The effect of competition on the height and length of the forest of genealogical trees of a large population

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April 17, 2012

Introduction

- Consider a Galton–Watson branching process X^m in continuous time with m ancestors at time $t = 0$: for $k \geq 1$, the process

  $$X^m \text{ jumps } \begin{cases} k \to k+1, & \text{at rate } \mu k; \\ k \to k-1, & \text{at rate } \lambda k. \end{cases}$$

  For any $0 \leq m < n$, $X^n \overset{(d)}{=} X^m + X^{n-m}$ (branching property).

- In order to model competition within the population described by X^m, we superimpose to each individual a death rate due to competition, equal at time t to γX_t^m. This gives a global death rate equal at time t to $\gamma (X_t^m)^2$.

- Set $m = [Nx]$, $\mu_N = 2N + \theta$, $\lambda_N = 2N$ and $\gamma_N = \gamma / N$, and $Z^{N,x} = \frac{X^{[Nx]}}{N}$. The "total population mass process" Z^N converges weakly to the solution of the Feller SDE with logistic drift

$$dZ_t^x = \left[\theta Z_t^x - \gamma (Z_t^x)^2\right] dt + 2\sqrt{Z_t^x} dW_t, \; Z_0^x = x. \qquad (1)$$

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- This model has been studied by Lambert (2005) and Pardoux-Wakolbinger (2011). The forest of those m trees is also finite a.s. . and for any x la solution Z^x of equation (1) goes extint in finite time *a.s.*.
- On can define the height and the length of the discrete forest of genealogical trees

$$H^m = \inf\{t > 0,\ X_t^m = 0\}, \quad L^m = \int_0^{H^m} X_t^m dt, \quad \text{for } m \geq 1,$$

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Branching process with polynomial competition

- We generalize the above models, replacing the death rate $\gamma(X_t^m)^2$ by $\gamma(X_t^m)^\alpha$, $\alpha > 0$.
- Set $m = [Nx]$, $\mu_N = 2N + \theta$, $\lambda_N = 2N$ and $\gamma_N = \gamma/N^{\alpha-1}$, and $Z^{N,x} := \frac{X^{[Nx]}}{N}$. We show that the process Z^N converges weakly to to a Feller SDE with a negative polynomial drift.

$$dZ_t^x = [\theta Z_t^x - \gamma(Z_t^x)^\alpha]\, dt + 2\sqrt{Z_t^x} dW_t, \;\; Z_0^x = x. \tag{2}$$

- We study the height and the length of the genealogical tree, as an (increasing) function of the initial population in the discrete and the continuous model.

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Results

- Both $\mathbb{E}[\sup_m H^m] < \infty$ and $\mathbb{E}[\sup_x T^x] < \infty$ if $\alpha > 1$, while $H^m \to \infty$ as $m \to \infty$ and $T^x \to \infty$ as $x \to \infty$ a. s. if $\alpha \leq 1$.
- Both $\mathbb{E}[\sup_m L^m] < \infty$ and $\mathbb{E}[\sup_x S^x] < \infty$ if $\alpha > 2$, while $L^m \to \infty$ as $m \to \infty$ and $S^x \to \infty$ as $x \to \infty$ a. s. if $\alpha \leq 2$.

This necessitates to define in a consistent way the population processes jointly for all initial population sizes, i. e. we will need to define the two–parameter processes $\{X_t^m,\ t \geq 0, m \geq 1\}$ and $\{Z_t^x,\ t \geq 0, x > 0\}$.

The model with asymetric effect of the competition

- The description of the process $(X_t^m, t \geq 0)$ is valid for one initial condition m, but it is not sufficiently precise to describe the joint evolution of $\{(X_t^m, X_t^n),\ t \geq 0\}$, $1 \leq m < n$.
- Modelize the effect of the competition in a asymmetric way. The idea is that the progeny X_t^m of the m "first" ancestors should not feel the competition due to the progeny $X_t^n - X_t^m$ of the $n - m$ "additional" ancestors which is present in the population X_t^n.
- We order the ancestors from left to right, this order being passed to their progeny.

