Spectral analysis of restricted put operators and RND recovery

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3. Spectral recovery method
   - Grid and market quotes
   - Continuous no-arbitrage constraints
   - Extended bid-ask constraints
   - Quadratic program

No arbitrage pricing theory refresher

Assets are modeled by semimartingales, most often diffusion processes (continuous-time Markov processes with continuous paths).

Theorem (First Fundamental Theorem of Asset Pricing)

(see [Musiela and Rutkowski, 2008, p.72]) market is arbitrage-free if and only if there exists a martingale measure $\mathbb{Q}$ (under which discounted price processes are martingales).

Theorem (Second Fundamental Theorem of Asset Pricing)

(see [Musiela and Rutkowski, 2008, p.73]) $\mathbb{Q}$ is unique if and only if the market is complete (any contingent claim can be replicated with a self-financing portfolio).

Notations

- Underlying S worth S_τ at maturity τ.
- European option pays off $\pi(S_\tau)$ at maturity τ.
- Assume the risk-neutral law of S_τ admits a density, $\mathbb{Q}(S_\tau \in dx) = q(x)dx$.
- Derivative price = discounted expected payoff under $\mathbb{Q}$ (see [Cox and Ross, 1976]).

$$P_\pi = e^{-r\tau}\mathbb{E}_{\mathbb{Q}}\pi(S_\tau) = e^{-r\tau}\int_{x\geq 0}\pi(x)q(x)dx,$$

Objective

Recover q from option prices

Why are we interested in q ?

$\Rightarrow$ Because q captures the subjective forward looking view of the market
See [Bahra, 1997] for the Bank of England.

- Assessing monetary credibility (ex. compare inflation target and implied forward inflation rate risk-neutral distribution).
- Assessing timing and effectiveness of monetary operations (ex. impact of money market operations on risk-neutral rates distribution)
- Identifying market anomalies (ex. does the market assume that a sudden price move today will last)

From prices to RND

Objective

Recover q from liquid vanilla options (efficiently priced)

- call options $\theta(S_\tau, \xi) = (S_\tau - \xi)^+$ (insurance against price going up).
- put options $\theta^*(S_\tau, \xi) = (\xi - S_\tau)^+$.

$$C : \xi \in \mathbb{R}^+ \mapsto C(\xi) = e^{-r\tau}\mathbb{E}_{\mathbb{Q}}\theta(S_\tau, \xi),$$
$$P(\xi) = e^{-r\tau}\mathbb{E}_{\mathbb{Q}}\theta^*(S_\tau, \xi)$$

Data set

[Breeden and Litzenberger, 1978]

$$e^{-r\tau}q(\xi) = \frac{\partial^2 P(\xi)}{\partial \xi^2} = \frac{\partial^2 C(\xi)}{\partial \xi^2}$$

- Price quotes in the cross section $P(\xi_i), 1 \leq i \leq n$, where n ranges from 5 to 50 depending on the underlying market.
- In fact an "ask" price (at which you can buy the option) and a "bid" price (at which you can sell).

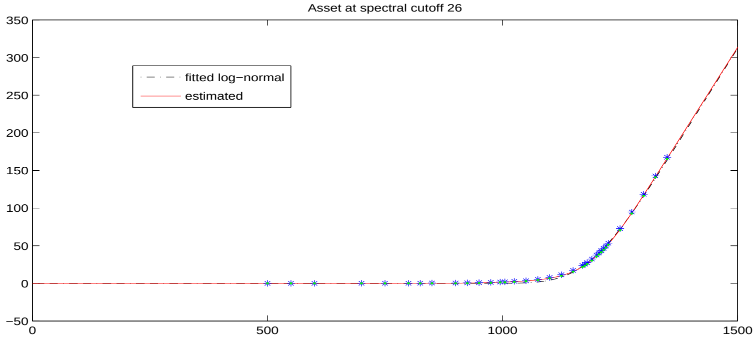
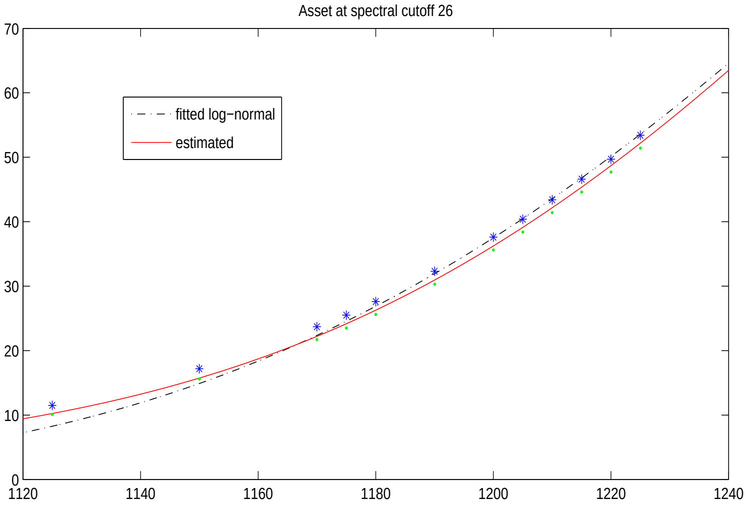


