Numerical approximation of two-scale SDEs

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Stochastic Volatility

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$$dS_t = b(S_t)dt + \sigma(S_t, v_t)dW_t$$

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Empirical studies on high frequency data (Ex: S&P [Fouqué et al. (98)] ; IBOVESPA [Souza et al. (06)]) support financial asset modeling using Stochastic Volatility with

- Mean reverting volatility
- Fast reverting process: mean reverting time in the order of days ($\ll$ maturity of instruments/derivatives).

A general two-scale SDE

We consider the following two-scale SDE

$$
\begin{aligned}
X_t^\epsilon &= x_0 + \int_0^t f\left(X_s^\epsilon, Y_s^\epsilon\right) ds + \int_0^t g\left(X_s^\epsilon, Y_s^\epsilon\right) dW_s \\
Y_t^\epsilon &= y_0 + \epsilon^{-1} \int_0^t b(X_s^\epsilon, Y_s^\epsilon) ds + \epsilon^{-1/2} \int_0^t \sigma(X_s^\epsilon, Y_s^\epsilon) d\tilde{W}_s,
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We will assume

- $\epsilon \ll 1$
- W and $\tilde{W}$ are independent
- $f(x, y)$ and $g(x, y)$ C^1 with linear growth in x, C_b^∞ in y
- $b(x, y)$, C_b^∞ in both x, y
- Non-degenerate: $\lambda_M > \sigma\sigma^*(x, y) > \lambda_m > 0$. And mean reverting
$$\lim_{|y| \to \infty} b(x, y) y = -\infty$$

Homogenization result

Theorem [Pardoux & Veretennikov (01, 03)]

Under the hypothesis the rescaled, frozen parameter diffusion

$$Y_t^x = y_0 + \int_0^t b(x, Y_s^x)ds + \int_0^t \sigma(x, Y_s^x)d\tilde{W}_s,$$

is ergodic with invariant measure μ^x. $X^\epsilon \xrightarrow{\mathcal{L}} X$ where X solves

$$X_t = x_0 + \int_0^t F(X_s)ds + \int_0^t G(X_s)dW_s \quad \text{(Effective equation)},$$

with $G = \sqrt{A}$, and F and A are averages of f and $a = gg^*$ with respect to μ^x.

$$F(x) = \int f(x, y)\mu^x(dy) \quad A(x) = \int g \cdot g^*(x, y)\mu^x(dy)$$

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- Problem: In general we do not know μ^x explicitly
- Our goal: Propose a numerical method to approximate the Effective equation

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- Discretize the approximated equation. Euler scheme: For $t_k = k/n$

$$\tilde{X}^n_{t_{k+1}} = \frac{1}{n}\tilde{F}^{(n)}\left(\tilde{X}^n_{t_k}\right) + \frac{1}{\sqrt{n}}\tilde{G}^{(n)}\left(\tilde{X}^n_{t_k}\right)U_{k+1}$$

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- Similar approach used for deterministic [Fatkullin, Vanden-Eijnden (04)] and stochastic setup [E et al. (04)] .
- Our approach: Choose estimators for the averages $\tilde{F}^{(n)}, \tilde{G}^{(n)}$ allowing us to develop a C.L.T like result for the strong error.

Invariant approximation algorithm: Decreasing Euler step

Decreasing Euler step

Let $\{\gamma_k\}$ be a sequence of decreasing positive reals tending to zero, and $\Gamma_M := \sum_{k=0}^{M} \gamma_k$, and $\bar{U}_k \sim \mathcal{N}(0,1)$

- Decreasing step Euler scheme:

$$\bar{Y}^x_{k+1} = \bar{Y}^x_k + \gamma_{k+1} b\left(x, \bar{Y}^x_k\right) + \sqrt{\gamma_{k+1}}\sigma\left(x, \bar{Y}^x_k\right)\bar{U}_{k+1},$$

- Average estimator:

$$\nu(f, x; M) := \frac{1}{\Gamma_M} \sum_{k=1}^{M} \gamma_k f\left(x, \bar{Y}^x_{k-1}\right)$$

