Deviation inequalities and moderate deviation principle for Bifurcating Markov Chains.

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joint work with
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Colloque Jeunes Probabilistes Statisticiens, CIRM 2012

Outline

Motivation

- Guyon, J. *Limit theorems for bifurcating markov chains. Application to the detection of cellular aging*. Ann. Appl. Probab., (2007), Vol. 17, No. 5-6, pp. 1538-1569.
- Escherichia Coli (E.Coli)

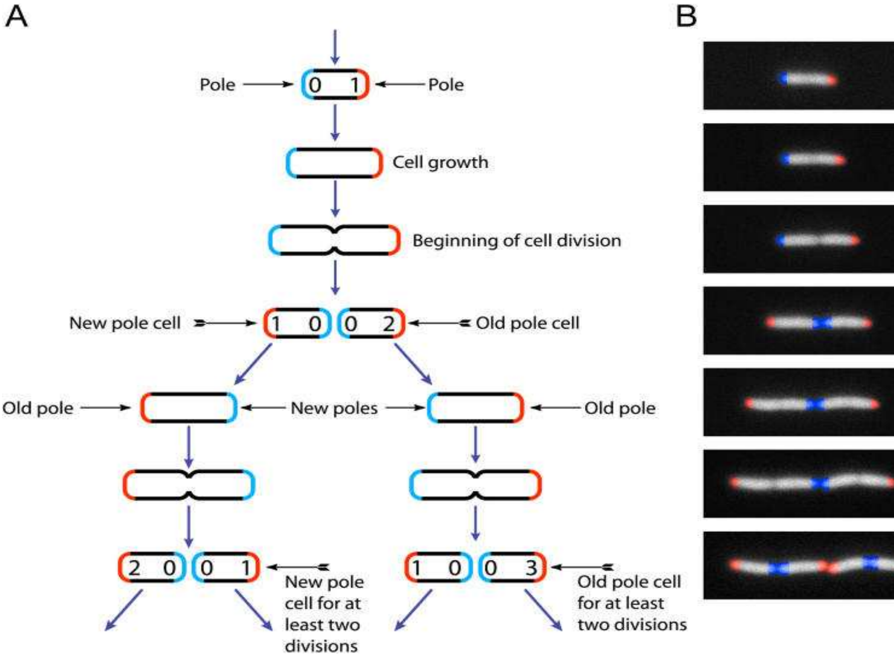
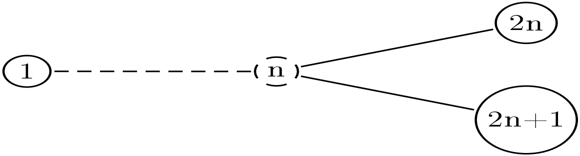


Figure: Cell division from E. J. Stewart and al.

Motivation

- Guyon, J. Bize, A. Paul, G. Stewart, E.J. Delmas, J.F. Taddéi, F. *Statistical study of cellular aging*. CEMRACS 2004 Proceedings, ESAIM Proceedings, (2005), 14, pp. 100-114.

First order bifurcating autoregressive process BAR(1)



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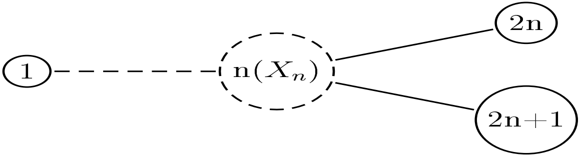

First order bifurcating autoregressive process BAR(1)

$$\mathcal{L}(X_1) = \nu, \quad \text{and} \quad \forall n \geq 1, \quad \begin{cases} X_{2n} = \alpha_0 X_n + \beta_0 + \varepsilon_{2n} \\ \\ X_{2n+1} = \alpha_1 X_n + \beta_1 + \varepsilon_{2n+1}, \end{cases} \tag{1}$$