Figure: S&P 500 put option prices, Jan. 5, 2005. S&P 500 Index closing level $= 1183.74$; Option expiration $= 03/18/2005$ (72 days); $r = 2.69\%$; $\delta = 1.70\%$.


Asset at spectral cutoff 26
fitted log−normal
estimated
70
60
50
40
30
20
10
0
1120
1140
1160
1180
1200
1220
1240

Warning

- Breeden-Litzenberger cannot be applied as such !!
- Several q are compatible with quoted prices, how to choose q ?
- How do people do in practice ? (see [Bahra, 1997])

Criterion

q chosen according to a criterion typically related to its smoothness or information content.

How to proceed ?

- Parametric methods,
- Nonparametric methods,
- Models of the underlying price process.

Each of them have their pros and cons and we focus on NONPARAMETRIC methods (non-implementable in [Bahra, 1997] !!)

- The expansion methods. It includes the Edgeworth (see [Jarrow and Rudd, 1982]), cumulant (see [Potters et al., 1998]) and orthonormal basis expansions.
- The kernel regression methods. Shape constrained local polynomial estimator of the RND. (Requires pre-processing of the average prices and information content or smoothness not controlled).
- The maximum entropy method. See [Buchen and Kelly, 1996, Stutzer, 1996], where the RND q is obtained via the maximization of an entropy criterion. (multi-modal estimates and convergence issues [Jackwerth and Rubinstein, 1996, p.1620])
- Other methods. Among others, smoothed implied volatility smile method (SML) as in [Figlewski, 2008]. Outperformed in term of accuracy and stability by simpler parametric methods in [Bu and Hadri, 2007].

2 Singular Value Decomposition

Set $\mathcal{I} = [0, B]$.

- Restricted call operator γ from $\mathbb{L}_2\mathcal{I}$ into itself.

$$(\gamma f)(\xi) = \int_{\mathcal{I}} \theta(\xi, x) f(x) dx, \qquad \xi \in \mathcal{I}, f \in \mathbb{L}_2\mathcal{I},$$
$$\theta(\xi, x) = (x - \xi)^+.$$

- Its adjoint γ^* is nothing else than the restricted put operator.

$$(\gamma^* f)(\xi) = \int_{\mathcal{I}} \theta^*(\xi, x) f(x) dx, \qquad \xi \in \mathcal{I}, f \in \mathbb{L}_2\mathcal{I},$$
$$\theta^*(\xi, x) = \theta(x, \xi).$$

which coincides with the regular put operator on $\mathcal{I}$.

Theorem (Spectral theorem $\Rightarrow$)

- *There exist two orthonormal bases (φ_k) and (ψ_k) of $\mathbb{L}_2\mathcal{I}$ and a postitive decreasing sequence of singular values λ_k, such that, for all $k \geq 0$,*

$$\gamma\varphi_k = \lambda_k\psi_k, \qquad \gamma^*\psi_k = \lambda_k\varphi_k.$$

- *Furthermore, they verify*

$$\lambda_k\partial_\xi^2\psi_k = \varphi_k,$$
$$\lambda_k\partial_\xi^2\varphi_k = \psi_k.$$

We have furthermore the following spectral decomposition,

$$f = \sum_{k\geq 0} \langle f, \psi_k \rangle \psi_k, \qquad f \in \mathbb{L}_2\mathcal{I},$$

$$\gamma^* f = \sum_{k\geq 0} \lambda_k \langle f, \psi_k \rangle \varphi_k, \qquad f \in \mathbb{L}_2\mathcal{I}.$$

Question

Can you see the estimation strategy coming ?

