Lie groups and Riemannian symmetric spaces

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Contents

Introduction

Warning: Under construction!

Convention

Unless otherwise specified, all manifolds, tensors, maps etc.. under consideration in this lecture will be assumed to be of $\mathcal{C}^\infty$ regularity for which smooth is a synonym. We recall it frequently, but we almost always consider connected manifolds, except for Lie groups which are not necessarily connected.

Prerequisites

Differentiable manifolds, Vector fields, Frobenius integrability theorem, Basics of Riemannian Geometry: Levi-Civita connexion, geodesics, exponential map, Riemann curvature tensor, sectional curvature.

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Chapter 1

Toolbox from Riemannian geometry and affine geometry

1.1 Linear connections, contractions

A linear connection ∇ is a connection on the tangent bundle TM of a smooth manifold M. We note $\mathcal{T}^{r,s}M = T^*M^{\otimes r} \otimes TM^{\otimes s}$ the vector bundle of tensors r-times covariant and s-times contravariant. A tensor T of type (r,s) is then a smooth section of $T : M \to \mathcal{T}^{r,s}M$.

Remark 1.1.1. Let T be a tensor of type (r,s) on M. Let (U,φ) be a local chart of M. Write $\varphi = (x^1,\dots,x^n)$. Then, $(\mathrm{d}x^1,\dots,\mathrm{d}x^n)$ is a local coframe on U, *i.e.* a framing of T^*M defined on U. Let $(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n})$ be the dual local frame. For $1 \leqslant i_1,\dots,i_r \leqslant n$ and $1 \leqslant j_1,\dots,j_s \leqslant n$

$$T^{i_1,\dots,i_s}_{j_1,\dots,j_r} = T\left(\frac{\partial}{\partial x^{j_1}},\dots,\frac{\partial}{\partial x^{j_r}},\mathrm{d}x^{i_1},\dots,\mathrm{d}x^{i_s}\right).$$

We obtain the local expression

$$T = \sum_{\substack{1\leqslant i_1,\dots,i_s\leqslant n\\ 1\leqslant j_1,\dots,j_r\leqslant n}} T^{i_1,\dots,i_s}_{j_1,\dots,j_r}\mathrm{d}x^{j_1}\otimes\cdots\otimes\mathrm{d}x^{j_r}\otimes\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_s}}.$$

Definition 1.1.1. Let $r,s \geqslant 1$ and let $T \in \Gamma(\mathcal{T}^{r,s}(M))$. Let $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$ be two indices. Consider the map $c^{i,j}(T) : \Gamma(TM)^{r-1} \times \Gamma(T^*M)^{s-1} \to \mathcal{C}^\infty(M)$ defined for all $X_1,\dots,\widehat{X_i},\dots,X_r$ and $\omega_1,\dots,\widehat{\omega_j},\dots,\omega_s$ by

$$c^{i,j}(T)(X_1,\dots,\widehat{X_i},\dots,X_r,\omega_1,\dots,\widehat{\omega_j},\dots,\omega_s) = \operatorname{Tr} u$$

where $u \in \operatorname{End}(TM) = T^*M \otimes TM$ is the $\mathcal{C}^\infty(M)$ bilinear map

$$(X,\omega) \in TM \times T^*M \mapsto T(X_1,\dots,X,\dots,X_r,\omega_1,\dots,\omega,\dots,\omega_s) \in \mathcal{C}^\infty(M).$$

It is straightforward that $c^{i,j}(T)$ is $\mathcal{C}^\infty(M)$-multilinear, hence a tensor of type $(r-1,s-1)$, called the (i,j)-*contraction* of T.

Example 1.1.1. 1. If $\omega \in \Omega^1(M)$ and $X \in \Gamma(TM)$, $T := \omega \otimes X \in \Gamma(\mathcal{T}^{1,1}(M))$, the only possible contraction is $c^{1,1}$ and we have $c^{1,1}(T) = \omega(X)$.

2. Let R be the $(3,1)$-Riemann curvature tensor of a (pseudo)-Riemannian manifold (see below for a reminder). Then, $c^{3,1}(R) = 0$ because for $X, Y \in \Gamma(TM)$, the map $\{u_{X,Y} : Z \to R(X,Y,Z)\}$ satisfies $g(u_{X,Y}(Z), T) + g(u_{X,Y}(T), Z) = 0$. It means that $u_{X,Y}$ is skew-symmetric with respect to g, in particular it is trace-free.

   The symmetries of R imply that $c^{1,1}(R) = -c^{2,1}(R)$. So, up to the sign, there is only one possibly non-zero contraction of R, and it is the *Ricci curvature.*

3. Finally, the only possible contraction of the Ricci curvature is the *scalar curvature.*

Remark 1.1.2. Let T be a tensor of type (r,s). Let $(U, (x^1, \dots, x^n))$ be a local coordinate system. Let $\{T^{i_1,\dots,i_s}_{j_1,\dots,j_r}\}$ be the corresponding local components. For any (i,j), the components of $c^{i,j}(T)$ are given by

$$\sum_k T^{i_1,\dots,k,\dots,i_s}_{j_1,\dots,k,\dots,j_r}$$

where the summation index k appears in position i in the lower indices, and position j in the upper indices.

Proposition 1.1.1. *Let ∇ be a linear connection on a manifold M. Then ∇ extends uniquely into a connection on all vector bundles $\mathcal{T}^{r,s}M$, $r, s \geqslant 0$, still denoted ∇, that satisfies the following criteria :*

1. *For all $X \in \Gamma(TM)$, ∇_X coincides with the Lie derivative $\mathcal{L}_X$ on $\Gamma(\mathcal{T}^{0,0}M) = \mathcal{C}^\infty(M)$.*
2. *∇_X commutes with contractions : for all $r, s \geqslant 1$ and $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$, we have for all $\sigma \in \Gamma(\mathcal{T}^{r,s}M)$, $c^i_j(\nabla_X \sigma) = \nabla_X c^i_j(\sigma)$.*
3. *∇_X satisfies a Leibniz rule for the tensor product : for all $\sigma_1 \in \Gamma(\mathcal{T}^{r,s}M)$ and $\sigma_2 \in \Gamma(\mathcal{T}^{r',s'}M)$, we have*

$$\nabla_X(\sigma_1 \otimes \sigma_2) = (\nabla_X \sigma_1) \otimes \sigma_2 + \sigma_1 \otimes (\nabla_X \sigma_2).$$

Using this characterization, we can for instance recover the usual formulas for the connection that ∇ induces on T^*M. Indeed, let $\omega \in \Omega^1(M) = \Gamma(T^*M)$ and $X, X_1 \in \Gamma(TM)$. By (3), we have $\nabla_X(\omega \otimes X_1) = (\nabla_X \omega) \otimes X_1 + \omega \otimes (\nabla_X X_1)$. Next, we apply the only possible contraction:

$$c^{1,1}(\nabla_X(\omega \otimes X_1)) = \nabla_X(c^{1,1}(\omega \otimes X_1)) = X.\omega(X_1),$$

the second equality coming from (1). Contracting the RHS we obtain:

$$X.\omega(X_1) = (\nabla_X \omega)(X_1) + \omega(\nabla_X X_1),$$

recovering the expression for $\nabla_X \omega(X_1) = X.\omega(X_1) - \omega(\nabla_X X_1)$.

Definition 1.1.2. A tensor T is said to be *parallel* if $\nabla T = 0$.

1.2 Riemannian manifolds

Definition 1.2.1. A *Riemannian metric* g on M is a section $g \in S^2(T^*M)$ such that for every $x \in M$, g_x is positive definite.

Theorem 1. *Given a Riemannian metric* g *on* M*, there exists a unique linear connection* $\nabla = \nabla^g$*, called the Levi-Civita connection of* g*, such that*

1. ∇ *is torsion-free, i.e. for every* $X, Y \in \Gamma(TM)$*, we have* $[X,Y] = \nabla_X Y - \nabla_Y X$.
2. $\nabla g = 0$*, i.e.* g *is parallel with respect to* ∇*, which concretely means that for all* $X, X_1, X_2 \in \Gamma(TM)$*, we have*
$$X.g(X_1, X_2) = g(\nabla_X X_1, X_2) + g(X_1, \nabla_X X_2).$$

1.3 Parallel transport, geodesics, exponential map

Let (M,g) be a Riemannian manifold.

Definition 1.3.1. Let $x \in M$. The injectivity radius of M at x, which we denote $\mathrm{inj}_M(x)$, is the suppremum of $r > 0$ such that $\exp_x$ is defined on $B(0,r) \subset T_xM$ and realizes a diffeomorphism onto its image. (The ball refers to the Euclidean ball in (T_xM, g_x)).

The (global) injectivity radius of (M,g) is $\mathrm{inj}_M = \inf_{x \in M} \mathrm{inj}_M(x)$.

Recall that $\mathrm{inj}_M(x) > 0$ by a standard argument of inverse mapping theorem. Of course, the global injectivity radius can be zero (consider for instance $\mathbf{R}^n \setminus \{0\}$).

1.4 Isometries and local isometries

Definition 1.4.1. An *isometry* of a Riemannian manifold (M,g) is a diffeomorphism $f \in \mathrm{Diff}(M)$ such that $f^*g = g$, *i.e.*

$$g_{f(x)}(\mathrm{d}_x f u, \mathrm{d}_x f v) = g_x(u,v)$$

for all $x \in M$ and $u, v \in T_xM$. The *isometry group* of a Riemannian manifold (M,g) is the group of all its isometries. We denote it by $\mathrm{Iso}(M,g)$.

Exercise 1.4.1. Show that if (M,g) is complete and $f : M \to M$ is a smooth injective map such that $f^*g = g$, then $f \in \mathrm{Iso}(M,g)$. Give a counter-example when M is not complete.

Thus, requiring $f \in \mathrm{Diff}(M)$ in the definition of an isometry is an important assumption.

Let d_g denote the *length* distance on M. Recall that it is defined as the infimum of length among piecewise smooth curves joining two points

$$d_g(x,y) = \inf\{\ell(\gamma) = \int_0^1 \|\gamma'(t)\| \mathrm{d}t,\ \gamma \text{ piecewise smooth },\gamma(0) = x,\ \gamma(1) = y\}.$$

The following property is central in Riemannian geometry.

Theorem 2. $\operatorname{Iso}(M,g) = \operatorname{Iso}(M,d_g)$, *where elements in the latter are seen as isometries of the metric space* (M,d_g).

Proposition 1.4.1. *An isometry is determined by its* 1*-jet at any point, i.e. given any point* $x_0 \in M$, *if* $\varphi_1,\varphi_2 \in \operatorname{Iso}(M,g)$ *are such that* $\varphi_1(x_0) = \varphi_2(x_0)$ *and* $\mathrm{d}_{x_0}\varphi_1 = \mathrm{d}_{x_0}\varphi_2$, *then* $\varphi_1 = \varphi_2$.

Proof. We may assume $\varphi_2 = \mathrm{id}$. We are reduced to prove that for a global isometry φ, if there exists a point x such that $\varphi(x) = x$ and $\mathrm{d}_x\varphi = \mathrm{id}$, then $\varphi = \mathrm{id}$. Consider $U = \{x \in M \mid \varphi(x) = x, \text{ and } \mathrm{d}_x\varphi = \mathrm{id}\}$. Then, U is closed, and open because of the linearization property **ref**. Hence, $U = M$ because $U \neq \varnothing$ and M is connected. ☐

Definition 1.4.2. We call *local isometry* of (M,g) any diffeomorphism $f : U \to V$ between two open subsets $U,V \subset M$ such that f is an isometry between $(U,g|_U)$ and $(V,g|_V)$.

Remark 1.4.1. This definition should be taken with precaution as other references in the literature call "local isometry" a *globally defined* smooth map $f : M \to N$ between Riemannian manifolds (M,g) and (N,h) such that $f^*h = g$. Such f are necessarily immersions from M to N and then local diffeomorphisms for $\dim M = \dim N$. Typically, one can have in mind the case of Riemannian covering $\pi : \tilde{M} \to M$: start with a Riemannian manifold (M,g) and a covering map $\pi : \tilde{M} \to M$. Define $\tilde{g} := \pi^*g$. Then $(\tilde{M},\tilde{g})$ is a Riemannian manifold and π a local isometry, in the sense that we will not use.

The need for local and global isometries is justified for instance when we consider quotient manifolds. Consider a Riemannian manifold (M,g) and let $\Gamma \subset \operatorname{Iso}(M,g)$ a group acting freely, properly discontinuously on M. Let $(\overline{M},\overline{g})$ be the quotient manifold, *i.e.* $\overline{M} = \Gamma\backslash M$ as a manifold, and if $\pi : M \to \overline{M}$ is the canonical map, $\overline{g}$ is the (unique) metric such that $\pi^*\overline{g} = g$.

In general, a global isometry f of M does *not* induce an isometry on $\overline{M}$ (or even a global map). Precisely, the question is answered by the following.

Proposition 1.4.2. *For every* $\overline{f} \in \operatorname{Iso}(\overline{M},\overline{g})$, *there exists* $f \in \operatorname{Iso}(M,g)$ *such that* $\overline{f}\circ\pi = \pi\circ f$. *Conversely, for* $f \in \operatorname{Iso}(M,g)$, *if there exists a map* $\overline{f} : \overline{M} \to \overline{M}$ *such that* $\overline{f}\circ\pi = \pi\circ f$, *then* $\overline{f}$ *is an isometry of* $\overline{M}$. *In this situation, we say that* $\overline{f}$ *is induced by* f.

An isometry $f \in \operatorname{Iso}(M,g)$ *induces an isometry on* $(\overline{M},\overline{g})$ *if and only if* f *normalizes* Γ *in the group* $\operatorname{Iso}(M,g)$.

Example 1.4.1. Consider $(M,g) = \mathbf{E}^n$ the Euclidean space and $\Gamma = \mathbf{Z}^n < \operatorname{Iso}(M,g)$ the group of translations with integer coordinates. Then, $(\overline{M},\overline{g})$ is a flat torus, and an isometry of $(\overline{M},\overline{g})$ is induced by a Euclidean motion normalizing $\mathbf{Z}^n$. Hence, $\operatorname{Iso}(\overline{M},\overline{g}) \simeq (O(n)\cap\mathrm{SL}_n(\mathbf{Z}))\ltimes\mathbf{T}^n$ and only finitely many elements of $O(n)$ descend to the quotient.

Recall that $\pi : M \to \overline{M}$ is a covering map. Let $\overline{x} \in \overline{M}$ and $U \ni \overline{x}$ be a trivializing open neighborhood. Let $x \in \pi^{-1}(\overline{x})$ and let $\tilde{U}$ be the connected component of $\pi^{-1}(U)$ containing x. Let $f \in \operatorname{Iso}(M,g)$ be arbitrary. Then, we can define $\overline{f} : U \to \pi(f(\tilde{U}))$ by $\overline{f} = \pi\circ f\circ(\pi|_{\tilde{U}})^{-1}$ and by construction, $\overline{f}$ is a local isometry between U and $\pi(f(\tilde{U}))$. Hence, an isometry of (M,g) always yields local isometries of $\overline{M}$, but they do not globalize in general.

Chapter 2

(Locally) Symmetric spaces: Definitions and first properties

2.1 Definitions and first examples

Definition 2.1.1. A (global, Riemannian) symmetric space is a Riemannian manifold (M, g) such that for every $x \in M$, there exists an isometry $s_x \in \mathrm{Iso}(M, g)$ such that $s_x(x) = x$ and $\mathrm{d}_x s_x = -\mathrm{id}$.

Remark that when it exists, such an isometry s_x

1. must be unique ;
2. is involutive.

Equivalently, s_x is characterized by the fact that $s_x(\exp_x(v)) = \exp_x(-v)$ for all tangent vector v such that $\|v\|_x < \mathrm{inj}_M(x)$.

In general, (M, g) being symmetric or not, one can always define a natural involution on the exponential neighborhood $N_x := \exp_x(B(0, \mathrm{inj}(x)))$ of any point $x \in M$ via the formula

$$f(\exp_x(v)) = \exp_x(-v).$$

Such f again is involutive, and is often called the *local geodesic involution* at x. An equivalent formulation of Definition 2.1.1 is the following.

Definition 2.1.2. A (global, Riemannian) symmetric space is a Riemannian manifold (M, g) such that for all $x \in M$, the geodesic involution at x is the restriction of a global isometry of (M, g).

Exercise 2.1.1. Verify that the two definitions of globally symmetric space are equivalent.

This leads naturally to the more general notion of locally symmetric spaces.

Definition 2.1.3. A (Riemannian) locally symmetric space is a Riemannian manifold (M, g) such that for all $x \in M$, the geodesic involution at x is a local isometry of (M, g).

Example 2.1.1 (Spaces of constant sectional curvature)**.** The first examples of globally symmetric spaces are *simply connected* manifolds of *constant sectional curvature.*

Recall **ref** that up to scale, these are isometric to either the real hyperbolic space $\mathbf{H}^n_{\mathbf{R}}$, the flat Euclidean space $\mathbf{R}^n$, or the sphere $\mathbf{S}^n = \{x_0^2 + \dots + x_n^2 = 1\}$ with the metric induced by the flat metric of $\mathbf{R}^{n+1}$. All of them are homogeneous, *i.e.* the identity component of their isometry group acts transitively (see **ref**). So, if we exhibit one global symmetry s_{x_0} at a point $x_0 \in M$, then it will follow that M is globally symmetric because for every $x \in M$, there exists an isometry $f \in \mathrm{Iso}(M, g)$ such that $x = f(x_0)$, and $f \circ s_{x_0} \circ f^{-1}$ is a geodesic symmetric at x.

1. For $(M, g) = \mathbf{H}^n_{\mathbf{R}}$, consider the hyperboloid model, *i.e.* $M = \{(x_1, \dots, x_{n+1}) \in \mathbf{R}^{n,1} \mid x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_0 > 0\}$ with the metric induced by the quadratic form $x_1^2 + \dots + x_n^2 - x_{n+1}^2$. Let $x = (0, \dots, 0, 1)$. Then, the diagonal matrix $\mathrm{diag}(-1, \dots, -1, 1) \in O(n, 1)$, and induces an isometry of M such that $s(x) = x$ and $\mathrm{d}_x s_x = -\mathrm{id}_{T_x M}$.
2. For $(M, g) = (\mathbf{R}^n, \mathrm{d}x_1^2 + \dots + \mathrm{d}x_n^2)$, if $x = 0$, then $s_x = -\mathrm{id}$ is an isometry and a geodesic symmetry.
3. For $\mathbf{S}^n$, let $x = (1, 0, \dots, 0)$. Then, the diagonal matrix $\mathrm{diag}(1, -1, \dots, -1) \in O(n+1)$ induces an isometry of $\mathbf{S}^n$ and is a geodesic symmetry at x.

Proposition 2.1.1. *Let (M, g) be a globally symmetric space. Then, it is homogeneous and geodesically complete.*

Remark 2.1.1. In fact, the *identity component* $G = \mathrm{Iso}(M, g)_0$ acts transitively. The latter is defined as the connected component of id in $\mathrm{Iso}(M, g)$ with respect to the compact-open topology. It is a closed (and open) normal subgroup (see later). For now, it is easily observed if we take for granted the reasonable fact: $x \in M \mapsto s_x \in \mathrm{Iso}(M, g)$ is continuous.