First order bifurcating autoregressive process BAR(1)

where ν is a distribution probability on $\mathbb{R}$, $\alpha_0, \alpha_1 \in (-1, 1)$; $\beta_0, \beta_1 \in \mathbb{R}$ and $\big((\varepsilon_{2n}, \varepsilon_{2n+1}), n \geq 1\big)$ forms a sequence of centered i.i.d bivariate random variables with covariance matrix

$$\Gamma = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \sigma^2 > 0, \quad \rho \in (-1, 1).$$

Motivation

$\theta = (\alpha_0, \beta_0, \alpha_1, \beta_1)$, σ and ρ.

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$H_0 = \{(\alpha_0, \beta_0) = (\alpha_1, \beta_1)\}$ against $H_1 = \{(\alpha_0, \beta_0) \neq (\alpha_1, \beta_1)\}$.

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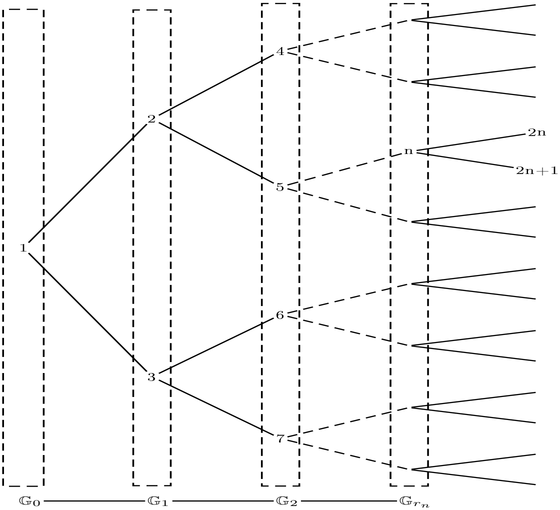
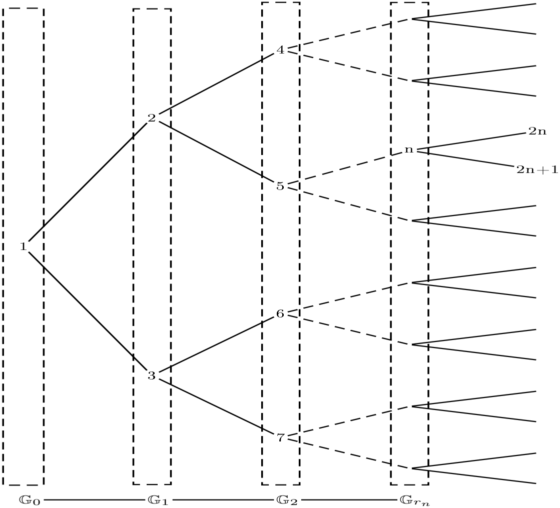
The binary tree $\mathbb{T}$



The binary tree $\mathbb{T}$

$$\mathbb{G}_q = \{2^q, 2^q + 1, \cdots, 2^{q+1} - 1\},\ \mathbb{T}_r = \bigcup_{q=0}^{r} \mathbb{G}_q,\ r_n = [\log_2 n].$$



The model of bifurcating Markov chain

- $(S, \mathcal{S})$

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- $(S, \mathcal{S})$
- $P : S \times \mathcal{S}^2 \to [0,1]$ such that
    - $P(., A)$ is measurable for all $A \in \mathcal{S}^2$,
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- For all $f \in \mathcal{B}(S^3)$,

$$Pf : x \mapsto Pf(x) = \int_{S^2} f(x, y, z) P(x, dydz). \qquad (2)$$

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- $(X_n, n \in \mathbb{T}) : (\Omega, \mathcal{F}, (\mathcal{F}_r, r \in \mathbb{N}), \mathbb{P}) \to (S, \mathcal{S})$