Theorem

The eigenvectors (φ_k) *of* $\gamma^*\gamma$ *and* (ψ_k) *of* $\gamma\gamma^*$ *are such that*

$$\varphi_k = h_{k,1} + h_{k,2}, \qquad \psi_k = h_{k,1} - h_{k,2}$$

where

$$h_{k,1}(\xi) = a_{k,1} e^{\rho_k \xi / B} + a_{k,2} e^{-\rho_k \xi / B}$$
$$h_{k,2}(\xi) = a_{k,3} \cos(\rho_k t / B) + a_{k,4} \sin(\rho_k \xi / B).$$

Theorem

Furthermore

$$\lambda_k = \left(\frac{B}{\rho_k}\right)^2,$$

where

$$\rho_k = \frac{\pi}{2} + k\pi + (-1)^k \beta_k, \qquad k \in \mathbb{N},$$

and, for all $k \in \mathbb{N}$, β_k is the smallest positive solution of the following fixed point equation in u,

$$\exp(\pi/2 + k\pi + (-1)^k u) = \frac{1 + \cos(u)}{\sin(u)}.$$

So that $\beta_k = O(e^{-k\pi})$.

Theorem

And finally

$$a_{k,1} = \frac{1}{\sqrt{B}} \frac{(-1)^k}{e^{\rho_k} + (-1)^k},$$

$$a_{k,2} = (-1)^k e^{\rho_k} a_{k,1} = \frac{1}{\sqrt{B}} \frac{1}{1 + (-1)^k e^{-\rho_k}},$$

$$a_{k,3} = -\frac{1}{\sqrt{B}},$$

$$a_{k,4} = \frac{1}{\sqrt{B}} \frac{1 - (-1)^k e^{-\rho_k}}{1 + (-1)^k e^{-\rho_k}}.$$

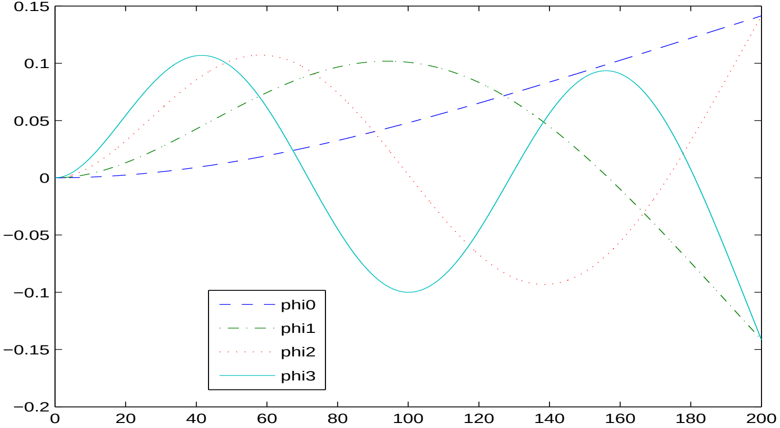


Figure: φ_k, $k = 0, \ldots, 3$

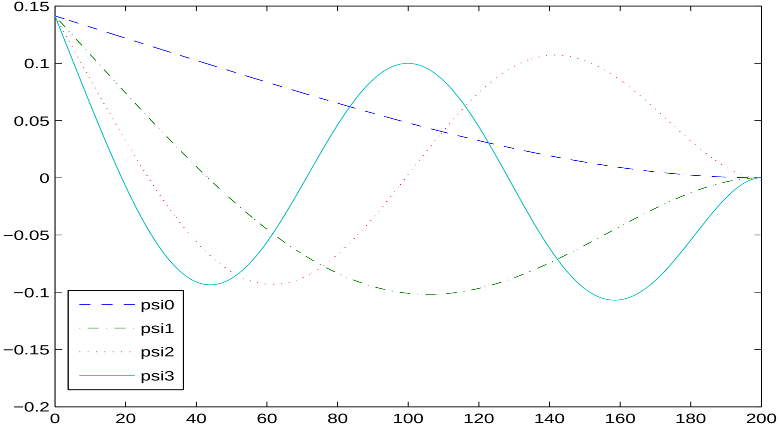


Figure: ψ_k, $k = 0, \ldots, 3$

3 Spectral recovery method

- Grid and market quotes
- Continuous no-arbitrage constraints
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From put prices to RND

Our strategy

Look for the smoothest RND whose corresponding put prices lie inside the bid-ask quotes.

