mathbb{E}[\log^+(d(Y_0, y))] < \infty$ *for all* $y \in E$.

We always assume the following condition :

(ST) The process (X_t) satisfying (2) exists. It is a stationary, non anticipative and ergodic process with finite logarithmic moments.

Continuous invertibility

- Using the relation $Z_t = \Sigma_t^{-1/2} \cdot X_t$, it is possible to study a new functional SRE driven by the observations (X_t) :
$$(h(\Sigma_k(\theta)))_{k \le t} = \phi_t((h(\Sigma_k(\theta)))_{k \le t-1}, \theta). \tag{3}$$
here ϕ_t is generated by $(\mathcal{G}_{t-1} = \sigma(X_{t-1}, X_{t-2}, \cdots))$.
- But only the $(X_t)_{t \ge 1}$ are observable, so we need to approximate ϕ_t by $\hat{\phi}_t$ by assuming all the $(X_t)_{t \le 0}$ are zero.

Definition 1

The model is continuously invertible if and only if

- *The solution of* (3)*, the function* $h(\Sigma_t(\cdot))$ *is continuous for all* $\theta \in \Theta$*.*
- *The solution of the SRE*
$$(h(\hat{\Sigma}_k(\theta)))_{k \le t} = \hat{\phi}_t((h(\hat{\Sigma}_k(\theta)))_{k \le t-1}, \theta) \qquad t \ge 1 \tag{4}$$
is convergent for any arbitrary initial values $h(\hat{\Sigma}_k)(\theta)_{k \le 0}$ *and such that* $\|\hat{\Sigma}_t(\theta) - \Sigma_t(\theta)\| \to 0$ *in probability as* $t \to \infty$*.*
- *For any* $\theta \in \Theta$ *there exists an* $\epsilon > 0$ *such that* $\hat{\Sigma}_t(\theta)$ *satisfying* (4) *satisfies*
$$\lim_t \sup_{\theta' \in \overline{B}(\theta, \epsilon) \cap \Theta} d(\hat{\Sigma}_t(\theta'), \Sigma_t(\theta')) \xrightarrow{e.a.s.} 0. \tag{5}$$

(CL) For any metric spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, a function $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathcal{Z}$ satisfies (**CL**) if there exists a continuous function $\Lambda_f : \mathcal{Y} \mapsto \mathbb{R}^+$ such that $\Lambda(f(\cdot, y)) \leq \Lambda_f(y)$ for all $y \in \mathcal{Y}$.

(CI) Assume that the SRE (4) holds with $\hat{\phi}_t$ satisfying **(CL)**. And assume that there exists an positve integr r such that $\mathbb{E}[\log \Lambda_{\phi_0}^{(r)}(\theta)] < 0$ and that $\mathbb{E}[\sup_\Theta \log^+ \Lambda_{\phi_0}^{(r)}(\theta)] < \infty$ and that for some $y \in E$ such that $\mathbb{E}[\sup_\Theta \log^+(d(\phi_0(y, \theta), y))] < \infty$.

Theorem 2 (Condition of Continuously Invertibility (Wintenberger and Cai 2011b))

Under conditions **(ST)** *and* **(CI)**, *and if the inverse of the function h is continuous, the model is continuously invertible.*

- We note ℓ the inverse of h.

  (IN) The (Z_t) are iid and $\mathbb{E}[Z_0^T Z_0]$ is the identity matrix.

  (IV) The function h and $\log(\det(\ell))$ are Lipschitz and $\det(\Sigma_0(\theta)) \geq C(\theta)$ for some continuous function $C : \Theta \mapsto (0, \infty)$.

Theorem 3 (Strong Consistency (Wintenberger and Cai 2011b))

Assume that **(ST)** *and* **(CI)** *are satisfied on the compact set* Θ*. If* **(IN)** *and* **(IV)** *are satisfied and the model is identifiable, i.e.* $h(\Sigma_0)(\theta) = h(\Sigma_0)$ *iff* $\theta = \theta_0$*, then* $\hat{\theta}_n \to \theta_0$ *a.s. for any* $\theta_0 \in \Theta$*.*