Proof. We prove first that (M, g) is complete. Let $x \in M$ and $v \in T_x M$. Suppose that the geodesic γ starting at x with velocity v is defined on an interval $[-a, a]$ for $a > 0$. We claim that it is automatically defined on $[-2a, 2a]$, and therefore on all of $\mathbf{R}$. Let $y = \gamma(a/2)$, let s_y be the global isometry inducing the geodesic involution at y, and let $\gamma_0(t) := s_y(\gamma(a/2 - t))$. Then, γ_0 is a geodesic defined on $[-a/2, 3a/2]$ (the post-composition of a geodesic by an isometry is a geodesic). Moreover, $\gamma_0(0) = y$ and $\gamma_0'(0) = \gamma'(a/2)$. Therefore, $\gamma_0(t) = \gamma(t + a/2)$ for all $t \in [-a/2, a/2]$ and

$$\tilde{\gamma}(s) = \begin{cases} \gamma(s) & \text{if } s \in [-a, a] \\ \gamma_0(s - a/2) & \text{if } s \in [0, 2a] \end{cases}$$

is a geodesic extending γ to $[-a, 2a]$. Applying the same construction at $\gamma(-a/2)$, the claim follows.

Now, since (M, g) is geodesically complete, for any two points $x, y \in M$, there exists a geodesic γ such that $\gamma(0) = x$ and $\gamma(1) = y$ (Hopf-Rinow). Then, $s_{\gamma(1/2)}$ is a global isometry such that $s_{\gamma(1/2)} \circ \gamma(t) = \gamma(1-t)$ for all $t \in \mathbf{R}$. So $s_{\gamma(1/2)}(x) = y$ and we see that (M, g) is homogeneous. $\square$

Hence, completeness or homogeneity are necessary conditions to *global* symmetry.

Example 2.1.2. Consider any (possibly non simply-connected nor complete) space of constant sectionnal curvature, for instance a flat torus, i.e. $M = \mathbf{R}^2/\mathbf{Z}^2$ endowed with the flat metric inherited from $\mathbf{R}^2$ or a complete space with one point removed. Then it is locally symmetric because its universal cover is (see the paragraph below Prop. 1.4.2). However, it won't be globally symmetric if non complete.

The following important proposition characterizes locally symmetric spaces at the infinitesimal scale. It shows that they are of "constant curvature", but in a more general sense than constant sectional curvature.

Proposition 2.1.2. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and Riemann curvature tensor R. Then, M is locally symmetric if and only if $\nabla R = 0$.*

Proof. Assume first that (M, g) is locally symmetric. Let $x \in M$ and let s be an isometric local geodesic involution at x, defined on some exponential neighborhood of x. Then, $s^*(\nabla R) = \nabla R$ on that same neighborhood because s is isometric. By definition of s, this relation implies at x that for any $u, v, w \in T_xM$, we have $(\nabla_u R)_x(v, w) = -(\nabla_u R)_x(v, w) = 0$. Hence R is parallel as announced.

Conversely, let us assume that $\nabla R = 0$ and prove that g is locally symmetric. Let $x \in M$ be a point. We want to prove the existence of a local isometry f defined near x such that $f(x) = x$ and $\mathrm{d}_x f = -\mathrm{id}$. We prove the more general

Lemma 2.1.1. *Let $(M^n, g), (N^n, h)$ be two Riemannian manifolds whose Riemann curvature tensors R^M and R^N are both parallel. Let $x \in M$, $y \in N$ and $\phi : (T_xM, g_x) \to (T_yN, g_y)$ be a linear isometry such that $\phi^*(R^N)_y = (R^M)_x$. Then, there exists a local isometry $f : U \to V \subset N$ defined on a neighborhood of x such that $f(x) = y$ and $d_x f = \phi$.*

Proof. By properties of the exponential map, reducing U if necessary, *if* it exists, f must be given by the formula

$$f(\exp_x(v)) = \exp_y(\phi(v)).$$

As before, such an expression surely defines a diffeomorphism between two exponential neighborhoods of x and y. The question is to see why it must be isometric under our assumptions. Let $p = \exp_x(v)$ and $q = \exp_y(\phi(v))$. Let $w \in T_xM$, seen as a tangent vector at v of this vector space. Differentiating this identity at v and applying it at w, we get

$$\mathrm{d}_p f \circ \mathrm{d}_v \exp_x(w) = \mathrm{d}_{\phi(v)} \exp_y(\phi(w)).$$

Let $\gamma(t) = \exp_x(tv)$, defined on a neighborhood of $[0,1]$. Let J be the Jacobi field along the geodesic γ such that $J(0) = 0$ and $J'(0) = w$. Then, we know that $J(1) = \mathrm{d}_v \exp_x w$. Let us introduce $(e_1(t), \dots, e_n(t))$ a parallel orthonormal frame field along γ, such that $e_1(0) = v$. Let $y_1(t), \dots, y_n(t)$ be such that $Y(t) = \sum y_k(t)e_k(t)$ for all t. Now, let $\eta(t) = \exp_y(t\phi(v))$ and let $(\varepsilon_1(t), \dots, \varepsilon_n(t))$ be the parallel orthonormal frame field along η, such that $\varepsilon_i(0) = \phi(e_i(0))$ for all i and define $J_N(t) \coloneqq \sum_k y_k(t)\varepsilon_k(t)$.

We claim that J_N is a Jacobi field along η satisfying $J_N'(0) = \sum y_k'(0)\varepsilon_k(0) = \phi(w)$. To see it, observe first that necessarily $e_1(t) = \gamma'(t)$ and $\varepsilon_1(t) = \eta'(t)$ for all t. Then we compute for all i:

$$\begin{aligned}
h(J_N''(t) + R^N(\eta'(t), J_N(t))\eta'(t), \varepsilon_i(t)) &= y_i''(t) + \sum_k h(R^N(\varepsilon_1(t), \varepsilon_k(t))\varepsilon_1(t)), \varepsilon_i(t)) \\
&= y_i''(t) + \sum_k h_y(R^N(\varepsilon_1(0), \varepsilon_k(0))\varepsilon_1(0), \varepsilon_i(0)) \\
&= y_i''(t) + \sum_k g_x(R^M(e_1(0), e_k(0))e_1(0), e_i(0)) \\
&= y_i''(t) + \sum_k g(R^M(e_1(t), e_k(t))e_1(t)), e_i(t)) \\
&= g(J''(t) + R^M(\gamma'(t), J(t))\gamma'(t), e_i(t)) = 0.
\end{aligned}$$

This calculation is justified by :

1. For all parallel vector fields X, Y along a curve c, $g_{c(t)}(X, Y)$ is constant.
2. For all parallel vector fields X, Y, Z along a curve c, the vector field $R(X,Y)Z$ also is parallel along c because $\nabla R = 0$.
3. $R(\varepsilon_1(0), \varepsilon_k(0))\varepsilon_1(0) = \phi(R(e_1(0), e_k(0))e_1(0))$ because $\phi^* R = R$
4. The fact that $\phi : T_x M \to T_y N$ is isometric.

Hence, J_N is a Jacobi vector field. It follows that $J_N(1) = \mathrm{d}_{\phi(v)} \exp_y .\phi(w) = \mathrm{d}_p f \circ \mathrm{d}_v \exp_x .w$. Moreover, since we have $\|J_N(t)\|_N^2 = \sum y_k(t)^2 = \|J(t)\|_M^2$ for all t, we finally get:

$$\|\mathrm{d}_p f(\mathrm{d}_v \exp_x(w))\|_N = \|\mathrm{d}_v \exp_x(w)\|_M,$$

proving that $\mathrm{d}_p f : T_p M \to T_{f(p)} N$ is isometric as expected. □

Observe now that we deduce easily that local geodesic symmetries are isometric when R is parallel: we can just apply this lemma to the case $M = N$, $x = y$ and $\phi = -\mathrm{id}$. It is straightforward that ϕ satisfies the hypothesis of the lemma. □

If we admit that Riemannian manifolds of constant sectional curvature are, up to scale, locally isometric to the sphere, the Euclidean space or the hyperbolic space of same dimension, then it is clear that constant sectional curvature implies parallel Riemann curvature tensor. It is nonetheless interesting to have a straight explanation. This is the object of the following.

Proposition 2.1.3. *Let (M, g) be a Riemannian manifold whose sectional curvature is invariant under parallel transport. Then, it is locally symmetric.*

Proof. Let $x, y \in M$ and let γ be a smooth curve such that $\gamma(0) = x$ and $\gamma(1) = y$. Let $\tau : T_xM \to T_yM$ be the parallel transport along γ. By assumption, we have for all $v_1, v_2, v_3, v_4 \in T_xM$

$$g_x(R_x(v_1, v_2)v_3, v_4) = g_y(R_y(\tau(v_1), \tau(v_2))\tau(v_3), \tau(v_4)).$$

Expressing that the metric tensor g is parallel, hence invariant under parallel transport, we have, for all $u, v \in T_xM$,

$$g_x(u, v) = g_y(\tau(u), \tau(v)).$$

Consequently, $g_y(R_y(\tau(v_1), \tau(v_2))\tau(v_3), \tau(v_4)) = g_y(\tau(R_x(v_1, v_2)v_3), \tau(v_4))$ for any $v_i \in T_xM$. It follows that $R_y(\tau(v_1), \tau(v_2))\tau(v_3) = \tau(R_x(v_1, v_2)v_3)$. Applying this to $y = \gamma_v(t)$ and taking derivative at $t = 0$, we obtain $(\nabla_v R)_x = 0$ for all $v \in T_xM$, and R is parallel, which implies that M is locally symmetric by Proposition 2.1.2. ☐

Finally, let us cite a "local to global" result.

Theorem 3. *Let (M, g) be a locally symmetric space. If (M, g) is simply-connected and complete, then it is globally symmetric.*

Proof. To do. ☐

Remark 2.1.2. If completeness is clearly a necessary requirement, simple-connectedness is not. For instance, the real projective space $\mathbf{R}P^n$ endowed with a metric of constant positive sectional curvature is globally symmetric (it is $\{\pm \mathrm{id}\}\backslash\mathbf{S}^n$ and $\{\pm \mathrm{id}\}$ is central in $\mathrm{Iso}(\mathbf{S}^n)$, so every isometry descends to the quotient), but has fundamental group of order 2.

If we consider now a general complete locally symmetric space (M, g), then its universal cover $\tilde{M}$ is globally symmetric due to the previous theorem. Hence, (M, g) is isometric to a quotient $\Gamma\backslash\mathbf{X}$, where $\mathbf{X}$ is a globally symmetric space, and $\Gamma \simeq \pi_1(M)$ is a group of isometries of $\mathbf{X}$ acting freely, properly discontinuously. So we focus now on globally symmetric spaces.

Let $G = \mathrm{Iso}(\mathbf{X})_0$ be the identity component of its isometry group, with respect to the compact-open topology. Because G acts transitively on $\mathbf{X}$, if we fix a point $x_0 \in \mathbf{X}$, we get a G-equivariant identification $\mathbf{X} = G/K$, where $K = G_{x_0}$ is the stabilizer of x_0. The *theorem of Myers-Steenrod* implies that G has a natural (unique) Lie group structure compatible with the compact-open topology and such that the action $G \times \mathbf{X} \to \mathbf{X}$ is smooth. Also, as a consequence of Arzela-Ascoli's theorem, $K < G$ is a compact Lie subgroup.

The last paragraph is a teaser for the rest of the lecture where we will define precisely what these words precisely mean. The general principle is to translate all the geometry of a globally symmetric space into algebraic data in the pair of Lie groups (G, K), and conversely to characterize which pairs (G, K) arise as the isometry group G of a globally symmetric space with isotropy K.

Chapter 3

Lie groups, Lie algebras and homogeneous spaces

3.1 Preliminaries on topological groups

Definition 3.1.1. A *topological group* is a group G endowed with a topology such that $(x,y) \in G \times G \mapsto xy^{-1} \in G$ is continuous.

Example 3.1.1. 1. Any group G can be given the discrete topology, or the trivial topology, which obviously makes it a topological group in both cases.

2. $(\mathbf{R},+)$, $(\mathbf{C}^*,\times)$, $\mathbf{S}^1 = \{z \in \mathbf{C} \mid |z| = 1\}$, $\mathrm{SU}(2) \simeq \mathbf{S}^3 = \{q \in \mathbf{H} \mid q\bar{q} = 1\}$, $\mathrm{GL}_n(\mathbf{R})$, $\mathrm{GL}_n(\mathbf{C})$ .. all these groups are naturally embedded into a finite dimensional vector space, whose (normed) topology induces a topological group structure.

3. If G_1 and G_2 are two topological groups, then their product, endowed with the product topology is a topological group.

4. If G is a topological group, and $H < G$ a subgroup, then the induced topology gives H the structure of a topological group.

5. Any linear group $G < \mathrm{GL}_N(\mathbf{C})$ can be endowed with induced standard topology of $M_N(\mathbf{C})$. Remark that although this can be an intuitive natural topology in many cases ($\mathrm{SL}_n(\mathbf{R})$, $\mathrm{SU}(p,q)$ etc..), it might be possible that the result is a pathological topology like that induced on a line with irrational slope embedded into the 2-torus $U(1) \times U(1) < \mathrm{GL}_2(\mathbf{C})$.

6. Let G be a topological group and let $H < G$ be a subgroup. Then its closure $\overline{H}$ is a subgroup of G.

7. Given a topological space X, the group $\mathrm{Homeo}(X) = \{f : X \to X,\ f \text{ homeomorphism}\}$ can be endowed with the *compact-open* topology. The latter is defined as the topology generated by the $W(C,U) = \{f \in \mathrm{Homeo}(X) \mid f(C) \subset U\}$, $C \subset X$ compact,

$U \subset X$ open. When X is a metric space, *i.e.* when its topology is defined by a distance $d : X \times X \to \mathbf{R}_{\geqslant 0}$, this topology coincides with that of uniform convergence over compact subsets of X.

We will especially be interested in the topology of the last example for $X = M$ a smooth manifold induced on subgroups of $\mathrm{Diff}(M) < \mathrm{Homeo}(M)$. The whole group of diffeomorphisms is far from looking like a Euclidean space: it is not locally compact.

Exercise 3.1.1. Let G be a topological group. Show that if a subgroup $H < G$ is open, then it is closed. Deduce that if G is connected and if U is a neighborhood of the identity $e \in G$, then G is generated by U.

Exercise 3.1.2. Show that a topological group G is Hausdorff if and only if $\{e\}$ is closed.

Definition 3.1.2. Let G be a topological group. We call *identity component* of G, and we note G_0, the connected component of e.

Proposition 3.1.1. *Let G be a topological group and let G_0 be its identity component. Then,*

1. *G_0 is a normal subgroup of G ;*
2. *The connected components of G are the right (or left, no matter) cosets of G_0 ;*
3. *If G_0 is open, then the group G/G_0 is discrete, when endowed with the natural quotient topology induced from that of G.*

Definition 3.1.3. We call this subgroup G_0 the *identity component* of G. A topological group G is said to be *totally discontinuous* if $G_0 = \{e\}$.

Example 3.1.2 (Ring of p-adic integers)**.** Let p be a prime number. Note $C_n := \mathbf{Z}/p^n\mathbf{Z}$ for all $n \geqslant 1$ and $\pi_n : \mathbf{Z} \to C_n$ the canonical projection. There is a natural surjective homomorphism $p_n : C_{n+1} \to C_n$ The *ring of p-adic integers* is defined as the projective limit $\mathbf{Z}_p := \varprojlim \mathbf{Z}/p^n\mathbf{Z}$. Concretely, this object can be define as a subset

$$\mathbf{Z}_p \subset \prod_{n \geqslant 1} \mathbf{Z}/p^n\mathbf{Z}$$

consisting of sequences $(\dots, x_n, \dots, x_1) \in \prod_{n\geqslant 1} C_n$ such that $p_n(x_{n+1}) = x_n$ for all $n \geqslant 1$.

If each C_n is given the discrete topology, then Tychonov's theorem implies that $\prod_{n \geqslant 1} C_n$ is compact for the product topology.

$\mathbf{Z}_p$ is totally disconnected, and moreover the $\{p^n \mathbf{Z}_p,\ n \geqslant 0\}$ form a fundamental system of neighborhood of e, and all of them are subgroups.

This phenomenon of admitting arbitrarily small non-trivial subgroup does not appear in any linear group over $\mathbf{R}$.

Exercise 3.1.3. Let G be a closed subgroup of a linear group $\mathrm{GL}_n(\mathbf{R})$. Prove that there exists a neighborhood U of $\{\mathrm{id}\}$ such that any subgroup H of G contained in U must be reduced to $\{\mathrm{id}\}$.

Definition 3.1.4. Let X be a topological space and G a topological group. We call continuous action of G on X any action $G \times X \to X$ which is continuous.

Note that for any $g \in G$, the map $\{x \in X \mapsto g.x \in X\}$ is an homeomorphism of X.

Proposition 3.1.2. *Let G be a locally compact, σ-compact topological group and X a locally compact topological space on which G acts continuously and transitively. Let $x \in X$ and let $H = G_x$ be the stabilizer of x. Let $\phi : G/G_x \to X$ be the orbital identification.*

Then, ϕ is an homeomorphism when G/G_x is given the quotient topology.

Proof. Clearly, ϕ is bijective and continuous by definition of the quotient topology on G/G_x. So, we are left to prove that it is open. Let U be an open neighborhood of e and let V be a compact neighborhood of e such that $V^{-1}V \subset U$. It is enough to prove that $U.x = \{u.x,\ u \in U\}$ has x as an interior point. Indeed, any other point $u.x$ will then be interior because the action of u is an homeormorphism, so $U.x$ will be open, and finally for any open subset $U' \subset G$ and $g \in U'$, $\phi(U') = U'.x = g(g^{-1}U'.x)$ will be open because the action of g is an homeomorphism and $g^{-1}U'$ is a neighborhood of e. Since $\bigcup_{v \in V} v^{-1}(V.x) \subset U.x$, we are left to prove that $V.x$ has *an* interior point $y = v_0.x$ because it will imply that x is an interior point of $v_0^{-1}(V.x) \subset U.x$.

To see it, assume by contradiction that $V.x$ has empty interior. Then, let $G = \bigcup K_n$ be an exhaustion of G by compact subsets. For any fixed n, $K_n \subset \bigcup_{1 \leqslant k \leqslant \ell_n} g_k^{(n)} V$ for $g_k^{(n)} \in K$ by compactness. Hence, there exists a sequence (g_n) such that $G = \bigcup_{n \geqslant 0} g_n V$. So, we get that $X = G.x = \bigcup_{n \geqslant 0} g_n(V.x)$ is a countable union of closed subsets with empty interior, because so is $V.x$ (recall that V is compact, so $V.x$ is compact, hence closed) and the action of g_n is an homeomorphism of X. This contradicts Baire's category theorem. ☐

Corollary 3.1.1. *Let G be a locally compact, countable at infinity topological group, H a locally compact topological group and $f : G \to H$ be a continuous group isomorphism. Then, f is an homeomorphism.*

Proof. Apply the proposition to $X = H$, the action $G \times X \to X$ given by $(g, x) \mapsto f(g)x$, and base point $x = e$. The stabilizer is trivial and the proposition gives that the orbital map at e, namely f, is an homeomorphism. ☐

Corollary 3.1.2. *Let G be a group. Then, two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on G which make it a locally compact, second countable topological group are equal.*

Otherwise stated, there exists at most one topology on a group which makes it a locally compact, σ-compact, topological group.

