Introduction to Magnetohydrodynamic Turbulence

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Contents

MHD approximation

- Crucial role of the magnetic field in geophysical and astrophysical fluid dynamics (stellar or solar wind, convective zone of stars, accretion discs, magnetic field generation by dynamo effect, ...) leads to explore properties of MHD, namely the interaction between an electrically conducting fluid and a magnetic field
  * motion of weakly conducting fluid; low magnetic Reynolds number
  $\rightarrow$ terrestrial MHD
  * motion in a highly conducting fluid; high magnetic Reynolds number
  $\rightarrow$ astrohysical MHD

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  * motion of weakly conducting fluid; low magnetic Reynolds number
  $\rightarrow$ terrestrial MHD
  * motion in a highly conducting fluid; high magnetic Reynolds number
  $\rightarrow$ astrohysical MHD
- MHD is a fluid approximation: does not describe the detailed processes of plasma physics which require description of individual motions of particles (plasma frequencies, frequencies and lenght scales for both ions and electrons, ..)

- MHD approximation is valid for some plasmas and liquid metals
  * quasi-neutral property ($\nabla \cdot \mathbf{E} \simeq 0$, with $\mathbf{E}$ the electric field)
  * fluid approximation: electrical conduction of the medium by electrons alone
  * non relastivistic limit (typical velocity $U \ll c$)
  * collisional plasma/fluid: conductivity is independent of U (time evolution of the fluid $\gg$ time between 2 collisions ions/electrons, fluid elements contain many ions & electons)
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  * solar wind, heliosphere, and Earth's magnetosphere (on large scales)
  * inertial range of plasma turbulence
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- MHD description is limited when
  * non-fluid or kinetic effects are important (dissipation in the turbulent solar wind, magnetic reconnection, small-scales dynamics in Earth's magnetosphere)
  * plasmas are fully ionized (solar photosphere/chromosphere, molecular clouds, protoplanetary disks, Earth's ionosphere, some laboratory plasmas)

MHD equations

- Maxwell's equations

  Faraday's law $\nabla \times \mathbf{E} + \partial \boldsymbol{B}/\partial t = 0$

  Ampère's law $\nabla \times \boldsymbol{B} = \mu_0[\boldsymbol{j} + \cancel{\epsilon_0 \partial \mathbf{E}/\partial t}]$ with $c^2 = (\mu_0 \epsilon_0)^{-1}$

  Coulomb's gauge $\nabla \cdot \boldsymbol{B} = 0$

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  where $\sigma \equiv$ electrical conductivity of the medium
- equations combine to yield an evolution equation for the magnetic field or the so-called induction equation

$$\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \Delta \boldsymbol{B}$$

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- note that $\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$ represents both advection and stretching of the field $\boldsymbol{B}$ (with $\nabla \cdot \boldsymbol{u} = 0 = \nabla \cdot \boldsymbol{B}$)

- Incompressible MHD equations

$$\partial \boldsymbol{u}/\partial t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p/\rho + \nu\Delta\boldsymbol{u} + (\boldsymbol{j}\times\boldsymbol{B})/\rho + \mathbf{F}$$
$$\partial \boldsymbol{B}/\partial t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} + \eta\Delta\boldsymbol{B}$$
$$\nabla\cdot\boldsymbol{u} = 0 = \nabla\cdot\boldsymbol{B} \quad \text{and} \quad B.C.$$

where $\boldsymbol{j}\times\boldsymbol{B} = (\nabla\times\boldsymbol{B})\times\boldsymbol{B}/\mu_0$ is the Lorentz force and $\mathbf{F}$ a body force (friction, gravity, Coriolis force,..). $\boldsymbol{B}$ can be replaced by the scaled magnetic field $\boxed{\boldsymbol{b} = \boldsymbol{B}/(\rho\mu_0)^{1/2} = \boldsymbol{v_a}}$ which has dimension of a velocity and is called the Alfvén velocity
although $\boldsymbol{B}$ is a pseudovector [1]

[1] $(\mathbf{i},\mathbf{j},\mathbf{k}) \to (\mathbf{i},\mathbf{j},-\mathbf{k})$, **A** vector : $(A_x, A_y, A_z) \to (A_x, A_y, -A_z)$ & **B** pseudovector : $(B_x, B_y, B_z) \to (-B_x, -B_y, B_z)$

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- dimensionless parameters
  * kinetic Reynolds number = advection / dissipation = $R_e = UL_0/\nu$
  * magnetic Reynolds number = induction / diffusion = $R_M = UL_0/\eta$
  * magnetic Prandtl numer = $P_M = \nu/\eta = R_M/R_e$

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$$P_M = \frac{kinematic\ viscosity}{magnetic\ diffusivity} = \frac{viscous\ dissipation\ rate}{magnetic\ diffusion\ rate} = \nu/\eta$$

- P_M small: $P_M \ll 1$ (highly conducting fluids)
  * liquid metals $P_M \sim 10^{-6}$... 10^{-5} (liquid sodium experiments $\sim 10^{-6}$)
  * stars $P_M \sim 10^{-7}$... 10^{-4}
  * protostellar disk $P_M \sim 10^{-7}$
  * sun (at the basis of the convection zone) $P_M \sim 10^{-2}$

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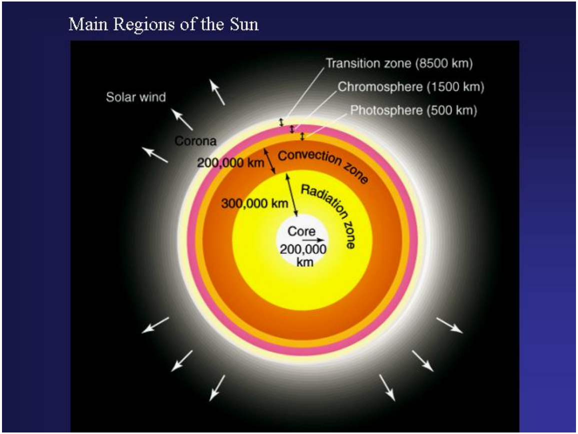
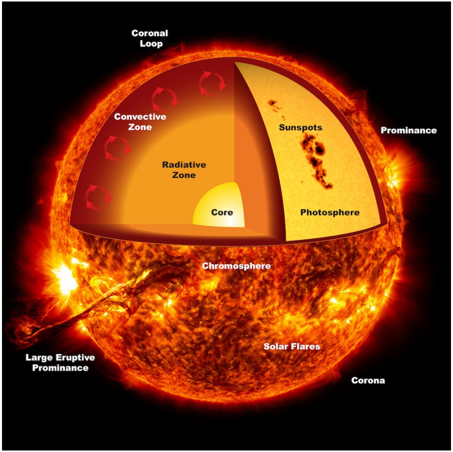
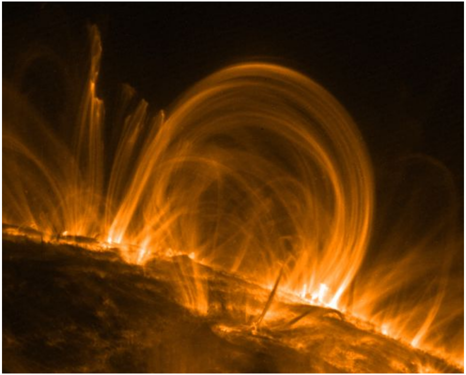
- P_M large: $P_M \gg 1$ (poorly conducting fluids)
  * protogalaxies, galaxies $P_M \sim 10^{14}$, clusters $P_M \sim 10^{29}$
  * as in solar wind or interstellar medium, ... $P_M \sim 10^{14} - 10^{19}$