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- $(X_n, n \in \mathbb{T}) : (\Omega, \mathcal{F}, (\mathcal{F}_r, r \in \mathbb{N}), \mathbb{P}) \to (S, \mathcal{S})$
-
  - **(a)** X_n is $\mathcal{F}_{r_n}$-measurable for all $n \in \mathbb{T}$,
  - **(b)** $\mathcal{L}(X_1) = \nu$,
  - **(c)** for all $r \in \mathbb{N}$ and for all family $(f_n, n \in \mathbb{G}_r) \subseteq \mathcal{B}_b(S^3)$

$$\mathbb{E}\left[\prod_{n \in \mathbb{G}_r} f_n(X_n, X_{2n}, X_{2n+1}) \Big/ \mathcal{F}_r\right] = \prod_{n \in \mathbb{G}_r} Pf_n(X_n).$$

Outline

1. Introduction
2. **Goals**
3. Framework for the results
4. Results
5. Application

For all $i \in \mathbb{T}$, set $\Delta_i = (X_i, X_{2i}, X_{2i+1})$.

$$\overline{M}_{\mathbb{T}_r}(f) = \frac{1}{|\mathbb{T}_r|} \sum_{i \in \mathbb{T}_r} f(X_i) \quad \text{if } f \in \mathcal{B}(S)$$

and

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Non asymptotic behavior for $\overline{M}_{\mathbb{T}_r}(f)$ ($f \in \mathcal{B}(S) or \mathcal{B}(S^3)$)

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Non asymptotic behavior for $\overline{M}_{\mathbb{T}_r}(f)$ ($f \in \mathcal{B}(S) or \mathcal{B}(S^3)$)

A moderate deviation principle for $\frac{M_{\mathbb{T}_r}(f)}{b_{|\mathbb{T}_r|}}$ (for $f \in \mathcal{B}(S^3)$ such that $Pf = 0$) where

$$M_{\mathbb{T}_r}(f) = \sum_{i\in\mathbb{T}_r} f(\Delta_i) \quad \text{if } f \in \mathcal{B}(S^3)$$

Outline

We will work with the subspace F of $\mathcal{B}(S)$ which verifies

- **(i)** F contains the constants,
- **(ii)** $F^2 \subset F$,
- **(iii)** $F \otimes F \subset L^1(P(x,.))$ for all $x \in S$, and $P(F \otimes F) \subset F$,
- **(iv)** there exists a probability μ on $(S, \mathcal{S})$ such that $F \subset L^1(\mu)$ and $\lim_{r \to \infty} Q^r f(x) = (\mu, f)$ for all $x \in S$ and $f \in F$,
- **(v)** for all $f \in F$, there exists $g \in F$ such that for all $r \in \mathbb{N}$, $|Q^r f| \leq g$,
- **(vi)** $F \subset L^1(\nu)$

Hypothesis

Two cases for the results

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(H1) Geometric ergodicity of Q: $\forall f \in F$ such that $(\mu, f) = 0$, $\exists g \in F$ such that $\forall r \in \mathbb{N}$ and $\forall x \in S$, $|Q^r f(x)| \leq \alpha^r g(x)$ for some $\alpha \in (0, 1)$.

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(H2) Uniform geometric ergodicity of Q**:** $\exists c > 0$ such that

$$|Q^r f(x)| \leq c\alpha^r \quad \text{for some } \alpha \in (0,1) \text{ and for all } x \in S,$$

Outline

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- S.Valère. Bitseki Penda, Hacène. Djellout and Arnaud. Guillin. *Deviation inequalities, Moderate deviations and some limit theorems for bifurcating Markov chains with application.* arXiv:1111.7303 (2011)

Theorem (Deviation inequalities I)