Proof. Consider $(G \times G, \mathcal{T}_1 \times \mathcal{T}_2)$ and $D = \{(g, g),\ g \in G\}$. Endow D with the induced topology. Then the projections $pr_1 : D \to (G, \mathcal{T}_1)$ and $pr_2 : D \to (G, \mathcal{T}_2)$ satisfy the hypothesis of the previous corollary, showing that the identity $(G, \mathcal{T}_1) \to (G, \mathcal{T}_2)$ is an homeomorphism, hence $\mathcal{T}_1 = \mathcal{T}_2$. ☐

3.2 Lie groups and Lie algebras: definition

Lie groups are the natural analog in the smooth category of topological groups.

Definition 3.2.1. A *real Lie group* is a group G endowed with a compatible smooth differentiable structure, *i.e.* such that the map $(g,h) \in G \times G \mapsto gh^{-1} \in G$ is smooth.

The question of the regularity of the Lie group is made pointless by the following important result, which gave a solution to Hilbert's fifth problem.

Theorem 4 (Gleason,Montgomery-Zippin (1952))**.** *Let G be a second countable, topological group which is locally homeomorphic to the Euclidean space, i.e. a topological manifold. Then, G admits a unique real-analytic atlas, compatible with its topology, and which makes it a Lie group with real-analytic regularity.*

Remark 3.2.1. For all $g \in G$, the left translation $L_g : G \to G$ is a diffeomorphism of G, and so is R_g the right translation by g.

Remark 3.2.2.

1. A Lie group G may not be connected. For instance any countable discrete group can be given uniquely the smooth atlas of dimension 0. These are precisely Lie groups of dimension 0.

2. $\mathbf{R}$, $\mathbf{C}^*$, $\mathbf{U} = \{z \in \mathbf{C} \mid |z| = 1\}$, $\mathbf{S}^3 = \{q \in \mathbf{H} \mid q\overline{q} = 1\}$ all are smooth submanifolds of a given vector space and product and inverse are immediately seen to be smooth. If not, exercise !

3. $\mathrm{GL}_n(\mathbf{R})$ which is open in $M_n(\mathbf{R})$, so has the smooth structure of an open set of this space. The product is clearly polynomial in the entries. As for the inverse, thanks to the formula $A\,{}^t\mathrm{Com}(A) = \det(A) I_n$, the inverse is given by everywhere defined rational entries.

4. We will see later (Cartan's Theorem) that in particular here, any *closed subgroup* of $\mathrm{GL}_n(\mathbf{R})$ is a Lie group.

Proposition 3.2.1. *A Lie group has trivial tangent bundle.*

In particular, $\mathbf{S}^2$ *cannot* be given a Lie group structure.

Definition 3.2.2. Let G_1, G_2 be two Lie groups. A *Lie group homomorphism* is a group homomorphism $f : G_1 \to G_2$ such that f is a smooth map. Isom, Autom.

Definition 3.2.3. Let G be a Lie group. We call Lie subgroup, or properly embedded Lie subgroup, any subgroup H which is also a properly embedded submanifold of G.

We call *immersed Lie subgroup*, or *integral subgroup* any pair (i, H) where H is a Lie group and $i : H \to G$ is an injective immersion.

Example 3.2.1. In the two torus, $\mathbf{S}^1 \times \mathbf{S}^1$, define $i(t) = (e^{it}, e^{i\alpha t})$. Then $i(\mathbf{R})$ is a Lie subgroup, isomorphic to $\mathbf{S}^1$ as long as $\alpha \in \mathbf{Q}$, and it is no longer properly embedded if $\alpha \notin \mathbf{Q}$ and $(i, \mathbf{R})$ is an immersed Lie subgroup.

Proposition 3.2.2.

- *A Lie group homomorphism has constant rank.*
- *A bijective Lie group homomorphism is a Lie group isomorphism.*

Definition 3.2.4. Let G be a Lie group. We call Lie group representation of G any pair (V, ρ) where V is a real finite-dimensional vector space and $\rho : G \to \mathrm{GL}(V)$ a Lie group homomorphism.

Definition 3.2.5. Let G be a Lie group. We call **adjoint representation** of G, and we note Ad, the representation

$$\mathrm{Ad} : G \to \mathrm{GL}(T_e G)$$

given by $\mathrm{Ad}(g) = \mathrm{d}_e i_g$, where $i_g : x \in G \mapsto gxg^{-1} \in G$ is the inner automorphism corresponding to g.

Note that the center $\mathcal{Z}(G) \subset \ker \mathrm{Ad}$, but that equality does not always hold. We will see later **ref** that it is the case when G is connected.

Example 3.2.2. For $G = \mathrm{GL}_n(\mathbf{R})$ or a closed subgroup, $T_e G \subset M_n(\mathbf{R})$ is a vector subspace invariant under conjugacy by elements of G and for all $X \in T_e G$ and $g \in G$, we have $\mathrm{Ad}(g)X = gXg^{-1}$.

3.3 Lie algebras, Lie algebra of a Lie group

We define Lie algebras over $\mathbf{R}$, but the definition naturally extends to any field.

Definition 3.3.1. A Lie algebra over $\mathbf{R}$ is the data of a real vector space $\mathfrak{g}$, together with a bilinear map $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

1. $\forall X, Y \in \mathfrak{g}$, $[X, Y] = -[Y, X]$ (skew-symmetry) ;
2. $\forall X, Y, Z \in \mathfrak{g}$, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

The bilinear map $[.,.]$ is called the Lie bracket of $\mathfrak{g}$.

Definition 3.3.2. Given two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, we call Lie algebra homomorphism any linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\forall X, Y \in \mathfrak{g}_1$, $f([X, Y]) = [f(X), f(Y)]$. An isomorphism of Lie algebras is a bijective homomorphism (note that the inverse is automatically bracket-preserving). An automorphism of a Lie algebra $\mathfrak{g}$ is an isomorphism $\mathfrak{g} \to \mathfrak{g}$.

Definition 3.3.3. Given a Lie algebra $\mathfrak{g}$, a Lie subalgebra of $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ closed under bracket, *i.e.* such that $\forall X, Y \in \mathfrak{h}$, $[X, Y] \in \mathfrak{h}$.

Definition 3.3.4. The center of a Lie algebra $\mathfrak{g}$ is the subalgebra

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g},\ [X, Y] = 0\}.$$

Example 3.3.1.

1. Any vector space $\mathfrak{g}$ can be endowed with the trivial bracket $[X,Y]=0$ for all X,Y. The result is called an *abelian* Lie algebra.
2. Let A be an associative $\mathbf{R}$-algebra. Then, the bilinear map defined by $[X,Y]:=XY-YX$ for all $X,Y\in A$ is a Lie bracket on A. A very natural instance for us will be that of $A=\mathrm{End}(V)$, for V a real vector space or $A=M_n(\mathbf{R})$.
3. Any Lie subalgebra of the previous examples. For instance,
   (a) $\mathfrak{g}=\mathfrak{sl}_n(\mathbf{R})=\{X\in M_n(\mathbf{R})\mid \mathrm{Tr}\,X=0\}$
   (b) $\mathfrak{g}=\mathfrak{so}(p,q)=\{X\in M_n(\mathbf{R})\mid {}^tXI_{p,q}+I_{p,q}X=0\}$
   (c) $\mathfrak{g}=\mathfrak{sp}_{2n}(\mathbf{R})=\{X\in M_{2n}(\mathbf{R})\mid {}^tXJ_{2n}+J_{2n}X=0\}$
4. The affine algebra $\mathfrak{aff}(\mathbf{R})=\mathrm{Span}(X,Y)$, where $[X,Y]=Y$.
5. The Heisenberg algebra of dimension $2n+1$: given a symplectic form ω on a $2n$-dimensional vector space V, define $\mathfrak{heis}(2n+1)=V\oplus\mathbf{R}.Z$ for Z an exterior element, with bracket such that
   (a) Z is central
   (b) $\forall X,Y\in V$, $[X,Y]=\omega(X,Y).Z$.

   Exercise: Show that this definition does not depend on (V,ω), *i.e.* that the resulting Lie algebra is well defined up to isomorphism.

   Example: For $n=1$, $\mathfrak{heis}(3)$ has a basis (X,Y,Z) satisfying the relations $[X,Y]=Z$, $[X,Z]=[Y,Z]=0$. This can be an alternative definition (verify it).
6. Let M be a smooth manifold. The space $\Gamma(TM)$ of all smooth vector fields, endowed with the bracket of vector fields, is a Lie algebra. It is of course infinite dimensional. For all local diffeomorphism $f:M\to N$, $f^*:\Gamma(TN)\to\Gamma(TM)$ is then a Lie algebra homomorphism.
7. Let G be a Lie group.

   Definition 3.3.5. A vector field $X\in\Gamma(TG)$ is said to be left-invariant if for all $g\in G$, $(L_g)^*X=X$. Similarly, it is said to be right-invariant if $(R_g)^*X=X$ for all g.

   According to the previous item, the space of all left-invariant (resp. right-invariant) vector fields on G is a Lie subalgebra of $\Gamma(TG)$. We note ${}^G\Gamma(TG)$ the Lie algebra of left-invariant vector fields, and $\Gamma(TG)^G$ the Lie algebra of right-invariant vector fields.

   Immediately, the mapping $\{X\in{}^G\Gamma(TG)\mapsto X(e)\in T_eG\}$ is a linear isomorphism and similarly for the right-invariant vector fields. Consequently, these Lie algebra are of finite dimension, equal to $\dim G$.

> **Exercise 3.3.1.** Show that the Lie algebras ${}^G\Gamma(TG)$ and $\Gamma(TG)^G$ are anti-isomorphic, *i.e.* that there exists a linear map such that $f([X,Y]) = -[f(X), f(Y)]$ for all X, Y.
>
> We will define a Lie algebra structure on T_eG below. It will seen to be isomorphic to that coming from ${}^G\Gamma(TG)$.

Exercise 3.3.2. Show that up to isomorphism, there are two Lie algebras of dimension 2.

Definition 3.3.6. Let (M, g) be a Riemannian manifold. A **Killing vector field** is a vector field X such that $\mathcal{L}_X g = 0$.

Recall that the Lie derivative extends to any $(r, 0)$-tensor on M. Here, we have that for all $Y, Z \in \Gamma(TM)$, $\mathcal{L}_X g(Y,Z) = X.g(Y,Z) - g([X,Y],Z) - g(Y,[X,Z])$.

Proposition 3.3.1. *Let (M, g) be a Riemannian manifold and X a vector field. Let ∇ denote the Levi-Civita connection of g. The following are equivalent:*

1. *X is a Killing vector field.*
2. *The local flow ϕ_X^t is formed of local isometries.*
3. *For all $x \in M$, the linear mapping $\{v \in T_xM \mapsto (\nabla_v X)_x \in T_xM\}$ is skew-symmetric with respect to g_x.*

We will note $(\nabla X)_x \in \mathfrak{so}(T_xM, g_x)$ this field of skew-symmetric endomorphisms.

Proof. Since g is parallel and without torsion, we obtain $g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = g(\nabla_X Y - \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) + \mathcal{L}_X g(Y,Z)$. So $\mathcal{L}_X g(Y,Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all vector fields Y, Z, and we proved (1) $\iff$ (3).

Recall that $(\mathcal{L}_X g)_x = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}((\phi_X^t)^* g)_x$, so (2) $\Rightarrow$ (1) is direct.

Suppose (1). For all $x \in M$ and $u, v \in T_xM$, $\alpha(t) = [(\phi_X^t)^* g]_x(u,v)$ is defined for t small enough, smooth and verifies $\alpha'(0) = 0$. Since $\phi_X^{t+s} = \phi_X^s \circ \phi_X^t$ on a small neighborhood of x and for t, s small, we obtain $\alpha'(t) = (\mathcal{L}_X g)_{\phi_X^t(x)}(\mathrm{d}_x\phi_X^t.u, \mathrm{d}_x\phi_X^t.v) = 0$ for t small. Hence α is constant on its interval of definition, meaning that ϕ_X^t is a local isometry. □

Immediately, we see that Killing vector fields of (M, g) form a Lie subalgebra of $\Gamma(TM)$ (recall that $\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$). We note $\mathrm{Kill}(M, g)$ this Lie subalgebra.

Proposition 3.3.2. *For (M, g) a Riemannian manifold, $\mathrm{Kill}(M, g)$ is finite dimensional, of dimension at most $\frac{n(n+1)}{2}$, where $n = \dim M$. In fact, for any fixed $x \in M$, the linear mapping*

$$X \in \mathrm{Kill}(M,g) \mapsto (X(x), (\nabla X)_x) \in T_xM \times \mathfrak{so}(T_xM, g_x)$$

is injective.

Remark 3.3.1. Compare with the property saying that an isometry with trivial 1-jet is locally trivial.

Proof. If $(X(x), (\nabla X)_x) = (0,0)$, then $\phi_X^t(x) = x$ where ϕ_X^t is the local flow of X defined for all $|t| < \varepsilon$ on an open neiborhood of x. Then, $(\nabla X)_x = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{d}_x \phi_X^t \in \mathfrak{gl}(T_x M)$. For t, s such that $|t| + |s| < \varepsilon$, we have $\phi_X^{t+s} = \phi_X^t \circ \phi_X^s$, so $\mathrm{d}_x \phi_X^{t+s} = \mathrm{d}_x \phi_X^t \circ \mathrm{d}_x \phi_X^s$ for any small enough s and t. Hence, for t small enough, $\mathrm{d}_x \phi_X^t = \exp(t(\nabla X)_x) = \mathrm{id}$.

So, $\mathrm{Kill}(M,g)$ is finite dimensional and $\dim \mathrm{Kill}(M,g) \leqslant n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$. ☐

Definition 3.3.7. We derivation of a Lie algebra $\mathfrak{g}$ a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ such that for all $X, Y \in \mathfrak{g}$, $\delta([X,Y]) = [\delta(X), Y] + [X, \delta(Y)]$.

Although it is not always true, we can think a derivation as the derivative of an automorphism.

Example 3.3.2. For any fixed $X \in \mathfrak{g}$, the linear mapping $\mathrm{ad}(X) : Y \in \mathfrak{g} \mapsto [X,Y] \in \mathfrak{g}$ is a derivation of $\mathfrak{g}$. The fact that $\mathrm{ad}(X)$ is a derivation for all $X \in \mathfrak{g}$ is in fact equivalent to requiring that the Lie bracket satisfies Jacobi's identity.

Proposition 3.3.3. *If* $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ *denotes the subspace of derivation of a Lie algebra* $\mathfrak{g}$*, then* $\mathrm{Der}(\mathfrak{g})$ *is a Lie subalgebra of* $\mathfrak{gl}(\mathfrak{g})$.

Definition 3.3.8. A representation of a Lie algebra is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a finite dimensional vector space.

Definition 3.3.9. We call adjoint representation of the Lie algebra $\mathfrak{g}$, and we note ad, the representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ defined by $\mathrm{ad} : X \mapsto \mathrm{ad}(X)$ as defined above.

Let G be a Lie group. Recall that we defined a Lie group representation $\mathrm{Ad} : G \to \mathrm{GL}(V)$.

Definition 3.3.10. For all $X \in T_e G$, define $\mathrm{ad}(X) = \mathrm{d}_e \mathrm{Ad}(X)$.

So far, it is just a linear mapping $T_e G \to \mathfrak{gl}(T_e G)$.

Proposition 3.3.4. *For all* $X, Y \in \mathfrak{g}$*, the bilinear map* $T_e G \times T_e G \to T_e G$ *defined by* $[X,Y] = \mathrm{ad}(X) Y$ *is a Lie bracket on the vector space* $T_e G$*, which then makes it a Lie algebra which is usually denoted by* $\mathfrak{g}$.

Proof. ☐

Proposition 3.3.5. *Let* G, H *be two Lie groups, with Lie algebra* $\mathfrak{g}, \mathfrak{h}$ *respectively. Then, for every Lie group homomorphism* $f : G \to H$*, its differential* $\mathrm{d}_e f : \mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra homomorphism.*

Proof. Let $X \in \mathfrak{g}$. Since f is a group homomorphism, for all $g \in G$, $f \circ i_g = i_{f(g)} \circ f$. It follows that $\mathrm{d}_e f \circ \mathrm{Ad}(g) = \mathrm{Ad}(f(g)) \circ \mathrm{d}_e f$. Therefore, for all $X \in \mathfrak{g}$, $\mathrm{d}_e f \circ \mathrm{ad}(X) = \mathrm{ad}(\mathrm{d}_e f . X) \circ \mathrm{d}_e f$. Applying this to any $Y \in \mathfrak{g}$, we obtain $\mathrm{d}_e f([X,Y]) = [\mathrm{d}_e f X, \mathrm{d}_e f Y]$ as expected. ☐

A very natural question is the following : conversely, given a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$, does there exists a Lie group homomorphism $f : G \to H$ such that $\phi = \mathrm{d}_e f$?

The answer is no in general as it can be seen fairly easily. Let $G = \mathbf{S}^1$ and $H = \mathbf{R}$ with their standard Lie group structures. They both have 1-dimensional, hence abelian, Lie algebra. A non-trivial linear map $\mathfrak{g} \to \mathfrak{h}$ would give rise to a non-trivial Lie (hence continuous) group homomorphism $\mathbf{S}^1 \to \mathbf{R}$, whose image would be a compact subgroup of $\mathbf{R}$, a contradiction.

We will see later that assuming G *simply-connected,* every Lie algebra homomorphism $f : \mathfrak{g} \to \mathfrak{h}$ gives rise to a Lie group homomorphism $G \to H$.

Proposition 3.3.6. *For every $g \in G$, $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$, that is the adjoint representation of G is by automorphisms of $\mathfrak{g}$.*

Proof. Differentiating the map $h \in G \mapsto \mathrm{Ad}(ghg^{-1}) = \mathrm{Ad}(g)\,\mathrm{Ad}(h)\,\mathrm{Ad}(g^{-1}) \in \mathrm{GL}(\mathfrak{g})$, we obtain

$$\forall X \in \mathfrak{g},\ \ \mathrm{ad}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g) \circ \mathrm{ad}(X) \circ \mathrm{Ad}(g)^{-1}. \tag{3.1}$$

Applying now both member of this identity to $\mathrm{Ad}(g)Y$ for $Y \in \mathfrak{g}$, and we obtain

$$[\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y] = \mathrm{Ad}(g)\left([X,Y]\right),$$

as expected. □

In fact, the adjoint representation preserves another object that will play a crucial role. It can be defined generally.

