Un nouveau résultat pour les EDSs rétrogrades de second ordre (2EDSRs) à croissance quadratique et ses applications

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Classical utility maximization [Hu et al. (2005)]

The financial market consists of one bond with interest rate zero and d stocks. The price process of stock i is given by the following SDE:

$$dS_t^i = S_t^i(b_t^i dt + \sigma_t^i dB_t), \quad i = 1, 2, \ldots, d,$$

where b^i (resp. σ^i) is a $\mathbb{R}$-valued (resp. $\mathbb{R}^{1\times m}$-valued) stochastic process.

A d-dimensional $\mathcal{F}_t$-progressively measurable process $\pi = (\pi_t)_{0\leq t\leq 1}$ is called trading strategy if $\int_0^1 ||\pi_t \sigma_t||^2 dt < \infty$, $\mathbb{P}$-a.s..

For $1 \leq i \leq d$, the process π_t^i describes the amount of money invested in stock i at time t. The number of shares is $\frac{\pi_t^i}{S_t^i}$. Suppose the trading strategies are self-financing. The wealth process X^π of a trading strategy π with initial capital x satisfies the equation

$$X_t^\pi = x + \sum_{i=1}^d \int_0^t \frac{\pi_{i,s}}{S_{i,s}} dS_{i,s} = x + \int_0^t \pi_s \sigma_s(\theta_s ds + dB_s),\ 0 \leq t \leq 1,$$

where $\theta_t = \sigma_t^{\mathrm{T}}(\sigma_t \sigma_t^{\mathrm{T}})^{-1} b_t$, $0 \leq t \leq 1$, where x is the initial wealth.

We suppose in addition that our investor has a liability ξ at time 1.

Let us recall that for a $c > 0$ the exponential utility function is defined as

$$U_c(x) = -\exp(-cx),\ x \in \mathbb{R}.$$

Definition (Admissible strategies with constraints)

Let $\tilde{C}$ be a closed set in $\mathbb{R}^d$. $\tilde{\mathcal{A}}$ denote the set of all admissible trading strategies $\pi = (\pi_t)_{0 \leq t \leq 1}$ which are taking values in $\tilde{C}$, as well as $\{\exp(-cX^\pi_\tau)\}_{\tau \in [0,T]}$ is a uniformly integrable family, for some $c > 0$.

The investor wants to solve the maximization problem

$$V_c^\xi(x) := \sup_{\pi \in \tilde{\mathcal{A}}} E\Big[-\exp\Big(-c\Big(x + \int_0^1 \pi_t \sigma_t (\theta_s ds + dB_s) - \xi\Big)\Big)\Big]. \qquad (1)$$

This problem has been studied by many authors (e.g. El Karoui & Rouge (2000)), but they suppose that the constraint is convex in order to apply convex duality. Hu et al. (2005) is a starting point to work on this utility maximization problem via the technique of BSDEs with quadratic growth.

Theorem (Theorem 7 in Hu et al. (2005))

Assume that the risk premium θ and the liability ξ are bounded. The value function of the maximization problem (1) is given by

$$V_c^{\xi}(x) = -\exp(-c(x - Y_0)),$$

where $(Y, Z) \in \mathcal{H}^{\infty}(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ is the unique solution of the following BSDE:

$$Y_t = \xi + \int_t^T f_s(Z_s)ds - \int_t^T Z_s dB_s,\ 0 \le t \le 1,$$

with

$$f_t(z) = \frac{c}{2}\mathrm{dist}^2\left(z + \frac{1}{c}, \tilde{C}\right) - z\theta_t - \frac{1}{2c}|\theta_t|^2.$$