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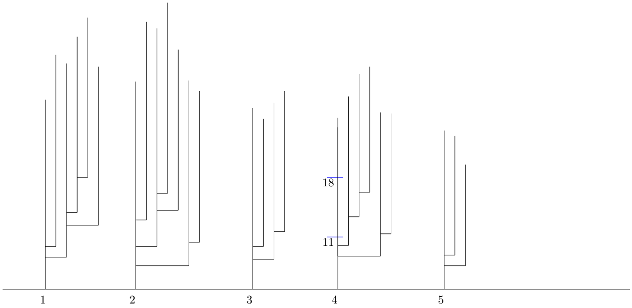
Order killing

- The individual placed at the position i at time t dies becuse of competition at rate $\gamma[\mathcal{L}_i(t)^\alpha - (\mathcal{L}_i(t) - 1)^\alpha]$, $\mathcal{L}_i(t)$ is the number of alive individuals at time t, who are located at his left on the planar tree.



Discret model in the asymetric competition picture

- $\{X_t^m,\ t \geq 0\}$ is a continuous time $\mathbb{Z}_+$–valued Markov process, which evolves as follows. X_t^m jumps to $\begin{cases} k+1, & \text{at rate } \mu k; \\ k-1, & \text{at rate } \lambda k + \gamma (k-1)^\alpha. \end{cases}$
- The above description specifies well the evolution of the $two_parametersprocess\{X_t^m,\ t \geq 0, m \geq 0\}$.
- For $\alpha = 1$, $\{X_t^m,\ m \geq 1\}$ is a Markov chain for fixed t
- For $\alpha \neq 1$, $\{X_t^m,\ m \geq 1\}$ is not a Markov chain for fixed t. The conditional law of X_t^{n+1} given X_t^n differs from its conditional law given $(X_t^1, X_t^2, \ldots, X_t^n)$.
- However, $\{X_\cdot^m,\ m \geq 0\}$ is a Markov chain with values in the space $D([0,\infty); \mathbb{Z}_+)$, which starts from 0 at $m = 0$.

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Discription of jointly laws

For arbitrary $0 \leq m < n$, let $V_t^{m,n} := X_t^n - X_t^m$, $t \geq 0$. Conditionally upon $\{X_\cdot^\ell,\ \ell \leq m\}$, and given that $X_t^m = x(t)$, $t \geq 0$, $\{V_t^{m,n},\ t \geq 0\}$ is a $\mathbb{Z}_+$–valued time inhomogeneous Markov process starting from $V_0^{m,n} = n - m$, whose time–dependent infinitesimal generator $\{Q_{k,\ell}(t),\ k, \ell \in \mathbb{Z}_+\}$ is given by

$$
\begin{aligned}
Q_{0,\ell}(t) &= 0, \quad \forall \ell \geq 1, \quad \text{and for any } k \geq 1, \\
Q_{k,k+1}(t) &= \mu k, \\
Q_{k,k-1}(t) &= \lambda k + \gamma(x(t) + k - 1)^\alpha, \\
Q_{k,\ell}(t) &= 0, \quad \forall \ell \notin \{k-1, k, k+1\}.
\end{aligned}
$$

Moreover we have

$$\langle V^{m,n}, X^m \rangle = 0.$$

The continuous Model

- $\{Z^x_t,\ t \ge 0,\ x \ge 0\}$ is such that for each fixed $x > 0$, $\{Z^x_t, t \ge 0\}$ is continuous process, solution of the SDE (2).
- For any $0 < x < y$, $\{V^{x,y}_t := Z^y_t - Z^x_t,\ t \ge 0\}$ solves the SDE

$$dV^{x,y}_t = \left[\theta V^{x,y}_t - \gamma\left\{(Z^x_t + V^{x,y}_t)^\alpha - (Z^x_t)^\alpha\right\}\right]dt + 2\sqrt{V^{x,y}_t}\,dW'_t,$$
$$V^{x,y}_0 = y - x,$$

  $\{W'_t,\ t \ge 0\}$ is independent from the Brownian motion W which drives the SDE (2) for Z^x_t.
- $\{Z^x_\cdot,\ x \ge 0\}$ is a Markov process with values in $C([0,\infty), \mathbb{R}_+)$, starting from 0 at $x = 0$.

Convergence result

Let $\{\tilde{Z}_t^{N,x},\ t \geq 0, x \geq 0\}$ be the process such that for fixed x, $\{\tilde{Z}_t^{N,x},\ t \geq 0\}$ is the linear interpolation of $\{Z_t^{N,x},\ t \geq 0\}$ between its jumps.