Definition 3.3.11. Let $\mathfrak{g}$ be a Lie algebra. Its *Killing form* is the symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ defined by

$$\forall X, Y \in \mathfrak{g},\ \ B(X,Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

Example 3.3.3. For $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{R})$, its elements are matrices and we have $B(X,Y) = 2n\,\mathrm{Tr}(XY) - \mathrm{Tr}(X)\,\mathrm{Tr}(Y)$.

Proposition 3.3.7. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let B be the Killing form of $\mathfrak{g}$. For all $g \in G$, $\mathrm{Ad}(g) \in O(B)$ is linear isometric with respect to B.*

For a general Lie algebra $\mathfrak{g}$, for every element $X \in \mathfrak{g}$, $\mathrm{ad}(X)$ is skew-symmetric with respect to B, that is:

$$\forall X, Y, Z \in \mathfrak{g},\ \ B(\mathrm{ad}(X)Y, Z) + B(Y, \mathrm{ad}(X)Z) = 0.$$

Proof. We simply have to apply (3.1) and invariance of the trace under conjugacy.

For the second point, note that it follows from the first if $\mathfrak{g}$ is the Lie algebra of a Lie group. This can nonetheless be obtained directly:

$$\begin{aligned} B([X,Y],Z) &= \mathrm{Tr}(\mathrm{ad}([X,Y]),Z) = \mathrm{Tr}((\mathrm{ad}(X)\,\mathrm{ad}(Y) - \mathrm{ad}(Y)\,\mathrm{ad}(X))\,\mathrm{ad}(Z) \\ &= \mathrm{Tr}(\mathrm{ad}(Y)\,\mathrm{ad}(Z)\,\mathrm{ad}(X) - \mathrm{ad}(Y)\,\mathrm{ad}(X)\,\mathrm{ad}(Z)) \\ &= B(Y,[Z,X]) = -B(Y,[X,Z]). \end{aligned}$$

□

3.4 One-parameter subgroups. Exponential map.

Lemma 3.4.1. *Let* $X \in \mathfrak{g}$ *and let* $\overline{X} \in^G \Gamma(TG)$ *be the corresponding left-invariant vector field. Then,* $\overline{X}$ *is complete.*

Proof. Let $x \in G$ and suppose that the $c(t)$, $t \in [-a, a]$, is an integral curve of $\overline{X}$ starting at x. Let $y = c(a/2)$, $g = yx^{-1}$. Because $(dL_g)^*\overline{X} = \overline{X}$, we get that $gc(t)$ is an integral curve of $\overline{X}$, starting at y. Necessarily, $gc(t) = c(t + a/2)$ for $|t| \leqslant a/2$, proving that c can be extended to $[-a, 3a/2]$. It follows that the integral curves are defined for all $t \in \mathbf{R}$. □

Definition 3.4.1. The exponential map of a Lie group is the map $\exp : \mathfrak{g} \to G$ defined for all $X \in \mathfrak{g}$ by $\exp(X) = \phi^1_{\overline{X}}(e)$ (time 1 of the flow).

Example 3.4.1. Consider $G = \mathbf{S}^1 = \{z \in \mathbf{C} \ : \ |z| = 1\}$. Then $\mathfrak{g} = i\mathbf{R}$ with abelian Lie bracket and for all $z \in \mathbf{S}^1$ we have $T_zG = iz\mathbf{R}$. If $X = ix\mathfrak{g}$, then the corresponding left-invariant vector field is $\overline{X}(z) = zX$ and its flow is $\phi^t_{\overline{X}}(z) = e^{ixt}z$ (where e^{ixt} refers here to the power-series definition) because $\frac{\mathrm{d}}{\mathrm{d}t}e^{ixt}z = ixe^{ixt}z = \overline{X}(e^{ixt}z)$.

Example 3.4.2. Consider $G = \mathrm{GL}_n(\mathbf{R})$. Now $\mathfrak{g} = M_n(\mathbf{R})$ and for all $X \in \mathfrak{g}$, we have $\overline{X}(g) = gX$ because the left multiplication on G is the restriction to G of a linear operator of $M_n(\mathbf{R})$. Now, it follows similarly that $\phi^t_{\overline{X}}(I_n) = e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}$.

Now, for a general Lie group, we will note indifferently $\exp(X)$ or e^X. Remark that due to change time for the flow, for all $t \in \mathbf{R}$, $\exp(tX) = \phi^1_{t\overline{X}}(e)$ and $\exp((t+s)X) = \exp(tX)\exp(sX)$.

Proposition 3.4.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Denote* exp *its exponential map.*

1. exp *is smooth and realizes a diffeomorphism from a neighborhood of* 0 *in* $\mathfrak{g}$ *onto a neighborhood of* e *in* G.
2. *The flow of* $\overline{X}$ *is given by* $\phi^t_{\overline{X}} = R_{e^{tX}}$.
3. $\{\exp(\mathfrak{g})\} = G_0$, *the identity component of* G.
4. *For all* $X \in \mathfrak{g}$, *the map* $\{t \in \mathbf{R} \mapsto e^{tX}\}$ *is a smooth homomorphism. Conversely, given* $\gamma : \mathbf{R} \to G$ *a smooth homomorphism, if* $X = \gamma'(0)$, *then* $\gamma(t) = e^{tX}$ *for all* $t \in \mathbf{R}$.
5. *If* $f : G \to H$ *is a Lie group homomorphism, then*
$$\forall X \in \mathfrak{g}, \ f(\exp_G(X)) = \exp_H(\mathrm{d}_e f X).$$
6. *For all* $X \in \mathfrak{g}$
$$\mathrm{Ad}(e^X) = e^{\mathrm{ad}(X)} = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}(X))^n}{n!}.$$

7. *For all $g \in G$ and $X \in \mathfrak{g}$,*

$$ge^X g^{-1} = e^{\mathrm{Ad}(g)X}.$$

Definition 3.4.2. Let G be a Lie group. A Lie group homomorphism $\gamma : \mathbf{R} \to G$ is called a *one-parameter subgroup.* According to the previous proposition, they all are of the form $\gamma(t) = e^{tX}$, $X = \gamma'(0) \in \mathfrak{g}$.

Remark 3.4.1. In fact, it is not necessary to require γ to be smooth or even differentiable. Any *continuous* group homomorphism $\gamma : \mathbf{R} \to G$ is automatically smooth, hence falls into the previous description. It will follow from Theorem **ref** below. However, it can be recovered directly (Exercise. *Hint: Use that any continuous additive map $f : \mathbf{R} \to \mathbf{R}$ is always linear*)

Example 3.4.3.

1. If $G < \mathrm{GL}_n(\mathbf{R})$ is a Lie subgroup, then $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbf{R})$ is a Lie algebra of matrices and $\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$ for all $X \in \mathfrak{g}$.
2. If $G = \mathbf{S}^1$, $\mathfrak{g} = i\mathbf{R} \subset \mathbf{C}$ and $\exp : G \to \mathbf{S}^1$ is $\{t \mapsto e^{it}\}$, it is not injective.

3.5 Cartan - Von Neumann theorem

Theorem 5. *Let G be a real Lie group. Then, any closed subgroup of G is a (properly embedded) Lie subgroup. For G a complex Lie group, an integral subgroup is closed if and only if it is properly embedded.*

Proof. We give the proof in the real case. **ref** for the complex case.

Let H be a closed subgroup of a real Lie group G. Define

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \forall t \in \mathbf{R},\ e^{tX} \in H\}.$$

We prove that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. For this, we use the following.

Lemma 3.5.1. *For all $X, Y \in \mathfrak{g}$,* $\left(e^{X/n}e^{Y/n}\right)^n \xrightarrow[n\to\infty]{} e^{X+Y}$.

Lemma 3.5.1 implies that $\mathfrak{h}$ is a vector subspace. Note that we use that H is closed for this first step.

Let now $\mathcal{U}_0 \subset \mathfrak{g}$ be a neighborhood of 0 such that $\exp : \mathcal{U}_0 \to U_0$ realizes a diffeomorphism onto its image $U_0 \subset G$. We claim that reducing $\mathcal{U}_0$ if necessary,

$$H \cap U_0 = \exp\left(\mathcal{U}_0 \cap \mathfrak{h}\right).$$

Once this is established, it will follow that for every $h \in H$, the composition of a left translation and the exponential map gives a local chart of H at h.

So let us prove the claim. We do it by contradiction. Let us assume that there exists a sequence (h_n) of elements of H such that $(h_n) \to e$ and $h_n \notin \exp(\mathcal{U}_0 \cap \mathfrak{h})$.

We choose an arbitrary supplementary subspace $V \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus V$. Considering the map $(X, Y) \in \mathfrak{h} \times V \mapsto e^X e^Y$, the inverse mapping theorem gives a neighborhood of $(0,0)$ in $\mathfrak{h} \times V$ and a neighborhood U_0 of e in G such that any $g \in U_0$ can be uniquely written $g = e^X e^Y$ for $(X,Y) \in \mathcal{U}_0$. Without loss of generality, we may assume that $h_n \in U_0$ for all n. Therefore, writing $h_n = e^{X_n} e^{Y_n}$, we want to prove that $Y_n = 0$ for n large enough.

Let us assume to the contrary that up to an extraction $Y_n \neq 0$ for all n. Since $e^{X_n} \in H$, we can assume $X_n = 0$. Now, write $Y_n = t_n Z_n$ for $\|Z_n\| = 1$ and $(t_n) \to 0$ where $\|.\|$ refers to an arbitrary norm on $\mathfrak{g}$. Up to another extraction, $(Z_n) \to Z \in V$, with $\|Z\| = 1$. Let $t \in \mathbf{R}$. Let $k_n = \lfloor \frac{t}{t_n} \rfloor$. Then, because $e^{Y_n} \in H$, we get

$$\underbrace{\left(e^{t_n Z_n}\right)^{k_n}}_{\in H} \xrightarrow[n \to \infty]{} e^{tZ}.$$

Using once more that H is closed, we get $e^{tZ} \in H$. And this for all t, so $Z \in \mathfrak{h}$ and then $Z \in V \cap \mathfrak{h} = \{0\}$: contradiction.

So, exp gives a local chart of G at e in which H is sent to a neighborhood of 0 in $\mathfrak{h}$. We need the same at every point of H. So let $h \in H$. Because $L_{h^{-1}}$ is an homeomorphism of G, there exists U a neighborhood of h in G such that $L_{h^{-1}}(U) = U_0$. Consequently, $L_{h^{-1}}(U \cap H) = U_0 \cap H = \exp(\mathcal{U}_0 \cap \mathfrak{h})$ and $\phi := (\exp|_{\mathcal{U}_0})^{-1} \circ L_{h^{-1}} : U \to \mathcal{U}_0$ gives the desired chart. ☐

As a consequence of Cartan - Von Neumann's theorem, we have the remarkable:

Proposition 3.5.1. *Let $f : G_1 \to G_2$ be an abstract group homomorphism. If f is continuous, then it is smooth, i.e. a Lie group homomorphim.*

Proof. In the Lie group product $G_1 \times G_2$, consider the graph $\mathcal{G}$ of f :

$$\mathcal{G} = \{(x, f(x)),\ x \in G_1\}.$$

Then, $\mathcal{G}$ is a subgroup of $G_1 \times G_2$ and it is closed because f is continuous. So it is a properly embedded subgroup of $G_1 \times G_2$ by Cartan - Von Neumann theorem. Let $p_1 : \mathcal{G} \to G_1$ and $p_2 : \mathcal{G} \to G_2$ be the projections on the first and second factors. Then p_1 is a bijective local diffeomorphism onto G_1, so it is a diffeomorphism, and $f = p_2 \circ (p_1|_{\mathcal{G}})^{-1}$ is smooth. ☐

3.6 Lie groups - Lie algebra correspondence

Theorem 6. *Let G be a Lie group. To any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, corresponds a unique integral subgroup H of G such that $T_e H = \mathfrak{h}$.*

Remark 3.6.1. Let us insist strongly: H may not be a (properly embedded) Lie subgroup. Consider (once more!) $G = \mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. Then $\mathfrak{g}$ is naturally identified with $\mathbf{R}^2$ with abelian bracket, and if we take $\mathfrak{h}$ a line with irrational slope, the corresponding integral subgroup will be non-properly immersed. Note that here, H is concrete: if $X \in \mathfrak{h}$ is non-zero, then $H = \{e^{tX},\ t \in \mathbf{R}\}$.

We may want to see more elaborate examples of non-properly embedded subgroups, for instance in non-abelian Lie groups. A standard construction is the following.

1. Nilpotent case. Consider $H = \mathrm{Heis}(3)$ and let
$$\zeta = \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \in \mathcal{Z}(H).$$
Let $\alpha \in \mathbf{R}$ such that $\alpha/\pi \notin \mathbf{Q}$. In $G = H \times \mathbf{S}^1$, consider the subgroup $\Gamma = \langle(\zeta, e^{i\alpha})\rangle$. Then, Γ is closed and normal (in fact central) in G. Therefore, the quotient G/Γ is a Lie group, with Lie algebra $\mathfrak{h} \oplus \mathbf{R}$, and the $\mathfrak{h}$ factor has a corresponding integral subgroup which is a non-closed immersed subgroup isomorphic to H.

2. Semi-simple case. Consider $H = \widetilde{\mathrm{SL}}_2(\mathbf{R})$. Let $\zeta \in H$ a generator of the center. Similarly as above, consider $G = H \times \mathbf{S}^1$ and $\Gamma = \langle(\zeta, e^{i\alpha})\rangle$ for $\alpha/\pi \notin \mathbf{Q}$. Then, the integral subgroup corresponding to $\mathfrak{h} \times \{0\}$ is a non-closed immersion of $\widetilde{\mathrm{SL}}_2(\mathbf{R})$.1

To finish let us cite, without proving, the following theorem.

Theorem 7 (Ado). *Any finite dimensional Lie algebra $\mathfrak{g}$ is linear, i.e. it admits a faithful linear representation $\mathfrak{g} \to \mathfrak{gl}(V)$, for some finite dimensional vector space V.*

It then follows directly from the two last results:

Corollary 3.6.1. *Let $\mathfrak{g}$ be a real, finite-dimensional Lie algebra. Then, there exists a connected Lie group G with Lie algebra $\mathfrak{g}$.*

Of course, the Lie group G is not unique: for instance $G = \mathbf{R}^n$ has Lie algebra $\mathfrak{g} = \mathbf{R}^n$ (with abelian bracket), but so does $G' = G/\mathbf{Z}^n$ the n-torus.

3.7 Topological considerations

Consider G a connected Lie group, and choose e as a base point. Let $\Gamma := \pi_1(G, e)$. Let $\pi : \tilde{G} \to G$ denote a universal covering of G, which we fix once and for all. Also, we choose $\tilde{e}$ in the fiber of $\pi^{-1}(e)$. Recall that we can identify Γ as the group of deck transformations of the universal cover $\pi : \tilde{G} \to G$.

Definition 3.7.1. A deck transformation of a covering map $p : M \to B$ is an homeomorphism $f : M \to M$ such that for all $x \in M$, $p(f(x)) = p(x)$. These maps form a group. The covering is said to be Galois if the group of deck transformations acts transitively on the fibers $p^{-1}(b)$, $b \in B$.

Recall the lifting property (for general coverings) :

Proposition 3.7.1. *Let $p : M \to B$ be a covering and let N be a simply connected manifold. Then, for any smooth map $f : N \to B$, and for every $x \in N$ and $y \in M$ such that $p(y) = f(x)$, there exists a unique smooth map $\tilde{p} : N \to M$ such that $f = p \circ \tilde{f}$ and $\tilde{f}(x) = y$.*

Let G_1 and G_2 be two Lie groups.

Definition 3.7.2. A Lie group covering of G_2 by G_1 is a covering map $p : G_1 \to G_2$ which is a Lie group homomorphism.

The standard way to produce Lie group coverings is by modding out by a discrete central subgroup. Let us explain how, and why it's in fact the only way.

Definition 3.7.3. Let Γ be a group acting continuously on a manifold M. The action is said to be *properly discontinuous* if for every compact subset $K \subset M$, $\{\gamma \in \Gamma \mid \gamma.K \cap K \neq \varnothing\}$ is finite. The action is said to be *free* if for all $x \in M$ and $\gamma \in \Gamma$, $\gamma.x = x \Rightarrow \gamma = e$.

Note that for M a compact manifold, a group acting properly discontinuously on M must be finite. The following property is useful in practice.

Proposition 3.7.2. *Let M be a smooth manifold and Γ a group acting smoothly, freely and properly discontinuously on M. Then, there exists a unique differential structure on the orbit space $\Gamma \backslash M$ such that the natural projection $M \to \Gamma \backslash M$ is covering map.*

Remark 3.7.1. Recall that the orbit space $\Gamma \backslash M$ is the quotient of M by the equivalence relation $x \sim y \iff \exists \gamma \in \Gamma,\ \gamma.x = y$.

Corollary 3.7.1. *Let G be a Lie group and let $\Gamma < G$ be a discrete subgroup. Suppose that Γ is normal in G. Then, G/Γ has a unique Lie group structure such that the natural projection $G \to G/\Gamma$ is a Lie group covering. Moreover, $\mathrm{d}_e p$ induces a Lie algebra isomorphism between $\mathfrak{g}$ and the Lie algebra of G/Γ.*

Proof. Consider the action of Γ on G by left translations. The orbit space is G/Γ and has a group structure such that the natural projection is group homomorphism since $\Gamma \lhd G$. We wish to see why it carries a unique differential structure turning it into a Lie group naturally covered by G. We apply of course the previous proposition.

The action of Γ is clearly free. Let us see why it is properly discontinuous. Let $K \subset G$ be a compact subset. By continuity of the map $(x, y) \in G \times G \mapsto xy^{-1}$, the subset $KK^{-1} = \{gh^{-1},\ g, h \in K\}$ is a compact subset of G. Let $\{\gamma_n\}$ be a sequence in Γ such that $\gamma_n K \cap K \neq \varnothing$. Thus, for all n, $\gamma_n \in KK^{-1}$. Hence, up to an extraction, $\{\gamma_n\}$ is convergent, so that $\gamma_n \gamma_{n+1}^{-1} \to e$. Since Γ is discrete, there exists a neighborhood $U \subset G$ of e such that $\Gamma \cap U = \{e\}$. Therefore, for n large enough, $\gamma_n = \gamma_{n+1}$. Hence, any sequence in $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \varnothing\}$ has a constant subsequence, meaning that this set is finite.