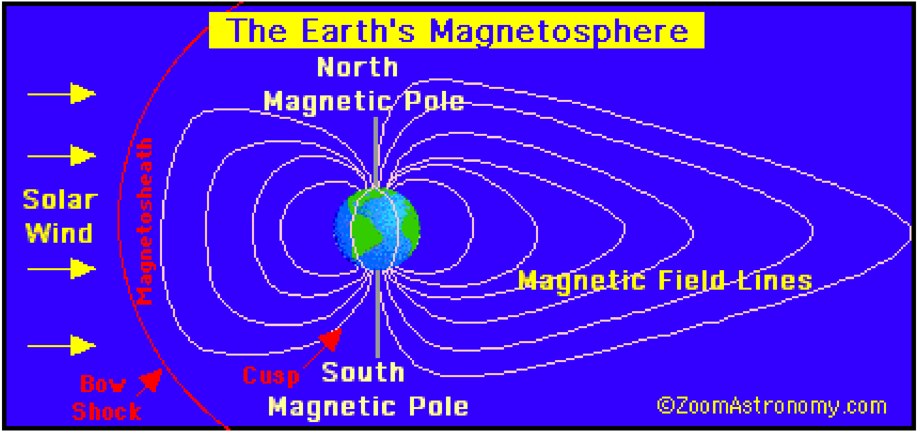
schematic view of the main regions of the Sun



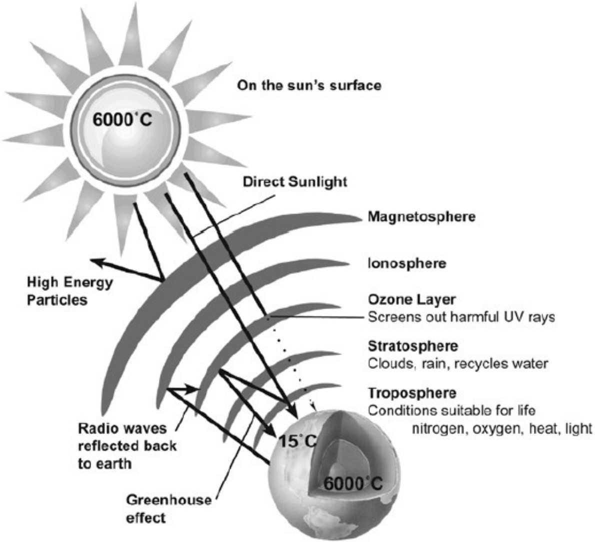
another one in 3D



coronal loops



the magnetosphere of Earth



cartoon of the configuration of sun-earth layers

Another important dimensionless parameter β

using $\nabla(\boldsymbol{B}^2)/2 = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \boldsymbol{B}\times(\nabla\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \boldsymbol{B}\times\mu_0\boldsymbol{j}$
thus, from Ampère's law, the Lorentz force can be written as

$$\boldsymbol{j}\times\boldsymbol{B} = \underbrace{\frac{(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}}{\mu_0}}_{magnetic\ \ tension} - \nabla \underbrace{(\frac{\boldsymbol{B}^2}{2\mu_0})}_{magnetic\ \ pressure}$$

and the parameter β is defined as

$$\boxed{\beta = \frac{plasma\ \ pressure}{magnetic\ \ pressure} = \frac{p}{\boldsymbol{B}^2/(2\mu_0)}}$$

- if $\beta \ll 1$, the magnetic field dominates;
  solar corona, tokamaks ($\beta \lessapprox 0.1$)

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- if $\beta \gg 1$, plasma pressure forces dominate; stellar interiors
- $\beta \sim 1$, pressure/magnetic forces are both important;
  solar chromosphere, parts of solar wind & interstellar medium, some laboratory plasma experiments

- Other formulation of the incompressible MHD equations

$$\partial \boldsymbol{u}/\partial t = -\nabla(p/\rho + \boldsymbol{u}^2/2) + \boldsymbol{u} \times \boldsymbol{\omega} + \nu\Delta\boldsymbol{u} + (\boldsymbol{j} \times \boldsymbol{B})/\rho + \mathbf{F}$$
$$\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta\Delta\boldsymbol{B}$$
$$\nabla \cdot \boldsymbol{u} = 0 = \nabla \cdot \boldsymbol{B}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is the vorticity, and the use of the vector relationship $\boldsymbol{u} \times \boldsymbol{\omega} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u} \cdot \nabla\boldsymbol{u} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla(\boldsymbol{u}^2/2)$

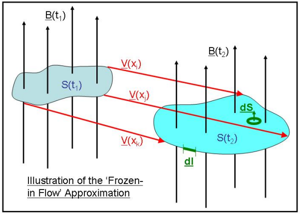
- several physical cases
  * $\boldsymbol{u} \times \boldsymbol{\omega} = 0 \rightarrow \boldsymbol{u}(\boldsymbol{x}) \parallel \boldsymbol{\omega}(\boldsymbol{x})$, the flow is completely helical ("Beltrami" flow)
  * $\boldsymbol{j} \times \boldsymbol{B} = 0 \rightarrow \boldsymbol{B}(\boldsymbol{x}) \parallel \boldsymbol{j}(\boldsymbol{x})$, it is "the force free field" case, the magnetic field does not act on the velocity, as the Lorentz force is null
  * $\boldsymbol{u} \times \boldsymbol{B} = 0 \rightarrow \boldsymbol{u}(\boldsymbol{x}) \parallel \boldsymbol{B}(\boldsymbol{x})$, means alignment between velocity and magnetic field everywhere in space

- Remark : if $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, where $\boldsymbol{A}$ is a magnetic potential (defined up to a gradient), then $\boldsymbol{j} = -\Delta\boldsymbol{A}$ and an equation for $\boldsymbol{A}$ can be derived

Frozen in Alfvén's theorem in ideal MHD

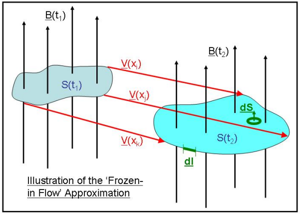
- Consider a surface $\mathcal{S}$ bounded by a close curve $\partial\mathcal{S} \equiv \mathcal{C}$, the magnetic flux through $\mathcal{S}$ is $\Phi = \int_{\mathcal{S}} \boldsymbol{B} \cdot d\mathbf{S}$ (where $d\mathbf{S} = \mathbf{n} dS$),
- the rate of change of the flux is given by

$$\frac{d\Phi}{dt} = \underbrace{\int_{\mathcal{S}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\mathbf{S}}_{(1)\ changes\ in\ \boldsymbol{B}} + \underbrace{\int_{\mathcal{S}} \boldsymbol{B} \cdot \frac{\partial \mathbf{S}}{\partial t}}_{(2)\ changes\ in\ \mathcal{S}}$$



