Conditional Markov chains and credit risk in the Lévy Libor model

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I. Libor models and credit risk – main problems and tools

Short overview

- In mathematical finance defaultable interest rate models are often obtained by adding the defaultable term structure to the existing default-free term structure models in an appropriate way

- Various defaultable extensions of the Heath–Jarrow–Morton (HJM) modeling methodology are found in the literature, whereas credit risk in Libor market models is far less studied (only papers by Lotz and Schlögl (2000), Schönbucher (2000), Eberlein, Kluge, and Schönbucher (2006), a series of papers by Brigo (2005, 2006, 2008), Li and Rutkowski (2010))

- None of the existing defaultable Libor market models incorporates ratings and credit migration

Libor market models – notation

Discrete tenor structure: $0 = T_0 < T_1 < \ldots < T_n = T^*$, with $\delta_k = T_{k+1} - T_k$

Default-free zero coupon bonds: $B(\cdot, T_1), \ldots, B(\cdot, T_n)$

Forward Libor rate at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$

$$L(t, T_k) = \frac{1}{\delta_k}\left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1\right)$$

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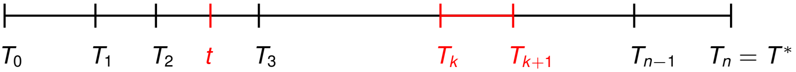


Defaultable zero coupon bonds: $B_C(\cdot, T_1), \ldots, B_C(\cdot, T_n)$

Defaultable forward Libor rate at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$, defined on the set $\{\tau > t\}$

$$L_C(t, T_k) = \frac{1}{\delta_k}\left(\frac{B_C(t, T_k)}{B_C(t, T_{k+1})} - 1\right)$$

Defaultable bonds with ratings

- Credit ratings identified with elements of a finite set $\mathcal{K} = \{1, 2, \ldots, K\}$, where 1 is the best possible rating and K is the default event

- Credit migration is modeled by a conditional Markov chain C with state space $\mathcal{K}$, where K is the absorbing state

- Default time τ: the first time when C reaches state K, i.e.

$$\tau = \inf\{t > 0 : C_t = K\}$$

- Defaultable bonds with credit migration process C and fractional recovery of Treasury value $q = (q_1, \ldots, q_{K-1})$ upon default:

$$B_C(t, T_k) = \sum_{i=1}^{K-1} B_i(t, T_k)\mathbf{1}_{\{C_t=i\}} + q_{C_{\tau-}} B(t, T_k)\mathbf{1}_{\{C_t=K\}}$$

Rating-dependent Libor rates

- The forward Libor rate for credit rating class i

$$L_i(t, T_k) := \frac{1}{\delta_k}\left(\frac{B_i(t, T_k)}{B_i(t, T_{k+1})} - 1\right), \quad i = 1, 2, \ldots, K-1$$

  We put $L_0(t, T_k) := L(t, T_k)$ (default-free Libor rates).

- The corresponding discrete-tenor forward inter-rating spreads

$$H_i(t, T_k) := \frac{L_i(t, T_k) - L_{i-1}(t, T_k)}{1 + \delta_k L_{i-1}(t, T_k)}$$

Libor modeling

- Modeling under forward martingale measures, i.e. risk-neutral measures that use zero-coupon bonds as numeraires
- On a given stochastic basis, construct a family of Libor rates $L(\cdot, T_k)$ and a collection of mutually equivalent probability measures $\mathbb{Q}_{T_k}$ such that

$$\left(\frac{B(t, T_j)}{B(t, T_k)}\right)_{0 \le t \le T_k \wedge T_j}$$

  are $\mathbb{Q}_{T_k}$-local martingales

- In addition model defaultable Libor rates $L_C(\cdot, T_k)$ such that

$$\left(\frac{B_C(t, T_j)}{B(t, T_k)}\right)_{0 \le t \le T_k \wedge T_j}$$

  are $\mathbb{Q}_{T_k}$-local martingales

Risk-free Libor models

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathbb{P}_{T^*})$ be a complete stochastic basis.

- As driving process take any special semimartingale X, which satisfies certain integrability conditions and has canonical decomposition

$$X_t = \int_0^t b_s \mathrm{d}s + \int_0^t \sqrt{c_s} \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x),$$

- W^{T^*} denotes a $\mathbb{P}_{T^*}$-standard Brownian motion and
  μ is the random measure of jumps of X with $\mathbb{P}_{T^*}$-compensator ν^{T^*}
- We assume $b = 0$.

