American put option in a model with discrete dividends

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April 18, 2012

- Introduction
- Reduction to simple problems without dividends
- Existence and continuity of the exercise boundary
  - Existence
  - Continuity
- Smooth-fit
- Equivalent near the dividend date
- Conclusion

Put option contract with maturity (or life-time) T and strike K on an underlying S :

- European type : at the end of the life-time of the option, the buyer gets $(K - S_T)^+$.
- American type : during all the life-time of the option, the buyer can ask to stop and gets $(K - S_t)^+$

Fair value of the put option contract with maturity (or life-time) T and strike K on an underlying S :

- European type : the fair value is $\mathbb{E}\left[e^{-rT}(K - S_T)^+\right]$.
- American type : the fair value is $\sup_{\tau\in[0,T]} \mathbb{E}\left[e^{-r\tau}(K - S_\tau)^+\right]$.

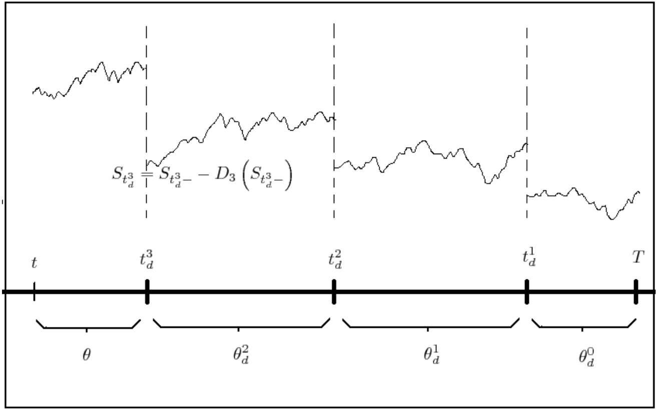


Figure: A trajectory of the stock price process

- Between dividend dates, dynamics of the stock price process S is the standard Black Scholes dynamics (parameters r and σ) i.e : $dS_t = rS_t dt + \sigma S_t dB_t$
- At a dividend date, t_d^i, $S_{t_d^i} = S_{t_d^i -} - D_i\left(S_{t_d^i -}\right)$ where D_i is a deterministic function satisfying the following assumptions $\forall i \in \{1, \ldots, I\}$:

$$\begin{cases} \text{(a) } D_i \text{ is } \uparrow \text{ and non-negative,} \\ \text{(b) } \rho_i : x \mapsto x - D_i(x) \text{ is } \uparrow \text{ and non-negative.} \end{cases}$$

Aim : Obtain the value of the American Put option with strike K and maturity T on an underlying paying discrete dividends. Since we are in a Markovian framework, the price can be characterized in terms of a value function depending of the time t and the stock price at time t.

As $t \mapsto (K - S_t)^+$ is upper semi-continuous, general optimal stopping time theory is valid (cf [EK81]) :

- $u(t,x) = \sup_{\tau \in [t,T]} \mathbb{E}\left[e^{-r(\tau - t)} (K - S_\tau)^+ \middle| S_t = x \right]$,
- $\tau^\star = \inf \{ v \geq t | u(v, S_v) \leq (K - S_v)^+ \} \wedge T$ is optimal.

Introduction

Reduction to simple problems without dividends

Existence and continuity of the exercise boundary
    Existence
    Continuity

Smooth-fit

Equivalent near the dividend date

Conclusion

Let $(\theta_d^i = t_d^i - t_d^{i+1})_{0\le i\le I-1}$ with the convention $t_d^0 = T$ denote the durations between the dividend dates. For non-negative values of θ and x, we define by induction

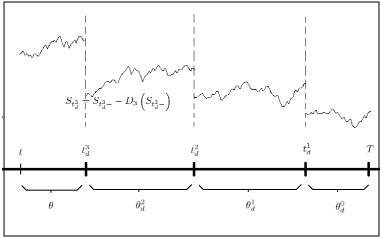
- value function : $u_0(\theta,x) = \sup_{\tau\in[0,\theta]} \mathbb{E}\left[e^{-r\tau}\left(K-\bar{S}_\tau^x\right)^+\right]$
  - exercise boundary : $c_0(\theta)$,
$\{x : u(0,x) > (K-x)^+\} = (c_0(\theta), +\infty)$,
  - $\bar{c}(\theta)$ exercise boundary associated to $K=1$.
  - $c_0(\theta) = \sup\{x | u_0(\theta,x) = (K-x)^+\} = K\bar{c}(\theta)$.
- $\forall i \in \{1,\dots,I\}$,

$$u_i(\theta,x) = \sup_{\tau\in[0,\theta]} \mathbb{E}\left[\begin{array}{l} e^{-r\tau}\left(K-\bar{S}_\tau^x\right)^+ \mathbf{1}_{\{\tau<\theta\}} \\ +e^{-r\theta}u_{i-1}(\theta_d^{i-1}, \bar{S}_\theta^x - D_i(\bar{S}_\theta^x))\mathbf{1}_{\{\tau=\theta\}} \end{array}\right].$$

Note that $u_i(0,x) = u_{i-1}(\theta_d^{i-1}, x - D_i(x))$.