- (1) represents the change in $\boldsymbol{B}$ with $\mathcal{S}$ held fixed & (2) the flux swept out by $\mathcal{C}$ as it moves with the fluid

[2] $\iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial\mathcal{S}} \mathbf{F} \cdot d\boldsymbol{\ell}$, flux of $\nabla \times \mathbf{F}$ through $\mathcal{S}$ = circulation of $\mathbf{F}$ along $\partial\mathcal{S} \equiv \mathcal{C}$



- (1) represents the change in $\boldsymbol{B}$ with $\mathcal{S}$ held fixed & (2) the flux swept out by $\mathcal{C}$ as it moves with the fluid
- using Faraday's law and Stokes'theorem [2], (1) gives $\int_{\mathcal{S}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\mathbf{S} = \int_{\mathcal{S}} (-\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\oint_{\mathcal{C}} \mathbf{E} \cdot d\boldsymbol{\ell}$

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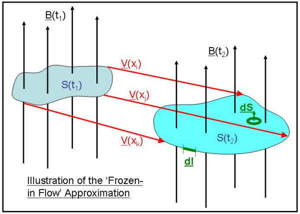

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- the incremental change in $\mathcal{S}$ due to movement of an element $\delta\boldsymbol{\ell}$ of $\mathcal{C}$ during δt is $\delta\mathbf{S} = \boldsymbol{u}\delta t \times \delta\boldsymbol{\ell}$ (area of a tiny parallelogram)

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- the flux throught this area is $\boldsymbol{B} \cdot d\mathbf{S} = \boldsymbol{B} \cdot (\boldsymbol{u} dt \times d\boldsymbol{\ell})$
  so that term (2) is $\int_{\mathcal{S}} \boldsymbol{B} \cdot \frac{\partial \mathbf{S}}{\partial t} = \oint_{\mathcal{C}} \boldsymbol{B} \cdot (\boldsymbol{u} \times d\boldsymbol{\ell}) = -\oint_{\mathcal{C}} (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell}$,
  by rearranging the triple product [3]

[3] $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = -\mathbf{C} \cdot (\mathbf{B} \times \mathbf{A})$

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- all together, the rate of change of the magnetic flux is given by

$$\frac{d\Phi}{dt} = -\oint_{\mathcal{C}} (\mathbf{E} + \boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell}$$

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- all together, the rate of change of the magnetic flux is given by

$$\frac{d\Phi}{dt} = -\oint_{\mathcal{C}} (\mathbf{E} + \boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell}$$

- but, in a perfectly conducting fluid (with an infinite electrical conductivity σ), i.e. "ideal MHD" ($\eta \sim 1/\sigma = 0$), the Ohm's law reads $\mathbf{E} + \boldsymbol{u} \times \boldsymbol{B} = 0$. This gives the Alfvén's theorem

$$\frac{d\Phi}{dt} = 0$$

[3] $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = -\mathbf{C} \cdot (\mathbf{B} \times \mathbf{A})$

- physical meanings

 * suppose $\mathcal{C}$ encircles an isolated flux tube at $t = 0$. Since the magnetic flux through $\mathcal{C}$ is conserved, the material curve $\mathcal{C}$ must continue to encircle the flux tube $\forall t$, and this is possible only if the tube itself moves with the fluid. $\boldsymbol{B}$-lines are material lines as $\mathcal{C}$ could be chosen as vanishingly small cross-section
 * if two fluid elements are initially connected by a magnetic field line, they remain connected by a magnetic field line $\forall t$; the magnetic topology (connectivity) is preserved in ideal MHD
 * the plasma cannot move accross magnetic field lines (however it remains free to move along the field)

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- remark

  if σ is finite ($\eta \neq 0$), then, with Ohm's law, the rate of change of the flux is

$$\frac{d\Phi}{dt} = -\oint_{\mathcal{C}} (\mathbf{E} + \boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell} = -\frac{1}{\sigma} \oint_{\mathcal{C}} \boldsymbol{j} \cdot d\boldsymbol{\ell}$$

Helicities and topology

Magnetic helicity H^m

- the magnetic helicity is defined as $H^m = \int_{\mathcal{V}} \boldsymbol{A} \cdot \boldsymbol{B} \, dV$ [4] where $\boldsymbol{A}$ is a vector potentiel for $\boldsymbol{B}$ ($\nabla \times \boldsymbol{A} = \boldsymbol{B}$) satisfying $\nabla \cdot \boldsymbol{A} = 0$

[4] $\int_{\mathcal{V}} stuff \ \ dV \equiv \iiint_{\mathcal{V}} stuff \ \ dV$ also denoted $< stuff >$ when $\mathcal{V}$ is the whole volume occupied by the fluid

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- let's derive the equation for the rate of change of H^m :
  * the induction equation reads : $\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \Delta \boldsymbol{B}$
  * "uncurling" gives : $\partial \boldsymbol{A}/\partial t = (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \varphi + \eta \Delta \boldsymbol{A}$
  (for some scalar field φ)

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- let's derive the equation for the rate of change of H^m :
  * the induction equation reads : $\partial \boldsymbol{B} / \partial t = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \Delta \boldsymbol{B}$
  * "uncurling" gives : $\partial \boldsymbol{A} / \partial t = (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \varphi + \eta \Delta \boldsymbol{A}$
  (for some scalar field φ)
- in a perfectly conducting fluid ($\eta = 0$), it follows that

$$\frac{\partial (\boldsymbol{A} \cdot \boldsymbol{B})}{\partial t} = \boldsymbol{A} \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \boldsymbol{B} \cdot (\boldsymbol{u} \times \boldsymbol{B}) - \boldsymbol{B} \cdot \nabla \varphi$$
$$= \boldsymbol{A} \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \cdot (\boldsymbol{B} \varphi)$$

[4] $\int_{\mathcal{V}} stuff \;\; dV \equiv \iiint_{\mathcal{V}} stuff \;\; dV$ also denoted $< stuff >$ when $\mathcal{V}$ is the whole volume occupied by the fluid

with the help of the vector relationships :
$\nabla\cdot((\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{A}) = \boldsymbol{A}\cdot\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - (\boldsymbol{u}\times\boldsymbol{B})\cdot(\nabla\times\boldsymbol{A}) = \boldsymbol{A}\cdot\nabla\times(\boldsymbol{u}\times\boldsymbol{B})$ [5] and $\nabla\cdot((\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{A}) = \nabla\cdot(\boldsymbol{B}(\boldsymbol{u}\cdot\boldsymbol{A}) - \boldsymbol{u}(\boldsymbol{A}\cdot\boldsymbol{B})]$, one obtains

$$\frac{\partial(\boldsymbol{A}\cdot\boldsymbol{B})}{\partial t} = \nabla\cdot[(\boldsymbol{A}\cdot\boldsymbol{u}-\varphi)\boldsymbol{B}] + \nabla\cdot(\boldsymbol{u}(\boldsymbol{A}\cdot\boldsymbol{B}))$$

the use of $\nabla\cdot(\boldsymbol{u}(\boldsymbol{A}\cdot\boldsymbol{B})) = (\boldsymbol{u}\cdot\nabla)(\boldsymbol{A}\cdot\boldsymbol{B})$ (recall $\nabla\cdot\boldsymbol{u}=0$) yields

$$\frac{D(\boldsymbol{A}\cdot\boldsymbol{B})}{Dt} = \frac{\partial(\boldsymbol{A}\cdot\boldsymbol{B})}{\partial t} + \boldsymbol{u}\cdot\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \nabla\cdot[(\boldsymbol{u}\cdot\boldsymbol{A}-\varphi)\boldsymbol{B}]$$