Construction of Libor rates (backward induction)

General step: for each T_k

(i) Define the forward measure $\mathbb{P}_{T_{k+1}}$ via

$$\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T^*}}\bigg|_{\mathcal{F}_t} = \prod_{l=k+1}^{n-1} \frac{1+\delta_l L(t,T_l)}{1+\delta_l L(0,T_l)} = \frac{B(0,T^*)}{B(0,T_{k+1})}\frac{B(t,T_{k+1})}{B(t,T^*)}.$$

(ii) The dynamics of the Libor rate $L(\cdot, T_k)$ under this measure

$$L(t,T_k) = L(0,T_k)\exp\left(\int_0^t b^L(s,T_k)\mathrm{d}s + \int_0^t \sigma(s,T_k)\mathrm{d}X_s^{T_{k+1}}\right), \tag{1}$$

where

$$X_t^{T_{k+1}} = \int_0^t \sqrt{c_s}\,\mathrm{d}W_s^{T_{k+1}} + \int_0^t\int_{\mathbb{R}^d} x(\mu - \nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)$$

with $\mathbb{P}_{T_{k+1}}$-Brownian motion $W^{T_{k+1}}$ and

$$\nu^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x) = \prod_{l=k+1}^{n-1}\left(\frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}(e^{\langle\sigma(s,T_l),x\rangle}-1)+1\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x).$$

The drift term $b^L(s,T_k)$ is chosen such that $L(\cdot,T_k)$ becomes a $\mathbb{P}_{T_{k+1}}$-martingale.

- The backward induction construction was first proposed in Musiela and Rutkowski (1997)

- The forward bond price processes

$$\left(\frac{B(t, T_j)}{B(t, T_k)}\right)_{0 \le t \le T_j \wedge T_k}$$

are martingales under the forward measure $\mathbb{P}_{T_k}$ associated with the date T_k by construction (for all $j, k = 1, \ldots, n$).

- The arbitrage-free price at time t of a contingent claim with payoff X at maturity T_k, π_t^X, is given by

$$\pi_t^X = B(t, T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[X|\mathcal{F}_t].$$

II. Rating based Lévy Libor model

Adding credit risk to the Lévy Libor model

- Take driving process X to be time-inhomogeneous Lévy process (independent, but not stationary increments) with triplet $(0, c, F^{T^*})$
$\Rightarrow$ Lévy Libor model (Eberlein and Özkan (2005))

- Enlarge probability space: $(\Omega, \mathcal{F}, \mathbb{P}_{T^*}) \to (\widetilde{\Omega}, \mathcal{G}, \mathbb{Q}_{T^*})$
and construct the credit migration process C as a conditional Markov chain with stochastic, $\mathbb{F}$-adapted infinitesimal generator $[\lambda_{ij}(t)]_{i,j=1,\dots,K}$
(canonical construction, Bielecki and Rutkowski (2002))

- The process C is a *conditional Markov chain* relative to $\mathbb{F}$, i.e. for every $0 \leq t \leq s$ and any function $h : \mathcal{K} \to \mathbb{R}$

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_t \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_t \vee \sigma(C_t)],$$

where $\mathbb{F}^C = (\mathcal{F}_t^C)$ denotes the filtration generated by C

- The enlarged filtration $(\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_t^C)_{0 \leq t \leq T^*}$ satisfies the $(\mathcal{H})$-hypothesis
(canonical construction + Brémaud and Yor (1978) or Elliot, Jeanblanc and Yor (2000))

Consequences:

- *X remains* a time-inhomogeneous Lévy process with respect to $\mathbb{Q}_{T^*}$ and $\mathbb{G}$ with the *same* characteristics

- Define on this space the forward measures $\mathbb{Q}_{T_k}$ by:

  for each tenor date T_k $\mathbb{Q}_{T_k}$ is obtained from $\mathbb{Q}_{T^*}$ in the same way as $\mathbb{P}_{T_k}$ from $\mathbb{P}_{T^*}$ ($k = 1, \ldots, n-1$)

- Then

$$\frac{\mathrm{d}\mathbb{Q}_{T_k}}{\mathrm{d}\mathbb{Q}_{T^*}} = \psi^k,$$

  where ψ^k is $\mathcal{F}_{T_k}$-measurable.