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- Average estimator:

$$\nu(f, x; M) := \frac{1}{\Gamma_M} \sum_{k=1}^{M} \gamma_k f\left(x, \bar{Y}^x_{k-1}\right) \approx \frac{1}{\Gamma_M} \int_0^{\Gamma_M} f\left(x, Y^x_s\right) ds$$

$$\approx \lim_{M\to\infty} \frac{1}{\Gamma_M} \int_0^{\Gamma_M} f\left(x, Y^x_s\right) ds = \int f(x, y)\mu^x(dy) = F(x)$$

Invariant approximation algorithm

Properties [Lamberton, Pagès (02)]

Under some hypothesis on the step sizes γ_k,

1. Almost sure convergence: For x fixed, $\nu(f, x; M) \xrightarrow{a.s.} F(x)$
2. C.L.T.: For x fixed,

$$\sqrt{\Gamma_M}\,(\nu(f, x; M) - F(x)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Xi(x))$$

Ξ depends on μ^x and the coefficients of the ergodic diffusion, but not on the choice of γ_k.

Effective equation approximation

$$\bar{X}^{(n)}_{t_{k+1}} = \frac{1}{n}\nu\left(f, \bar{X}_{t_k}; M(n)\right) + \frac{1}{\sqrt{n}}\sqrt{\nu\left(gg^*, \bar{X}_{t_k}; M(n)\right)}\,U_{k+1}$$

i.e. it is an Euler scheme for which we use independent realizations of the decreasing Euler estimator at each discretization step.

1. We fix $\gamma_k = k^{-\theta}$ for $1/3 < \theta < 1$.
2. We relate the two parameters M and n. The most efficient way to do so is by fixing $M(n)$ such that

$$\Gamma_M \propto n$$

Result 1 : Strong convergence

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{0\leq t\leq T}\left|X_t - \bar{X}_t^{(n)}\right|^2\right) \to 0$$

Sketch of the proof

Stability technique

- A priori bounds
- Obtain a global L_2 error control from step-wise L_2 error control (Burkholder maximal inequality + Gronwall lemma)

Result 2 : Limit error distribution

Limit distribution

If the effective diffusion is non-degenarte, fixing $\Gamma_M = n$

$$n^{1/2}\left(X_t - \bar{X}^{(n)}\right) \Rightarrow \zeta$$

with ζ defined as the solution of

$$\begin{aligned}\zeta_t = \int_0^t \partial_x F(X_s)\zeta(s)ds + \int_0^t \partial_x G(X_s)\zeta(s)dW_s \\ + \frac{1}{\sqrt{2}}\int_0^t \partial_x G(X_s)G(X_s)dB_s^1 + \int_0^t \sqrt{\Xi(X_s)}dB_s^2\end{aligned}$$

with B^1 and B^2 are two independent standard Brownian motions.

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Result 2 : Limit error distribution - outline of the proof

$$n^{1/2}\left(X_t - \bar{X}^{(n)}\right) =: \zeta^n \Rightarrow \zeta$$

Sketch of the proof

- Tightness
  - Obtain an SDE for the error $\zeta^n = \tilde{\zeta}^n + R^n$
  - Prove $R^n \xrightarrow{L_2} 0$
  - Classical results (Kurtz and Protter) to deduce tightness of $\tilde{\zeta}^n$ from convergence in law of the tuple of coefficients of related SDE.
  - Convergence of the tuple: Use C.L.T of decreasing Euler, independence and convergence of the quadratic variations.
- Identification is straightforward thanks to strong convergence.

Result 3 : Romberg extrapolation

Let $l \in \mathbb{N}$, $l \geq 2$. Let $c_1, \ldots, c_l \in \mathbb{R}$ satisfy the linear system

$$\sum_{i=1}^{l} c_i = 1 \qquad \sum_{i=1}^{l} \left(\frac{c_i}{\Gamma_{iM}} \sum_{j=1}^{iM} \gamma_j^r \right) = 0 \text{ for } r = 2, \ldots, l.$$

Define the approximation function

$$\hat{\nu}(x, F; M, l) = \sum_{i=1}^{l} c_i \nu(x, F; iM)$$

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Convergence and limit error results using this approximation are unchanged (up to a constant), but in this case we may take

$$\frac{1}{2l+1} < \theta < 1$$

Efficiency analysis

Define

$$\tau := \# \text{ of operations}$$

Our algorithm with n steps requires

$$\tau(n) = Kn^{\frac{2-\theta}{1-\theta}}$$

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For a fixed strong error tolerance Δ (recall $\Delta(n) := Kn^{-1/2}$) :