- integrating over a material volume $\mathcal{V}$ (always consisting of the same fluid particles) for which $\boxed{\boldsymbol{B}\cdot d\mathbf{S} = \boldsymbol{B}\cdot\mathbf{n}dS = 0}$ (with $\partial\mathcal{V}\equiv\mathcal{S}$) gives the conservation law for the magnetic helicity H^m wich is topological in nature

[5] noting that $(\boldsymbol{u}\times\boldsymbol{B})\cdot\boldsymbol{B} = \boldsymbol{u}\cdot(\boldsymbol{B}\times\boldsymbol{B}) = 0$

- namely :

$$\frac{dH^m}{dt} = \int_{\mathcal{V}} \frac{D}{Dt}(\boldsymbol{A} \cdot \boldsymbol{B}) dV = \oiint_{\mathcal{S}} (\mathbf{n} \cdot \boldsymbol{B})(\boldsymbol{u} \cdot \boldsymbol{A} - \varphi) dS = 0$$

by means of the Divergence (or Gauss's or Ostrogradsky's) theorem

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{F}) dV = \oiint_{\mathcal{S} \equiv \partial \mathcal{V}} (\mathbf{F} \cdot \mathbf{n}) dS$$

note that if **F** represents the flow of a fluid, $\nabla \cdot \mathbf{F}$ represents the expansion or compression of the fluid. The divergence theorem says that the total expansion of the fluid inside a given volume $\mathcal{V}$ necessarily equals the total outgoing flux of the fluid through the volume's boundary $\mathcal{S} \equiv \partial \mathcal{V}$.

Topological interpretation of H^m

suppose that the magnetic field $\boldsymbol{B}$ is identically zero except in one (or two) closed flux tube(s);

- suppose a single flux loop with $\boldsymbol{B}$-lines in the tube being "parallel" circles ("untwisted" or non-helical), $H^m = 0$

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- suppose a single flux loop with a flux Φ, 2π-twisted around itself, & imagine this twisted tube as built up through the addition of incremental fluxes $d\varphi$, the total helicity is $H^m = \pm 2\int_0^{\Phi} \varphi d\varphi = \pm\Phi^2$ (with $\pm$ chosen according as the twist is right- or left-handed). If the flux rope twists around itself T times then $H^m = T\Phi^2$.

- suppose two interlinked untwisted flux tubes carrying fluxes Φ_1 & Φ_2; $H^m = \int_{\mathcal{V}_1} \boldsymbol{A} \cdot \boldsymbol{B} dV + \int_{\mathcal{V}_2} \boldsymbol{A} \cdot \boldsymbol{B} dV$;

$$\int_{\mathcal{V}_1} \boldsymbol{A} \cdot \boldsymbol{B} dV = \int_{\mathcal{V}_1} \boldsymbol{A} \cdot \boldsymbol{B} d\ell dS = \int_{\mathcal{V}_1} \boldsymbol{A} \cdot B d\boldsymbol{\ell} dS$$
$$= \oint_{\mathcal{C}_1} \boldsymbol{A} \cdot (\iint_{\delta_{\mathcal{S}_1}} B dS) d\boldsymbol{\ell} = \oint_{\mathcal{C}_1} \Phi_1 \boldsymbol{A} \cdot d\boldsymbol{\ell} = \Phi_1 \oint_{\mathcal{C}_1} \boldsymbol{A} \cdot d\boldsymbol{\ell}$$

with $d\boldsymbol{\ell}$ a short portion of curve $\mathcal{C}_1$ (the contour of the tube), $\delta_{\mathcal{S}_1}$ the section's tube; and $\oint_{\mathcal{C}_1} \boldsymbol{A} \cdot d\boldsymbol{\ell} = \iint_{\mathcal{S}_1} \nabla \times \boldsymbol{A} \cdot \mathbf{n} dS = \iint_{\mathcal{S}_1} \boldsymbol{B} \cdot \mathbf{n} dS = \Phi_2$ (with $\mathcal{S}_1$ the surface defined by the closed tube 1); one finally gets $H^m = \Phi_1\Phi_2 + \Phi_2\Phi_1 = 2\Phi_1\Phi_2$ if the linkage is right-handed and $H^m = -2\Phi_1\Phi_2$ if the linkage is left-handed

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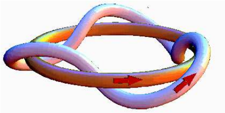
- similarly it can be shown that if 2 untwisted flux tubes have n linkages, then $H^m = \pm 2n\Phi_1\Phi_2$



Remark: helicities are pseudo-scalars, they change sign from a right-handed to a left-handed frame of reference : $(\mathbf{i},\mathbf{j},\mathbf{k}) \rightarrow (\mathbf{i},\mathbf{j},-\mathbf{k})$, $H_R \rightarrow H_L = -H_R$

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- topological interpretation of H^c: consider a thin isolated vortex tube (T_1) and a thin isolated magnetic flux tube (T_2) in a same region of space; H^c is a measure of the linkage of these two tubes
  * if the tubes are not interlinked $H^c = 0$
  * if the tubes are interlinked (with linking number 1) $H^c = \pm\Phi_1\Phi_2$
  where Φ_2 is the magnetic flux in T_2 and Φ_1 is the vorticity flux in T_1

MHD equations in Elsässer variables

The velocity $\boldsymbol{u}$ and magnetic field $\boldsymbol{b}$ can be combined into the Elsässer fields $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$ (remember $\boldsymbol{b} = \boldsymbol{B}/(\rho\mu_0)^{1/2}$), to obtain the following symmetric equations :

$$(\partial_t + \boldsymbol{z}^{\mp} \cdot \nabla)\boldsymbol{z}^{\pm} = \nu_1 \Delta \boldsymbol{z}^{\pm} + \nu_2 \Delta \boldsymbol{z}^{\mp} - \nabla P_* + \boldsymbol{f}^{\pm}$$

where $\nabla \cdot \boldsymbol{z}^{\pm} = 0$, and $P_* = (p/\rho + \boldsymbol{b}^2/2)$ is the total pressure, $\nu_1 = \frac{1}{2}(\nu + \eta)$, $\nu_2 = \frac{1}{2}(\nu - \eta)$

Note that

- the $\boldsymbol{z}^+$ fluctuations are advected by the $\boldsymbol{z}^-$ ones and conversely
- $\boldsymbol{u} = (\boldsymbol{z}^+ + \boldsymbol{z}^-)/2$, $\boldsymbol{b} = (\boldsymbol{z}^+ - \boldsymbol{z}^-)/2$
- $\boldsymbol{u} \cdot \boldsymbol{b} = (\boldsymbol{z}^{+2} - \boldsymbol{z}^{-2})/4$, $\boldsymbol{u}^2 - \boldsymbol{b}^2 = \boldsymbol{z}^+ \cdot \boldsymbol{z}^-$
- $\boldsymbol{\omega}^{\pm} = \nabla \times \boldsymbol{z}^{\pm} = \boldsymbol{\omega} \pm \boldsymbol{j}$

