A new statistical procedure for testing the presence of a significative correlation in the residuals of stable autoregressive processes

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CIRM - Marseille
16-20 avril 2012

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- **Significance of the serial correlation in the residuals.**
- **Autoregressive process of order p.**
  - For all $n \geq 1$,

$$\begin{cases} X_n = \theta_1 X_{n-1} + \ldots + \theta_p X_{n-p} + \varepsilon_n \\ \varepsilon_n = \rho \varepsilon_{n-1} + V_n \end{cases}$$

    where $X_0 = \varepsilon_0$ are square-integrable and $X_{-1}, X_{-2}, \ldots, X_{-p} = 0$.

- **Preliminary study.**
  - The empirical process is asymptotically stationary.
  - (X_n) has an autoregressive behavior.
- **Objectives.**
  - Sharp analysis on the least squares estimators of θ and ρ.
  - Statistical procedure for testing $\mathcal{H}_0$: "$\rho = 0$" vs $\mathcal{H}_1$: "$\rho \neq 0$".
  - Comparison of the empirical power of the test with commonly used procedures.
  - Use of the Durbin-Watson statistic.
- **ArXiv : 1203.1871.**

- **Causality of the model.**
  - Compact expression, for all $1 \leq t \leq N$,
  $$\mathcal{A}(B)X_t = \varepsilon_t$$
  where the polynomial $\mathcal{A}(z) = 1 - \beta_1 z - \ldots - \beta_p z^p + \theta_p \rho z^{p+1}$ and
  $$\beta = \begin{pmatrix} \theta_1 + \rho & \theta_2 - \theta_1 \rho & \ldots & \theta_p - \theta_{p-1}\rho \end{pmatrix}'.$$
  - $\mathcal{A}$ is causal : $\mathcal{A}(z) \neq 0$ for all $z \in \mathbb{C}$ such that $|z| \leq 1$.
  - $\|\theta\|_1 < 1$ and $|\rho| < 1$ imply causality.
  - On $\mathbb{N}$, causality often coincides with asymptotic stationarity.

- **Autoregressive structure of order at least p.**
  - $\theta_p \neq 0$.

- **Moments on the noise.**
  - (V_n) i.i.d. such that $\mathbb{E}[V_1] = 0$, $\mathbb{E}[V_1^2] = \sigma^2$ and $\mathbb{E}[V_1^4] = \tau^4$.
  - Possible to do without i.i.d. assumption.

- **Serial correlation stability.**
  - $|\rho| < 1$ translates moments properties from (V_n) to (ε_n).
- **Autoregression stability.**
  - $\|\theta\|_1 < 1$ translates moments properties from (ε_n) to (X_n).

Lemma (Stability, M. Duflo)

Assume that (V_n) is a sequence of independent and identically distributed random variables such that, for some $a \geq 1$, $\mathbb{E}[|V_1|^a]$ is finite. Then,

$$\sum_{k=0}^{n} |X_k|^a = O(n) \quad \text{a.s.} \qquad \text{and} \qquad \sup_{0 \leq k \leq n} |X_k| = o(n^{1/a}) \quad \text{a.s.}$$

- **More generally.**
  - For all $j \geq 0$,

$$\sum_{k=1}^{n} X_{k-j} X_k = O(n) \quad \text{a.s.}$$

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- **Least squares estimator.**
  - For all $n \geq 0$, denote by
    $$\Phi_n^p = \begin{pmatrix} X_n & X_{n-1} & \dots & X_{n-p+1} \end{pmatrix}'$$
    and
    $$S_n = \sum_{k=0}^{n} \Phi_k^p \Phi_k^{p\,\prime} + S.$$
  - In the AR(p) structure $X_n = \theta' \Phi_{n-1}^p + \varepsilon_n$, for all $n \geq 1$,
    $$\widehat{\theta}_n = (S_{n-1})^{-1} \sum_{k=1}^{n} \Phi_{k-1}^p X_k.$$
  - S is a symmetric positive definite matrix added to ensure the invertibility of S_n.

- **Convergence.**

Theorem

We have the almost sure convergence

$$\lim_{n \to \infty} \widehat{\theta}_n = \theta^* \qquad \text{a.s.}$$

- The limiting value is given by

$$\theta^* = \alpha(I_p - \theta_p \rho J_p)\beta$$

where I_p is the identity matrix of order p, J_p is the exchange matrix of order p and

$$\alpha = (1 - \theta_p \rho)^{-1}(1 + \theta_p \rho)^{-1}.$$

- Strong consistency when $\rho = 0$ under the stability conditions (Lai and Wei, 1983).

