Stochastic expansions for model combining local and stochastic volatility

Romain Bompis (CMAP)
joint work with
Emmanuel Gobet (CMAP)

CIRM Marseille, 16 April 2012

Summary

Summary

Aim and applications

Aim: analytic L^p-approximation of $(X_t)_{0\leq t\leq T}$ solving:

$$\begin{aligned} dX_t =& \sigma_t(X_t)\sqrt{V_t}dW_t,\ X_0 = x_0 \\ dV_t =& \alpha_t dt + \xi_t\sqrt{V_t}dBt,\ V_0 = v_0 > 0, \\ \mathrm{d}\langle W, B\rangle_t =& \rho_t \mathrm{d}t. \end{aligned}$$

Application: price approximations for European options:

$$\mathbb{E}[h(X_T)],$$

- Easily computable,
- With a precise error estimate.

Motivations

Motivations:

- Explicit calculus in closed form not possible in general and need to use a numerical method (PDE, Monte Carlo...).
- Real time computations in finance: fastness is an important operational constraint, find a compromise between rapidity and good adequateness of the model with the reality.
- In particular: fast calibration.
- Control the errors in function of the model parameters in order to choose the best model according to the behaviour of the asset.

Summary

The BGM approach

BGM approach (Gobet, Miri and all (09)): non-asymptotic approach in time-dependent framework:

- Choose a proxy model X^P,
- Find an artificial parameterization to link X and X^P,
- Propose the generic approximation structure:

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^P)] + \sum_i \eta_i \partial_\theta^i \mathbb{E}[h(X_T^P)] + Error,$$

  where $\partial_\theta^i \mathbb{E}[h(X_T^P)]$ are sensibilities in the proxy model and η_i depends of the model parameters,
- Estimate the error according to the model parameters.

Proxy model

Gaussian Proxy model:

$$\begin{aligned} dX_t^P &= \sigma_t(x_0)\sqrt{v_t}\mathrm{d}W_t,\ x_0, \\ dv_t &= \alpha_t \mathrm{d}t,\ v_0. \end{aligned}$$

Justification:

- $|\xi|_\infty$ small and $\sigma_t(.)$ has small variations.
- $|\sigma|_\infty$ small.
- T small.

Parameterization

Parameterization: $\epsilon \in [0,1]$**:**

$$\begin{aligned} dX_t^\epsilon &= \epsilon\sigma_t(X_t^\epsilon)\sqrt{V_t^\epsilon}\mathrm{d}W_t,\ x_0, \\ \mathrm{d}V_t^\epsilon &= \alpha_t \mathrm{d}t + \epsilon\xi_t\sqrt{V_t^\epsilon}\mathrm{d}B_t,\ v_0. \end{aligned}$$

Link between X **and** X^P**:**

- $X_t^1 = X_t$ and $V_t^1 = V_t$,
- $X_t^0 = x_0$ and $V_t^0 = v_t$,
- $X_t^P = x_0 + \underbrace{''\partial_\epsilon(X_t^\epsilon)|_{\epsilon=0}''}_{X_{1,t}}$,

$$dX_{1,t} = \sigma_t(x_0)\sqrt{v_t}\mathrm{d}t,\ X_{1,0} = 0.$$

Summary

Technical hypotheses

Technical hypotheses:

- (E): $\exists C_\sigma \in]0,1]$, $\int_0^T \sigma_t^2(x_0) v_t \mathrm{d}t \geq C_\sigma |\sigma|_\infty^2 T > 0$.
- (R_2): $\sigma \in C^2$ with:
  - $M_1 = max(|\sigma^{(1)}|_\infty, |\sigma^{(2)}|_\infty) < \infty$,
  - $M_0 = max(|\sigma|_\infty, M_1) < \infty$.
- (P): α and ξ are bounded and positive with $\xi_{\inf} > 0$ and $2(\frac{\alpha}{\xi^2})_{\inf} \geq 1$.

Main result

Theorem

Assume (E), (R_2) and (P). Then we have:

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^P)] + C_{1,T}^l \mathcal{G}_3^h(X_T^P) + \frac{C_{1,T}^s}{2} \mathcal{G}_3^h(X_T^P) + Error_{2,h},$$

where:

$$C_{1,T}^l = \int_0^T \sigma_t^2(x_0) v_t \Big(\int_t^T \sigma_s(x_0) \sigma_s^{(1)}(x_0) v_s \mathrm{d}s \Big) \mathrm{d}t,$$

$$C_{1,T}^s = \int_0^T \rho_t \xi_t \sigma_t(x_0) v_t \Big(\int_t^T \sigma_s^2(x_0) \mathrm{d}s \Big) \mathrm{d}t,$$

$$\mathcal{G}_i^h(X_T^P) = \partial_{x^i}^i (\mathbb{E}[h(X_T^P + x)])\Big|_{x=0}.$$

Formal Taylor expansions

Perform formal Taylor expansion twice:

$$X_T = X_T^{\epsilon=1} = X_T^{\epsilon=0} + \partial_\epsilon X_T^\epsilon|_{\epsilon=0} + \frac{1}{2}\partial^2_{\epsilon^2} X_T^\epsilon|_{\epsilon=0} + \dots \quad ,$$

$$= \underbrace{x_0 + X_{1,T}}_{X_T^P} + \underbrace{\frac{X_{2,T}}{2}}_{double\ Wiener\ integral} + \dots \quad .$$