Robust utility maximization

In the classical utility maximization problem, the probability measure $\mathbb{P}$ under consideration is exogenous, that is, the investor knows the probability $\mathbb{P}$ from the historical data. In reality, the investor may have some uncertainty on the probability that means for the investor there can be a collection of probability measures to be considered. Thus, some authors introduced the robust utility maximization problem which can be formulated as follows:

$$V^{\xi}(x) := \sup_{\pi \in \tilde{\mathcal{A}}} \inf_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[U(X_T^{\pi} - \xi)].$$

- classical model, $\mathcal{P}_H$ contains only one probability measure $\mathbb{P}$;
- dominated models, $\forall \mathbb{P} \in \mathcal{P}_H$, $\mathbb{P} << \mathbb{P}_0$, drift uncertainty;
- non-dominated models, volatility uncertainty:
  - Denis & Kervarec (2007), established a duality theory for robust utility maximization and show that there exists a least favorable probability.
  - Matoussi et al. (2012), considered the $\mathcal{P}_H$ of mutually singular probability measure, and solved the problem via 2BSDE technique, under some restrictive assumptions on ξ or $\tilde{C}$.

In this paper, we generalize the result to the robust utility maximization problem obtained in Matoussi et al. (2012) without these assumptions.

The class of probability measures

Let $\Omega := \{\omega : \omega \in \mathcal{C}([0,1], \mathbb{R}^d), \omega(0) = 0\}$ be the canonical space equipped with the uniform norm $||\omega||_1^\infty := \sup_{0 \le t \le 1} |\omega_t|$, B the canonical process, $\mathcal{F}$ the filtration generated by B, $\mathcal{F}^+$ the right limit of $\mathcal{F}$.

By Soner et al. (2010), $\mathbb{P}$ is said to be a local martingale measure if B is a local martingale under $\mathbb{P}$. We can define the quadratic variation of B universally for all local martingale measure $\mathbb{P}$ and its density as

$$\hat{a}_t := \overline{\lim_{\varepsilon \downarrow 0}} \frac{1}{\varepsilon}(\langle B \rangle_t - \langle B \rangle_{t-\varepsilon}),\ \mathbb{P} - a.s..$$

We denote $\overline{\mathcal{P}}_W$ the collection of all local martingale measure $\mathbb{P}$ such that $\langle B \rangle_t$ is absolutely continuous in t and $\hat{a}$ takes values in $\mathbb{S}_d^+$, $\mathbb{P}$-a.s.. It is easy to verify that the stochastic integral under $\mathbb{P}$,

$$W_t^{\mathbb{P}} := \int_0^t \hat{a}_t^{-1/2} dB_s,\ 0 \le t \le 1,$$

defines a $\mathbb{P}$-Brownian motion. Moreover, we denote by $\overline{\mathcal{P}}_S$ a subclass of the measures induced by strong formulation in $\overline{\mathcal{P}}_W$.

The nonlinear generator

The nonlinear generator is a map $F_t(\omega, y, z, a) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_{F_t(y,z)} \to \mathbb{R}$, where $D_{F_t(y,z)} \subset \mathbb{S}_d^+$ is the domain of F in a for a fixed (t, ω, y, z). For simplicity, we set

$$\hat{F}_t(y,z) := F_t(y, z, \hat{a}_t), \textit{ and } \mathbb{F}_t^0 := \hat{F}_t(0,0).$$

(A1) *$D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z);*
(A2) *F is $\mathcal{F}$-progressively measurable, and uniformly continuous in ω under the uniform norm;*
(A3) *F is continuous and has a quadratic growth, i.e. there exists a triple $(\alpha, \beta, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ such that*

$$|F_t(\omega, y, z, a)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|a^{1/2}z|^2;$$

(A4) *F is uniform Lipschitz in y, i.e. there exists a $\mu > 0$, such that*

$$|F_t(\omega, y, z, a) - F_t(\omega, y', z, a)| \leq \mu|y - y'|;$$

(A5) *F is local Lipschitz in z, i.e. there exist a $C > 0$ such that*

$$|F_t(\omega, y, z, a) - F_t(\omega, y, z', a)| \leq C(1 + |a^{1/2}z| + |a^{1/2}z'|)|a^{1/2}(z - z')|.$$