Ideal invariants in homogeneous MHD turbulence

- $\partial_t E^T = -\nu < \boldsymbol{\omega}^2 > -\eta < \boldsymbol{j}^2 >$, for $\nu = \eta = 0$ $\rightarrow$ total energy (kinetic + magnetic) is a conserved quantity

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  for $\nu = \eta = 0 = \nu_1 = \nu_2 \;\rightarrow$ conservation of the z^+ and the z^- energies
- note that: 1) helicities are pseudo-scalars, 2) in a perfect MHD fluid, the mutual topologies of tubes are conserved

Alfvén waves

- Linearization of incompressible MHD eqs around a uniform magnetic field $\boldsymbol{b}_0$ with $\rho_0 = cst, p_0 = cst, \boldsymbol{u}_0 = 0$ (ν and η neglected) leads to :

$$\partial_t \boldsymbol{z}^+ - (\boldsymbol{b}_0 \cdot \nabla)\boldsymbol{z}^+ = 0$$
$$\partial_t \boldsymbol{z}^- + (\boldsymbol{b}_0 \cdot \nabla)\boldsymbol{z}^- = 0$$

  Looking for a solution of plane-wave type for perturbations

$$\boldsymbol{z}^\pm = \boldsymbol{z}_k{}^\pm \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \bar{\omega}^\pm t)}$$

  gives: $\bar{\omega}^+ = -(\boldsymbol{b}_0 \cdot \boldsymbol{k})$ and $\bar{\omega}^- = +(\boldsymbol{b}_0 \cdot \boldsymbol{k})$ with $\boldsymbol{k} \cdot \boldsymbol{z}_k^+ = 0$ and $\boldsymbol{k} \cdot \boldsymbol{z}_k^- = 0$ (incompressibility).

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- $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ are the so-called Alfvén waves : transverse waves ($\boldsymbol{z}_k^\pm \perp \boldsymbol{k}$) with group velocity $v_g = \pm b_0$ and phase velocity $v_\phi = \pm b_0 k_\parallel / k$ (semi-dispersives waves), where $k_\parallel$ is the component of $\boldsymbol{k} \parallel \boldsymbol{b}_0$.

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- oppositely travelling waves: $\boldsymbol{z}^-$ travels in the $\boldsymbol{b}_0$-direction while $\boldsymbol{z}^+$ is backward travelling, with group velocity $\boldsymbol{b}_0$, the so-called Alfvén velocity denoted $\boldsymbol{v_a}$

* A uniform magnetic field $\boldsymbol{b}_0$ (or a local one at scale larger than a given ℓ in the inertial range, or at the largest scale) has a significant dynamical effect for energy transfers : z^+ and z^- blob disturbances (wavepackets) only interact when they collide $\longrightarrow$ weakening of the transfer of energy between scales (i.e. weak nonlinearity)
* Multiple collisions are needed to pass energy in the blobs to smaller scales
* This is the basic idea of "IK" phenomenology (Iroshnikov 63, Kraichnan 65): interplay between turbulent eddies and Alfvén waves travelling along a mean field $\longrightarrow$ crucial difference between hydrodynamic and conducting fluids
* Does Kolmogorov's approach still work ? Does it need to be modified ? Alfvén waves and correlation between $\boldsymbol{u}$ and $\boldsymbol{b}$ fields (cross helicity) are crucial and lead to a lack of universality for inertial MHD spectra

Phenomenologies

Let's take $P_M \sim 1$ from now on.
Suppose $|\boldsymbol{b}| \ll |\boldsymbol{b}_0|$, the IK phenomenology is based on weak nonlinear interactions and many collisions, say N, between z^+ and z^- wavepackets of similar size ℓ, are needed to pass energy to smaller scales. For simplicity, ignore anisotropy ($\ell_{||} \sim \ell_\perp \sim \ell$) and suppose zero cross helicity $H^C \sim 0$ ($z_\ell^+ \sim z_\ell^- \sim z_\ell$). Disturbances are sheared by an amount

$$\delta z_\ell \sim (z_\ell z_\ell/\ell)(\ell/b_0) \longrightarrow \delta z_\ell/z_\ell \sim z_\ell/b_0$$

- $t_a \sim \ell/b_0 \equiv \ell/v_a$ is the interaction time for one collision (Alfvén time) at scale ℓ, i.e. characteristic time for wave propagation over distance ℓ

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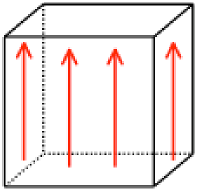
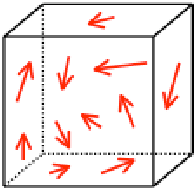
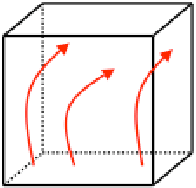
Regimes in MHD turbulence



$\boldsymbol{B} = \boldsymbol{B}_0 + \varepsilon \boldsymbol{b}$: Wave turbulence



$\boldsymbol{B} = \boldsymbol{b}$: Isotropic turbulence



$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$: Anisotropic turbulence

Isotropic descriptions

(here b_0 might be b_{rms} or b at the largest scale in the inertial range)

- uncorrelated case $< \boldsymbol{u} \cdot \boldsymbol{b} > \sim < (\boldsymbol{z}^+)^2 - (\boldsymbol{z}^-)^2 > \sim 0$, $z_\ell^+ \sim z_\ell^- \sim z_\ell$

  * K41, $b_0 \sim 0$

  $t_{tr}^+ \sim t_{tr}^- \sim t_\ell \sim \ell / z_\ell \longrightarrow \epsilon_\ell \sim \epsilon \sim {z_\ell}^3 / \ell$ within inertial range

  $K_\ell^\pm \sim {z_\ell}^2 \sim kE(k)$, $\epsilon_\ell \sim [kE(k)]^{3/2} k \sim \epsilon$ $\boxed{E(k) \sim \epsilon^{2/3} k^{-5/3}}$

  * dissipation scale $t_{tr} \sim t_\nu \sim \ell^2 / \nu \rightarrow \ell_\nu \sim (\nu^3 / \epsilon)^{1/4}$ $(P_M \sim 1)$

  * IK $b_0 \gg u_\ell \sim b_\ell$, $t_a \ll t_\ell$

  $t_{tr}^+ \sim t_{tr}^- \sim t_{tr} \sim {t_\ell}^2 / t_a \sim \ell b_0 / z_\ell^2 \rightarrow \epsilon_\ell^+ \sim \epsilon_\ell^- \sim \epsilon_\ell \sim \epsilon \sim z_\ell^4 / \ell b_0$

  $K_\ell^\pm \sim {z_\ell}^2 \sim kE(k)$, $\epsilon_\ell \sim [kE(k)]^{4/2} k \sim \epsilon$ $\boxed{E(k) \sim (b_0 \epsilon)^{1/2} k^{-3/2}}$

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- correlated case $< \boldsymbol{u} \cdot \boldsymbol{b} > \sim < (\boldsymbol{z}^+)^2 - (\boldsymbol{z}^-)^2 > \nsim 0$, $z_\ell^+ \nsim z_\ell^-$