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^P + \frac{X_{2,T}}{2} + \dots)] = \mathbb{E}[h(X_T^P)] + \mathbb{E}[h^{(1)}(X_T^P)\frac{X_{2,T}}{2}] + \dots \quad .$$

Transformation of the corrective term in sensibilities

- **Employ regularization argument if h not smooth enough,**
- **Use Malliavin duality relationship and integration by parts formulas (see Nualart (06)) to obtain:**

$$\mathbb{E}[h^{(1)}(\underbrace{X_T^P}_{Wiener\ integral})\underbrace{\frac{X_{2,T}}{2}}_{double\ Wiener\ integral}] = C^l_{1,T}\mathbb{E}[h^{(3)}(X_T^P)] + \frac{C^s_{1,T}}{2}\mathbb{E}[h^{(3)}(X_T^P)]$$

$$= C^l_{1,T}\mathcal{G}_3^h(X_T^P) + \frac{C^s_{1,T}}{2}\mathcal{G}_3^h(X_T^P).$$

Summary

Main result

Theorem

Assume that (R_2), (E), (P) *hold and that* h *is bounded, a.e. differentiable with bounded derivative. Then we have:*

$$|\mathrm{Error}_{2,\mathrm{h}}| \leq_c |h^{(1)}|_\infty \, |\sigma|_\infty [|\xi|_\infty^2 + M_1(M_0 + |\xi|_\infty)] \, T^{\frac{3}{2}}.$$

proof relies on (heavy) technical Lemmas. Main tools are Malliavin integration by parts formula and small noise perturbation to smooth h and to overcome degeneracy problems.

Approximation of Call options for the Piterbarg model

Analytical approximation of:

$$\begin{aligned} C(T,K) =& \mathbb{E}[(S_T - K)_+], \\ dS_t =& \lambda \sqrt{V_t}((1-\beta) + \beta S_t)\mathrm{d}W_t, \; S_0 = 1, \\ dV_t =& \kappa(1 - V_t)\mathrm{d}t + \xi \sqrt{V_t}\mathrm{d}B_t, \; V_0 = 1, \\ \mathrm{d}\langle W_t, B_t\rangle =& \rho \mathrm{d}t. \end{aligned}$$

If $\beta \approx 1$ and ξ small, lognormal proxy model:

$$e^{X_T^P},$$

$$dX_T^P = \sigma(x_0)\sqrt{v_t}\mathrm{d}W_t - \frac{1}{2}\sigma^2(x_0)v_t\mathrm{d}t, \; X_0 = x_0 = 0,$$

$$\sigma(x) = \lambda((1-\beta)e^{-x} + \beta).$$

Third order approximation accuracy: parameters

Table: Set of maturities and strikes used for the numerical tests.

T/K							
3M	70%	80%	90%	100%	110%	125%	135%
1Y	55%	75%	90%	100%	115%	140%	180%
3Y	35%	55%	80%	100%	125%	175%	270%
10Y	15%	35%	65%	100%	150%	275%	365%

Values of model parameters:

- $\lambda = 0.2$,
- $\beta = 0.8$,
- $\kappa = 3$,
- $\xi = 1.5$,
- $\rho = -20\%$.

Table: Call prices of the closed formula, of the third order approximation formula and related errors (in bp):

3M	30.00%	20.09%	10.80%	3.93%	0.85%	0.12%	0.01%
	30.00%	20.09%	10.80%	3.93%	0.85%	0.12%	0.04%
	0.13	**-0.26**	**-0.27**	**0.33**	**0.38**	**-0.03**	**-0.23**
1Y	45.04%	25.81%	13.68%	7.82%	2.75%	0.33%	0.01%
	45.04%	25.83%	13.67%	7.80%	2.74%	0.34%	0.01%
	0.32	**-2.04**	**0.93**	**2.12**	**1.67**	**-0.43**	**-0.45**
3Y	65.07%	45.77%	25.15%	13.61%	5.49%	0.68%	0.004%
	65.06%	45.79%	25.13%	13.57%	5.47%	0.69%	0.005%
	0.23	**-2.20**	**1.42**	**3.96**	**1.88**	**-1.34**	**-0.43**
10Y	85.19%	66.52%	43.27%	24.79%	10.77%	1.42%	0.38%
	85.20%	66.54%	43.23%	24.72%	10.76%	1.46%	0.40%
	-0.34	**-1.93**	**4.26**	**7.61**	**1.63**	**-3.32**	**-1.76**

Summary

Conclusion

- The accuracy of the approximations is very satisfying: the errors do not exceed few bps for various maturities and strikes.
- The computational cost is very cheap in comparison to Fourier inversion, PDE approach or Monte Carlo simulations.
- Extensions of this expansion methodology for path-dependent payoffs like barrier options is left for further research.

Summary

Bibliography

E. Benhamou, E. Gobet, and M. Miri. Smart expansion and fast calibration for jump diffusion. Finance and Stochastics, 13(4) :563-589, 2009.

E. Benhamou, E. Gobet, and M. Miri. Expansion formulas for European options in a local volatility model. International Journal of Theoretical and Applied Finance, 13(4) :603-634, 2010.

E. Benhamou, E. Gobet, and M. Miri. Time dependent Heston model. SIAM Journal on Financial Mathematics, 1 :289-325, 2010.

D. Nualart. Malliavin Calculus and related topics. Springer-Verlag, Berlin, second edition, 2006. 2011.

Thank you for your attention!