- The market provides us with $y_1^{Ask}, \ldots, y_s^{Ask}$ and $y_1^{Bid}, \ldots, y_s^{Bid}$ at strike prices $\xi_1 < \xi_2 < \ldots < \xi_s$, where $5 \leq s \leq 50$.
- Define a new (denser) grid of strikes $\xi_1 < \xi_2 < \ldots < \xi_n$ such that $\xi_1 = 0$, $\xi_n = B$. We denote by $I = \{i_1, \ldots, i_s\}$ the subset of $\{1, \ldots, n\}$ corresponding to the indexes of the initial quoted strikes.

No-arbitrage constraints

$$\max(0, \xi e^{-r\tau} - S_0 e^{-\delta\tau}) \leq P(\xi) \leq \xi e^{-r\tau}, \quad (1)$$
$$0 \leq \partial_\xi P(\xi) \leq e^{-r\tau}, \quad (2)$$
$$0 \leq \partial_\xi^2 P(\xi). \quad (3)$$

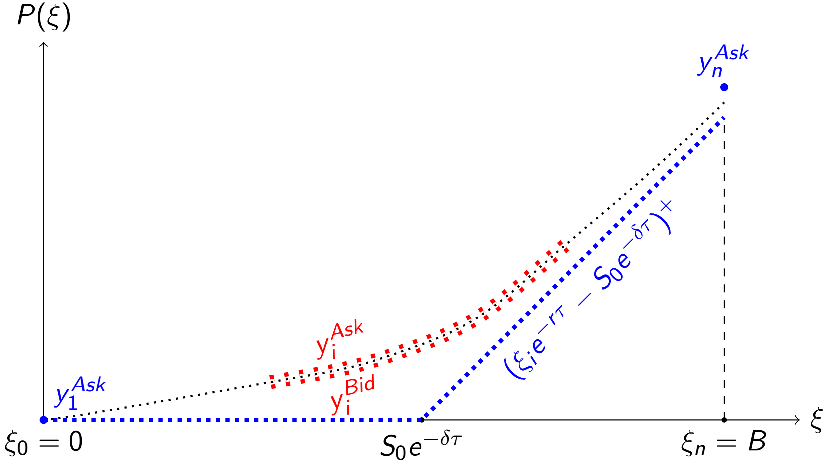
- Translate into AFFINE CONSTRAINTS on put prices $m_1, \ldots, m_n$ on the grid $\xi_1 < \xi_2 < ... < \xi_n$ (see [Aït-Sahalia and Duarte, 2003]).
- $$Am \leq b_p,$$

  where A stands for a $2n \times n$ matrix, m is the $n \times 1$ vector such that $m^T = \begin{bmatrix} m_1 & \ldots & m_n \end{bmatrix}$ and b_p is a $2n \times 1$ vector.

Bid-ask constraints

- We must have $0 = P(0) = m_1$, so that we define $y_1^{Ask} = 0$.
- $P(\xi)$ cannot grow at a rate faster than $e^{-r\tau}$, so that we can define y_n^{Ask} to be the corresponding linear extrapolation of the right-most market quote $y_{i_s}^{Ask}$, meaning $y_n^{Ask} = y_{i_s}^{Ask} + e^{-r\tau}(\xi_n - \xi_{i_s})$.
- Boils down to $2s + 2$ constraints,

$$\begin{aligned} m_i &\leq y_i^{Ask}, & i &\in I \cup \{1, n\}, \\ -m_i &\leq -y_i^{Bid}, & i &\in I. \end{aligned}$$


$P(\xi)$
y_n^{Ask}
y_i^{Ask}
y_i^{Bid}
y_1^{Ask}
$(\xi_i e^{-r\tau} - S_0 e^{-\delta\tau})^+$
ξ
$\xi_0 = 0$
$S_0 e^{-\delta\tau}$
$\xi_n = B$