Let $f \in F$ *such that* $(\mu, f) = 0$. *We assume hypothesis* ***(H1)****.*
Then for all $r \in \mathbb{N}$

$$\mathbb{P}\left(|\overline{M}_{\mathbb{T}_r}(f)| > \delta\right) \leq \begin{cases} \frac{c'}{\delta^4}\left(\frac{1}{4}\right)^{r+1} & \textit{if} \quad \alpha^2 < \frac{1}{2} \\ \frac{c'}{\delta^4} r^2 \left(\frac{1}{4}\right)^{r+1} & \textit{if} \quad \alpha^2 = \frac{1}{2} \\ \frac{c'}{\delta^4}\alpha^{4r+4} & \textit{if} \quad \alpha^2 > \frac{1}{2} \end{cases} \qquad (3)$$

where the positive constant c' *depends on* α *and* f*.*

When f depends on the mother-daughters triangle (Δ_i)

Theorem (Deviation inequalities II)

*We assume that (**H1**) is fulfilled. Let* $f \in \mathcal{B}(S^3)$ *such that* Pf *and* Pf^2 *exists and belong to* F *and* $(\mu, Pf) = 0$. *Then for all* $\delta > 0$ *and all* $r \in \mathbb{N}$

$$\mathbb{P}\Big(\big|\overline{M}_{\mathbb{T}_r}(f)\big| > \delta\Big) \leq \begin{cases} \frac{c'}{\delta^2}\left(\frac{1}{2}\right)^{r+1} & \text{if} \quad \alpha^2 < \frac{1}{2}; \\ \frac{c'}{\delta^2} r \left(\frac{1}{2}\right)^{r+1} & \text{if} \quad \alpha^2 = \frac{1}{2}; \\ \frac{c'}{\delta^2}\alpha^{2(r+1)} & \text{if} \quad \alpha^2 > \frac{1}{2}, \end{cases} \tag{4}$$

where the positive constant c' *depends on* f *and* α.

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If $Pf = 0$,

$$\mathbb{P}\left(\left|\overline{M}_{\mathbb{T}_r}(f)\right| > \delta\right) \leq \frac{c'}{\delta^4}\left(\frac{1}{4}\right)^{r+1} \tag{5}$$

Ideas for the proofs

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Markov inequality
+ control of second and fourth order moment of $\overline{M}_{\mathbb{T}_r}(f)$ using (**H1**) and hypothesis (i)-(vi) on F.

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Markov inequality
+ control of second and fourth order moment of $\overline{M}_{\mathbb{T}_r}(f)$ using (**H1**) and hypothesis (i)-(vi) on F.

- Notice that the dichotomy around the value $\alpha^2 = \frac{1}{2}$ naturally appears in the calculus.

Under the stronger assumption of uniform geometric ergodicity of Q, we have the following more sharp estimations for the above empirical mean.

Theorem: Exponential probability inequalities

Let $f \in \mathcal{B}_b(S)$ such that $(\mu, f) = 0$. Assume that **(H2)** is satisfied. Then for all $\delta > 0$ we have

$$\mathbb{P}\Big(\overline{M}_{\mathbb{T}_r}(f) > \delta\Big) \leq \begin{cases} \exp\left(c''\delta\right)\exp\left(-c'\delta^2|\mathbb{T}_r|\right), \ \forall r \in \mathbb{N}, \text{ if } \alpha < \frac{1}{2} \\ \\ \exp\left(2c'\delta(r+1)\right)\exp\left(-c'\delta^2|\mathbb{T}_r|\right), \ \forall r \in \mathbb{N}, \text{ if } \alpha = \frac{1}{2} \\ \\ \exp\left(-c'\delta^2|\mathbb{T}_r|\right), \ \forall r > \frac{\log(\delta/c_0)}{\log\alpha}, \text{ if } \frac{1}{2} < \alpha < \frac{\sqrt{2}}{2} \\ \\ \exp\left(-c'\delta^2\frac{|\mathbb{T}_r|}{r+1}\right), \ \forall r > \frac{\log(c_0/\delta)}{\log\sqrt{2}}, \text{ if } \alpha = \frac{\sqrt{2}}{2}, \\ \\ \exp\left(-c'\delta^2\left(\frac{1}{\alpha^2}\right)^{r+1}\right), \ \forall r > \frac{\log(\delta/c_0)}{\log\alpha}, \text{ if } \alpha > \frac{\sqrt{2}}{2} \end{cases} \tag{6}$$

where c_0, c' and c'' depend on α, $\|f\|_\infty$ and c.