  * IK $t_a \ll t_\ell^\pm$ $t_\ell^+ \sim \ell / z_\ell^-$, $t_\ell^- \sim \ell / z_\ell^+$

  $t_{tr}^+ \sim {t_\ell^+}^2/t_a \sim \ell b_0 / {z_\ell^-}^2 \rightarrow \epsilon_\ell^+ \sim {z_\ell^+}^2/t_{tr}^+ \sim {z_\ell^+}^2 {z_\ell^-}^2/\ell b_0$

  $t_{tr}^- \sim {t_\ell^-}^2/t_a \sim \ell b_0 / {z_\ell^+}^2 \rightarrow \epsilon_\ell^- \sim {z_\ell^-}^2/t_{tr}^- \sim {z_\ell^-}^2 {z_\ell^+}^2/\ell b_0$

$K_\ell^+ \sim {z_\ell^+}^2 \sim kE^+(k) \rightarrow z_\ell^+ \sim \sqrt{kE^+(k)}$,
$K_\ell^- \sim {z_\ell^-}^2 \sim kE^-(k) \rightarrow z_\ell^- \sim \sqrt{kE^-(k)}$
within the inertial range $k_0 \ll k \ll k_\nu$
$\epsilon_\ell^+ \sim \epsilon_\ell^- \sim \epsilon \sim [kE^+(k)][kE^-(k)]k/b_0$ $\boxed{E^+(k)E^-(k) \sim (b_0\epsilon)k^{-3}}$
suppose that $E^+(k) \sim k^{-m^+}$ and $E^-(k) \sim k^{-m^-} \longrightarrow$ $\boxed{m^+ + m^- = 3}$
* dissipation scales ($P_M \sim 1$)
$t_{tr}^+ \sim t_\nu^+ \sim {\ell^+}^2/\nu \rightarrow \ell_\nu^+ \sim \nu b_0/{z_{\ell_\nu^+}^-}^2$, $t_{tr}^- \sim t_\nu^- \sim {\ell^-}^2/\nu \rightarrow \ell_\nu^- \sim \nu b_0/{z_{\ell_\nu^-}^+}^2$
it can be showned that $k_\nu^+ \sim k_\nu^-$ which leads to $k_\nu \sim (\epsilon/b_0\nu^2)^{1/3} \sim 1/\ell_\nu$

* K41 $b_0 \sim 0$, a similar analysis
$t_{tr}^+ \sim t_\ell^+$, $t_{tr}^- \sim t_\ell^-$ and $\epsilon_\ell^+ \sim \epsilon_\ell^- \sim \epsilon$ leads to $m^+ = m^- = 5/3$
* with dissipation wave number $k_\nu \sim (\epsilon/\nu^3)^{1/4}$

Anisotropic descriptions

Here, let's write $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$, where $\boldsymbol{B}_0$ is an ambient magnetic field.

It is possible to take into account anisotropy within *weak turbulence theory* (weak nonlinearity) using resonant triad waves interactions [6] theory; waves satisfy conditions:

$\boldsymbol{k}^{(1)} + \boldsymbol{k}^{(2)} = \boldsymbol{k}^{(3)}$, $\bar{\omega}^{(1)} + \bar{\omega}^{(2)} \approx \bar{\omega}^{(3)}$, with dispersion relationship $\bar{\omega} = \pm v_a k_\parallel$

As only oppositely travelling waves interact, the 3 waves must satisfy

$k_\parallel^{(1)} + k_\parallel^{(2)} = k_\parallel^{(3)}$ and $v_a k_\parallel^{(1)} - v_a k_\parallel^{(2)} \approx \pm v_a k_\parallel^{(3)}$,

the only possibilities are

$$k_\parallel^{(1)} \approx k_\parallel^{(3)}, \quad k_\parallel^{(2)} \approx 0, \quad \bar{\omega}^{(2)} \approx 0$$

$$k_\parallel^{(2)} \approx k_\parallel^{(3)}, \quad k_\parallel^{(1)} \approx 0, \quad \bar{\omega}^{(1)} \approx 0$$

- modes $k_\parallel \approx 0$, $\bar{\omega} \approx 0$ are not really waves but rather quasi-2D fluctuations highly elongated along $\boldsymbol{B}_0$
- wave (1), for ex., interacts with a quasi-static quasi-2D disturbance and the generated wave (3) has $\sim k_\parallel^{(1)}$, so a negligible change in $\ell_\parallel$ from the collision

[6] strict resonance is not required for the non-linear interactions between 3 waves of form $\boldsymbol{z}_k \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \bar{\omega} t)}$

- For sake of simplicity, we still suppose $P_M \sim 1$ and zero cross helicity ($H^c \sim 0$), thus $z^+_{\boldsymbol{\ell}} \sim z^-_{\boldsymbol{\ell}} \sim z_{\boldsymbol{\ell}}$

- For sake of simplicity, we still suppose $P_M \sim 1$ and zero cross helicity ($H^c \sim 0$), thus $z_\ell^+ \sim z_\ell^- \sim z_\ell$
- IKa, $B_0 \gg b_{rms}$
  * $t_a \ll t_\ell$
  energy transfer time
  $t_{tr} \sim t_\ell^2/t_a \sim (\ell_\perp/z_\ell)^2/(\ell_\parallel/B_0) \sim (k_\parallel B_0)/(k_\perp^2 z_\ell^2)$
  energy flux down through the inertial range
  $\epsilon_\ell^+ \sim \epsilon_\ell^- \sim \epsilon_\ell \sim \epsilon \sim z_\ell^2/t_{tr} \sim k_\perp^2 z_\ell^4/k_\parallel B_0 \longrightarrow z_\ell \sim (\epsilon k_\parallel B_0/k_\perp^2)^{1/4}$
  which leads to $\epsilon \sim k_\perp^2 (k_\parallel k_\perp E(k_\perp, k_\parallel))^2/(k_\parallel B_0)$ and
  $\boxed{E(k_\perp, k_\parallel) \sim (\epsilon B_0)^{1/2} k_\parallel^{-1/2} k_\perp^{-2}}$ (Ng & Bhattacharjee, 1997)

  * $t_a \ll \varepsilon t_\ell$, asymptotic analytical result within Alfvén waves turbulence theory $\boxed{E(k_\parallel, k_\perp) \sim C_k f(k_\parallel) {k_\perp}^{-2}}$ ($k_\parallel \neq 0$)
  (no energy transfer along $\boldsymbol{B}_0$) (Galtier et al., 2000)

