Look-down model and Λ-Wright-Fisher SDE

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Introduction

- We consider a new version of the look-down construction with selection. We assume that two types of individuals coexist in the population: individuals with the wild-type allele b and individuals with the advantageous allele B. This selective advantage is modeled by a death rate α for the type b individuals.
- We assume that individuals are placed at time 0 on levels $1, 2, \ldots,$ each one being, independently from the others, b with probability x, B with probability $1 - x$, for some $0 < x < 1$. For any $t \geq 0, i \geq 1$, let

$$\eta_t(i) = \begin{cases} 1 & \text{if the i-th individual is b at time } t \\ 0 & \text{if the i-th individual is B at time } t. \end{cases}$$

The population evolves as follows :

Introduction

- *Births:* Let Λ be an arbitrary finite measure on $[0,1]$ such that $\Lambda(\{0\}) = 0$. Consider a Poisson random measure on $\mathbb{R}_+\times]0,1]\times]0,1]$,

$$m = \sum_{k=1}^{\infty} \delta_{t_k,x_k}$$

with intensity measure $dt \otimes x^{-2}\Lambda(dx)$. let $(t,x) \in m$, for each level $i \geq 1$, we define $Z_i \simeq Bernoulli(x)$. Let

$$I_{t,x} = \{i \geq 1 : Z_i = 1\} \quad \ell_{t,x} = \inf\{i \in I_{t,x} : i > \min I_{t,x}\}.$$

$$\eta_t(i) = \begin{cases} \eta_{t^-}(i), & \text{if } \; i < \ell_{t,x} \\ \eta_{t^-}(\min I_{t,x}), & \text{if } \; i \in I_{t,x} \setminus \{\min I_{t,x}\} \\ \eta_{t^-}(i - (\#\{I_{t,x} \cap [1,\dots,i]\} - 1)), & \text{otherwise} \end{cases}$$

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- *Deaths:* Any type b individual dies at rate α. If the level of the dying individual is i, then for all $j > i$, the individual at level j replaces instantaneously the individual at level $j-1$.

$$\eta_t(k) = \begin{cases} \eta_{t^-}(k) & \text{for } k < i \\ \eta_{t^-}(k+1) & \text{if } k \geq i \end{cases}$$

Construction of our process

- First, we introduces some variables

$$\lambda_{p,k} = \int_0^1 x^{k-2}(1-x)^{p-k}\Lambda(dx) \qquad \phi(N) = \sum_{k=2}^{N}(k-1)\binom{N}{k}\lambda_{N,k}.$$

Lemma

$$N^{-1}\phi(N) \uparrow \int_0^1 x^{-1}\Lambda(dx)$$

In the rest of this section, we suppose that Λ fulfills the following condition

$$\mu_{-1} = \int_0^1 x^{-1}\Lambda(dx) > \alpha.$$

Construction of our process

For each N, consider the process $\{\eta_t^N(i), i \geq 1, t \geq 0\}$, obtained by applying only the arrows between $1 \leq i < j \leq N$, and the crosses on levels 1 to N. In other words, we disregard all the arrows pointing to levels above N, as well as all the crosses on levels above N.
Let δ, γ be two real numbers such that $\mu_{-1} > (1+\delta) > (1+\gamma) > \alpha$

Proposition

There exists $N_0 \geq 1, C > 1$, $\gamma < \gamma' < \delta$ and $0 < \beta < \frac{\delta - \gamma'}{1+\delta}$ such that , for each $N \geq N_0$

$$\mathbb{P}\left(\exists 1 \leq i \leq N, k \geq 1, t > 0 \ : \ \eta_t^{(1+\gamma)N}(i) \neq \eta_t^{(1+\gamma)N+k}(i)\right)$$
$$\leq C((1+\gamma)N+1)\left(\frac{1+\gamma'}{1+\gamma'+\beta(1+\delta)}\right)^{N\gamma}$$

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Exchangeability

- For each $n \geq 1$ and $t \geq 0$, let

$$I(n,t) = \{k \geq 1 : t_k \in [0,t] \text{ and } I_{t_k,x_k} \cap [n] \geq 2\}.$$

We have

Lemma

For each $n \geq 1$ and $t \geq 0$,

$$\#I(n,t) < \infty \quad a.s$$

Proposition

If $(\eta_0(i))_{i\geq 1}$ are exchangeable random variables, then $t > 0$, $(\eta_t(i))_{i\geq 1}$ are exchangeable.