The spaces and the norms

We consider a restricted class of probability measures $\mathcal{P}_H \subset \bar{\mathcal{P}}_S$ defined by the following:

Definition (Collection of the probability measures)

Let $\mathcal{P}_H$ denote the collection of all those $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that

$$\underline{a} \leq \hat{a} \leq \overline{a} \text{ and } \hat{a}_t \in D_{F_t},\ \lambda \times \mathbb{P} - a.s.,$$

for some $\underline{a}$, $\overline{a} \in \mathbb{S}_d^+$.

Definition (Quasi surely)

A property holds $\mathcal{P}_H$-quasi surely if it holds $\mathbb{P}$-almost surely for all $\mathbb{P} \in \mathcal{P}_H$.

Let $\mathbb{L}_H^\infty$ denote the space of all $\mathcal{F}_1$-measurable scalar random variable ξ with

$$||\xi||_{\mathbb{L}_H^\infty} := \sup_{\mathbb{P}\in\mathcal{P}_H} ||\xi||_{L^\infty(\mathbb{P})} < +\infty.$$

We denote by $UC_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \to \mathbb{R}$ with respect to the uniform norm, and we denote by $\mathcal{L}_H^\infty$ the closure of $UC_b(\Omega)$ under the norm$|| \cdot ||_{\mathbb{L}_H^\infty}$.

Let $\mathbb{D}_H^\infty$ denote the space of all $\mathbb{R}$-valued $\mathcal{F}^+$-progressively measurable process Y which satisfies

$$\mathcal{P}_H - q.s.\ c\grave{a}dl\grave{a}g\ and\ ||Y||_{\mathbb{D}_H^\infty} := \sup_{0\le t\le 1} ||Y_t||_{\mathbb{L}_H^\infty} < +\infty,$$

and $\mathbb{H}_H^2$ denote the space of all $\mathbb{R}^d$-valued $\mathcal{F}^+$-progressively measurable process Z which satisfies

$$||Z||^2_{\mathbb{H}_H^2} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt\right] < +\infty.$$

BMO martingale generator under $\mathcal{P}_H$

Definition (BMO martingale under $\mathcal{P}_H$)

Let $\mathcal{M}^2(\mathcal{P}_H)$ be the collection of all $\mathcal{P}_H$-square integrable martingale on $[0,1]$ i.e. for each process $H \in \mathcal{M}^2(\mathcal{P}_H)$,

$$\sup_{\mathbb{P}\in\mathcal{P}} \sup_{0\le t\le 1} \mathbb{E}^{\mathbb{P}}[H_t^2] < +\infty \text{ and } H \text{ is a } \mathbb{P}-\text{martingale}, \ \forall \mathbb{P} \in \mathcal{P}_H.$$

Furthermore, a process $H \in \mathcal{M}^2(\mathcal{P}_H)$ is said to be a BMO($\mathcal{P}_H$)-martingale if

$$||H||_{BMO(\mathcal{P}_H)} := \sup_{\mathbb{P}\in\mathcal{P}_H} \sup_{\tau\in\mathcal{T}_0^1} ||\mathbb{E}^{\mathbb{P}}[\langle H\rangle_1 - \langle H\rangle_\tau | \mathcal{F}_\tau]||_{L^\infty(\mathbb{P})} < +\infty.$$

where $\mathcal{T}_0^1$ is the collection of all $\mathcal{F}$-stopping times $0 \le \tau \le 1$.