- **Central limit theorem.**

Theorem

Assume that (V_n) has a finite moment of order 4. Then, we have the asymptotic normality

$$\sqrt{n}\left(\widehat{\theta}_n - \theta^*\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \Sigma_\theta).$$

- The asymptotic covariance matrix is given by

$$\Sigma_\theta = \alpha^2 \left(I_p - \theta_p \rho J_p\right) \Delta_p^{-1} \left(I_p - \theta_p \rho J_p\right)$$

  where Δ_p is the positive definite almost sure limiting matrix of $\sigma^{-2} S_n / n$.

- Δ_p is the covariance matrix of the standard stationary process built from (X_n).
- When $\rho = 0$, $\Sigma_\theta = \Delta_p^{-1}$ (see e.g. Brockwell and Davis, 1991).

- **Quadratic strong law.**

Theorem

Assume that (V_n) has a finite moment of order 4. Then, we have the quadratic strong law

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \left(\widehat{\theta}_k - \theta^*\right)\left(\widehat{\theta}_k - \theta^*\right)' = \Sigma_\theta \qquad \text{a.s.}$$

- **Law of iterated logarithm.**

Theorem

Assume that (V_n) has a finite moment of order 4. Then, we have the law of iterated logarithm

$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log n}\right)\left(\widehat{\theta}_n - \theta^*\right)\left(\widehat{\theta}_n - \theta^*\right)' = \Sigma_\theta \qquad \text{a.s.}$$

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- **Estimated residual set.**
  - For all $1 \leq k \leq n$, let
$$\widehat{\varepsilon}_k = X_k - \widehat{\theta}_n{}' \Phi^p_{k-1}.$$
- **Least squares estimator.**
  - In the AR(1) structure $\varepsilon_n = \rho \varepsilon_{n-1} + V_n$, for all $n \geq 1$,
$$\widehat{\rho}_n = \frac{\sum_{k=1}^n \widehat{\varepsilon}_k \widehat{\varepsilon}_{k-1}}{\sum_{k=1}^n \widehat{\varepsilon}^2_{k-1}}.$$
- **Convergence.**

Theorem

We have the almost sure convergence
$$\lim_{n \to \infty} \widehat{\rho}_n = \rho^* \qquad \text{a.s.}$$

- The limiting value is given by
$$\rho^* = \theta_p \rho \theta^*_p$$
where θ^*_p is the $p-$th element of θ^*.
- When $\rho = 0$, $\widehat{\rho}_n$ converges a.s. to 0.

- **Central limit theorem.**

Theorem

Assume that (V_n) has a finite moment of order 4. Then, we have the joint asymptotic normality

$$\sqrt{n}\begin{pmatrix}\widehat{\theta}_n - \theta^* \\ \widehat{\rho}_n - \rho^*\end{pmatrix} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \Gamma).$$

In particular,

$$\sqrt{n}\Big(\widehat{\rho}_n - \rho^*\Big) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \sigma_\rho^2)$$

where $\sigma_\rho^2 = \Gamma_{p+1,\,p+1}$ is the last diagonal element of Γ.

- The asymptotic covariance matrix is given by

$$\Gamma = \begin{pmatrix} \Sigma_\theta & \theta_p \rho J_p \Sigma_\theta e \\ \theta_p \rho e' \Sigma_\theta J_p & \sigma_\rho^2 \end{pmatrix}$$

where

$$\sigma_\rho^2 = P_L' \Delta_p P_L - 2\alpha^{-1}\theta_p^* \Lambda_p^{1\,\prime} J_p P_L + \left(\alpha^{-1}\theta_p^*\right)^2 \lambda_0.$$

- When $\rho = 0$, $\sigma_\rho^2 = \theta_p^2$.