Proposition

Suppose that $t < t_d^i < t_d^{i-1} < \cdots < t_d^1 < T$ *and set* $\theta = t_d^i - t$, $\theta_d^0 = T - t_d^1$, *and for* $j = 1 \dots i-1$, $\theta_d^j = t_d^j - t_d^{j+1}$, *then the value at time* t *when the spot price of the stock is equal to* x *of the American put option with strike* K *and maturity* T *is given by* $u_i(\theta, x)$.



Introduction

Reduction to simple problems without dividends

Existence and continuity of the exercise boundary
  - Existence
  - Continuity

Smooth-fit

Equivalent near the dividend date

Conclusion

Here are Lemmas from [JV11].

Lemma

For each $\theta \geq 0$*, the mapping* $x \mapsto u_i(\theta, x)$ *is non-increasing and* $x \mapsto x + u_i(\theta, x)$ *is non-decreasing.*

Corollary (Exercise boundary)

For any $\theta \geq 0$*, it exists* $c_i(\theta) \in [0, K)$ *such that :*

$u_i(\theta, x) > (K - x)^+ \Leftrightarrow x > c_i(\theta)$

$c_i(\theta) = \sup \{x \geq 0 | u_i(\theta, x) = (K - x)^+\}$

Lemma

The mapping $(\theta, x) \mapsto u_i(\theta, x)$ *is continuous on* $\mathbb{R}_+ \times \mathbb{R}_+$

- Upper semi-continuity of the exercise boundary is a direct consequence of the global continuity of the value function.
- Right continuity ($\lim_{\theta' \downarrow \theta} c_i(\theta') = c_i(\theta)$) comes from a comparaison with respect to the case without dividends.

$$
\begin{aligned}
u_i(\theta + t, x) =& \mathbb{E}\left[e^{-r\tau} \left(\mathbf{1}_{\{\tau < t\}} \left(K - \bar{S}^x_\tau\right)^+ + \mathbf{1}_{\{\tau = t\}} u_i(\theta, \bar{S}^x_t) \right) \right] \\
\leq& \mathbb{E}\left[e^{-r\tau} \left(\mathbf{1}_{\{\tau < t\}} \left(K - \bar{S}^x_\tau\right)^+ + \mathbf{1}_{\{\tau = t\}} \left(K - \left[c_i(\theta) \wedge \bar{S}^x_t\right]\right)^+ \right) \right. \\
\leq& \mathbb{E}\left[e^{-r\tau} \left(K - \left[c_i(\theta) + (K - c_i(\theta))(1 - e^{-r(t-\tau)}) \wedge \bar{S}^x_\tau \right] \right)^+ \right] \\
=& (K - c_i(\theta)) e^{-rt} + \mathbb{E}\left[e^{-r\tau} \left(c_i(\theta) + (K - c_i(\theta))(1 - e^{-r(t-\tau)}) - \bar{S}^x_\tau \right)^+ \right] \\
\leq& (K - c_i(\theta)) e^{-rt} + \mathbb{E}\left[e^{-r\tau} \left(c_i(\theta) + (K - c_i(\theta))(1 - e^{-rt}) - \bar{S}^x_\tau \right)^+ \right] \\
\leq& (K - c_i(\theta)) e^{-rt} + \left(c_i(\theta) + (K - c_i(\theta))(1 - e^{-rt}) \right) v\left(t, \frac{x}{c_i(\theta) + (K - c_i(\theta))(1 - e^{-rt})} \right)
\end{aligned}
$$

$$u_i(\theta+t,x)\leq(K-c_i(\theta))e^{-rt}+\Big(c_i(\theta)+(K-c_i(\theta))(1-e^{-rt})\Big)v\left(t,\frac{x}{c_i(\theta)+(K-c_i(\theta))(1-e^{-rt})}\right)$$

- The previous equation gives us exactly $c_i(\theta+t)\geq\left(c_i(\theta)+(K-c_i(\theta))(1-e^{-rt})\right)\bar{c}(t)$.