Theorem

As $N \to \infty$,

$$\{\tilde{Z}_t^{N,x},\ t \geq 0, x \geq 0\} \Rightarrow \{Z_t^x,\ t \geq 0, x \geq 0\}$$

in $D([0,\infty); C([0,\infty); \mathbb{R}_+))$, equipped with the Skohorod topology of the space of càdlàg functions of x, with values in the space $C([0,\infty); \mathbb{R}_+)$ equipped with the topology of locally uniform convergence.

Height of the discrete tree

Theorem

- *If* $0 < \alpha \leq 1$, *then*

$$\sup_{m \geq 1} H^m = +\infty \qquad a.\ s.$$

- *If* $\alpha > 1$, *then*

$$\mathbb{E}\left[\sup_{m \geq 1} H^m\right] < \infty.$$

Proof of the Theorem[Height of the discrete tree] for $\alpha > 1$

Let $H_1^m = \inf\{s \geq 0; X_s^m = 1\}$.

Proposition

For $\alpha > 1$, $\lambda = 0$, $\forall\ m \geq 1$, $\mathbb{E}(H_1^m)$ is given by

$$\mathbb{E}(H_1^m) = \sum_{k=2}^{m} \frac{1}{\gamma(k-1)^\alpha} \sum_{n=0}^{\infty} \frac{\mu^n}{\gamma^n} \frac{1}{[k(k+1)\cdots(k+n-1)]^{\alpha-1}} < \infty.$$

Moreover we have

$$H^m \leq H_1^m + GH_1^2 + \sum_{i=1}^{G} T_i,$$

where G is geometric variable with parameter $\frac{\lambda}{\lambda+\mu}$ and T_i is exponential with mean $1/(\lambda+\mu)$.

Proof of the Theorem[Height of the discrete tree] for $\alpha \leq 1$

We have $X_t^{\alpha,m} \geq X_t^{1,m}$, for all $m \geq 1$, $t \geq 0$, a. s..
$\{X_t^m,\ t \geq 0\}$ is the sum of m mutually independent copies of $\{X_t^1,\ t \geq 0\}$. The result follows from the fact that: for all $t > 0$ $\mathbb{P}(H^1 > t) > 0$ and $\mathbb{P}(H^m < t) = (1 - \mathbb{P}(H^1 > t))^m$.

Length of the discrete tree

- Time change of X^m:

$$A_t^m := \int_0^t X_r^m dr, \quad \eta_t^m = \inf\{s > 0;\ A_s^m > t\}.$$

$$U^m := X^m \circ \eta^m, \quad \text{and} \quad S^m = \inf\{r > 0; U_r^m = 0\}.$$

- We have $L^m = S^m$ since $S^m = \int_0^{H^m} X_r^m dr$.

Length of the discrete tree

$$X_t^m = m + P_1\left(\int_0^t \mu X_r^m dr\right) - P_2\left(\int_0^t [\lambda X_r^m + \gamma(X_r^m - 1)^\alpha]\, dr\right),$$

$$U_t^m = m + P_1(\mu t) - P_2\left(\int_0^t \left[\lambda + \gamma(U_r^m)^{-1}(U_r^m - 1)^\alpha\right] dr\right).$$

On the interval $[0, S^m)$, $U_t^m \geq 1$, and consequently we have

$$m - P_2\left(\int_0^t \left[\lambda U_r^m + \gamma(U_r^m - 1)^{\alpha-1}\right] dr\right) \leq U_t^m$$
$$\leq m + P_1\left(\int_0^t \mu U_r^m dr\right) - P_2\left(\int_0^t \left[\frac{\gamma}{2}(U_r^m - 1)^{\alpha-1}\right] dr\right).$$