Definition (BMO martingale generator under $\mathcal{P}_H$)

A process $Z \in \mathbb{H}^2_H$ is said to be a BMO($\mathcal{P}_H$)-martingale generator if

$$\begin{aligned} ||Z||^2_{\mathbb{H}^2_{BMO(\mathcal{P}_H)}} &:= \sup_{\mathbb{P}\in\mathcal{P}_H} \left|\left| \int_0^\cdot Z_t dB_t \right|\right|_{BMO(\mathbb{P})} \\ &= \sup_{\mathbb{P}\in\mathcal{P}_H} \sup_{\tau\in\mathcal{T}_0^1} \left|\left| \mathbb{E}^{\mathbb{P}}_\tau \left[\int_\tau^1 |\hat{a}_t^{1/2} Z_t|^2 dt \right] \right|\right|_{L^\infty(\mathbb{P})} < +\infty. \end{aligned}$$

We denote by $\mathbb{H}^2_{BMO(\mathcal{P}_H)}$ the collection of all BMO($\mathcal{P}_H$)-martingale generators.

Formulation of 2BSDEs

Definition (Solution to 2BSDE)

We say $(Y, Z) \in \mathbb{D}^{\infty}_{H} \times \mathbb{H}^2_H$ is a solution to the following 2BSDE:

$$Y_t = \xi + \int_t^1 \hat{F}_s(Y_s, Z_s)ds - \int_t^1 Z_s dB_s + K_1 - K_t,\ 0 \le t \le 1,\ \mathcal{P}_H - q.s.. \quad (2)$$

if the following conditions are satisfied:
- $Y_1 = \xi$, $\mathcal{P}_H$-q.s.;
- The process $K^{\mathbb{P}}$ is defined as below, for all $\mathbb{P} \in \mathcal{P}_H$,

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \le t \le 1,\ \mathbb{P} - a.s., \quad (3)$$

and it has non-decreasing paths $\mathcal{P}_H$-q.s.;
- The family $\{K^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_H\}$ satisfies the minimum condition, for all $\mathbb{P} \in \mathcal{P}_H$,

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[K_1^{\mathbb{P}'}],\ 0 \le t \le 1,\ \mathbb{P} - a.s.. \quad (4)$$

Representation theorem

We first introduce a lemma in Possamai and Zhou (2012). The parallel version of the lemma for BSDEs plays a very important role to show the connection between the boundness of Y and the BMO property of the martingale part. We note that the following lemma only depends on assumption of the quadratic growth in coefficients, i.e. (A3).

Lemma

We assume (A1)-(A3) and $\xi \in \mathbb{L}^\infty_H$. *If* $(Y, Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ *is a solution to 2BSDE (2), then* $Z \in \mathbb{H}^2_{BMO(\mathcal{P}_H)}$.

By the procedure in Soner et al. (2011), we consider under each $\mathbb{P}$ the classical BSDE:

$$y^{\mathbb{P}}_s = \eta + \int_s^t \hat{F}_u(y^{\mathbb{P}}_u, z^{\mathbb{P}}_u)du - \int_s^t z^{\mathbb{P}}_u dB_u,\ 0 \le s \le t,\ \mathbb{P} - a.s., \tag{5}$$

where $0 \le t \le 1$ and η is a $\mathcal{F}_t$-measurable random variable in $L^\infty(\mathbb{P})$. Under (A1)-(A5), the BSDE (5) admits a unique solution $(y^{\mathbb{P}}(t, \eta), z^{\mathbb{P}}(t, \eta))$. (cf. Kobylanski (2000), Hu et al. (2005) and Morlais (2009)).

Theorem (Representation theorem)

Let (A1)-(A5) hold. Assume that $\xi \in \mathbb{L}^\infty_H$ *and* $(Y, Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ *is a solution to 2BSDE (2). Then, for all* $\mathbb{P} \in \mathcal{P}_H$ *and* $0 \leq t_1 \leq t_2 \leq 1$,

$$Y_{t_1} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})} y^{\mathbb{P}'}_{t_1}(t_2, Y_{t_2}), \ \mathbb{P} - a.s.. \tag{6}$$

Consequently, the 2BSDE (2) has at most one solution in $\mathbb{D}^\infty_H \times \mathbb{H}^2_H$.