- K41a, $B_0 \sim b_{rms}$

  "strong" turbulence regime, i.e. strong non-linear collisions of z^+ and z^- propagating waves to pass energy to smaller scales, with the so called *critical balance* assumption $t_\ell \sim t_a$, i.e. equilibrium between inertial forces and Maxwell stresses (Goldreich & Sridhar, 1995)
  * nonlinear interaction time = interaction time of 2 oppositely travelling waves (as only 1 collision is needed): $t_a \sim \ell_{||}/B_0$
  * flux of energy though inertial rang: $\epsilon_\ell \sim z_\ell^2/t_a \sim z_\ell^2/t_\ell \sim z_\ell^3/\ell_\perp$
  * this yields $z_\ell^2 \sim \epsilon^{2/3}\ell_\perp^{2/3} \rightarrow z_\ell^2 \sim k_\perp E(k_\perp) \sim \epsilon^{2/3}{k_\perp}^{-2/3}$ and thus

  $$\boxed{E(k_\perp) \sim \epsilon^{2/3}{k_\perp}^{-5/3}}$$

  Remarks:
  - $\ell_{||} \sim B_0\ell_\perp/z_\ell \sim (B_0/\epsilon^{1/3})\ell_\perp^{2/3}$
  - $z_\ell^2 \sim \epsilon^{2/3}\ell_\perp^{2/3} \sim \epsilon\ell_{||}/B_0 \longrightarrow E(k_{||}) \sim (\epsilon/B_0)k_{||}^{-2}$
  - within IK theory ($t_a \ll t_\ell$), assuming $E(k_\perp, k_{||}) \sim k_\perp^{-a}k_{||}^{-b}$, it can be show that $3a + 2b = 7$, thus $a = 5/3$, $b = 1$ for K41a & $a = 2$, $b = 1/2$ for IKa, and, if $t_a(\ell_{||})/t_\ell(\ell_\perp) \sim cst$, $\ell_{||} \sim (B_0/\epsilon_{IKa}^{1/3})\ell_\perp^{2/3}$ (Galtier et al., 2005)

von Kármán-Howarth equations

To obtain such von Kármán-Howarth (VKH) equations:

1) write the two-point (at $\boldsymbol{x}$ & $\boldsymbol{x}+\boldsymbol{r}$) correlations for the different components of given fields ($\boldsymbol{u}$,$\boldsymbol{b}$, $z^{\pm}$, ..), or their respective increments, namely 1st, 2nd and 3rd order correlations, reduce the associated tensors (or pseudo-tensors) using incompressibility condition, homogeneity and isotropy assumptions, finally write the tensor coefficients in terms of u_p longitudinal component ($\parallel$ to $\boldsymbol{r}$) and u_{n1}, u_{n2} lateral components ($\perp$ to $\boldsymbol{r}$)

2) write the movement equations at two different spatial locations, $\boldsymbol{x}$ & $\boldsymbol{x}+\boldsymbol{r}$, derive the time evolution of the two-point second order correlation of the fields ($\boldsymbol{u}$,$\boldsymbol{b}$, $z^{\pm}$, ..) and, using homogeneity, obtain the equations for the tensor coefficients

- VKH eq. (1938) for homogeneous fully isotropic NS turbulence

$$\frac{\partial}{\partial t} < u_p(\mathbf{x}) u_p(\mathbf{x}+\mathbf{r}) >= \frac{1}{r^4}\frac{\partial}{\partial r}[r^4 < u_p^2(\mathbf{x}) u_p(\mathbf{x}+\mathbf{r}) >] + 2\nu \frac{1}{r^4}\frac{\partial}{\partial r}[r^4 \frac{\partial}{\partial r} < u_p(\mathbf{x}) u_p(\mathbf{x}+\mathbf{r}) >] \quad (1)$$

a VKH eq. for helical flows (skew isotropy) can be derived (Gomez et al., 2000)

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- VKH for homogeneous isotropic MHD turbulence
for sake of simplicity, let consider $P_M = 1$ (Politano & Pouquet, 1998)

$$\frac{\partial}{\partial t} < z_p^{\pm}(\boldsymbol{x}) z_p^{\pm}(\boldsymbol{x}+\boldsymbol{r}) >= \frac{1}{r^4}\frac{\partial}{\partial r}[r^4 < z_p^{\pm}(\boldsymbol{x}) z_p^{\mp}(\boldsymbol{x}) z_p^{\pm}(\boldsymbol{x}+\boldsymbol{r}) >] + 2\nu \frac{1}{r^4}\frac{\partial}{\partial r}[r^4 \frac{\partial}{\partial r} < z_p^{\pm}(\boldsymbol{x}) z_p^{\pm}(\boldsymbol{x}+\boldsymbol{r}) >] \quad (2)$$

a VKH eq. for magnetic helicity can be obtained (Politano et al. 2003)

Laws for third-order correlation of increments

- Kolmogorov "4/5" law

  * consider velocity increments $\delta u_i(\boldsymbol{r}) = u_i(\boldsymbol{x}+\boldsymbol{r}) - u_i(\boldsymbol{x})$ and the 2nd order $< \delta u_i(\boldsymbol{r}) \delta u_j(\boldsymbol{r}) >$ and 3rd order $< \delta u_i(\boldsymbol{r}) \delta u_j(\boldsymbol{r}) \delta u_k(\boldsymbol{r}) >$ structure functions. Let replace them in VKH eq. (1), and use $\partial_t E = -\epsilon = \frac{1}{2}\partial_t < u_i(\boldsymbol{x}) u_i(\boldsymbol{x}) >= \frac{3}{2}\partial_t < u_p^2(\boldsymbol{x}) >$ (by isotropy)

  * under hypothesis : i) $t \to \infty$ (stationary state) and ϵ is finite per unit mass (ν fixed) and ii) $\nu \to 0$ (ϵ still fixed), one obtains

$$< (\delta u_p(\boldsymbol{r}))^3 >= -\frac{4}{5}\epsilon r \qquad \text{within the inertial range}$$

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- MHD "4/3" law

  a similar approach for MHD VKH eq. (2) gives in the inertial range

$$< (\delta \boldsymbol{z}^{\pm} \cdot \delta \boldsymbol{z}^{\pm}) \delta z_p^{\mp}(\boldsymbol{r}) > = -\frac{4}{3} \epsilon^{\pm} r$$

(Politano & Pouquet, 1998)

Structures fonctions and scaling exponents

Two-points statistics can be described in terms of moments of velocity increments (or "structure functions") of order p, for $\ell \ll \ell_0$, namely

$$\delta v_{\boldsymbol{\ell}}^p \equiv < [u(\boldsymbol{x} + \boldsymbol{\ell}) - u(\boldsymbol{x})]^p >$$

where, here, u is the field component, say velocity, in the direction of the separation vector $\boldsymbol{\ell} = (\ell, 0, 0)$ (longitudinal component).

Suppose a scaling law within the inertial range $\ell_\nu \ll \ell \ll \ell_0$; $\boxed{\delta v_{\boldsymbol{\ell}}^p \sim \ell^{\xi_p}}$

exact results

* if $\boldsymbol{u}$-fluctuations are bounded then $\xi_{2p+2} \geqslant \xi_{2p}$ (p =1, 2, 3 ..) (Frisch 91)
* Schwartz inequality gives $\xi_{p+q} \geqslant (\xi_{2p} + \xi_{2q})/2$ (for all positive p, q)
* hence, $d^2\xi_p/dp^2 \leqslant 0$ and ξ_p is a concave function of p ($\forall p > 0$)

linear behavior of ξ_p predicted from phenomenology

* K41 approach $\xi_p = p/3$
* IK approach $\xi_p = p/4$ (uncorrelated case)

Experimental results

* many analysis of observational and numerical data show departure from a linear behavior of the scaling exponents, ξ_p, and this departure becomes larger as $p \nearrow$ ... something is going wrong with the original K41 theory