So, G/Γ has a unique differential structure such that $p : G \to G/\Gamma$ is a smooth covering. We are left to observe that $(\overline{x}, \overline{y}) \in G/\Gamma \times G/\Gamma / \mapsto \overline{x}.\overline{y}^{-1} \in G/\Gamma$ is smooth. This is a local question. Let U and V be trivializing neighborhoods of $\overline{x}$ and $\overline{y}$ respectively. Let $x \in \pi^{-1}(\overline{x})$ and $y \in \pi^{-1}(\overline{y})$ and let $\mathcal{U} \subset G$ and $\mathcal{V} \subset G$ be neighborhoods of x and y respectively such that $p|_{\mathcal{U}}$ and $p|_{\mathcal{V}}$ are diffeomorphisms onto U and V respectively. Then, for all $x' \in U$ and $y' \in V$,

$$x'(y')^{-1} = p\left((p|_U)^{-1}(x')\left((p|_V)^{-1}(y')\right)^{-1}\right).$$

This expression shows that the map is smooth at (x', y') as expected. Finally, considering the restriction of p to a small enough neighborhood of e, we obtain that $\mathrm{d}_e p$ realizes a Lie algebra isomorphism from T_eG to $T_{\overline{e}}G/\Gamma$. □

The bad (?) news is that there are only few such subgroups Γ:

Proposition 3.7.3. *Let G be a connected Lie group and let $\Gamma \lhd G$ be a discrete normal subgroup. Then, $\Gamma \subset \mathcal{Z}(G)$ is central.*

Proof. Let $\gamma \in \Gamma$. The map $g \in G \mapsto g\gamma g^{-1} \in \Gamma$ has connected image. By discreteness of Γ, its image must be $\{\gamma\}$, *i.e.* γ is central. □

Hence, we can only mod out by discrete central subgroups to obtain Lie group coverings.

Example 3.7.1.

1. Let $G = \mathbf{R}^n$ with its standard Lie group structure. Let $\Gamma = \mathrm{Span}_{\mathbf{Z}}(v_1, \ldots, v_k)$ for $v_1, \ldots, v_k$ $\mathbf{R}$-linearly independent vectors. Then Γ satisfies the hypothesis of the previous corollary (G is abelian so $\mathcal{Z}(G) = G$ and we let the reader verify that Γ is discrete). A particular case is when Γ is a lattice in $\mathbf{R}^n$, i.e. when $k = n$. A special case is $\Gamma = \mathbf{Z}^n$. In this situation, G/Γ is an n-torus.

2. Let $G = \mathrm{SL}_2(\mathbf{R})$. Its center is $\{\pm \mathrm{id}\}$, so the only example we can produce is the covering $\mathrm{SL}_2(\mathbf{R}) \to \mathrm{PSL}_2(\mathbf{R})$.

3. Let $G = \mathrm{Heis}(3)$, *i.e.*

   $$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},\ x, y, z \in \mathbf{R} \right\}.$$

   Then the center of G is the subgroup (isomorphic to $\mathbf{R}$) corresponding to matrices with $x = y = 0$. Hence, we can consider

   $$\Gamma = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ n \in \mathbf{Z} \right\}.$$

   Then Γ satisfies the hypothesis and $G \to G/\Gamma$ is a Lie group covering. Interestingly, it can be proved that G/Γ is non-linear, *i.e.* it has no faithful finite dimensional representation.

Proposition 3.7.4. *Let $p : G_1 \to G_2$ a covering of Lie groups. Then, there exists $\Gamma < \mathcal{Z}(G_1)$ discrete and a Lie group isomorphism $f : G_2 \to G_1/\Gamma$ such that $f \circ p = \pi$, where $\pi : G_1 \to G_1/\Gamma$ is the natural projection.*

Proof. Consider $\Gamma = \operatorname{Ker} p = p^{-1}(e)$. It is a discrete subgroup of G (it is the fiber of a covering map). It also is normal as kernel of a group homomorphism, hence it is central. By definition, p induces an isomorphism $f : G_1/\Gamma \to G_2$ such that $f \circ p = \pi$. It is again straightforward to verify that f is smooth (restrict to a trivializing neighborhood). Therefore, it is a Lie group homomorphism which is bijective, so it must be a Lie group isomorphism in due to **ref**. ☐

Theorem 8. *Let G be a Lie group and let $\pi : \tilde{G} \to G$ be a universal covering and let $\tilde{e}$ be an element in the fiber of the neutral element $e \in G$. Then, there exists a unique Lie group structure on $\tilde{G}$ whose neutral element is $\tilde{e}$ and such that for all $x, y \in \tilde{G}$, $\pi(xy) = \pi(x)\pi(y)$.*

Corollary 3.7.2. *The fundamental group of a Lie group is abelian.*

This gives an interesting restriction on tolologies that can be made Lie groups.

Example 3.7.2. $\pi_1(\mathbf{S}^1) = \mathbf{Z}$, $\pi_1(\mathbf{T}^n) = \mathbf{Z}^n$, $\pi_1(\mathrm{SO}(3)) = \mathbf{Z}/2\mathbf{Z}$, $\pi_1(\mathrm{SL}_2(\mathbf{R})) = \mathbf{Z}$, $\pi_1(\mathrm{SO}_0(2,2)) = \mathbf{Z}^2$, $\pi_1(\mathrm{Sp}_{2n}(\mathbf{R})) = \mathbf{Z}$.

3.8 Homogeneous spaces

Let G be a Lie group and X a manifold. We say G acts smoothly on X if the action $G \times X \to X$ is smooth. We denote by $G_x < G$ the stabilizer of a point x in X.

Proposition 3.8.1. *The stabilizer G_x is always a closed, hence properly embedded Lie subgroup, of G.*

Definition 3.8.1. Consider a Lie group G acting smoothly on a manifold X. The action is said to be

1. *faithful* if $e \in G$ is the only element acting trivially on X.
2. *transitive* if it is transitive in the sense of general group actions.
3. *proper* if for all compact subset $K \subset X$ such that $\{g \in G \mid g.K \cap K \neq \varnothing\}$ is compact in G. Note that in general, this subset set is closed in G.

Definition 3.8.2. A *homogeneous space* is the data of a transitive Lie group action $G \curvearrowright M$ of a Lie group G on a manifold M.

As sets, we have a G-equivariant identification $M \simeq G/G_x$ for any base point $x \in M$, given by the orbital map at x. The aim of this section is to define a natural differential structure on a general quotient G/H, where H is a closed subgroup of G, whose underlying topology is the quotient topology[1] and such that the previous identification becomes a diffeomorphism. First, let us examine the quotient topology. We prove first the general:

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[1] The finest topology on $G\backslash M$ such that the canonical projection $\pi : M \to G\backslash M$ is continuous.

Proposition 3.8.2. *Let M be a manifold on which a Lie group G acts smoothly and properly. Then, the quotient topology on the orbit space $G\backslash M$ is Hausdorff and second countable.*

Proof. Let $\pi : M \to G\backslash M$ be the canonical projection. Observe first that π is open. Indeed, by definition this is equivalent to saying that for any open subset $U \subset M$, the saturated subset $G.U$ is open. But this is true regardless G acts properly or not: $G.U = \bigcup_{g \in G} g.U$ and $g.U$ is open because G acts (in particular) by homeomorphisms of M. Therefore, π being open and M second countable, so is $G\backslash M = \pi(M)$.

Lemma 3.8.1. *Let K be a compact subset of M. Then, $G.K := \{g.x,\ g \in G,\ x \in K\}$ is closed in M. In particular, G-orbits are closed.*

Proof. Let (g_n) be a sequence in G and (x_n) a sequence in K such that $(g_n.x_n) \to y \in M$. Up to an extraction, we may assume $(x_n) \to x \in M$. Let L be a compact subset of M such that $x, y \in \mathring{L}$. For n large enough, $x_n \in L$ and $g_n x_n \in L$. So, $g_n \in \{g \in G \mid g.L \cap L \neq \varnothing\}$. Hence, up to another extraction, $(g_n) \to g \in G$, showing that $y = g.x$. ☐

Let $x, y \in M$ such that $\pi(x) \neq \pi(y)$. This means $x \notin G.y$. By the previous lemma, $G.y$ is closed. So, we have a neighborhood U of x, which we may assume relatively compact, such that $\overline{U} \cap G.y = \varnothing$. Equivalently, $y \notin G.\overline{U}$. The same lemma gives that $G.\overline{U}$ is closed. So we get a relatively compact neighborhood V of y such that $V \cap G.\overline{U} = \varnothing$, or equivalently $G.V \cap G.\overline{U} = \varnothing$. Hence, $\pi(U)$ and $\pi(V)$ are disjoint open neighborhoods of $\pi(x)$ and $\pi(y)$ respectively. ☐

Although there are general properties of smooth Lie group actions, we restrict to the homogeneous setting. Observe now:

Proposition 3.8.3. *Let G be a Lie group and H be a subgroup. Then the left action $H \curvearrowright G$ is proper if and only if H is closed.*

Proof. If this action is proper, then all orbits are closed due to the previous proposition. Hence, $H = H.e$ is closed.

Conversely, if H is closed, and if $K \subset G$ is a compact subset, then $\{h \in H \mid h.K \cap K \neq \varnothing\} = KK^{-1} \cap H$ is compact in H because KK^{-1} is compact in G by continuity of xy^{-1}. ☐

Theorem 9. *Let G be a Lie group and H a Lie subgroup (embedded). Then, there exists a unique differential structure on G/H such that the canonical projection $\pi : G \to G/H$ is a smooth submersion. Moreover,*

1. *The action of G on G/H by left-translations is smooth.*
2. *If $H \lhd G$, then G/H with this smooth structure becomes a Lie group. The Lie algebra $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and the Lie algebra of G/H is $\mathfrak{g}/\mathfrak{h}$ with the induced bracket.*
3. $\dim G/H = \dim G - \dim H$.
4. $\operatorname{Ker} \mathrm{d}_e\pi = \mathfrak{h}$, *so* $\mathrm{d}_e\pi$ *induces an isomorphism* $\mathfrak{g}/\mathfrak{h} \to T_{\pi(e)}G/H$.

5. *For all manifold N, a map $f : G/H \to N$ is smooth if and only if $f \circ \pi : G \to N$ is smooth. This is crucial for concrete use!*

We recommend to admit this result at first reading and learn how to apply it in practice (although it's not a difficult theorem). It is important to retain the following corollary.

Corollary 3.8.1. *Let G be a Lie group acting transitively on a set X. Assume that there exists $x \in X$ such that the stabilizer G_x is closed in G (equivalently the same is true for every point in X). Then, there exists a unique differential structure on X such that the action is smooth and the natural application $\overline{\varphi_x} : G/G_x \to X$ is a smooth diffeomorphism.*

In particular, if X was a manifold from the beginning, then this atlas coincides with the differential structure given by the previous corollary.

Proof. The existence is immediate: $\overline{\varphi_x}$ is a bijection. So X can be given the push-forward of the atlas of G/G_x built in the previous theorem.

So we are left to observe that if X has a manifold structure with respect to which the G-action is smooth, then it must be diffeomorphic to the previous one. In this situation, $\overline{\varphi_x}$ is smooth because $\varphi_x = \overline{\varphi_x} \circ \pi$ is smooth. We claim that it has constant rank. Indeed, for all $g \in G$, $\overline{\varphi_x} \circ L_g = \alpha(g) \circ \overline{\varphi_x}$, where $\alpha(g) \in \mathrm{Diff}(X)$ is the diffeomorphism corresponding to the action of g and $L_g : G/G_x \to G/G_x$ denotes (abusively) the action induced by left multiplication. Differentiating this identity at $e.H$, we obtain $\mathrm{d}_{g.H}\overline{\varphi_x} \circ \mathrm{d}_{e.H}L_g = d_x\alpha(g) \circ \mathrm{d}_{e.H}\overline{\varphi_x}$. Thus, $\mathrm{d}_{g.H}\overline{\varphi_x}$ and $\mathrm{d}_{e.H}\overline{\varphi_x}$ have the same rank because $\alpha(g)$ and L_g are diffeomorphisms. But it can't be a "strict" submersion because $\overline{\varphi_x}$ is injective and the normal form of submersions. Also, if it was a "strict" immersion, we would have a neighborhood V of e in G such that $\overline{\varphi_x}$ sends $\pi(V)$ onto a submanifold of X of positive codimension, hence of empty interior. Thus, considering a dense sequence (g_n), X would be covered by the $\overline{\varphi_x}(\pi(g_nV)) = \alpha(g_n)\overline{\varphi_x}(\pi(V))$, $n \in \mathbf{N}$, which are all of empty interior, contradicting Baire category theorem. □

Example 3.8.1.

1. Consider the n-sphere $\mathbf{S}^n \subset \mathbf{R}^{n+1}$. Let $G = O(n+1)$, acting naturally and transitively on $\mathbf{S}^n$. Then, for $x_0 = (1, 0, \dots, 0) \in \mathbf{S}^n$, the stabilizer is

$$G_{x_0} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \ A \in O(n) \right\} \simeq O(n)$$

   Therefore, hence we obtain a diffeomorphism $\overline{\varphi_{x_0}} : G/G_{x_0} \to \mathbf{S}^n$ which is G-equivariant. It is quite common to phrase this by saying $\mathbf{S}^n = O(n+1)/O(n)$.

2. Consider the Euclidean space $\mathbf{E}^n$ and $G = \mathrm{SO}(n) \ltimes \mathbf{R}^n$ the group of direct Euclidean motions, and $x_0 = 0$. Clearly G is transitive and $G_{x_0} = \mathrm{SO}(n)$. Hence, E^n is diffeomorphic to $(\mathrm{SO}(n) \ltimes \mathbf{R}^n)/\mathrm{SO}(n)$.

3. Consider the hyperbolic space $\mathbf{H}^n = \{(x_0,\dots,x_n)\in\mathbf{R}^{n+1} \mid x_0>0 \text{ and } -x_0^2+x_1^2+\cdots+x_n^2=-1\}$. The Lorentzian metric $-\mathrm{d}x_0^2+\mathrm{d}x_1^2+\cdots+\mathrm{d}x_n^2$ induces a Riemannian metric on $\mathbf{H}^n$ which is complete and has constant sectional curvature -1. The group $G=O(1,n)_0$ preserves $\mathbf{H}^n$ and acts by isometries of this metric. The action is transitive and if $x_0=(1,0,\dots,0)$, then the stabilizer is

$$G_{x_0}=\left\{\begin{pmatrix}1&0\\0&A\end{pmatrix},\ A\in\mathrm{SO}(n)\right\}\subset O(1,n)_0.$$

Hence, $\mathbf{H}^n$ is diffeomorphic to $O(1,n)_0/\mathrm{SO}(n)$. NB: $O(1,n)_0\neq\mathrm{SO}(1,n)$.

4. Let $X=\{\mathbf{R}_+.v,\ v\in\mathbf{R}^{n+1}\smallsetminus\{0\}\}$ be the set of rays emanating from 0 in $\mathbf{R}^{n+1}$. Let $G=\mathrm{SL}_{n+1}(\mathbf{R})$ which acts naturally and faithfully on X. Then, X is easily seen to be diffeomorphic to $\mathbf{S}^n$ and the action of G is smooth. Let $x_0=\mathbf{R}_+.e_1$. Then its stabilizer is

$$G_{x_0}=\left\{\begin{pmatrix}\det(A)^{-1}&v\\0&A\end{pmatrix},\ A\in\mathrm{GL}_n(\mathbf{R}),\ \det(A)>0,\ v\in\mathbf{R}^n\right\}\subset\mathrm{SL}_{n+1}(\mathbf{R}).$$

Hence, $G_{x_0}\simeq\mathrm{GL}_n^+(\mathbf{R})\ltimes\mathbf{R}^n$ and $\mathbf{S}^n$ is diffeomorphic to $\mathrm{SL}_{n+1}(\mathbf{R})/(\mathrm{GL}_n^+(\mathbf{R})\ltimes\mathbf{R}^n)$. Hence, a given manifold M can be diffeomorphic to G/H and G'/H' where G,G' are not related, nor have the same action.

5. Let $X=\{\Gamma<\mathbf{R}^n,\ \Gamma \text{ unimodular lattice}\}$ be the set of all unimodular lattices in $\mathbf{R}^n$. Recall that a lattice of $\mathbf{R}^n$ is an additive subgroup of the form $\langle v_1,\dots,v_n\rangle$ where $(v_1,\dots,v_n)$ is a basis of $\mathbf{R}^n$ such that $|\det(v_1,\dots,v_n)|=1$. Let $G=\mathrm{SL}_n(\mathbf{R})$ and let G act on X via $g.\Gamma=\Gamma'$ where $\Gamma'=\langle gv_1,\dots,gv_n\rangle$ whenever $\Gamma=\langle v_1,\dots,v_n\rangle$. Let $x_0=\mathbf{Z}^n$. Then, $G_{x_0}=\mathrm{SL}_n(\mathbf{Z})$, showing that the homogeneous space $\mathrm{SL}_n(\mathbf{R})/\mathrm{SL}_n(\mathbf{Z})$ is in a natural bijection with X.

Definition 3.8.3. Let G be a Lie group acting smoothly on a manifold M. Let $x\in M$ and let G_x be the stabilizer x. The *isotropy representation* of G_x at x is the linear representation

$$\begin{array}{rccc}\rho_x: & G_x & \to & \mathrm{GL}(T_xM)\\ & g & \mapsto & \mathrm{d}_x\alpha(g)\end{array}$$

where $\alpha(g)\in\mathrm{Diff}(M)$ is the diffeomorphism defined by $\alpha(g)(x)=g.x$.

Proposition 3.8.4. *Let X be a homogeneous space, under the action of a Lie group G. Let $x\in X$ and let $H=G_x$. Then, T_xX is identified to the vector space $\mathfrak{g}/\mathfrak{h}$ via $\mathrm{d}_{e.H}\overline{\varphi_x}$. Moreover, it conjugates the isotropy representation at x to the representation*

$$\begin{array}{rccc}\overline{\mathrm{Ad}}: & H & \to & \mathrm{GL}(\mathfrak{g}/\mathfrak{h})\\ & h & \mapsto & \mathrm{Ad}(h)\ \textit{mod. }\mathfrak{h}\end{array}$$

It means that for all $h\in H$, $\rho_x(h)\circ\mathrm{d}_{e.H}\overline{\varphi_x}=\mathrm{d}_{e.H}\overline{\varphi_x}\circ\overline{\mathrm{Ad}}(h)$.

Proof. Let $X \in \mathfrak{g}$ and let $\overline{X} \in \mathfrak{g}/\mathfrak{h}$ its projection. Then $\mathrm{d}_{e.H}\overline{\varphi_x}\overline{X} = \mathrm{d}_e\varphi_x X = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi_x(e^{tX})$. So, for all $h \in H$, $\rho_x(h).\mathrm{d}_{e.H}\overline{\varphi_x}\overline{X} = (\mathrm{d}_x\alpha(h) \circ \mathrm{d}_e\varphi_x)(X) = \mathrm{d}_e(\alpha(h) \circ \varphi_x)(X)$. Now for all $g \in G$, $\alpha(h) \circ \varphi_x(g) = h.(g.x) = (hgh^{-1}).x = \varphi_x \circ i_h(g)$ because $h.x = x$ by definition. Therefore, $\mathrm{d}_x\alpha(h) \circ \mathrm{d}_e\varphi_x = \mathrm{d}_e\varphi_x \circ \mathrm{Ad}(h)$, giving the desired conjugacy. $\square$

Chapter 4

Myers-Steenrod Theorem

In this chapter, we give a detailed proof of:

Theorem 10. *Let* (M, g) *be a Riemannian manifold. Then its isometry group* $\mathrm{Iso}(M, g)$ *has a unique Lie group structure such that the natural action of* $\mathrm{Iso}(M, g)$ *is smooth. Moreover, the underlying topology on* $\mathrm{Iso}(M, g)$ *is the compact-open topology.*

The main idea is to imitate the proof of Cartan - Von Neumann Theorem. We will realize $\mathrm{Iso}(M, g)$ as a closed submanifold of the *orthonormal frame bundle* $\mathcal{O}(M)$, via any of its orbits, yielding the desired differential structure. Although $\mathcal{O}(M)$ has not a Lie group structure itself in general, certain important object and notions are transferable to this setting (notably an exponential map), which will allow to mimic the proof of the Lie group case.