  By [KS91], it gives us right-continuity.
- Tools to get right-continuity :
  - continuity of the value function (u.s.c)
  - exponential model.

- Left-continuity is a consequence of :

Proposition

Under (A), the property

$$(P_i) : \textit{For any } \theta > 0 \textit{ and } x \geq 0 \textit{ one has}$$
$$x > c_i(\theta) \Longleftrightarrow 1 + \partial_x u_i(\theta, x) > 0$$

holds for any $i \in \{0, \cdots, I\}$.

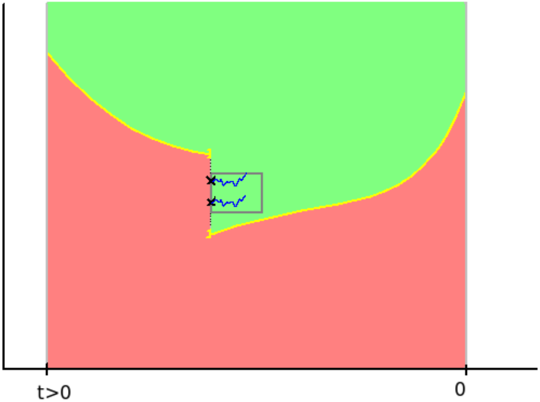


Figure: Proof of left continuity

Introduction

Reduction to simple problems without dividends

Existence and continuity of the exercise boundary
- Existence
- Continuity

Smooth-fit

Equivalent near the dividend date

Conclusion

Proposition (Smooth-fit)

For all $\theta > 0$, $u_i(\theta, \bullet)$ is $\mathcal{C}^1$.

It relies on the two following Lemmas :

Lemma

$(\theta, x) \mapsto u(\theta, x)$ is a viscosity supersolution of $\min(\partial_\theta u_i(\theta, x) - \mathcal{A}u_i(\theta, \bullet)(x), u_i(\theta, x) - (K - x)^+) = 0$ with $u_i(0, x) = u_{i-1}(\theta_d^{i-1}, \rho_i(x))$

Lemma

For any $i \geq 0$, $\theta > 0$ and $x \geq 0$ one has for some constant C

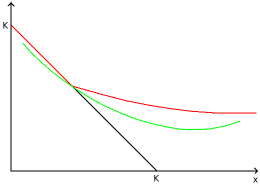
$$\limsup_{\theta' \to \theta} \left| \frac{u_i(\theta', x) - u_i(\theta, x)}{\theta' - \theta} \right| \leq (1 + x) \left(C + \frac{\sigma}{\sqrt{2\pi\theta}} \right)$$

Moreover $\partial_x u_i(\theta, x)$ admits a right-hand limit at $c_i(\theta)$ denoted by $\partial_x u_i(\theta, c_i(\theta)^+)$ and $\partial_x u_i(\theta, c_i(\theta)^+) \in [-1, 0]$.

Standard viscosity argument

$$\Phi(\theta+t,x) =$$
$$e^{-\beta t}\left(K - c_i(\theta) + p(x-c_i(\theta)) + \tfrac{1}{2\varepsilon}\left(x-c_i(\theta)\right)^2\right)$$

$\min_V u_i - \Phi = u_i - \Phi|_{(\theta,c_i(\theta))} = 0$



viscosity super-solution property implies :

$$-\partial_t \Phi + \mathcal{A}\Phi|_{(\theta,c_i(\theta))} \leq 0$$
$$-(\beta+r)(K - c_i(\theta)) + rpc_i(\theta) + \frac{\sigma^2}{2}\frac{c_i(\theta)^2}{\varepsilon} \leq 0$$

Introduction

Reduction to simple problems without dividends

Existence and continuity of the exercise boundary
  Existence
  Continuity

Smooth-fit

Equivalent near the dividend date

Conclusion

Lemma

Assume that $c_i(0) > 0$, $\liminf_{x \downarrow c_i(0)} \frac{D_i(x)}{x - c_i(0)} > 0$ *then* $c_i(\theta) - c_i(0) \sim_{\theta \downarrow 0} -\sigma c_i(0)\sqrt{\theta\,|\ln \theta|}$*.*

This result generalizes the following result from [JV11]

Lemma

Assume that D_i *is concave and* $c_i(0) = 0$*, then if* $\frac{x}{D_i(x)}$ *admits a finite right-hand limit at* $x = 0$*,* $c_i(\theta) \sim_{\theta \to 0^+} rK\theta \lim_{x \to 0^+} \frac{x}{D_i(x)}$*.*