A priori estimate

Lemma

Let (A1)-(A3) hold, and assume that $\xi \in \mathbb{L}^\infty_H$ and that $(Y, Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a solution to 2BSDE (2). Then, there exists a $C > 0$ such that

$$||Y||_{\mathbb{D}^\infty_H} \leq C(1 + ||\xi||_{\mathbb{L}^\infty_H}).$$

We note that in the proof of Lemma 3.1 in Possamai and Zhou (2012),

$$\mathbb{E}^{\mathbb{P}}_\tau\left[\int_\tau^1 |\hat{a}_t^{1/2} Z_t|^2\right] \leq \frac{1}{\gamma^2} e^{4\gamma||Y||_{\mathbb{D}^\infty_H}}(1 + 2\gamma(\alpha + \beta||Y||_{\mathbb{D}^\infty_H})),$$

for arbitrage $\mathbb{P} \in \mathcal{P}_H$ and $0 \leq \tau \leq 1$. Thus, we have for some $C > 0$,

$$||Z||^2_{\mathbb{H}^2_{BMO(\mathcal{P}_H)}} \leq Ce^{4\gamma||\xi||_{\mathbb{L}^\infty_H}}(1 + ||\xi||_{\mathbb{L}^\infty_H}).$$

Furthermore, from the proof of Representation theorem, for $p \geq 1$,

$$\sup_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[(K_1^{\mathbb{P}})^p] \leq c_p(1 + ||\xi||^p_{\mathbb{L}^\infty_H} + ||Y||^p_{\mathbb{D}^\infty_H} + ||Z||^p_{\mathbb{H}^2_{BMO(\mathcal{P}_H)}} + ||Z||^{2p}_{\mathbb{H}^2_{BMO(\mathcal{P}_H)}}).$$

Lemma

Let (A1)-(A5) hold, and assume that $\xi^i \in \mathbb{L}_H^\infty$, $i = 1, 2$ *and that* $(Y^i, Z^i) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$, $i = 1, 2$, *are two solution to 2BSDE (2). Denote*

$$\delta\xi := \xi^1 - \xi^2,\ \delta Y := Y^1 - Y^2, \delta Z := Z^1 - Z^2,\ \text{and } \delta K^{\mathbb{P}} := (K^1)^{\mathbb{P}} - (K^2)^{\mathbb{P}}.$$

Then, there exists a $C > 0$ *such that*

$$||\delta Y||_{\mathbb{D}_H^\infty} \leq C||\delta\xi||_{\mathbb{L}_H^\infty},$$

$$||\delta Z||_{H^2_{BMO(\mathcal{P}_H)}} \leq C||\delta\xi||^2_{\mathbb{L}_H^\infty} \sum_{i=1}^{2} (1 + e^{4\gamma||\xi^i||_{\mathbb{L}_H^\infty}})(1 + ||\xi^i||_{\mathbb{L}_H^\infty}),$$

and

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[\sup_{0\leq t\leq 1} |\delta K_t^{\mathbb{P}}|^p] \leq C_p||\delta\xi||^p_{\mathbb{L}_H^\infty} \sum_{i=1}^{2} (1 + e^{4p\gamma||\xi^i||_{\mathbb{L}_H^\infty}})(1 + ||\xi^i||^p_{\mathbb{L}_H^\infty}).$$

Existence result

Following the procedure in Soner et al. (2010) and Possamai & Zhou (2012), and with the help of the technique so-called regular conditional probability distributions, we can construct a solution to the 2BSDE (2) pathwisely when the terminal condition belongs to the space $UC_b(\Omega)$. Thus, we have

Theorem

Under (A1)-(A5) and for $\xi \in UC_b(\Omega)$, the 2BSDE (2) has a unique solution $(Y, Z) \in \mathbb{D}_H^{\infty} \times \mathbb{H}_H^2$.