In what follows, we denote by (M, g) a connected, Riemannian manifold of dimension $n \geqslant 1$.

4.1 Preliminaries

We recall two results that are key ingredients in the proof. Let $d = d_g$ denote the *length* distance on M. Recall that it is defined by

$$d_g(x, y) = \inf\{\ell(\gamma),\ \gamma \text{ piecewise smooth },\gamma(0) = x,\ \gamma(1) = y\}.$$

Theorem 11. $\mathrm{Iso}(M, g) = \mathrm{Iso}(M, d)$*, where elements in the latter are seen as isometries of the metric space* (M, d_g).

Proof. □

Theorem 12 (Arzela-Ascoli)**.** *Let* $(f_\alpha)_{\alpha\in\mathcal{A}}$ *be a family of maps of a complete metric space* (X, d)*. Suppose that*

1. (f_α) *is equicontinuous, i.e.* $\forall x_0 \in X$, $\forall \varepsilon > 0$, $\exists U \ni x_0$ *open neighborhood such that* $\forall x \in U$, $\forall \alpha \in \mathcal{A}$, $d(f_\alpha(x), f_\alpha(x_0)) < \varepsilon$,

2. $\forall x \in X$, $\{f_\alpha(x),\ \alpha \in \mathcal{A}\}$ *is relatively compact in* X*, i.e. has compact closure in* X*.*

Then, $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ *is relatively compact in* $\mathcal{C}^0(X, X)$ *with respect to the compact-open topology.*

4.2 The orthonormal frame bundle and the Cartan connection

Let $V = \mathbf{R}^n$ with its standard Euclidean structure. We denote by

$$\mathcal{O}(M) = \{u : V \to T_x M,\ x \in M,\ u \text{ linear isometry}\}$$

the *othonormal frame bundle* of (M, g). We denote by $\pi : \mathcal{O}(M) \to M$ the natural projection. For $g \in O(n)$ and $u \in \mathcal{O}(M)$, we define $u.g = u \circ g \in \mathcal{O}(M)$. We obtain a right-action of $O(n)$ on $\mathcal{O}(M)$ such that the $O(n)$-orbits are the fibers of π, *i.e.* for all $u \in \mathcal{O}(M)$, if $x = \pi(u)$, then $u.O(n) = \pi^{-1}(x)$. We refer to [6] for the proof of the following proposition.

Proposition 4.2.1. *Given a Riemannian manifold* (M, g)*, its orthonormal frame bundle* $\mathcal{O}(M)$ *is an* $O(n)$*-principal bundle over* M*, i.e. it has a unique smooth manifold structure such that the free right-action of* $O(n)$ *defined above is smooth and* M *is diffeomorphic to* $\mathcal{O}(M)/O(n)$*.*

Any isometry $f \in \mathrm{Iso}(M, g)$ acts on $\mathcal{O}(M)$ via $f.u = \mathrm{d}_x f \circ u$, where $x = \pi(u)$. By **ref**, we obtain a *free* action of $\mathrm{Iso}(M, g)$ on $\mathcal{O}(M)$, which commutes to the right-action of $O(n)$. In fact, any principal bundle automorphism $F : \mathcal{O}(M) \to \mathcal{O}(M)$ is obtained in this way.

The existence of the Levi-Civita connection of (M, g) is equivalent to that of a 1-form $\omega \in \Omega^1(\mathcal{O}(M), \mathfrak{so}(n) \ltimes \mathbf{R}^n)$, called the Cartan connection. For clarity, we note $P = \mathcal{O}(M)$ and $\mathfrak{g} = \mathfrak{so}(n) \ltimes \mathbf{R}^n$.

Theorem 13. *There exists a unique* 1*-form* $\omega \in \Omega^1(P, \mathfrak{g})$*, such that*

- $\omega_u : T_u P \to \mathfrak{g}$ *is a linear isomorphism for all* $u \in P$ *;*
- $\omega_u(A^*) = A$*, for all* $A \in \mathfrak{so}(n)$ *;*
- $(R_h)^* \omega = \mathrm{Ad}(h^{-1})\omega$ *for all* $h \in O(n)$*.*

Furthermore, for all isometry $f : M \to M$*, its lift* $\mathrm{d}f : P \to P$ *preserves* ω*.*

Definition 4.2.1. Let $X \in \mathfrak{g}$. We denote by $\tilde{X}$ the vector field of P such that $\omega_u(\tilde{X}) = X$. Such vector fields are called *ω-constant.*

These ω-constant vector fields can be compared to left-invariant vector fields on a Lie group. Similarly, they define an exponential map, but only on a neighborhood of the zero section of $TP \simeq P \times \mathfrak{g}$.

Proposition 4.2.2. *There exists an open neighborhood U of $P \times \{0\}$ in $P \times \mathfrak{g}$ such that for all $(u,X) \in U$, the flow of $\tilde{X}$ starting at u is defined for all $|t| \leqslant 1$, and if $V_u = \{X \in \mathfrak{g} \mid (u,X) \in U\}$, then the map $X \in V_u \mapsto \phi^1_{\tilde{X}}(u)$ is a diffeomorphism onto its image.*

We call *exponential map at $u \in P$* the diffeomorphism $\exp_u : X \in V_u \mapsto \phi^1_{\tilde{X}}(u)$.

Lemma 4.2.1. *If $\exp_u$ is defined on $V \subset \mathfrak{g}$ and if $f \in \mathrm{Iso}(M,g)$, then $\exp_{f(u)}$ is defined on V and*

$$f \circ \exp_u = \exp_{f(u)}$$

on V.

Proof. As the action of f on P preserves the Cartan connection ω, for every ω-constant vector field $\tilde{X}$, we have $f^*\tilde{X} = \tilde{X}$. So, if the flow of $\tilde{X}$ at u is defined up to time t, then the same is true at $f(u)$ and the flow is the post-composition by f. The claim follows. □

4.3 Closedness property

We prove here a fact that will play the role of the assumption of a closed subgroup in Cartan-Von Neumann Theorem.

Proposition 4.3.1. *Let (f_k) be a sequence in $\mathrm{Iso}(M,g)$. Assume that there exists $u \in P$ such that $(f_k(u)) \to v \in P$. Then, there exists $f \in \mathrm{Iso}(M,g)$ such that $(f_k) \to f$ in the $\mathcal{C}^1$-topology.*

Proof. □

4.4 H-orbits in P

Let us denote $\mathrm{Iso}(M,g)$ by H, and let $\mathfrak{h}$ denote the Lie algebra of *complete* Killing vector fields of (M,g). At the end of the proof, we will see that H is a Lie group whose Lie algebra is isomorphic to $\mathfrak{h}$, but so far H is just a topological group. We fix a reference frame $u \in P$. Using the action of their flow on P, elements $\mathfrak{h}$ can be seen as vector fields of P. We will use implicitly this identification.

Choose $V \subset \mathfrak{g}$ an arbitrary vector subspace such that $\{\omega_u(X_u),\ X \in \mathfrak{h}\} \oplus V = \mathfrak{g}$. Similarly to the construction of the exponential map, there exists a neighborhood $\mathcal{V}_1$ of 0 in $\mathfrak{h}$, and a neighborhood of $\mathcal{V}_2$ of 0 in V such that

$$(X,Y) \in \mathcal{V}_1 \times \mathcal{V}_2 \mapsto \exp_{\phi^1_X(u)}(Y) = \phi^1_X(\exp_u(Y)) \in P$$

is a diffeomorphism onto its image U.

Proposition 4.4.1. *Restricting the open subsets if necessary, $H.u \cap U = \{\phi^1_X(u),\ X \in \mathcal{V}_1\}$. In particular, $H.u$ is a submanifold of P.*

Proof. We prove it by contradiction. So, we assume that there exists a sequence (h_n) in H such that $(h_n.u) \to u$, with $h_n.u = \phi^1_{X_n}(\exp_u(Y_n))$ with $X_n \in \mathcal{V}_1$ and $Y_n \in \mathcal{V}_2$ such that $(X_n) \to 0$, $Y_n \neq 0$, $(Y_n) \to 0$. Translating h_n by $(\phi^1_{X_n})^{-1}$, we may assume $X_n = 0$. Write $Y_n = t_n Z_n$ with $\|Z_n\| = 1$, for an arbitrary norm. Up to an extraction, $(Z_n) \to Z \in \mathcal{V}$. Let $r > 0$ be such that $B(0,r) \subset \mathcal{V}_2$ for the same norm. Let $t < r$ and $k_n = \lfloor \frac{t}{t_n} \rfloor$. Since $k_n t_n < r$, we get (by finite induction)

$$h_n^{k_n}.u = \exp_u(k_n t_n Z_n) \to \exp_u(tZ).$$

According to Proposition 4.3.1, there exists $h^t \in H$ such that $h_n^{k_n}.u \to h^t.u$. Hence,

$$h^t.u = \exp_u(tZ)$$

for all $t \leqslant r$, and h^t is uniquely determined by this property. Thus, if $s,t \in]-r,r[$ are such that $|s+t| < r$, we have $h^{s+t} = h^s h^t$ because $\exp_u((t+s)Z) = \exp_{\exp_u(sZ)}(tZ)$. Consequently, we can upgrade h^t to a one-paramater subgroup of H by defining $h^t = (h^{t/n})^n$ for n large enough (it is well-defined because of the local property $h^{s+t} = h^s h^t$). Note that we do not make any claim about the regularity of this one-parameter subgroup. The regularity of its trajectories on P is enough for our purpose.

Lemma 4.4.1. *For all $v \in P$, the curve $\{t \in \mathbf{R} \mapsto h^t.v\}$ is smooth.*

Proof. Since $h^{t+s} = h^t h^s$, it is enough to verify it at $t = 0$. This is true at $v = u$ by definition of h^t.

Suppose now that it is true at some $v \in P$. Then, for $X \in \mathfrak{g}$ in a small enough ball, we have

$$h^t.\exp_v(X) = \exp_{h^t.v}(X)$$

showing that the property is also true in every exponential neighborhood of v by regularity of the exponential map. Let now $v \in P$ in the connected component of u. If we pick γ a curve joining u and v, then, covering it by finitely many exponential domains $\exp_{u_k}(V_k)$ with $u_k \in \gamma \cap \exp_{u_{k-1}}(V_{k-1})$, we obtain that the property is true in the connected component of u. Since it is immediately true in the fiber $u.O(n)$ of u, the lemma is established by connectedness of M. $\square$

Consider now the vector field X_h of P defined by

$$X_h(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} h^t(v).$$

It is a smooth vector field because for all $v \in P$ and $X \in \mathfrak{g}$ small enough, $X_h(\exp_v(X)) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp_{h^t.v}(X)$ and by regularity of the exponential map. Its flow is complete and is given by h^t. Therefore, $X_h \in \mathfrak{h}$, and this is a contradiction because $\omega_u(X_h) = \omega_u(\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp_u(tZ)) = \omega_u(\tilde{Z}(u)) = Z \in V \setminus \{0\}$, whereas V intersects $\omega_u(\mathfrak{h})$ trivially. $\square$

The proof is now almost completed. Since the action of H on P is free, the orbital map $\varphi_u : h \in H \mapsto h.u \in H.u$ is a bijection. We give H the differential structure of the submanifold $H.u$ of P. We claim now that the underlying topology coincides with the C^1-topology on H. It is the same as saying that φ_u is an homeomorphism, when $H.u \subset P$ is given the induced topology. Since φ_u is bijective and continuous, we are left to prove that it is open. Let $H' \subset H$ be a closed subset. Suppose a sequence $h_n(u) \in \varphi_u(H')$ converges to a point $v \in \varphi_u(H')$. Then, Proposition 4.3.1 implies that (h_n) converges in the C^0 topology to an element $h \in H$ (the C^0 topology because we see H acting on P), and since H' is closed in H (for this topology), $h \in H'$, hence $v \in \varphi_u(H')$, showing that the latter is closed as claimed.

4.5 Compactness of stabilizers

We finish with this proposition, which will be used later on.

Proposition 4.5.1. *Let (M, g) be a Riemannian manifold and let $x \in M$. Then, the stabilizer G_x of x is a compact Lie subgroup of the group of isometries $G = \mathrm{Iso}(M, g)$.*

Proof. We give a proof assuming that (M, g) is complete which is enough for our purpose here.

Clearly, G_x is closed in G. By Cartan - Von Neumann Theorem, we are left to prove that it is compact. Let (g_n) be a sequence in G_x. We wish to apply Arzela-Ascoli's theorem to (g_n). To that end, we need to show that for every $y \in M$, $\{g_n(y),\ n \in \mathbf{N}\}$ is relatively compact in M. Let d denote the length distance on M. Then, for all $n \in \mathbf{N}$, $d(x, g_n(y)) = d(x, y)$. By Hopf-Rinow theorem, closed balls in (M, d) are compact. So $(g_n(y))$ has compact closure.

Hence, there exists a continuous map $g : M \to M$ such that, up to an extraction, $(g_n) \to g$ uniformly over compact subsets. In particular, since $d(g_n(x), g_n(y)) = d(x, y)$, it follows that g preserves the distance d. Now, apply the same argument to the sequence (g_n^{-1}). We get another isometric map $h : X \to X$ such that, up to another extraction, $(g_n^{-1}) \to h$ uniformly over compact subsets. Consequently, $g \circ h = h \circ g = \mathrm{id}_M$, proving that $g \in \mathrm{Iso}(M, d)$. By **ref**, we obtain that g is in fact *smooth* and preserves the metric tensor of M. Finally, since $g_n(x) = x$, we get $g \in G_x$. $\square$

Chapter 5

Lie theoretic interpretation of symmetric spaces

Let (M,g) be a globally symmetric space. Let $G = \mathrm{Iso}(M,g)_0$. According to Myers-Steenrod theorem, it has a unique natural Lie group structure for which the action of G on M is smooth. By **ref**, G acts transitively on M, turning it into a G-homogeneous space. Let $x_0 \in M$. As it follows from Ascoli's theorem, the stabilizer $K := G_{x_0}$ is a *compact* Lie subgroup of G. So M is G-equivariantly diffeomorphic to G/K.

Exercise 5.0.1. Show that given a Lie group G and a compact subgroup $K < G$, there exists a G-invariant Riemannian metric on $M := G/K$.

The question is now to see what is special with symmetric spaces, among Riemannian homogeneous space. Of course, we have symmetries. Let s denote the geodesic symmetry at x_0 and define $\sigma \in \mathrm{Aut}(G)$ to be the involution

$$\begin{array}{rccc} \sigma : & G & \to & G \\ & g & \mapsto & sgs \end{array} .$$

Define $G^\sigma = \{g \in G \ : \ \sigma(g) = g\}$.

Proposition 5.0.1. *We have* $(G^\sigma)_0 < K < G^\sigma$.

Proof. By uniqueness of geodesic symmetries, for any $g \in G$, $g s_x g^{-1} = s_{gx}$. Consequently, for any $k \in K$, we have $sks = k$. Conversely, any element of G^σ will preserve the set of fixed points of s. Since the fixed points of a geodesic symmetry are isolated, if $\{\phi^t\}$ is any one parameter subgroup of G^σ, we deduce that $\phi^t x_0 = x_0$, and then $(G^\sigma)_0 \subset K$. $\square$

Let us now define an automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$ by $\theta = T_e\sigma$, which, as the above observations indicate, is a very important object in our symmetric space.

Definition 5.0.1. We call θ a *Cartan involution* of $\mathfrak{g}$. (note that it depends on the choice of a base point)

Because $\theta^2 = \mathrm{id}$, we can decompose $\mathfrak{g}$ with respect to the eigenspaces of θ:

$$\mathfrak{g} = \underbrace{\{\theta = \mathrm{id}\}}_{=\mathfrak{k}=\mathrm{Lie}(K)} \oplus \underbrace{\{\theta = -\mathrm{id}\}}_{:=\mathfrak{p}}.$$

This decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the *Cartan decomposition* of $\mathfrak{g}$ (with respect to $\mathbf{X}$ and its fixed origin x_0). Proposition 5.0.1 indicates that $\mathfrak{k}$ is exactly the Lie algebra of K because $(G^\sigma)_0$ and G^σ have the same Lie algebra.

Proposition 5.0.2.

1. *The Cartan decomposition is* $\mathrm{Ad}(K)$*-invariant.*
2. $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ *and* $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.
3. *The restriction of the Killing to* $\mathfrak{k}$ *is negative definite:* $B|_{\mathfrak{k}\times\mathfrak{k}} < 0$. *And the spaces* $\mathfrak{k}$ *and* $\mathfrak{p}$ *are orthogonal with respect to the Killing form.*

Proof.

1. Let $k \in K$. Because k fixes x_0, $ksk^{-1} = s$. Therefore, $\sigma(kgk^{-1}) = skgk^{-1}s = k\sigma(g)k^{-1}$, and differentiating this relation in g, we get $\theta(\mathrm{Ad}(k)X) = \mathrm{Ad}(k)\theta(X)$ for all $k \in K$ and $X \in \mathfrak{g}$. Hence, θ and $\mathrm{Ad}(k)$ commute, and $\mathrm{Ad}(k)$ must preserve the eigenspaces of θ.
2. If $X, Y \in \mathfrak{k}$, then $\theta([X,Y]) = [\theta(X), \theta(Y)] = [X,Y]$, so $[X,Y] \in \mathfrak{k}$. The other inclusions follow similarly.
3. Since K is compact, we can find by an averaging argument an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{g}$, denoted by $\langle .,. \rangle$. So for all $X \in \mathfrak{k}$, $\mathrm{ad}(X)$ is skew-symmetric with respect to $\langle .,. \rangle$, and for any orthonormal basis $(X_1, \dots, X_r)$ of $\mathfrak{g}$ with respect to this scalar product, we have

$$\begin{aligned} B(X,X) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(X)) &= \sum \langle \mathrm{ad}(X) \circ \mathrm{ad}(X)(X_i), X_i \rangle \\ &= -\sum \langle [X, X_i], [X, X_i] \rangle. \end{aligned}$$

   Thus, $B(X,X) \leqslant 0$ for all $X \in \mathfrak{k}$, and $B(X,X) = 0$ if and only if X is central in $\mathfrak{g}$. It follows that for all $t \in \mathbf{R}$, e^{tX} centralizes all of G because for all $Y \in \mathfrak{g}$, $e^Y e^{tX} e^{-Y} = \exp(t\,\mathrm{Ad}(e^Y)X) = \exp(te^{\mathrm{ad}(Y)}X) = e^{tX}$ because $(\mathrm{ad}(Y)^k X) = 0$ if $k > 0$ and X if $k = 0$ (recall that G is connected by definition and that $G = \langle \exp(\mathfrak{g}) \rangle$ for every connected Lie group). It follows that e^{tX} acts trivially: if $k \in K$ is central and $x \in M$, because there is $g \in G$ such that $x = g.x_0$, we have $k.x = (kg).x_0 = g.x_0 = x$. By definition, G acts faithfully, so we must have $e^{tX} = \mathrm{id}$ for all t, hence $X = 0$, proving that $B|_{\mathfrak{k}\times\mathfrak{k}}$ is definite.