Asymptotic Normality

- AV $\mathbb{E}(\|Z_0 Z_0^T\|^2) < \infty$ and that the functions ℓ and ϕ_t are 2-times continuously differentiable.
- MM $\mathbb{E}[\|\nabla s_0(\theta_0)\|^2] < \infty$ and $\mathbb{E}[\|\mathbb{H}s_0(\theta_0)\|] < \infty$. Where $\mathbb{H}s_0$ is the hessian matrix.
- LI The components of the vector $\nabla h(\Sigma_0(\theta_0))$ are linearly independent.
- DL In $\mathcal{V}$, a neighborhood of θ_0, the partial derivatives $\Phi_t = D_x(\phi_t), = D_\theta(\phi_t), = D^2_{x^2}(\phi_0), = D^2_{\theta,x}(\phi_0)$ or $= D^2_{\theta^2}(\phi_0)$ satisfy **(CL)** for stationary (Λ_{Φ_t}) with $\mathbb{E}[\sup_{\mathcal{V}} \log(\Lambda_{\Phi_0})] < \infty$. Assume there exists $y \in E$ such that $\mathbb{E}[\sup_{\mathcal{V}}(\log^+(d(\phi_0(y,\theta),y)))^2] < \infty$.
- LM Assume that $y \to \nabla \ell^{-1}(y)$ and $y \to \nabla \log(\det(\ell(y)))$ are Lipschitz functions.

Theorem 4 (Asymptotic Normality (Wintenberger and Cai 2011b))

Under the assumptions of Theorem 3, **(AV)**, **(MM)**, **(LI)**, **(DL)** *and* **(LM)** *then*

$$\sqrt{n}(\hat{\theta}_n - \hat{\theta}_0) \xrightarrow{d} \mathcal{N}(0, \mathbf{V}) \quad \textit{if} \quad \theta_0 \in \overset{\circ}{\Theta} .$$

Where $\mathbf{V} = \mathbf{P}^{-1}\mathbf{Q}\mathbf{P}^{-1}$ *with* $\mathbf{P} = \mathbb{E}[\mathbb{H}s_0(\theta_0)]$ *and* $\mathbf{Q} = \mathbb{E}[\nabla s_0(\theta_0)\nabla s_0(\theta_0)^T]$.

Stationarity and ergodicity of EGARCH$(1,1)$

- Let us recall the definiton of EGARCH(1,1) model.

EGARCH$(1,1)$

$$X_t = \sigma_t Z_t \quad \text{with} \quad \log \sigma_t^2 = \alpha_0 + \beta_0 \log \sigma_{t-1}^2 + \gamma_0 Z_{t-1} + \delta_0 |Z_{t-1}| \tag{6}$$

The condition of the stationarity of EGARCH model is simple.

Proposition 2 (Solution of EGARCH$(1,1)$)

$\mathbb{E}[\log^+(|Z_0|)] < \infty$ *and* $|\beta_0| < 1$, *then there exists a unique, ergodic, stationary and non anticipative solution* (X_t) *to SRE* (6) .

Then we can express log-volatility as a MA(∞) process.

$$\log \sigma_t^2 = \alpha_0 (1 - \beta_0)^{-1} + \sum_{k=1}^{\infty} \beta_0^{k-1} W_{t-k}(\theta_0).$$

Continuous Invertibility of EGARCH(1, 1)

Z_t is not observable, we need to approximate volatility using the following SRE with $(X_0, X_1, ...)$

$$\log \hat{\sigma}_t^2(\theta) = \begin{cases} \phi_t(\log \hat{\sigma}_{t-1}^2(\theta), \theta), & \text{if } t \geq 1 \\ 0 & \text{if } t = 0 \end{cases} \tag{7}$$

we also consider the SRE

$$\log \sigma_t^2(\theta) = \phi_t(\log \sigma_{t-1}^2(\theta), \theta), \quad \forall t \tag{8}$$

where the link function $\phi_t(\cdot;\theta) : s \mapsto \alpha + \beta s + (\gamma X_{t-1} + \delta |X_{t-1}|) \exp(-s/2)$.