For each $\xi \in \mathcal{L}_H^{\infty}$, we can find a sequence $\{\xi^n\}_{n \geq 0} \subset UC_b(\Omega)$, such that $||\xi^n - \xi||_{\mathbb{L}_H^{\infty}} \to 0$ as $n \to +\infty$. Thanks to the prior estimates, we can obtain the following main result.

Theorem

Under (A1)-(A5) and for $\xi \in \mathcal{L}_H^{\infty}$, the 2BSDE (2) has a unique solution $(Y, Z) \in \mathbb{D}_H^{\infty} \times \mathbb{H}_H^2$.

The main difference in our proof to the existence result from those by other authors is that we prove the following lemma, which shows the relation between the solution to (5) and that on a shifted space.

Lemma

Assume (A1)-(A5) holds. For a fixed $\mathbb{P}$, $y^{\mathbb{P}}(1,\xi)$ *is a solution to (5). Then, we have*

$$y_t^{\mathbb{P}}(1,\xi)(\omega) = y_t^{\mathbb{P}^{t,\omega},t,\omega}(1,\xi), \text{ for } \omega \in \Omega, \ \mathbb{P}-a.s.. \tag{7}$$

Remark: Notice that Soner et al. (2010) assumed that the generator F satisfies the Lipschitz condition, and the existence result in Possamai & Zhou (2012) is based on the assumptions given by Tevzadze (2008). Under their assumptions, the solution of BSDEs can be constructed via Picard iteration, one can easily verify this lemma by replacing both sides of (7) by their representations in the form of conditional expectation.

Application to finance

Recall the robust utility maximization problem we mentioned in the first section:

$$V^{\xi}(x) := \sup_{\pi \in \tilde{\mathcal{A}}} \inf_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[U(X_T^{\pi} - \xi)].$$

We assume that the $\mathcal{P}_H$ is a set containing of some probability measures $\mathbb{P} \in \bar{\mathcal{P}}_S$ which are mutually singular and for some $\underline{a}$ and $\overline{a}$, $\underline{a} \leq \hat{a} \leq \overline{a}$, $\lambda \times \mathbb{P}$-a.s..

The financial market consists of one bond with zero interest rate and d stocks. The price process of the stocks is give by the following stochastic differential equations:

$$dS_t^i = S_t^i(b_t^i dt + dB_t^i),\ 0 \leq t \leq 1,\ i = 1, 2, \ldots, d,\ \mathcal{P}_H - q.s.,$$

where b^i is an $\mathbb{R}$-valued uniformly bounded process which is uniformly continuous in ω under the uniform norm, $i = 1, 2, \ldots, d$. Indeed, under each $\mathbb{P} \in \mathcal{P}_H$, $dB_s = \hat{a}_t^{1/2} dW_t^{\mathbb{P}}$, thus, $\hat{a}^{1/2}$ plays the role of volatility. The difference of $\hat{a}^{1/2}$ under each $\mathbb{P}$ allows us to model the volatility uncertainty.

We give in addition some assumptions stronger than uniformly integrability on trading strategies, that is, $\pi \in \mathbb{H}^2_{BMO(\mathcal{P}_H)}$. Then, we have the following definition to the set of all admissible trading strategies.

Definition

Let $\tilde{C}$ be a closed set in $\mathbb{R}^d$. The set of admissible trading strategies $\tilde{A}$ consists of all d-dimensional progressively measurable processes $\pi = \{\pi_t\}_{0 \le t \le 1}$ taking values in $\tilde{C}$, $\lambda \otimes \mathcal{P}_H$-q.s. and satisfy $\pi \in \mathbb{H}^2_{BMO(\mathcal{P}_H)}$.