   Finally, for $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, in the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, their adjoint actions read:

$$\mathrm{ad}(X) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \text{ and } \mathrm{ad}(Y) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

   Hence, $\mathrm{ad}(X)\,\mathrm{ad}(Y)$ is trace-free as expected.

□

From now on, we identify the tangent space $T_{x_0}M \simeq \mathfrak{p}$ in the following way:

$$X \in \mathfrak{p} \mapsto v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{tX}.x_0 \in T_{x_0}M.$$

Otherwise said, $T_{x_0}M \simeq \mathfrak{g}/\mathfrak{k}$ via the orbital map, and $\mathfrak{g}/\mathfrak{k}$ is identified to the supplementary subspace $\mathfrak{p}$ via the natural projection. Under this identification, the isotropy representation at x_0 is conjugate to

$$\mathrm{Ad}\,|_{\mathfrak{p}} : K \to \mathrm{GL}(\mathfrak{p}).$$

(recall Proposition 3.8.4 and that the Cartan decomposition is $\mathrm{Ad}(K)$ invariant).

The geometric interpretation of the $+1$-eigenspace $\mathfrak{k}$ of θ is clear (Lie algebra of the stabilizer G_{x_0}). For the -1-eigenspace $\mathfrak{p}$ we introduce the notion of transvection.

Definition 5.0.2. A *transvection* of M is an isometry ϕ such that there exists a geodesic $c(t)$ and a number t_0 such that

1. $\phi(c(t)) = c(t + t_0)$;
2. $\mathrm{d}\phi$ induces the parallel transport along c.

Remark 5.0.1. c may not be unique: for instance, if M is the Euclidean space, then its transvections are translations, and any straight line parallel to axis of translation will work.

Proposition 5.0.3. *If $X \in \mathfrak{p}$, then $\{e^{tX}\}$ is a one-parameter group of transvections of M along the geodesic $\{t \mapsto \exp_{x_0}(tX)\}$ (in the second case, X is seen as an element of $T_{x_0}M$).*

In fact, any one parameter group of transvections of M is of the form $\{ge^{tX}g^{-1}\}$, with $g \in G$ and $X \in \mathfrak{p}$.

The idea that the parallel transport is induced by the differential action of one parameters subgroups selected by the Cartan involution will finish the implementation of the geometry of M into algebraic data in G. Indeed :

Proposition 5.0.4. *Let R be the $(3,1)$-curvature tensor of M, and let $X, Y, Z \in \mathfrak{p} \simeq T_{x_0}M$. Then, $R_{x_0}(X,Y,Z) = -[\underbrace{[X,Y]}_{\in\mathfrak{k}}, Z]$.*

Proof. We use another identification: to any $X \in \mathfrak{g}$ corresponds a Killing vector field of M, say $\overline{X} \in \mathrm{Kill}(M)$, naturally given by

$$\overline{X}_x = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{tX}.x,$$

i.e. $\overline{X}$ is the infinitesimal generator of the one-parameter group of isometries $\{e^{tX}\}$. Then, it is a general fact that $[\overline{X}, \overline{Y}] = -\overline{[X,Y]}$ (the bracket on the left is the one of vector fields).

Now, if $X \in \mathfrak{p}$ and T is any vector field on M, then the key point is that

$$\begin{aligned}(\nabla_X T)_{x_0} &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \|^{x_0}_{\phi^t_{\overline{X}}(x_0)} \, T_{\phi_{\overline{X}}(x_0)} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\phi^{-t}_{\overline{X}}\right)_* T_{\phi_{\overline{X}}(x_0)} \\ &= [\overline{X}, T](x_0).\end{aligned}$$

In particular, if T is of the form $T = \overline{Y}$, with $Y \in \mathfrak{p}$, then $(\nabla_X \overline{Y})_{x_0} = 0$ since $[\overline{X}, \overline{Y}] = -\overline{[X,Y]}$ and $[X,Y] \in \mathfrak{k}$. Then, since R is tensorial, we have that for any $X, Y, Z \in \mathfrak{p} \simeq T_{x_0}M$

$$R_{x_0}(X,Y,Z) = (\nabla_{\overline{X}} \nabla_{\overline{Y}} \overline{Z} - \nabla_{\overline{Y}} \nabla_{\overline{X}} \overline{Z} - \nabla_{[\overline{X},\overline{Y}]} \overline{Z})_{x_0}.$$

Note that the last term is 0 since $[\overline{X}, \overline{Y}]_{x_0} = 0$. For the others, we have $\nabla_{\overline{X}} \nabla_{\overline{Y}} \overline{Z} = [\overline{X}, \nabla_{\overline{Y}} \overline{Z}] = \nabla_{[\overline{X},\overline{Y}]} \overline{Z} + \nabla_{\overline{Y}} [\overline{X}, \overline{Z}]$ - by some sort of Leibniz rule for the Lie derivation in directions given by Killing fields. Again, the first term is 0, and we obtain that

$$\begin{aligned}R_{x_0}(X,Y,Z) &= (\nabla_{\overline{Y}} [\overline{X}, \overline{Z}])_{x_0} - (\nabla_{\overline{X}} [\overline{Y}, \overline{Z}])_{x_0} \\ &= [\overline{Y}, [\overline{X}, \overline{Z}]]_{x_0} - [\overline{X}, [\overline{Y}, \overline{Z}]]_{x_0} \\ &= [[\overline{Y}, \overline{X}], \overline{Z}]_{x_0} \text{ (Jacobi)} \\ &= [\overline{[X,Y]}, \overline{Z}]_{x_0} = -\overline{[[X,Y],Z]}_{x_0}.\end{aligned}$$

□

It remains to determine the link between the metric on M and the induced quadratic form on $\mathfrak{p} \simeq T_{x_0}M$. This will depend on the *type* of the symmetric space, which will be defined in the setting of irrecudible, simply-connected symmetric spaces. This quadratic form will be a multiple of the restriction of the *Killing form* of $\mathfrak{g}$.

From now on, M is assumed to be simply-connected.

Definition 5.0.3. A Riemannian manifold (M,g) is said to be *irreducible* if it is *not* isometric to a non-trivial product $(M_1 \times M_2, g_1 \oplus g_2)$ where (M_1, g_1) and (M_2, g_2) are Riemannian manifolds of positive dimension.

Definition 5.0.4. Let (M,g) be a Riemannian manifold and let $x \in M$. We define the holonomy group at x as the subgroup $\{\|^c_1,\ c : [0,1] \to M \text{ loop based at } x\} \subset O(T_xM, g_x)$, where $\|^c_t : T_xM \to T_{c(t)}M$ is the parallel transport along c. We denote it by $\mathrm{Hol}(x)$.

Proposition 5.0.5. *Let (M,g) be a complete, simply-connected Riemannian manifold and $x \in M$. Then, it is irreducible if and only if its holonomy group $\mathrm{Hol}(x)$ acts irreducibly on T_xM.*

Proof. We just give a rough idea: if $T_xM = V_1 \oplus V_2$ is a non-trivial $\mathrm{Hol}(x)$-invariant decomposition, then there are Δ_1, Δ_2 two parallel distributions on M such that $\Delta_1(x) = V_1$ and $\Delta_2(x) = V_2$. They are integrable because $[X,Y] = \nabla_X Y - \nabla_Y X$ (∇ is torsion free) and $\nabla_X Y$ is determined by the parallel transport. Considering $\mathcal{F}_1$ and $\mathcal{F}_2$ the leaves integrating Δ_1 and Δ_2 and passing through x respectively, and if $M_1 \to \mathcal{F}_1$ and $M_2 \to \mathcal{F}$ are the universal coverings (with respect to the manifold structure of leaves), then the immersion $\mathcal{F}_1 \times \mathcal{F}_2 \to M$ yields a Riemannian covering $M_1 \times M_2 \to M$ which must be an isometry because M is simply-connected. □

The following de Rham decomposition theorem asserts that complete, simply connected Riemannian manifolds always split into a product of irreducible manifolds.

Theorem 14. *Let (M,g) be a complete, simply-connected Riemannian manifold. Then, there exists $k \geqslant 0$ and $(M_1,g_1),\dots,(M_\ell,g_\ell)$ complete, simply-connected, irreducible Riemannian manifolds non-isometric to $\mathbf{R}$, such that (M,g) is isometric to the Riemannian product $\mathbf{R}^k \times M_1 \times \cdots \times M_\ell$ with the metric product $g_{flat} \oplus g_1 \oplus \cdots \oplus g_\ell$.*

Proof. Admitted (for the existence, an induction on $\dim M$). □

Starting from an *a priori* non-irreducible (simply connected) symmetric space (M,g), we can consider its de Rham decomposition into a product of irreducible manifolds. Each factor will be symmetric (they must be complete and recall that $s_x(\exp_x(v)) = \exp_x(-v)$, so the geodesic symmetries will preserve each factor). The next proposition characterizes the type of each irreducible factor.

Proposition 5.0.6. *Assume that (M,g) is an irreducible (and simply-connected) symmetric space. Let G denote as usual the identity component of its isometry group. Let B be the Killing form of its Lie algebra $\mathfrak{g}$. Then:*

1. *There exists $\alpha \in \mathbf{R}$ such that $B_{\mathfrak{p}\times\mathfrak{p}} = \alpha g_{x_0}(.,.)$ (where g denotes the metric of M).*
2. *The sectional curvature κ of M is given by:*

$$\kappa(P) = \begin{cases} \frac{1}{\alpha} B([X,Y],[X,Y]) & \text{if } \alpha \neq 0; \\ 0 & \text{else,} \end{cases}$$

   where $P \subset T_{x_0}M$ is a 2-plane, and $X,Y \in \mathfrak{p} \simeq T_{x_0}M$ form an orthonormal basis of P. In particular, κ has constant sign (but may vanish, and often does), which is the opposite of the sign of α.
3. *(a) If $\alpha = 0$, then M is isometric to $\mathbf{R}$.*
   (b) If $\alpha < 0$, then M and $G = \mathrm{Iso}(M)_0$ are compact, and G has finite center.
   (c) If $\alpha > 0$, then G has trivial center.

Proof. The key point is to see that the holonomy $\mathrm{Hol}(x_0) < O(T_{x_0}M)$ is contained in the isotropy K. This follows from the fact that for any loop γ based at x_0, the parallel transport along γ can be approximated by that along piecewise-geodesic loops based at x_0, and the parallel transport along such loops belongs to K (because it is realised by a finite product of transvections), and K is closed.

Therefore, by irreducibility, $\mathrm{Ad}(K)$ must act irreducibly on $\mathfrak{p}$ because it is conjugate to the isotropy representation of K on $T_{x_0}M$, which contains the holonomy.

1. We have two $\mathrm{Ad}(K)$-invariant quadratic forms on $\mathfrak{p}$: first the restriction of the Killing form (this is general and easy to see on the definition), second the inner product corresponding to the metric g_{x_0}. By irreducibility of $\mathrm{Ad}(K)$, they must be proportional.

2. This follows from the formula on $R_{x_0}(X, Y, Z)$ and the previous point.

3. (a) If $\alpha = 0$, then $\kappa \equiv 0$ by homogeneity and M is isometric to some Euclidean space, which must be 1-dimensional by irreducibility.
   (b) If $\alpha < 0$, we prove that the Ricci curvature is positive and conclude with Myers' theorem.
   (c) If $\alpha > 0$, then $\kappa \leqslant 0$ and M is CAT(0). An element g in $Z(G)$ is such that $x \mapsto d(x, gx)$ is constant (Clifford map). If this constant is non-zero, g is hyperbolic and since the minimum is realized everywhere, M splits into the product of all axes of translation of g, contradicting irreducibility.

□

Definition 5.0.5. Let M be a simply connected symmetric space, and let $M = \mathbf{R}^k \times M_1 \times \cdots \times M_s$ be its De Rham decomposition.

1. M is said to be of **compact type** if $k = 0$ and all $M_i's$ are in the case $\alpha < 0$ of the last proposition.

2. M is said to be of **non-compact type** if $k = 0$ and all $M_i's$ are in the case $\alpha > 0$ in the last proposition.

Recall that we have a finite group F and a short exact sequence

$$1 \to \mathrm{Iso}(\mathbf{R}^k) \times \mathrm{Iso}(M_1) \times \mathrm{Iso}(M_s) \to \mathrm{Iso}(M) \to F \to 1$$

Corollary 5.0.1. *If M is of non-compact type, then it has non-positive sectional curvature $\kappa \leqslant 0$ and $G = \mathrm{Iso}(M)_0$ is semi-simple without compact factor. In particular, M is diffeomorphic to $\mathbf{R}^{\dim M}$.*

Proof. The sign of the sectional curvature is clear by the last proposition. For the group, its Lie algebra splits into $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ where $\mathfrak{g}_i$ is the Lie algebra of $\mathrm{Iso}(X_i)$, and each has a Cartan decomposition $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ and the Killing form is < 0 on $\mathfrak{k}_i$ and > 0 on $\mathfrak{p}_i$. This proves that the Killing form of $\mathfrak{g}$ is non-degenerate, which is enough by Cartan's criterion of semi-simplicity. □

5.1 Brief detour on algebraic structure of semi-simple Lie algebras

Let $\mathfrak{g}$ be a finite dimensional, real Lie algebra. For $A, B \subset \mathfrak{g}$ two subsets, we note $[A, B] = \mathrm{Span}([X, Y],\ X \in A,\ Y \in B\}$ the Lie subalgebra spanned by brackets of elements of A and B. We define two sequences of subalgebras

1. The *derived series* $\mathfrak{d}^n(\mathfrak{g})$, by $\mathfrak{d}^0(\mathfrak{g}) = \mathfrak{g}$ and $\mathfrak{d}^{n+1}(\mathfrak{g}) = [\mathfrak{d}^n(\mathfrak{g}), \mathfrak{d}^n(\mathfrak{g})]$ for all $n \geqslant 0$.
2. The *lower central series* $\mathfrak{c}^n(\mathfrak{g})$, by $\mathfrak{c}^0(\mathfrak{g}) = \mathfrak{g}$ and $\mathfrak{c}^{n+1}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{c}^n(\mathfrak{g})]$.

Definition 5.1.1. The Lie algebra $\mathfrak{g}$ is said to be *solvable* if there exists $n \geqslant 0$ such that $\mathfrak{d}^n(\mathfrak{g}) = 0$. It is said to be *nilpotent* if there exists $n \geqslant 0$ such that $\mathfrak{c}^n(\mathfrak{g}) = 0$.

Remark 5.1.1. Nilpotent $\Rightarrow$ Solvable.

Example 5.1.1. The Lie subalgebra of $\mathfrak{gl}_n(\mathbf{R})$ formed of upper-triangular matrices

$$\mathfrak{g} = \left\{ \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \subset \mathfrak{gl}_n(\mathbf{R})$$

is solvable.

The Lie subalgebra of strictly upper-triangular matrices

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \right\} \subset \mathfrak{g}$$

is nilpotent.

Any abelian Lie algebra is nilpotent. The Heisenberg Lie algebra (seen in exercises) is another example of nilpotent Lie algebra.

Definition 5.1.2. An *ideal* of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h}$ such that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. We note $\mathfrak{h} \lhd \mathfrak{g}$ in this situation.

In fact, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ if and only is the corresponding integral subgroup H is normal in G.

Proposition 5.1.1. *There is a unique Lie algebra structure on $\mathfrak{g}/\mathfrak{h}$ such that the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is a Lie algebra homomorphism.*

Proposition 5.1.2. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be Lie algebras and suppose that we have*

$$0 \to \mathfrak{a} \to \mathfrak{b} \to \mathfrak{c} \to 0$$

Òa short exact sequence of Lie algebras. Then, $\mathfrak{b}$ is solvable if and only if $\mathfrak{a}$ and $\mathfrak{c}$ are solvable.

If $\mathfrak{a}, \mathfrak{b} \lhd \mathfrak{g}$ are solvable ideals, then $\mathfrak{a} + \mathfrak{b}$ is a solvable ideal.

The last observation enables to define:

Definition 5.1.3. We call *solvable radical* of $\mathfrak{g}$ the largest solvable ideal of $\mathfrak{g}$, and denote it by $\operatorname{rad}(\mathfrak{g})$.

Example 5.1.2. Let $n \geqslant 3$ and let $\mathfrak{g}$ be the Lie algebra of the Euclidean motion group in $\mathbf{R}^n$: $G = O(n) \ltimes \mathbf{R}^n$. Then, the solvable radical of $\mathfrak{g}$ is the abelian Lie algebra corresponding to the translation factor $\mathbf{R}^n$.

Definition 5.1.4. A Lie algebra $\mathfrak{g}$ is said to be semi-simple if $\operatorname{rad}(\mathfrak{g}) = \{0\}$, *i.e.* if does not contain any non-trivial solvable ideal.

A Lie algebra $\mathfrak{g}$ is said to be simple if it is non-abelian and does not contain any non-trivial ideal.

Exercise 5.1.1. Show that if $\mathfrak{g}$ is simple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Proposition 5.1.3. *For all Lie algebra $\mathfrak{g}$, the quotient Lie algebra $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ is semi-simple.*

Definition 5.1.5. Let $\mathfrak{a}, \mathfrak{b}$ be two Lie algebras, and let $\pi : \mathfrak{a} \to \operatorname{Der}(\mathfrak{b})$ be a Lie algebra homomorphism (possibly trivial). We define a Lie algebra structure on the Cartesian product $\mathfrak{a} \times \mathfrak{b}$ by declaring that it extends that of $\mathfrak{a}$ and $\mathfrak{b}$ and for all $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$, $[X, Y] := \pi(X)(Y)$ and $[Y, X] := -[X, Y]$.

This structure is called *semi-direct product* and noted $\mathfrak{a} \ltimes_\pi \mathfrak{b}$.

The following is a structure result for Lie algebras, which we cite for cultural reasons. In one sentence, any Lie algebra splits into a semi-simple factor and a solvable factor.

Theorem 15 (Levi decomposition)**.** *Let $\mathfrak{g}$ be a Lie algebra. Then, there exists a semi-simple Lie algebra $\mathfrak{s} \subset \mathfrak{g}$ such that $\mathfrak{g} \simeq \mathfrak{s} \ltimes_\pi \operatorname{rad}(\mathfrak{g})$, for some $\pi : \mathfrak{s} \to \operatorname{Der}(\operatorname{rad}(\mathfrak{g}))$.*

Exercise 5.1.2. Show that it is the same as stating that the short exact sequence

$$\operatorname{rad}(\mathfrak{g}) \hookrightarrow \mathfrak{g} \xrightarrow{p} \mathfrak{g}/\operatorname{rad}(\mathfrak{g})$$

is split, *i.e.* that there exists an homomorphism $s : \mathfrak{g}/\operatorname{rad}(\mathfrak{g}) \to \mathfrak{g}$ such that $p \circ s = \mathrm{id}$.