- **Quadratic strong law.**

Theorem

Assume that (V_n) has a finite moment of order 4. Then, we have the quadratic strong law

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \left(\widehat{\rho}_k - \rho^*\right)^2 = \sigma_\rho^2 \qquad \text{a.s.}$$

- **Law of iterated logarithm.**

Theorem

Assume that (V_n) has a finite moment of order 4. Then, we have the law of iterated logarithm

$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log n}\right) \left(\widehat{\rho}_n - \rho^*\right)^2 = \sigma_\rho \qquad \text{a.s.}$$

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- **Introduced by Durbin and Watson.**
  - Middle of last century.
  - Testing residual autocorrelation in standard regression analysis.
  - Biased in the dependent framework.
  - For all $n \geq 1$,

$$\widehat{D}_n = \frac{\sum_{k=1}^{n} \left(\widehat{\varepsilon}_k - \widehat{\varepsilon}_{k-1}\right)^2}{\sum_{k=0}^{n} \widehat{\varepsilon}_k^2}.$$

- **Properties.**
  - Asymptotic equivalence

$$\widehat{D}_n \underset{+\infty}{\sim} 2\left(1 - \widehat{\rho}_n\right) \qquad \text{a.s.}$$

  - Convergence

$$\lim_{n \to \infty} \widehat{D}_n = 2(1 - \rho^*) = D^* \qquad \text{a.s.}$$

  - Central limit theorem

$$\sqrt{n}\left(\widehat{D}_n - D^*\right) \xrightarrow[n \to \infty]{\mathcal{L}} \mathcal{N}(0, 4\sigma_\rho^2).$$

  - Rate of convergence

$$\left(\widehat{D}_n - D^*\right)^2 = O\left(\frac{\log \log n}{n}\right) \qquad \text{a.s.}$$

- **Test statistic.**
  - For all $n \geq 1$,

$$\widehat{Z}_n = \frac{n}{4\widehat{\theta}_{p,n}^2} \left(\widehat{D}_n - 2\right)^2$$

where $\widehat{\theta}_{p,n}^2$ is the $p-$th element of $\widehat{\theta}_n$.

Theorem

Assume that (V_n) *has a finite moment of order 4,* $\theta_p \neq 0$ *and* $\theta_p^* \neq 0$*. Then, under the null hypothesis* $\mathcal{H}_0$: "$\rho = 0$",

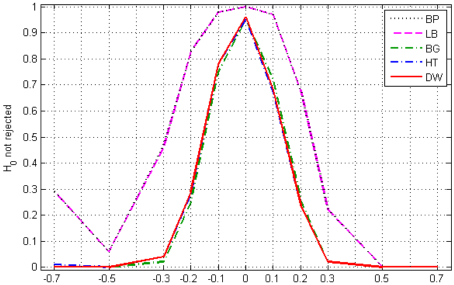
$$\widehat{Z}_n \xrightarrow[n\to\infty]{\mathcal{L}} \chi^2$$

where χ^2 *has a Chi-square distribution with one degree of freedom. In addition, under the alternative hypothesis* $\mathcal{H}_1$: "$\rho \neq 0$",

$$\lim_{n\to\infty} \widehat{Z}_n = +\infty \qquad \text{a.s.}$$

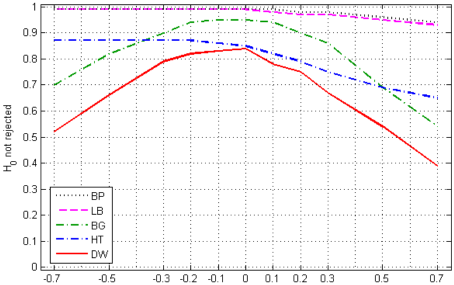
- Assumption $\{\theta_p^* \neq 0\}$ not restrictive : $\{\theta_p \neq 0\} \cap \{\theta_p^* = 0\} \Rightarrow \{\rho \neq 0\}$.
- $\mathcal{H}_0$ not rejected if $\widehat{Z}_n \leq z_a$, for a risk $0 < a < 1$.

- **Empirical power on simulated data.**
  - (V_n) i.i.d. $\mathcal{N}(0,1)$, $p = 3$, $\theta = \begin{pmatrix} 0.1 & -0.2 & 0.6 \end{pmatrix}'$, $n = 500$.



- **On large samples.**
  - More powerful than the *portmanteau* tests of Box-Pierce and Ljung-Box.
  - Wrong variance in the asymptotic normality : the *portmanteau* tests overestimate $\mathcal{H}_0$.
  - Equivalent to the *h-test* of Durbin and to the Breusch-Godfrey test.

- **Empirical power on simulated data.**
  - (V_n) i.i.d. $\mathcal{N}(0,1)$, $p = 3$, $\theta = \begin{pmatrix}0.1 & -0.2 & 0.6\end{pmatrix}'$, $n = 30$.



- **On small-sized samples.**
  - More powerful than all other tests, except under $\mathcal{H}_0$.
  - Always more sensitive to the presence of correlation in the residuals.
  - Performs pretty well.

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Thank you for your attention !

- **References.**

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