We still consider the case that the utility is in form of an exponential function, i.e.

$$U_c(x) = -\exp(-cx), \; x \in \mathbb{R}.$$

Then, the maximization problem can be rewritten as

$$V_c^\xi(x) := \sup_{\pi \in \tilde{\mathcal{A}}} \inf_{\mathbb{P} \in \mathcal{P}_H} E^{\mathbb{P}}\Big[-\exp\Big(-c\Big(x + \int_0^1 \pi_t \sigma_t (b_s ds + dB_s) - \xi\Big)\Big)\Big], \quad (8)$$

where x is the initial wealth.

Similar to that in Matoussi et al. (2012), we have the following theorem:

Theorem

Assume that ξ is an $\mathcal{F}_1$-measurable random variable in $\mathcal{L}_H^\infty$. The value function of the maximization problem (8) is given by

$$V_c^\xi(x) = -\exp(-c(x - Y_0)),$$

where Y_0 is defined by the unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ of the following 2BSDE:

$$Y_t = \xi + \int_t^T \hat{F}_s(Z_s) ds - \int_t^T Z_s dB_s + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H, \tag{9}$$

where for $(\omega, t, z) \in \Omega \times [0,1] \times \mathbb{R}^d$ and a fixed $a \in \mathbb{S}_d^+$, $\underline{a} \leq a \leq \overline{a}$,

$$\hat{F}_t(\omega, z, a) := \frac{c}{2} \operatorname{dist}^2 \left(a^{1/2} z + \frac{1}{c} a^{-1/2} b_t(\omega), a_t^{1/2} \tilde{C} \right) - z b_t(\omega) - \frac{1}{2c} |a^{-1/2} b_t(\omega)|^2. \tag{10}$$

Moreover, there exists an optimal trading strategy $\pi^* \in \tilde{\mathcal{A}}$ with

$$\hat{a}_t^{1/2}\pi_t^* \in \Pi_{\hat{a}_t^{1/2}\tilde{C}}\left(\hat{a}_t^{1/2}Z_t + \frac{1}{c}\hat{a}_t^{-1/2}b_t\right),\ 0 \le t \le 1,\ \mathcal{P}_H - q.s., \tag{11}$$

where $\Pi_A(r)$ denote the collection of some elements in the closed set A realizing the minimal distance to the point r.

Sketch of the proof:

Step 1: That F satisfies (A1) is obvious. From Lemma 11 in Hu et al. (2005) and by the properties of the process b we deduce that (A2) holds true. Because for all $a \in \mathbb{S}_d^+$, $\underline{a} \le a \le \overline{a}$, there exist a $K > 0$ depends on $\overline{a}$ and $\tilde{C}$ such that

$$\inf\{|r| : r \in a^{1/2}\tilde{C}\} \le K.$$

Then, for all $(t, z) \in [0,1] \times \mathbb{R}^d$,

$$\text{dist}^2\left(a^{1/2}z + \frac{1}{c}a^{-1/2}b_t, a^{1/2}\tilde{C}\right) \le 2|a^{1/2}z|^2 + 2\left(\frac{1}{c}|a^{-1/2}b_t| + K\right)^2,$$

from which we have for some α, $\gamma > 0$,

$$|F_t(\omega, z, a)| \le \alpha + \frac{\gamma}{2}|a^{1/2}z|^2.$$

That is to say (A3) is satisfied.

For all $(t, z^1, z^2) \in [0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ and $a \in \mathbb{S}_d^+$, $\underline{a} \leq a \leq \overline{a}$,

$$\begin{aligned}
&F_t(\omega, z^1, a) - F_t(\omega, z^2, a) \\
&= \frac{c}{2}\Bigg(\operatorname{dist}^2\left(a^{1/2} z^1 + \frac{1}{c} a^{-1/2} b_t, a^{1/2} \tilde{C} \right) \\
&- \operatorname{dist}^2\left(a^{1/2} z^2 + \frac{1}{c} a^{-1/2} b_t, a^{1/2} \tilde{C} \right)\Bigg) \\
&- (z^1 - z^2) b_t.
\end{aligned}$$