Proposition 3 (Continuous Invertibility of EGARCH(1, 1))

Let $\Lambda(\phi_t(\cdot,\theta))$ be the Lipschitz coefficients of the link function $\phi_t(\cdot,\theta_0)$. Suppose that the compact parameter set $\Theta \subset \mathbb{R} \times \mathbb{R}^+ \times \{\gamma \geq |\delta|\}$, If

$$\mathbb{E} \log \Lambda(\phi_t(\cdot,\theta)) < 0, \quad \forall \theta \in \Theta,$$

then there exists an $\epsilon > 0$ such that $\log \hat{\sigma}_t^2(\theta)$ satisfying (7) satisfies

$$\lim \sup_{\theta \in \overline{B}(\theta_0,\epsilon)\cap\Theta} d(\log \sigma_t^2(\theta)), \log \hat{\sigma}_t^2(\theta)) \xrightarrow{e.a.s.} 0. \tag{9}$$

Moreover,

$$\log \hat{\sigma}_t^2(\theta_0) \longrightarrow_{a.s.} \log \sigma_t^2 \quad \forall \theta_0 \in \Theta.$$

We also have $\Lambda(\phi_t(\cdot,\theta)) \leq \max\{\beta, 2^{-1}(\gamma X_{t-1} + \delta|X_{t-1}|)\exp(-2^{-1}\alpha/(1-\beta)) - \beta)\}$.

Region of Continuous Invertibility

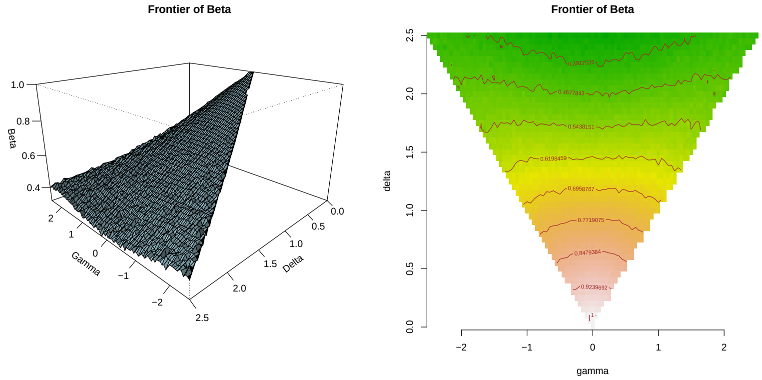

Figure: Perspective and contour plots of the admissible region for region for stability.

Asymptotics of QMLE

Theorem 5 (Strong consistence of $\hat{\theta}_n$ (Wintenberger and Cai 2011a))

Under the condition of stationarity, and the condition of Continuously Invertibility, $\hat{\theta}_n \to \theta_0$ *a.s. for any* $\theta_0 \in \Theta$.

Theorem 6 (Asymptotic Normality of $\hat{\theta}_n$ (Wintenberger and Cai 2011a))

Under the condition of the Theorem 5, and

$$\mathbf{H2} \quad \mathbb{E}[Z_0^4] < \infty, \mathbb{E}[(\beta_0 - 2^{-1}(\gamma_0 Z_0 + \delta_0 |Z_0|)^2] < 1,$$

then

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \to_d \mathcal{N}(0, \mathbf{V}^{TH})$$

with an explicit (but very complicated) invertible matrix $\mathbf{V}^{TH}$ *for any* $\theta_0 \in \mathring{\Theta}$.

Straumann and Mikosch (2006) study the asymptotics of a degenerate EGARCH ($\beta_0 = 0$) under stronger conditions.

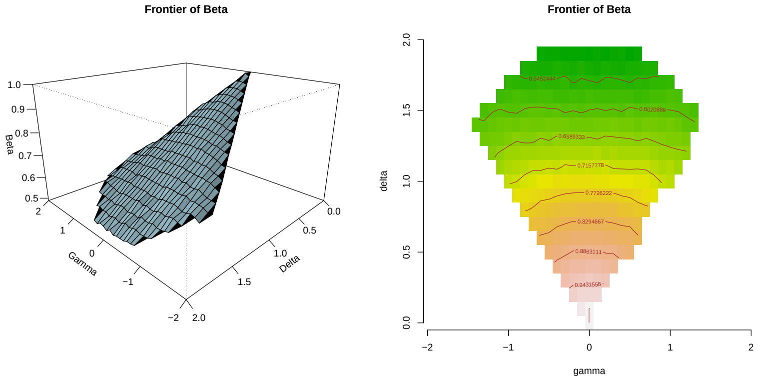

Figure: Perspective and contour plots of the admissible region for the asymptotic normality.