Conclusion

Proposition

Assume that $c_i(0) < c_{i-1}(\theta_d^{i-1})$*, that* D_i *is the difference of two convex functions, and that the positive part of the Jordan-Hahn decomposition of the measure* D_i'' *is absolutely continuous with respect to the Lebesgue measure.*
Assume moreover that, if g_i *denotes the density of the absolutely continuous part of the measure* D_i''*, it exists* $\varepsilon \in (0, c_{i-1}(\theta_d^{i-1}) - c_i(0))$ *and* $C_1 \in [0, +\infty)$ *such that*

$$\text{on } (c_i(0), c_i(0)+\varepsilon], \qquad -rD_i(x) + rxD_i'(x) + \frac{\sigma^2 x^2}{2} g_i(x) \leq rK - \varepsilon,$$

$$\forall x > c_i(0)+\varepsilon,\ g_i(x) \leq C_1 x^{C_1}.$$

Then it exists a neighborhood of $(0, c_i(0))$ *in* $\mathbb{R}_+ \times \mathbb{R}_+$ *such that* u_i *is non-increasing w.r.t* θ *in this neighborhood. Moreover the exercise boundary* c_i *is non-decreasing in a neighborhood of* 0*.*

Proposition

Assume that $c_i(0) < c_{i-1}(\theta_d^{i-1})$, *that* D_i *is the difference of two convex functions, and that the negative part of the Jordan-Hahn decomposition of the measure* D_i'' *is absolutely continuous with respect to the Lebesgue measure.*
Assume moreover that, if g_i *denotes the density of the absolutely continuous part of the measure* D_i''*, it exists* $\varepsilon \in (0, c_{i-1}(\theta_d^{i-1}) - c_i(0))$ *and* $C_1 \in [0, +\infty)$ *such that*

$$\text{on } (c_i(0), c_i(0)+\varepsilon],\ D_i \text{ is } C^2$$

$$\text{and such that} - rD_i(x) + rxD_i'(x) + \frac{\sigma^2 x^2}{2} g_i(x) \geq rK + \varepsilon,$$

$$\forall x > c_i(0) + \varepsilon,\ g_i(x) \leq -C_1 x^{C_1}.$$

Then it exists a neighborhood of $(0, c_i(0))$ *in* $\mathbb{R}_+ \times \mathbb{R}_+$ *such that* u_i *is non-decreasing w.r.t* θ *in this neighborhood. Moreover the exercise boundary* c_i *is non-increasing in a neighborhood of* 0*.*

Introduction

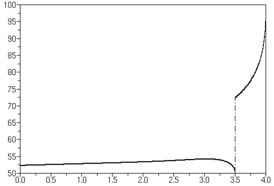
Reduction to simple problems without dividends

Existence and continuity of the exercise boundary
  Existence
  Continuity

Smooth-fit

Equivalent near the dividend date

Conclusion



(a) $D_1(x) = \min\left(\frac{1125}{32}, \frac{8}{1125}\left((x-50)^+\right)^2\right)$

(b) $D_1(x) = 0.05(x-50)^+$

Figure: Exercise boundaries of an American put option of maturity 4 with one dividend time at 3.5 for different dividend functions. Strike is 100, diffusion parameters are $r = 0.04$ and $\sigma = 0.3$.

(a) Maturity is 2 with one dividend time at 1.7; $D_1(x) = \frac{1}{5}\left((x-20)^+ - (x-30)^+\right)$

(b) Maturity is 0.1 with one dividend time at 0.05; $D_1(x) = \min\left(\frac{9}{8}, \frac{2}{9}\left((x-20)^+\right)^2\right)$

Figure: Exercise boundaries of an American put option with different maturities for different dividend functions. Strike is 100, diffusion parameters are $r = 0.04$ and $\sigma = 0.3$.

N. El Karoui.
Les aspects probabilistes du contrôle stochastique.
In *Ninth Saint Flour Probability Summer School—1979 (Saint Flour, 1979)*, volume 876 of *Lecture Notes in Math.*, pages 73–238. Springer, Berlin, 1981.

M. Jeunesse and B. Jourdain.
Regularity of the American put option in the Black Scholes model with general discrete dividends.
Stochastic Processes and their Applications, To appear.

B. Jourdain and M. Vellekoop.
Regularity of the Exercise Boundary for American Put Options on Assets with Discrete Dividends.
SIAM Journal on Mathematical Finance, 2:538–561, 2011.

I. Karatzas and S. E. Shreve.
Methods of mathematical finance.
Springer, 1991.

Thank you for your attention.

Questions ?