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where c_0, c' and c'' depend on α, $\|f\|_\infty$ and c.

If $f \in \mathcal{B}_b(S^3)$ such that $(\mu, Pf) = 0$ then we have the same conclusions for $\overline{M}_{\mathbb{T}_r}(f)$.

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Chernoff inequality + successive conditioning and successive applications of Azuma-Bennet-Hoeffding using **(H2).**

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- Once again, notice that the dichotomy around $\alpha = \frac{1}{2}$ and $\alpha^2 = \frac{1}{2}$ in (6) naturally appears from the calculations.

Moderate deviation principle for $M_{\mathbb{T}_r}(f)$

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Let (b_n) be an increasing sequence of positive real numbers such that

(I) $\frac{b_n}{\sqrt{n}} \longrightarrow +\infty$,

(II) if $\alpha^2 < \frac{1}{2}$, the sequence (b_n) is such that $\dfrac{b_n}{n} \longrightarrow 0$,

(III) if $\alpha^2 = \frac{1}{2}$, the sequence (b_n) is such that $\dfrac{b_n \log n}{n} \longrightarrow 0$,

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- The conditions (II)-(IV) come from deviation inequalities (6).

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Let $f \in \mathcal{B}_b(S^3)$ such that $Pf = 0$. $(Z_r) = \left(\frac{1}{b_{|\mathbb{T}_r|}} M_{\mathbb{T}_r}(f)\right)$

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Assume that **(H2)** *is satisfied. Let* (b_n) *be a sequence of real numbers satisfying the above assumptions (I)-(IV), then* (Z_r) *satisfies a MDP in* $\mathbb{R}$ *with the speed* $\frac{b^2_{|\mathbb{T}_r|}}{|\mathbb{T}_r|}$ *and rate function* $I(x) = \frac{x^2}{2(\mu, Pf^2)}$.

Particularly, for all $\delta > 0$, *we have*

$$\lim_{r\to\infty} \frac{|\mathbb{T}_r|}{b^2_{|\mathbb{T}_r|}} \log \mathbb{P}\left(\frac{1}{b_{|\mathbb{T}_r|}} |M_{|\mathbb{T}_r|}(f)| > \delta\right) = -I(\delta). \qquad (7)$$

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$$\mathbb{P}\left(\frac{1}{b_{\mathbb{T}_r}} |M_{\mathbb{T}_r(f)}| \geq \delta\right) \sim \exp\left(-\frac{b^2_{\mathbb{T}_r}}{\mathbb{T}_r} \frac{\delta^2}{2(\mu, Pf^2)}\right) \qquad (7')$$

Outline

We consider the BAR(1) process

$$\mathcal{L}(X_1) = \nu, \text{ and} \forall n \geq 1, \begin{cases} X_{2n} = \alpha_0 X_n + \beta_0 + \varepsilon_{2n} \\ \\ X_{2n+1} = \alpha_1 X_n + \beta_1 + \varepsilon_{2n+1}, \end{cases}$$

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The noise values in a compact set.

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(**H2**) are automatically satisfied with $\alpha = \max(|\alpha_0|, |\alpha_1|)$.