Using the Lipschitz property of the distance function from a closed set, we obtain the estimate

$$\begin{aligned}
&|F_t(\omega, z^1, a) - F_t(\omega, z^2, a)| \\
&\leq c_1(1 + |a^{-1/2} b_t|)|a^{1/2}(z^1 - z^2)| \\
&+ c_2(|a^{1/2} z^1| + |a^{1/2} z^2|)|a^{1/2}(z^1 - z^2)| \\
&\leq C(1 + |a^{1/2} z^1| + |a^{1/2} z^2|)|a^{1/2}(z^1 - z^2)|,
\end{aligned}$$

from which (A5) is verified, then, the 2BSDE (9) admits a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$.

Step 2: We construct a family $\{R^\pi\}_{\pi \in \tilde{\mathcal{A}}}$ which satisfies the following properties:

- $R_1^\pi = -\exp(-c(X_1^\pi - \xi))$ for all $\pi \in \tilde{\mathcal{A}}$;
- $R_0^\pi = R_0(x)$ is a constant for all $\pi \in \tilde{\mathcal{A}}$;
- for all $\pi \in \tilde{\mathcal{A}}$,

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[-\exp(-c(X_1^\pi - \xi))] \leq R_t^\pi, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H,$$

  and there exist a $\pi^* \in \tilde{\mathcal{A}}$ such that

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[-\exp(-c(X_1^{\pi^*} - \xi))] = R_t^{\pi^*}, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H,$$

It follows that

$$\begin{aligned} &\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}'}[-\exp(-c(X_1^\pi - \xi))] \leq R_0(x) \\ =&\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}'}[-\exp(-c(X_1^{\pi^*} - \xi))] = V_c(x). \end{aligned}$$

From the comparison above, π^* is the desired optimal strategy.

To construct this family, we set for all $\pi \in \tilde{\mathcal{A}}$,

$$R_t^\pi = -\exp(-c(X_t^\pi - Y_t)),\ 0 \le t \le 1,$$

where $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ is the unique solution define by the 2BSDE (9). We write R^π as a product of M^π and A^π, where

$$M_t^\pi := e^{-c(x-Y_0)} \exp\left(\int_0^t c(Z_s - \pi_s)dB_s - \frac{1}{2}\int_0^t c^2|\hat{a}_s^{1/2}(Z_s - \pi_s)^2|ds - cK_t^{\mathbb{P}}\right),$$
$$\mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H.$$

Comparing R^π and $M^\pi A^\pi$ yields

$$A_t^\pi := -\exp\left(\int_0^t \nu(s, Z_s, \pi_s)ds\right),$$

with

$$\nu(t, z, p) := \frac{1}{2}c^2|\hat{a}_t^{1/2}(z - p)|^2 - cpb_t - c\hat{F}_t(z).$$

By completing the square, we can determine the form of $\hat{F}$ (10), such that,

- for a optimal strategy π^* satisfies (11), $\nu(t, Z_t, \pi_t^*) = 0$, $0 \le t \le 1$;
- otherwise, $\nu(t, Z_t, \pi_t) \ge 0$, $0 \le t \le 1$.

Step 3: We verify that

- The optimal strategy $\pi^* \in \mathbb{H}^2_{BMO(\mathcal{P}_H)}$;
- M^π satisfies the maximum condition for all $\pi \in \tilde{\mathcal{A}}$, i.e.

$$\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[M_1^\pi] = M_t^\pi, \ \mathbb{P}-a.s., \ \forall \mathbb{P} \in \mathcal{P}_H;$$

- For all $\pi \in \tilde{\mathcal{A}}$,

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[-\exp(-c(X_T^\pi - \xi))] \le R_t^\pi, \ \mathbb{P}-a.s., \ \forall \mathbb{P} \in \mathcal{P}_H.$$

We complete the proof. $\square$

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