Spectral recovery method

Notice that $P_N = \gamma^* e^{-r\tau} q_N$, where,

$$P_N = \sum_{k=0}^{N} \omega_k \varphi_k, \qquad q_N = e^{r\tau} \sum_{k=0}^{N} \lambda_k^{-1} \omega_k \psi_k.$$

As a consequence, $\omega^\star$ is solution of,

$$\arg \min_{\omega \in \mathbb{R}^{N+1}} \underbrace{\|\partial_\xi^2 q_N\|_{\mathbb{L}_2}^2}_{\text{smoothness criterion}} \quad \text{subject to} \begin{cases} [P_N]_{I \cup \{1,n\}} & \leq y^{Ask}_{I \cup \{1,n\}}, \\ -[P_N]_I & \leq -y^{Bid}_I, \\ A P_N & \leq b_p, \\ q_N(0) & = 0. \end{cases} \qquad \text{(P1')}$$

where $P_N^T = \begin{bmatrix} P_N(\xi_1) & \dots & P_N(\xi_n) \end{bmatrix}$.

Canonical form

$$\arg \min_{\omega \in \mathbb{R}^{N+1}} \frac{1}{2} \omega^T \Omega_N^4 \omega \qquad \text{subject to} \begin{cases} \Phi_{I \cup \{1,n\}} \omega & \leq y_{I \cup \{1,n\}}^{Ask}, \\ -\Phi_I \omega & \leq -y_I^{Bid}, \\ A \Phi \omega & \leq b_p, \\ \psi_{0,N}(0)^T \Omega_N \omega & = 0. \end{cases}$$

Denote by $\varphi_{0,N}(\xi)^T = \begin{bmatrix} \varphi_0(\xi) & \dots & \varphi_N(\xi) \end{bmatrix}$ and, similarly, write $\psi_{0,N}(\xi)^T$. Then we have $[P_N]_i = \varphi_{0,N}(\xi_i)^T \omega$ and $q_N(\xi) = \psi_{0,N}(\xi)^T \Omega_N \omega$, where

$$\Omega_N = Diag(\lambda_0^{-1}, \dots, \lambda_N^{-1}), \tag{4}$$

And Φ the matrix whose rows are constituted by the $\varphi_{0,N}(\xi_i)^T$, $i = 1, \dots, n$.

- If the set of constraints is feasible in $\mathbb{L}_2\mathcal{I}$, then the QP admits a solution for N large enough since the family (φ_k) is complete in $\mathbb{L}_2\mathcal{I}$.
- We look for the smoothest density q satisfying the constraints. The higher k, the more ψ_k oscillates. It is natural to think that the smaller N, the smoother q_N. We choose N to be the smallest integer for which the QP admits a solution.

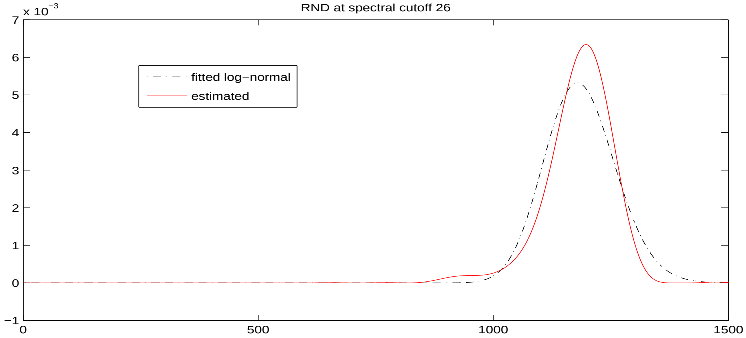


Figure: Here we plot the RND $q^{\star}_{26}$ (solid line) estimated from the real price quotes on the S&P500. We choose $B = 1.4 * F_0 = 1.4 * S_0 * e^{(r-\delta)\tau} = 1660$ for that plot. We display the full left tail of the RND $q^{\star}_{26}$.