* p.d.f.s of velocity increments have less and less Gaussian forms as $\ell \searrow$; for $\ell \sim \ell_0$ the p.d.f of this increments is essentially indistinguishable from a Gaussian, at inertial range separations, it develops almost expontial wings, and at even smaller scales, it takes form of "stretched exponential". This is probably due to the strong localization of the strong fluctuations

Interpretation and modeling

- Source of the fundamental problem with K41 : within a volume $\mathcal{V}_{\boldsymbol{\ell}}$, it is not the mean rate of dissipation ϵ that is relevant but rather the local dissipation $\epsilon(\boldsymbol{x}, t) = \nu/2(\partial_j u_i + \partial_i u_j)^2$ averaged over $\mathcal{V}_{\boldsymbol{\ell}}$ (for ex. a sphere of center $\boldsymbol{x}$ and radius ℓ) : $\epsilon_{\boldsymbol{\ell}}(\boldsymbol{x}, t) \equiv < \epsilon(\boldsymbol{x}, t) >_{\mathcal{V}_{\boldsymbol{\ell}}}$
$< \epsilon_{\boldsymbol{\ell}}^p(\boldsymbol{x}, t) >$ will depend upon $\boldsymbol{\ell}$ (homogeneity) and let's suppose
$\boxed{< \epsilon_{\boldsymbol{\ell}}^p(\boldsymbol{x}, t) > \sim \boldsymbol{\ell}^{\tau_p}}$ $(\boldsymbol{\ell}_\nu \ll \boldsymbol{\ell} \ll \boldsymbol{\ell}_0)$

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- $(\delta v_{\boldsymbol{\ell}})^p \sim (\boldsymbol{\ell}\epsilon_{\boldsymbol{\ell}})^{p/3} \sim \boldsymbol{\ell}^{p/3}\boldsymbol{\ell}^{\tau_{p/3}} \sim \boldsymbol{\ell}^{\xi_p} \longrightarrow \quad \xi_p = p/3 + \tau_{p/3}$

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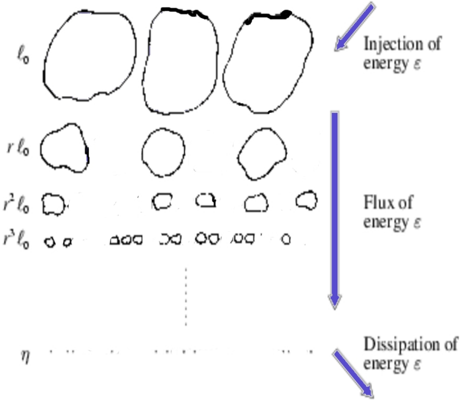
- $(\delta v_\ell)^p \sim (\ell\epsilon_\ell)^{p/3} \sim \ell^{p/3}\ell^{\tau_{p/3}} \sim \ell^{\xi_p} \longrightarrow \quad \xi_p = p/3 + \tau_{p/3}$
- many attempts to take into account the influence of possibly strong fluctuations in ϵ (or intermittency) with a modeling of $\tau_{p/3}$ exponent retaining the central concept of energy cascade through an extended inertial range (Log-normal model (Kolmogorov-Obukhov 62), $\beta-$model (Frisch et al. 78), Log-Poisson model (She-Lévêque 94))

Scenario of modified Richardson's cascade



sporadic energy tranfer through inertial range: only a small fraction of eddies of size $\ell \ll \ell_0$ is involved in the energy transfer to smaller scales, the other ℓ-eddies stay at rest (excitation on scale ℓ is confined, eddies are thus no more space-filling)

Log-Poisson model

The Log-Poisson model (She-Lévêque (SL),1994) currently provides the best fit for the ξ_p-exponents computed from experimental or numerical data.

- essential assumption: existence of a hierarchy of successive moments of energy dissipation at a given scale ℓ with a power law exponent, β, of the hierarchy ($0 < \beta < 1$)
- scaling exponent, α, for the characteristic time to dissipate the maximum amount of energy in the most intermittent dissipative structures; $t_\ell \sim \ell^\alpha$ (one can set a value for α in accordance with some phenomenology)
- C_0 codimension of the dissipative structures; $C_0 = \alpha/(1-\beta)$, and as $C_0 \leqslant D$ (where D is the dimension of space) $\rightarrow \beta \leqslant 1 - \alpha/D$

The model is thus a two-parameter model (for a general formulation of the model see Politano & Pouquet, 1995).

- SL HD
$$\xi_p = \frac{p}{3} + \alpha\left(\frac{1-\beta^{p/3}}{1-\beta} - \frac{p}{3}\right)$$
"standard" model: $\alpha = 2/3$ (K41) and $C_0 = 2$ codimension of tube-like dissipative structures $\rightarrow \beta = 2/3$ (original SL model, 1994)

- SL MHD IK, case $H^c \sim 0$ & $P_M \sim 1$,
$$\xi_p = \frac{p}{4} + \alpha\left(\frac{1-\beta^{p/4}}{1-\beta} - \frac{p}{4}\right)$$
"standard" model: $\alpha = 1/2$ (IK) and $C_0 = 1$ codimension of sheet-like dissipative structures $\rightarrow \beta = 1/2$ (Grauer et al., 1994)

- SL MHD K41, case $H^c \sim 0$ & $P_M \sim 1$,
$$\xi_p = \frac{p}{3} + \alpha\left(\frac{1-\beta^{p/3}}{1-\beta} - \frac{p}{3}\right)$$
"standard" model: $\alpha = 2/3$ (K41) and $C_0 = 1$ codimension of sheet-like dissipative structures $\rightarrow \beta = 1/3$ (Horbury & Balogh, 1997)

In the case of anisotropic MHD see, for ex., W.-C. Müller, in Lecture Notes in Physics, vol. 756, 2009

References: some books and references therein

G. K. Batchelor, *The Theory of Homogeneous Turbulence*, Cambridge University Press, 1953

A. Brandenburg & A. Nordlund, *Astrophysical Turbulence Modeling*, Rep. Prog. Phys, 74, 2011

P. A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge University Press, 2001 & *Turbulence in Rotating and Electrically Conducting Flow*, Cambridge Univ. Press, 2013

U. Frisch, *Turbulence*, Cambridge Univ. Press, 2nd edition, 1996

J. P. Goedbloed & S. Poedts, *Principles of Magnetohydrodynamics*, Cambrige Univ. Press, 2004

J. P. Goedbloed, R. Keppens & S. Poedts, *Advanced Magnetohydrodynamics*, Cambridge Univ. Press, 2009

R. M. Kulsrud, *Plasma Physics for Astrophysics*, Princeton University Press 2005

L. D. Landau & E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, Oxford, 1987

H. K. Moffatt, *Magnetic Field generation in electrically Conducting Fluids*, Cambridge Univ. Press, 2nd edition, 1983

S. Molokov, R. Moreau, H. K. Moffatt (Eds.), *Magnetohydrodynamics, Historical Evolution and Trends*, Springer, 2007 (contains review articles)

S. B. Pope, *Turbulent Flow*, Cambridge Univ. Press, 2000

E. R. Priest,*Solar Magnetohydrodynamics*, Springer Netherlands, 1982

P. H. Roberts, *An introduction to magnetohydrodynamics*, American Elsevier Pub. Co, 1967