Conditional Markov chain C *under forward measures*

Theorem

Let C be a canonically constructed conditional Markov chain with respect to $\mathbb{Q}_{T^*}$*. Then C is a conditional Markov chain with respect to every forward measure* $\mathbb{Q}_{T_k}$ *and*

$$p_{ij}^{\mathbb{Q}_{T_k}}(t,s) = p_{ij}^{\mathbb{Q}_{T^*}}(t,s)$$

i.e. the matrices of transition probabilities under $\mathbb{Q}_{T^*}$ *and* $\mathbb{Q}_{T_k}$ *are the same.*

Theorem

The $(\mathcal{H})$*-hypothesis holds under all* $\mathbb{Q}_{T_k}$*, i.e. every* $(\mathbb{F}, \mathbb{Q}_{T_k})$*-local martingale is a* $(\mathbb{G}, \mathbb{Q}_{T_k})$*-local martingale.*

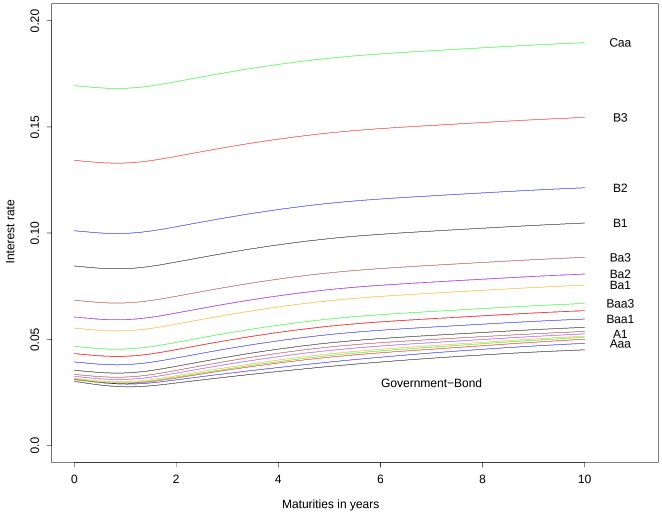
Recall that the Libor rate for the rating *i* can be expressed as

$$
\begin{aligned}
1+\delta_k L_i(t,T_k) &= (1+\delta_k L_{i-1}(t,T_k))(1+\delta_k H_i(t,T_k)) \\
&= \underbrace{(1+\delta_k L(t,T_k))}_{\text{default-free Libor}} \prod_{j=1}^{i} \underbrace{(1+\delta_k H_j(t,T_k))}_{\text{spread } j-1\to j}
\end{aligned}
$$

Hence: model $H_j(\cdot, T_k)$ as exponential semimartingales and thus ensure automatically the *monotonicity* of Libor rates w.r.t. the credit rating:

$$L(t,T_k) \le L_1(t,T_k) \le \cdots \le L_{K-1}(t,T_k)$$

$\Longrightarrow$ worse credit rating, higher interest rate



Interest rate
0.0
0.05
0.10
0.15
0.20
Caa
B3
B2
B1
Ba3
Ba2
Ba1
Baa3
Baa1
A1
Aaa
Government−Bond
0
2
4
6
8
10
Maturities in years

Pre-default term structure of rating-dependent Libor rates

For each rating i and tenor date T_k we model $H_i(\cdot, T_k)$ as

$$H_i(t, T_k) = H_i(0, T_k) \exp\left(\int_0^t b^{H_i}(s, T_k) \mathrm{d}s + \int_0^t \gamma_i(s, T_k) \mathrm{d}X_s^{T_{k+1}} \right) \tag{2}$$

with initial condition

$$H_i(0, T_k) = \frac{1}{\delta_k} \left(\frac{B_i(0, T_k) B_{i-1}(0, T_{k+1})}{B_{i-1}(0, T_k) B_i(0, T_{k+1})} - 1 \right).$$

$X^{T_{k+1}}$ is defined as earlier and $b^{H_i}(s, T_k)$ is the drift term (we assume $b^{H_i}(s, T_k) = 0$, for $s > T_k \Rightarrow H_i(t, T_k) = H_i(T_k, T_k)$, for $t \geq T_k$).

$\Rightarrow$ the forward Libor rate $L_i(\cdot, T_k)$ is obtained from relation

$$1 + \delta_k L_i(t, T_k) = (1 + \delta_k L(t, T_k)) \prod_{j=1}^{i} (1 + \delta_k H_j(t, T_k)).$$

Theorem

Assume that $L(\cdot, T_k)$ and $H_i(\cdot, T_k)$ are given by (1) and (2). Then:

(a) *The rating-dependent forward Libor rates satisfy for every T_k and $t \leq T_k$*

$$L(t, T_k) \leq L_1(t, T_k) \leq \cdots \leq L_{K-1}(t, T_k),$$

i.e. Libor rates are monotone with respect to credit ratings.