Length of the discrete tree

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Length of the discrete tree

Theorem

If $\alpha \leq 2$, then

$$\sup_{m \geq 0} L^m = \infty \quad a.\ s..$$

If $\alpha > 2$, then

$$\mathbb{E}\left[\sup_{m \geq 0} L^m\right] < \infty.$$

Height of the continuous tree

Consider again $\{Z^x_t,\ t \geq 0\}$ solution of (2). We have

Theorem

- *If* $0 < \alpha < 1$, $0 < \mathbb{P}(T^x = \infty) < 1$ *if* $\theta > 0$, *while* $T^x < \infty$ *a. s. if* $\theta = 0$.
- *If* $\alpha = 1$, $T^x < \infty$ *a. s. if* $\gamma \geq \theta$, *while* $0 < \mathbb{P}(T^x = \infty) < 1$ *if* $\gamma < \theta$.
- *If* $\alpha > 1$, $T^x < \infty$ *a. s.*

Height of the continuous tree

Theorem

- *It* $\alpha \leq 1$, *then* $T^x \to \infty$ *a. s., as* $x \to \infty$.
- *If* $\alpha > 1$, *then* $\mathbb{E}\left[\sup_{x>0} T^x\right] < \infty$.

Proof of of Theorem 5 for $\alpha > 1$

We first need to establish some preliminary results on SDEs with infinite initial condition. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz and such that

$$\lim_{x\to\infty} \frac{|f(x)|}{x^\alpha} = 0. \tag{3}$$

Theorem

Let $\alpha > 1$, $\gamma > 0$ and f satisfy the assumption (3). Then there exists a minimal $X \in C((0,+\infty);\mathbb{R})$ which solves

$$\begin{cases} dX_t = [f(X_t) - \gamma(X_t)^\alpha]\mathbf{1}_{\{X_t\geq 0\}}dt + dW_t; \\ X_t \to \infty, \ \text{as } t \to 0. \end{cases} \tag{4}$$

Moreover, if $T_0 := \inf\{t > 0, \ X_t = 0\}$, then $\mathbb{E}[T_0] < \infty$.

Proof of the Theorem for $\alpha > 1$

The process $Y_t^x := \sqrt{Z_t^x}$ solves the SDE

$$dY_t^x = \left[\frac{\theta}{2} Y_t^x - \frac{\gamma}{2}(Y_t^x)^{2\alpha-1} - \frac{1}{8Y_t^x}\right] dt + dW_t, \;\; Y_0^x = \sqrt{x}.$$

By a well–known comparison theorem, $Y_t^x \le U_t^x$, where U_t^x solves

$$dU_t^x = \left[\frac{\theta}{2} U_t^x - \frac{\gamma}{2}(U_t^x)^{2\alpha-1}\right] dt + dW_t, \;\; U_0^x = \sqrt{x}.$$

The result follows from the previous Theorem.

Proof of the Theorem for $\alpha < 1$

The result is equivalent to the fact that the time to reach 1, starting from $x > 1$, goes to ∞ as $x \to \infty$. But when $Z_t^x \geq 1$, a comparison of SDEs for various values of α shows that it suffices to consider the case $\alpha = 1$. But in that case, T^n is the maximum of the extinction times of n mutually independent copies of Z_t^1, hence the result.

Length of the continuous tree

Theorem

If $\alpha \leq 2$, *then* $S^x \to \infty$ *a. s. as* $x \to \infty$.
If $\alpha > 2$, *then* $\mathbb{E}\left[\sup_{x>0} S^x\right] < \infty$.

Proof of the Theorem for $\alpha > 2$

- Time change of Z^x:

$$A_t = \int_0^t Z_s^x ds, \;\; \eta(t) = \inf\{s > 0, \; A_s > t\}. \; t \geq 0, \; \text{and} \; U_t^x = Z^x \circ \eta(t)$$

- The process U^x solves the SDE

$$dU_t^x = \left[\theta - \gamma (U_t^x)^{\alpha - 1}\right] dt + 2dW_t, \;\; U_0^x = x. \tag{5}$$

- Let $\tau^x := \inf\{t > 0, \;\; U_t^x = 0\}$. It follows from the above that $\eta(\tau^x) = T^x$, hence $S^x = \tau^x$.

The result follows for $\alpha > 2$.

It suffice to consider the case $\alpha = 2$. In that case, we have

$$U_t^x = e^{-\gamma t} x + \int_0^t e^{-\gamma(t-s)}[\theta ds + 2dW_s],$$

hence

$$S^x = \inf \left\{ t > 0, \int_0^t e^{\gamma s}(\theta ds + 2dW_s) \leq -x \right\},$$

which clearly goes to infinity, as $x \to \infty$.

Thanks!