Monte Carlo Simulations

Monte Carlo simulation of 1000 replications of the sample path of different sizes $T = 512, 1034$ or 2048. We assume the innovations that drive EGARCH model are iid standard normal.

Table: Statistical inference and normal approximation

	T=	512			1024			2048		
θ	θ_0	mean	rmse	napp	mean	rmse	napp	mean	rmse	napp
α	-0.399	-.381	.127	.059	-.393	.041	.042	-.396	.030	.030
β	.9	.874	.170	.023	.897	.017	.016	.899	.012	.011
γ	-.3	-.300	.057	.045	-.301	.033	.032	-.299	.023	.023
δ	.5	.488	.097	.075	.492	.052	.053	.496	.038	.038

Co-variance Matrix

- Under the condition of Theorem 6, we have two ways of estimating the asymptotic co-variance matrix of our estimator
  - Plug-in $\mathbf{V}^{TH}(\hat{\theta}_n)$.
  - Use the SRE satisfied by $\nabla \log \hat{\sigma}_t(\hat{\theta}_n)$.
- Distance of co-variance matrix: $d(A, B) = \sqrt{\sum_{k=1}^{4} \log^2 \nu_k(AB^{-1})}$, where $\nu_k(A)$ is the k-th eigenvalue of matrix A.
- 100 random parameters in Θ. For each θ_k we simulate 3 path of different length and compare.

n	$d(\mathbf{V}^{TH}(\theta_k), \mathbf{V}^{SRE}(\theta_k))$	$d(\mathbf{V}^{TH}(\theta_k), \mathbf{V}^{TH}(\hat{\theta}_k))$	$d(\mathbf{V}^{TH}(\theta_k), \mathbf{V}^{SRE}(\hat{\theta}_k))$
512	.074	.788	.924
1024	.065	.767	.780
2048	.064	.426	.457

Forecasting S&P 500 volatilities

- Standard & Poor's 500 data from Jan 4th, 2000 to Jul 22th, 2003 ($n = 890$).
- Competing models:
  - GARCH(1,1): $\sigma_t^2 = \omega + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2$,
  - APGARCH(1,1): $\sigma_t^\delta = \omega + \alpha(|X_{t-1}| - \gamma X_{t-1})^\delta + \beta \sigma_{t-1}^\delta$,
  - Rolling Volatility (60days): the moving average $1/60 \sum_{i=1}^{60} X_{t-i}^2$,
  - Riskmetrics (Exponentially weighted moving average model) $\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda) X_{t-1}^2$ where $lambda = .94$,
- Volatility Proxy: squared return and realized volatilities.
- QLIK criteria is

$$\mathrm{QLIK}(\hat{\sigma}^2) = \sum_{t=1}^{n} \log(\hat{\sigma}_t^2) + \frac{RV_t}{\hat{\sigma}_t^2}.$$

which is consistent, see (Patton 2011).

Forecasting S&P 500 volatilities

- $\hat{\theta}^{\text{In Sample}} = \{-0.312, .976, -0.122, 0.122\}$.
- $\hat{\theta}^{\text{ Out Sample}} = \{-0.324, 0.974, -0.123, 0.123\}$.

Table: In sample performances of the different forecasts

QLIK	X_t^2	5min RV	15min RV	65min RV
GARCH(1,1)	-7.438	-7.517	-7.592	-7.607
GARCH(1,1) Student	-7.439	-7.516	-7.589	-7.604
APGARCH(1,1)	**-7.489**	-7.528	-7.613	-7.618
Rolling Volatility	-7.444	-7.473	-7.542	-7.570
Riskmetrics	-7.429	-7.510	-7.583	-7.597
EGARCH(1,1)	-7.487	**-7.537**	**-7.619**	**-7.626**

Table: Out of sample performances of the different forecasts

QLIK	X_t^2	5min RV	15min RV	65min RV
GARCH(1,1)	**-8.285**	-8.153	**-8.192**	**-8.170**
GARCH(1,1) Student	-8.226	-8.131	-8.153	-8.111
APGARCH(1,1)	-8.226	-8.130	-8.152	-8.111
Rolling Volatility	-8.195	-8.096	-8.128	-8.111
Riskmetrics	-8.053	-7.978	-7.998	-7.977
EGARCH(1,1)	-8.272	**-8.155**	-8.184	-8.135

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Thank you for your attentions !