- Simple SDE ($\theta > 1/3$) :

$$\tau(\Delta) = K\Delta^{-2-\frac{2}{1-\theta}} \geq K\Delta^{-5}$$

- Interpolated SDE ($\theta > \frac{1}{2l+1}$):

$$\tau(\Delta) = K_l\Delta^{-2-\frac{2}{1-\theta}} \geq K_l\Delta^{-4-\frac{1}{l}}.$$

Numerical test

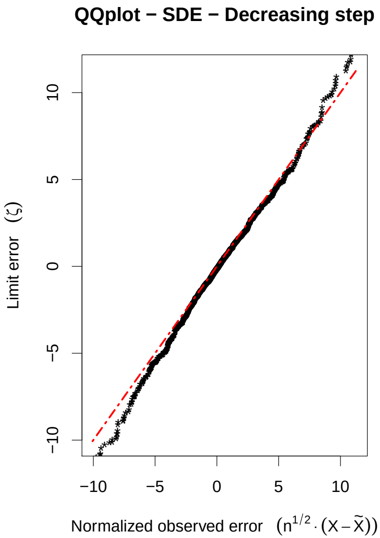
Test problem:

$$dX_t^\epsilon = X_t^\epsilon dt + X_t^\epsilon Y_t^\epsilon dW_t$$

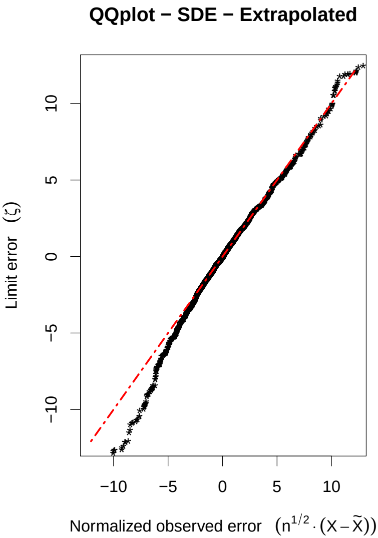
$$dY_t^\epsilon = \epsilon^{-1}\left(\sqrt{\frac{1}{2(1+(X_t^\epsilon)^2)}} - Y_t^\epsilon\right) dt + \epsilon^{-1/2}\sqrt{\frac{2(X_t^\epsilon)^2+1}{(X_t^\epsilon)^2+1}} dW_t$$

We test the algorithm with

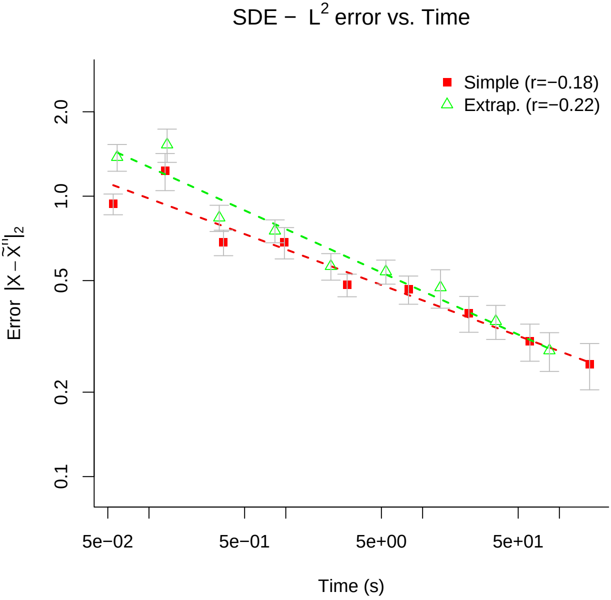
- $\gamma_k = k^{-0.35}$ for the simple version ($\theta \approx 1/3$)
- $\gamma_k = k^{-0.225}$ for the extrapolated version ($\theta \approx 1/5$)



QQplot − SDE − Decreasing step



QQplot − SDE − Extrapolated


SDE − L^2 error vs. Time
Simple (r=−0.18)
Extrap. (r=−0.22)
2.0
1.0
0.5
0.2
0.1
Error $|X - \tilde{X}^n|_2$
5e−02
5e−01
5e+00
5e+01
Time (s)