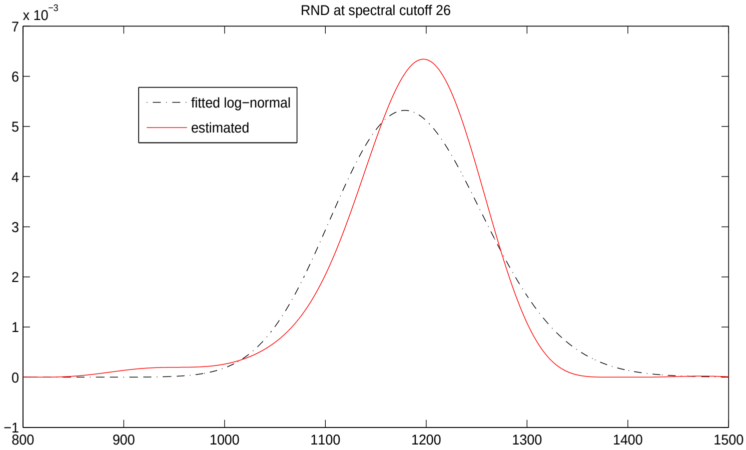


Figure: We zoom in on the fat left tail of the estimated RND distribution.

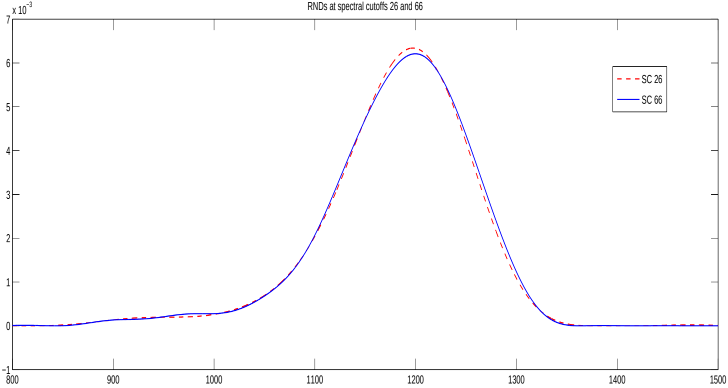


Figure: We choose $B = 2 * F_0 = 2 * S_0 * e^{(r-\delta)\tau} = 2372$ for that plot. We superimpose $q^{\star}_{66}$ (solid line) with $q^{\star}_{26}$ (dashed line) obtained for an other choice of B. Smoothness goes hand in hand with low spectral cutoff.

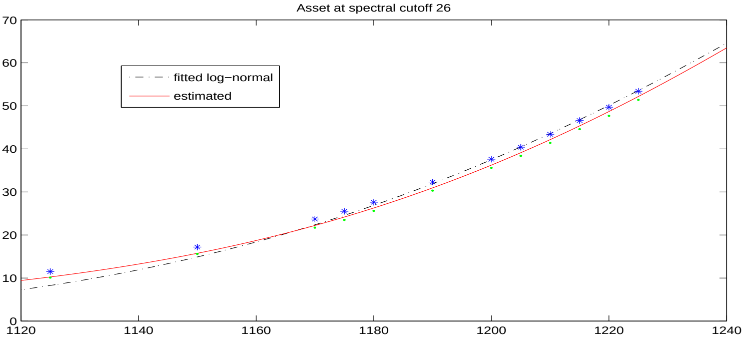


Figure: S&P 500 put option prices, Jan. 5, 2005. S&P 500 Index closing level = 1183.74; Option expiration = 03/18/2005 (72 days); $r = 2.69\%$; $\delta = 1.70\%$.

What's next ?

- What about other simple payoffs ? binary options, etc.
- Generalization to bivariate RND estimation via spread option prices ? (partial Radon transform, copulae)

Thank you !

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- (3) translates into $n-2$ constraints $1 \leq i \leq n-2$,

$$[Am]_i := \frac{m_{i+1} - m_i}{\xi_{i+1} - \xi_i} - \frac{m_{i+2} - m_{i+1}}{\xi_{i+2} - \xi_{i+1}} \leq 0 := [b_p]_i,$$

- lhs of (1) corresponds to n additional constraints $1 \leq i \leq n$,

$$[Am]_{i+n-2} := -m_i \leq -\max(0, \xi_i e^{-r\tau} - S_0 e^{-\delta\tau}) := [b_p]_{i+n-2}, \quad (5)$$

rhs of (1) need not be taken into account at this stage.

- (2) reduces to two additional constraints,

$$[Am]_{2n-1} := \frac{m_n - m_{n-1}}{\xi_n - \xi_{n-1}} \leq e^{-rT} := [b_p]_{2n-1},$$
$$[Am]_{2n} := m_1 - m_2 \leq 0 := [b_p]_{2n}.$$