The least square estimator $\hat{\theta}^r = \left(\hat{\alpha}_0^r, \hat{\beta}_0^r, \hat{\alpha}_1^r, \hat{\beta}_1^r\right)$ of $\theta = (\alpha_0, \beta_0, \alpha_1, \beta_1)$ is given by, for $\eta \in \{0, 1\}$

$$\begin{cases} \hat{\alpha}_\eta^r = \dfrac{|\mathbb{T}_r|^{-1} \sum\limits_{i \in \mathbb{T}_r} X_i X_{2i+\eta} - \left(|\mathbb{T}_r|^{-1} \sum\limits_{i \in \mathbb{T}_r} X_i\right)\left(|\mathbb{T}_r|^{-1} \sum\limits_{i \in \mathbb{T}_r} X_{2i+\eta}\right)}{|\mathbb{T}_r|^{-1} \sum\limits_{i \in \mathbb{T}_r} X_i^2 - \left(|\mathbb{T}_r|^{-1} \sum\limits_{i \in \mathbb{T}_r} X_i\right)^2} \\ \\ \hat{\beta}_\eta^r = |\mathbb{T}_r|^{-1} \sum\limits_{i \in \mathbb{T}_r} X_{2i+\eta} - \hat{\alpha}_\eta^r |\mathbb{T}_r|^{-1} \sum\limits_{i \in \mathbb{T}_r} X_i. \end{cases} \tag{8}$$

$\mu_1 : \Theta \to \mathbb{R}$ and $\mu_2 : \Theta \times \mathbb{R}_+^* \to \mathbb{R}$ by writing

$$(\mu, \mathbf{x}) = \mu_1(\theta) \quad \text{and} \quad (\mu, \mathbf{x}^2) = \mu_2(\theta, \sigma^2), \qquad (9)$$

where $\theta = (\alpha_0, \beta_0, \alpha_1, \beta_1) \in \Theta = (-1, 1) \times \mathbb{R} \times (-1, 1) \times \mathbb{R}$, and μ is the stationary distribution of Q.

Theorem

$\forall \delta > 0$ and $\forall \gamma < \min\left(\frac{c_1 b}{1+\delta}, \frac{c_1 b}{1+\sqrt{\delta}}, \frac{c_1 b}{1+\sqrt[4]{\delta}}\right)$, where $c_1 = c_1(\mu_1, \mu_2)$ we have $\mathbb{P}\left(\left\|\hat{\theta}^r - \theta\right\| > \delta\right) \leq$

$$\begin{cases} \exp\left(c''(\gamma\delta)^{1-p/2}\right)\exp\left(-c'(\gamma\delta)^{2-p}|\mathbb{T}_r|\right), \forall r \in \mathbb{N}, \text{ if } \alpha < \frac{1}{2} \\ \exp\left(c'(\gamma\delta(r+1))^{1-p/2}\right)\exp\left(-c'(\gamma\delta)^{2-p}|\mathbb{T}_r|\right), \forall r \in \mathbb{N}, \text{ if } \alpha = \frac{1}{2} \\ \exp\left(-c'(\gamma\delta)^{2-p}|\mathbb{T}_r|\right), \forall r > \frac{\log\left((\gamma\delta)^{1-p/2}/c_0\right)}{\log \alpha}, \text{ if } \frac{1}{2} < \alpha < \frac{\sqrt{2}}{2} \\ \exp\left(-c'(\gamma\delta)^{2-p}\frac{|\mathbb{T}_r|}{r+1}\right), \forall r > \frac{\log\left(c_0/(\gamma\delta)^{1-p/2}\right)}{\log\sqrt{2}}, \text{ if } \alpha = \frac{\sqrt{2}}{2} \\ \exp\left(-c'(\gamma\delta)^{2-p}\left(\frac{1}{\alpha^2}\right)^{r+1}\right), \forall r > \frac{\log\left((\gamma\delta)^{1-p/2}/c_0\right)}{\log \alpha}, \text{ if } \alpha > \frac{\sqrt{2}}{2}, \end{cases} \tag{10}$$

where c' and c'' depend on α, $\|f\|_\infty$ and c, c_0 depends on α, $\|f\|_\infty$, c and γ, and $p \in \{0, 1, 3/2\}$.

End.