(b) *The dynamics of the Libor rate $L_i(\cdot, T_k)$ under $\mathbb{P}_{T_{k+1}}$ is given by*

$$L_i(t, T_k) = L_i(0, T_k) \exp\left(\int_0^t b^{L_i}(s, T_k)\mathrm{d}s + \int_0^t \sqrt{c_s}\sigma_i(s, T_k)\mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} S_i(s, x, T_k)(\mu - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x) \right).$$

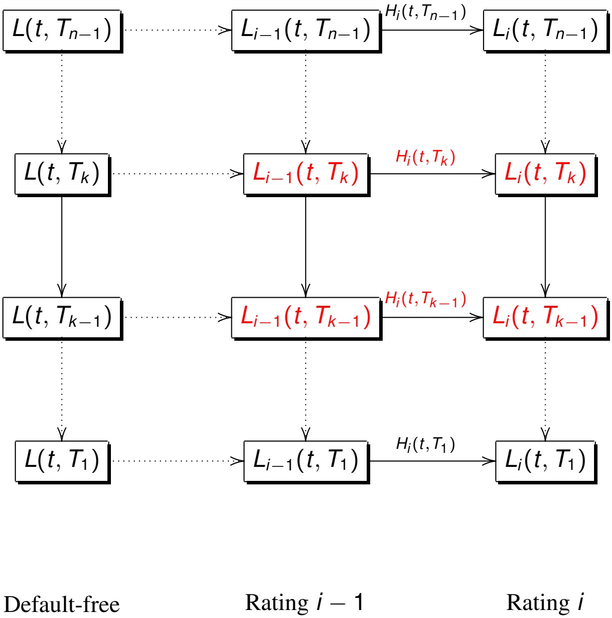


Figure: Connection between subsequent Libor rates

No-arbitrage condition for the rating based model

Recall the defaultable bond price process with fractional recovery of Treasury value q

$$B_C(t, T_k) = \sum_{i=1}^{K-1} B_i(t, T_k)\mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}} B(t, T_k)\mathbf{1}_{\{C_t=K\}}.$$

Note: the forward bond price process

$$\frac{B_C(\cdot, T_k)}{B(\cdot, T_j)}$$

is a $\mathbb{Q}_{T_j}$-local martingale for every $k, j = 1, \ldots, n-1$

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iff the forward bond price process

$$\frac{B_C(\cdot,T_k)}{B(\cdot,T_j)} = \frac{B_C(\cdot,T_k)}{B(\cdot,T_k)} \underbrace{\frac{B(\cdot,T_k)}{B(\cdot,T_j)}}_{\left.\frac{\mathrm{d}\mathbb{Q}_{T_k}}{\mathrm{d}\mathbb{Q}_{T_j}}\right|_{\mathcal{G}.}}$$

is a $\mathbb{Q}_{T_k}$-local martingale for every $k = 1,\dots,n-1$.

No-arbitrage condition

Theorem

Let T_k be a tenor date. Assume that the processes $H_j(\cdot, T_k)$, $j = 1, \ldots, K-1$, are given by (2). Then the process $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ is a local martingale with respect to the forward measure $\mathbb{Q}_{T_k}$ and filtration $\mathbb{G}$ iff:
for almost all $t \leq T_k$ on the set $\{C_t \neq K\}$

$$b^{\mathbb{H}}(t, T_k, C_t) + \lambda_{C_t}(t) = \left(1 - q_{C_t} \frac{e^{-\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}{\mathbb{H}(t-, T_k, C_t)}\right) \lambda_{C_t K}(t) + \sum_{j=1, j \neq C_t}^{K-1} \left(1 - \frac{\mathbb{H}(t-, T_k, j) e^{\int_0^t \lambda_j(s)\mathrm{d}s}}{\mathbb{H}(t-, T_k, C_t) e^{\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}\right) \lambda_{C_t j}(t). \tag{3}$$

Sketch of the proof: Use the fact that the jump times of the conditional Markov chain C do not coincide with the jumps of any $\mathbb{F}$-adapted semimartingale, use martingales related to the indicator processes $\mathbf{1}_{\{C_t = i\}}$, $i \in \mathcal{K}$, and stochastic calculus for semimartingales.

Pricing and defaultable forward measures

- Defaultable forward measure $\mathbb{Q}_{C,T_k}$ for the date T_k is defined on $(\Omega, \mathcal{G}_{T_k})$ by

$$\left.\frac{\mathrm{d}\mathbb{Q}_{C,T_k}}{\mathrm{d}\mathbb{Q}_{T_k}}\right|_{\mathcal{G}_t} := \frac{B(0,T_k)}{B_C(0,T_k)} \frac{B_C(t,T_k)}{B(t,T_k)}$$

  This corresponds to the choice of $B_C(\cdot, T_k)$ as a numeraire.

- Defaultable Libor rates are martingales wrt corresponding defaultable forward measures

- Pricing formulae for defaultable bonds, credit default swaps, caps/floors on the defaultable Libor rates (under defaultable forward measures)

The talk is based on:

E. Eberlein and Z. Grbac (2010). Rating-based Lévy Libor model. To appear in *Mathematical Finance*.