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Tightness and Convergence to the Λ-Wright-Fisher SDE with selection

- For $N \geq 1$ and $t \geq 0$, denote by X_t^N the proportion of type b individuals at time t among the first N individuals, i.e.

$$X_t^N = \frac{1}{N}\sum_{i=1}^{N}\eta_t(i)$$

- *Remark : For any $N \geq 1$, the process $\{X_t^N, t \geq 0\}$ is not a Markov process. Indeed, the past values $\{X_s^N, 0 \leq s < t\}$ give us some clue as to what the values of $\eta_t(N+1), \eta_t(N+2), \ldots$ may be, and this influences the law of the future values $\{X_{t+r}^N, r > 0\}$.*

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Tightness

- As a consequence of the de Finetti theorem, we have the following asymptotic property for fixed t of the sequence $(X_t^N)_{N\geq 1}$

Corollaire

For each $t \geq 0$,

$$X_t = \lim_{N\to\infty} X_t^N \quad \textit{exist a.s.}$$

- *Remark : $\{X_t, t \geq 0\}$ is a Markov process.*

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$(X^N, N \geq 1)$ is tight in $D([0,\infty])$.

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Convergence to the Λ-Wright-Fisher SDE with selection

Let

$$M = \sum_{k=1}^{\infty} \delta_{t_k, u_k, x_k}$$

Poisson point process on $\mathbb{R}_+ \times]0,1] \times]0,1]$ with intensity $dtdux^{-2}\Lambda(dx)$.
For every $u \in]0,1[$ and $r \in [0,1]$, we introduce the elementary function

$$\Psi(u,r) = \mathbf{1}_{u \le r} - r.$$

Definition

We shall call Λ-Wright-Fisher SDE with selection the following Poissonian stochastic differential equation

$$\begin{cases} X_t = r - \alpha \int_0^t X_s(1-X_s)ds + \int_{[0,t]\times]0,1[\times]0,1[} x\Psi(u, X_{s^-})\bar{M}(ds,du,dx), \\ X_0 = r, \quad 0 < r < 1, \end{cases} \quad (1)$$

The solution $\{X_t, t \ge 0\}$ is supermartingal which takes values in $[0,1]$.

Convergence to the Λ-Wright-Fisher SDE with selection

Theorem

Suppose that $X_0^N \to r$ *a.s, as* $N \to \infty$*. Then the* $[0,1]-$*valued process* $\{X_t, t \geq 0\}$ *is the (unique in law) solution of the* Λ*-Wright-Fisher SDE (1).*

Fixation and non-fixation in the Λ-Wright-Fisher SDE

- We say the Λ-coalescent comes down from infinity ($\Lambda \in$ **CDI**) if $\mathbb{P}(\#\Pi_t < \infty) = 1$ for all $t > 0$.
- We say it stays infinite ($\Lambda \notin$ **CDI**) if $\mathbb{P}(\#\Pi_t = \infty) = 1$ for all $t > 0$.
- $$\textit{if} \int_0^1 x^{-1}\Lambda(dx) < \infty \text{ then } \Lambda \notin \textbf{CDI} \qquad \text{(J. Pitman (1999)).}$$
- Let $\phi(n) = \sum_{k=2}^{n}(k-1)\binom{n}{k}\lambda_{N,k}$. The Λ-coalescent comes down from infinity if and only if

$$\sum_{n=2}^{\infty} \frac{1}{\phi(n)} < \infty \qquad \text{(J. SCHWEINSBERG 2000)} \qquad (2)$$

Fixation and non-fixation in the Λ-Wright-Fisher SDE

Theorem main-result 4 show that X_t is a bounded supermartingale. Consequently

$$X_\infty = \lim_{t \to \infty} X_t \in \{0, 1\}.$$

Theorem

If $\Lambda \in$ **CDI**, *then one of the type (b or B) fixates in finite time, i.e.*

$$\exists\, \zeta < \infty \; a.s : X_\zeta = X_\infty \in \{0, 1\}$$

If $\Lambda \notin$ **CDI**, *then*

$$\forall t \geq 0,\; 0 < X_t < 1 \; a.s.$$

Fixation and non-fixation in the Λ-Wright-Fisher SDE

- We suppose that $\Lambda \notin$ **CDI**. Let x be the proportion of type b individuals at time 0, $0 < x < 1$. We have

Proposition

If $\alpha = 0$, *then*

$$\mathbb{P}(X_\infty = 1) = 1 - \mathbb{P}(X_\infty = 0) = x$$

If $\alpha > 0$ *and* $\int_0^1 x^{-1}\Lambda(dx) \leq \alpha$ *,then*

$$\mathbb{P}(X_\infty = 1) = \mathbb{E}(X_\infty) = 0.$$

Thank you for your attention !