Finally, the following (admitted) characterizations of semi-simplicity will be useful for us.

Theorem 16 (Cartan semi-simplicity criterion)**.** *Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent:*

1. *$\mathfrak{g}$ is semi-simple.*
2. *There exists $\mathfrak{g}_1, \dots, g_r$ simple Lie algebras such that $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$.*
3. *$\mathfrak{g}$ admits no non-trivial abelian ideal.*
4. *The Killing form B of $\mathfrak{g}$ is non-degenerate.*

We will be especially interested in the equivalence (1) $\iff$ (4).

Classification of complex semi-simple Lie algebras

Theorem 17. *Up to isomorphism, there are* 4 *infinite families of complex, simple Lie algebras:*

1. $\mathfrak{sl}_{n+1}(\mathbf{C})$, $n \geqslant 1$ *(type* A_n*).*
2. $\mathfrak{so}_{2n+1}(\mathbf{C})$, $n \geqslant 2$ *(type* B_n*).*
3. $\mathfrak{sp}_{2n}(\mathbf{C})$, $n \geqslant 3$ *(type* C_n*).*
4. $\mathfrak{so}_{2n}(\mathbf{C})$, $n \geqslant 4$ *(type* D_n*).*

Apart from these regular families, there are 5 *additional exceptional complex simple Lie algebras:*

$$\mathfrak{e}_6(\mathbf{C}), \quad \mathfrak{e}_7(\mathbf{C}), \quad \mathfrak{e}_8(\mathbf{C}), \quad \mathfrak{f}_4(\mathbf{C}), \quad \mathfrak{g}_2(\mathbf{C}).$$

of respective types E_6, E_7, E_8, F_4, G_2.

Remark 5.1.2. Note that $\mathfrak{so}_2(\mathbf{C}) \simeq \mathbf{C}$ is not semi-simple.

For low values of n, we have exceptional isomorphisms:

- $\mathfrak{sp}_2(\mathbf{C}) \simeq \mathfrak{so}_3(\mathbf{C}) \simeq \mathfrak{sl}_2(\mathbf{C})$ $(C_1 = B_1 = A_1)$.
- $\mathfrak{sp}_4(\mathbf{C}) \simeq \mathfrak{so}_5(\mathbf{C})$ $(C_2 = B_2)$.
- $\mathfrak{so}_4(\mathbf{C}) \simeq \mathfrak{so}_3(\mathbf{C}) \oplus \mathfrak{so}_3(\mathbf{C})$, $(D_2 = B_1 \oplus B_1)$.
- $\mathfrak{so}_6(\mathbf{C}) \simeq \mathfrak{sl}_4(\mathbf{C})$ $(D_3 = A_3)$.

We give a brief idea of the structure of this classification, especially the "types" most commonly named "*root-system*" of the Lie algebra.

Definition 5.1.6. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. A *Cartan subalgebra* is an abelian Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\forall X \in \mathfrak{h}$, $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is a $\mathbf{C}$-diagonalizable, and such that $\mathfrak{h}$ is maximal with these properties.

Proposition 5.1.4. *Let* $\mathfrak{g}$ *be a complex semi-simple Lie algebra.*

1. *There exist Cartan subalgebras of* $\mathfrak{g}$.
2. *All Cartan subalgebras of* $\mathfrak{g}$ *are conjugate in* $\mathrm{Aut}(\mathfrak{g})$*: for all Cartan subalgebras* $\mathfrak{h}, \mathfrak{h}' \subset \mathfrak{g}$, *there exists* $\varphi \in \mathrm{Aut}(\mathfrak{g})$ *such that* $\mathfrak{h}' = \varphi(\mathfrak{h})$.

This fundamental result invites the following:

Definition 5.1.7. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. We call *complex rank* of $\mathfrak{g}$, and denote by $\mathrm{Rk}_{\mathbf{C}}(\mathfrak{g})$, the common dimension of all its Cartan subalgebras.

In Theorem 17, the index n of the root-system is the complex-rank of the Lie algebra. For instance, $\mathfrak{sl}_n(\mathbf{C})$ has rank $n-1$, $\mathfrak{so}_n(\mathbf{C})$ has rank $n/2$ if n is even and $(n-1)/2$ if n is odd. We will explicit this below.

Roots, root-systems, root-spaces Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Because $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism, by definition, $\{\mathrm{ad}(X),\ X \in \mathfrak{h}\}$ is a vector subspace of pairwise commuting, $\mathbf{C}$-diagonalisable endomorphisms.

Hence, there exists a finite family $\Delta \subset \mathfrak{h}^* \smallsetminus \{0\}$ such that, if we define for $\alpha \in \mathfrak{h}^*$

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h},\ [H,X] = \alpha(H)X\},$$

then $\mathfrak{h} = \mathfrak{g}_0 = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h},\ [H,X] = 0\} = \mathfrak{z}(\mathfrak{h})$ and

- $\forall \alpha \in \Delta,\ \mathfrak{g}_\alpha \neq \{0\}$;
- and we have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha. \tag{5.1}$$

Definition 5.1.8. We call $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ the *root-system* of $\mathfrak{g}$ corresponding to the Cartan subalgebra $\mathfrak{h}$. Elements of Δ are called *roots*. For all root α, the subspace $\mathfrak{g}_\alpha$ is called the *root-space* associated to α. The direct sum (5.1) is called the *root-space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Proposition 5.1.5. *Let $\mathfrak{h}$ be a Cartan subalgebra of a complex semi-simple Lie algebra $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. Then,*

1. *The restriction to $\mathfrak{h}$ of the Killing form B of $\mathfrak{g}$ is non-degenerate. It does not mean that $\mathfrak{h}$ is semi-simple!*
2. *The linear span* $\mathrm{Span}\{\alpha,\ \alpha \in \Delta\} = \mathfrak{h}^*$.

For all $\lambda \in \mathfrak{h}^*$, there exists a unique $H_\lambda \in \mathfrak{h}$ such that $\lambda(H) = B(H_\lambda, H)$ for all $H \in \mathfrak{h}$. Let $V := \mathrm{Span}_{\mathbf{R}}\{\alpha,\ \alpha \in \Delta\}$. Then

3. *$\mathfrak{h}^* = V \oplus iV$ (so $\dim_{\mathbf{R}} V = n$).*
4. *$\langle \lambda, \mu \rangle := \lambda(H_\mu) = \mu(H_\lambda)$ defines a Euclidean scalar product on V.*
5. *$\forall \alpha \in \Delta$ and $t \in \mathbf{R}$, $t\alpha \in \Delta \iff t = \pm 1$.*
6. *$\forall \alpha, \beta \in \Delta,\ n_{\alpha,\beta} = 2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} \in \mathbf{Z}$.*
7. *$\forall \alpha, \beta \in \Delta,\ s_\beta(\alpha) = \alpha - 2\frac{\langle \alpha,\beta \rangle}{\langle \beta,\beta \rangle}\beta = \alpha - n_{\beta,\alpha}\beta \in \Delta$.*

Proof. 1. Let $H_0 \in \mathfrak{h}$ be an element orthogonal to $\mathfrak{h}$, *i.e.* such that $\forall H \in \mathfrak{h},\ B(H_0, H) = 0$. Let $\alpha \in \Delta$ and let $H \in \mathfrak{h}$ be an element such that $\alpha(H) \neq 0$. Then, for all $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha(H)B(H_0, X_\alpha) = B(H_0, [H, X_\alpha]) = -B([H, H_0], X_\alpha) = 0$ using the fact that $\mathrm{ad}(H)$ is skew-symmetric with respect to B. Hence, $B(H_0, X_\alpha) = 0$, proving that H_0 is orthogonal to every root-space $\mathfrak{g}_\alpha$. So, H_0 is orthogonal to $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha = \mathfrak{g}$. This implies $H_0 = 0$ because B is non-degenerate, proving that $B|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate.

2. Consider an element $H \in \mathfrak{h}$ such that $\forall \lambda \in \operatorname{Span}_{\mathbf{C}} \Delta$, $\lambda(H) = 0$ (*i.e.* $H \in \operatorname{Span}(\Delta)^{\perp}$ in the duality sense). Then, for all $\alpha \in \Delta$, $\mathfrak{g}_\alpha \subset \mathfrak{z}(H)$, the centralizer of H. Since $\mathfrak{h} \subset \mathfrak{z}(H)$, we obtain $\mathfrak{g} \subset \mathfrak{z}(H)$, *i.e.* H is central. The center $\mathfrak{z}(\mathfrak{g}) \lhd \mathfrak{g}$ is an abelian ideal, hence trivial by semi-simplicity. So $H = 0$, and it follows $\operatorname{Span}_{\mathbf{C}}(\Delta) = \mathfrak{h}^*$.

□

Remark 5.1.3. Every root-space $\mathfrak{g}_\alpha$ is totally isotropic with respect to B.

Definition 5.1.9. Given a Euclidean space $(V, \langle ., . \rangle)$ of dimension $n \geqslant 1$, a finite family $\Sigma \subset V$ satisfying the following conditions is called a **reduced root-system** of rank n.

1. $V = \operatorname{Span}_{\mathbf{R}} \Sigma$.
2. $\forall \alpha \in \Delta$ and $t \in \mathbf{R}$, $t\alpha \in \Delta \iff t = \pm 1$.
3. $\forall \alpha, \beta \in \Delta, \mathfrak{n}_{\alpha,\beta} = 2\dfrac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z}$.
4. $\forall \alpha, \beta \in \Delta,\ s_\beta(\alpha) = \alpha - 2\dfrac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}\beta = \alpha - n_{\beta,\alpha}\beta \in \Delta$.

Definition 5.1.10. A reduce-root system (V, Σ) is said to be irreducible if there does not exist an orthogonal decomposition $V = V_1 \oplus^{\perp} V_2$ such that $\Sigma = (V_1 \cap \Sigma) \cup (V_2 \cap \Sigma)$.

Two root systems (V_1, Σ_1) and (V_2, Σ_2) are said to be isomorphic if there exists a similarity $\varphi : V_1 \to V_2$ carrying Σ_1 to Σ_2.

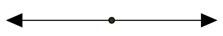
We will not discuss further the following

Proposition 5.1.6. *Reduced, irreducible root-systems are completely classified. They fall into four infinite families, named* $(A_n)_{n \geqslant 1}$, $(B_n)_{n \geqslant 2}$, $(C_n)_{n \geqslant 3}$, $(D_n)_{n \geqslant 4}$, *and five exceptions:* E_6, E_7, E_8, F_4, G_2.

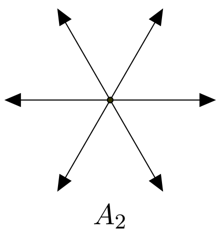
In fact, for every such label, we have a concrete root-system in an n-dimensional Euclidean space such that, non of them are pairwise isomorphic, and such that every reduced, irreducible root-system Σ is isomorphic to one of them.

Example 5.1.3. Let's now give explicit instances of the root-systems:

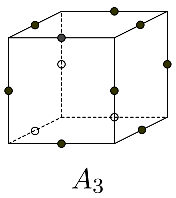
1. A_n. Let $V = \{x_1 + \cdots + x_{n+1} = 0\} \subset \mathbf{R}^{n+1}$ with standard inner product and $(e_1, \ldots, e_{n+1})$ the standard basis. Then, a root-system of type A_n is a root-system isomorphic to
$$\Delta = \{e_i - e_j,\ i \neq j\} \subset V.$$



A_1



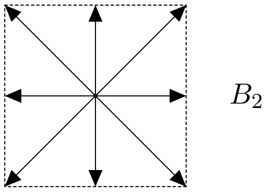
A_2



A_3

2. B_n. A root-system of type B_n is a root-system isomorphic to

$$\Delta = \{\pm e_i \pm e_j,\ i \neq j\} \cup \{\pm e_k\} \subset V = \mathbf{R}^n.$$



3. C_n. A root-system of type C_n is a root-system isomorphic to

$$\Delta = \{\pm e_i \pm e_j,\ i \neq j\} \cup \{\pm 2e_k\} \subset V = \mathbf{R}^n.$$

4. D_n. A root-system of type D_n is a root-system isomorphic to

$$\Delta = \{\pm e_i \pm e_j,\ i \neq j\} \subset V = \mathbf{R}^n.$$

Now, in our complex, simple Lie algebra, a reduced root-system $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$ appears in a real form of $\mathfrak{h}^*$ and its inner product is obtained by restricting the Killing form. But it comes with its associated root-spaces. Here are a couple properties about them:

Proposition 5.1.7. *Let $\mathfrak{h}$ be a Cartan subalgebra of a semi-simple, complex Lie algebra $\mathfrak{g}$. Let Δ be the corresponding root-system and for all $\alpha \in \Delta$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ denote as usual the root-space attached to α. Then,*

1. *For all $\alpha \in \Delta$, $\dim_{\mathbf{C}} \mathfrak{g}_\alpha = 1$.*
2. *For all $\alpha, \beta \in \Delta$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

As seen in the previous proof, opposite root-spaces generate copies of $\mathfrak{sl}_2(\mathbf{C})$ in $\mathfrak{g}$. These are called $\mathfrak{sl}_2$-*triples.* They play a key role in the theoretical understanding of the structure of $\mathfrak{g}$ and its finite dimensional representations. Somehow, any complex, simple Lie algebra is made of $\mathfrak{sl}_2(\mathbf{C})$ building blocks, and the root-system of $\mathfrak{g}$ explains how to assemble them (Serre relations).

Theorem 18. *For any $n \geqslant 1$, for any reduced, irreducible root-system Σ of rank n, there exists a unique (up to isomorphism) complex, simple Lie algebra $\mathfrak{g}$ such that*

1. $\mathrm{Rk}_{\mathbf{C}}\, \mathfrak{g} = n$ *;*
2. $\forall \mathfrak{h} \subset \mathfrak{g}$ *Cartan subalgebra,* $\Delta(\mathfrak{g}, \mathfrak{h}) \simeq \Sigma$.

Classification of real-simple Lie algebras Strategy : Of course, based on the classification in the complex case. Throughout this section, we denote by $\mathfrak{g}_0$ a real, simple Lie algebra. Let us first define the complexification and realification process.

Proposition 5.1.8. *Let $\mathfrak{g}_0$ be a real Lie algebra. Then, we can define a natural Lie algebra structure on $\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$ such that $X \in \mathfrak{g}_0 \mapsto X \otimes 1 \in \mathfrak{g}_0^{\mathbf{C}}$ is a (real) Lie algebra embedding. It is called the complexification of $\mathfrak{g}_0$ and denoted by $\mathfrak{g}_0^{\mathbf{C}}$.*

Let $\mathfrak{g}$ be a complex Lie algebra. Then, the underlying real vector space of $\mathfrak{g}$, endowed with the bracket of $\mathfrak{g}$, is a real Lie algebra, called the realification of $\mathfrak{g}$ and denoted by $\mathfrak{g}^{\mathbf{R}}$.

Proof. By the universal property of tensor products, the map

$$\begin{array}{ccc} \mathfrak{g}_0 \times \mathbf{C} \times \mathfrak{g}_0 \times \mathbf{C} & \to & \mathfrak{g}_0 \otimes \mathbf{C} \\ (X_1, z_1, X_2, z_2) & \mapsto & [X_1, X_2] \otimes z \end{array}$$

induces an $\mathbf{R}$-bilinear map $b : \mathfrak{g}_0^{\mathbf{C}} \times \mathfrak{g}_0^{\mathbf{C}} \to \mathfrak{g}_0^{\mathbf{C}}$ such that $b(X_1 \otimes z_1, X_2 \otimes z_2) = [X_1, X_2] \otimes z_1 z_2$ for all $X_1, X_2 \in \mathfrak{g}_0$ and $z_1, z_2 \in \mathbf{C}$. Hence, b is in fact $\mathbf{C}$-bilinear and extends the bracket on $\mathfrak{g}_0$. It is direct to observe that b is a Lie bracket on $\mathfrak{g}_0^{\mathbf{C}}$, which we still denote $[.,.]$. ☐

Remark 5.1.4. Complexification and realification are not inverse operations, the complexification of $\mathfrak{g}^{\mathbf{R}}$ is a complex Lie algebra with complex dimension *twice* the complex dimension of $\mathfrak{g}$.

Definition 5.1.11. Given a complex Lie algebra $\mathfrak{g}$, we say that a real Lie algebra $\mathfrak{g}_0$ is a *real form* of $\mathfrak{g}$ if $\mathfrak{g} \simeq \mathfrak{g}_0^{\mathbf{C}}$.

We wish to see what remains after complexifying or realifying. Our best friend is the Killing form.

Proposition 5.1.9. *Let $\mathfrak{g}_0$ be a real Lie algebra and $\mathfrak{g} = \mathfrak{g}_0^{\mathbf{C}}$. Let B be the Killing form of $\mathfrak{g}$ and B_0 that of $\mathfrak{g}_0$. Then, B is non-degenerate if and only if B_0 is non-degenerate. Hence, if $\mathfrak{g}_0$ is semi-simple, then so is its complification $\mathfrak{g}$. Conversely, if $\mathfrak{g}$ is complex semi-simple, then any real-form of $\mathfrak{g}$ is real semi-simple.*

Let $\mathfrak{g}$ be a complex Lie algebra, with Killing form B. Let $B^{\mathbf{R}}$ be the Killing form of $\mathfrak{g}^{\mathbf{R}}$. Then,

$$B^{\mathbf{R}} = 2\,\mathrm{Re}(B).$$

Hence, B and $B^{\mathbf{R}}$ are either both non-degenerate, or both degenerate and it follows that $\mathfrak{g}$ is semi-simple if and only if $\mathfrak{g}^{\mathbf{R}}$ is semi-simple.

Proof. If $(X_1, \ldots, X_n)$ is an $\mathbf{R}$-basis of $\mathfrak{g}_0$, then it also is a $\mathbf{C}$-basis of $\mathfrak{g}_0^{\mathbf{C}}$. So, if B_0 denotes the Killing form of $\mathfrak{g}_0$ and B the Killing form of $\mathfrak{g} = \mathfrak{g}_0^{\mathbf{C}}$, we have $B(X,Y) = B_0(X,Y)$ for all $X, Y \in \mathfrak{g}_0$, simply because the matrix in this basis of $\mathrm{ad}(X)\,\mathrm{ad}(Y)$ is the same, be it seen as an element of $\mathrm{End}_{\mathbf{C}}(\mathfrak{g})$ or an element of $\mathrm{End}_{\mathbf{R}}(\mathfrak{g}_0)$. It follows that B is non-degenerate if and only if B_0 is non-degenerate. Indeed, if B is non-degenerate and if $X \in \mathrm{Ker}\, B_0$, then for all $Y, Z \in \mathfrak{g}_0$, we have $B(X, Y + iZ) = 0$, so $X \in \mathrm{Ker}\, B = \{0\}$.

Conversely, if B_0 is non-degenerate, and if $X = Y + iZ \in \operatorname{Ker} B$, then for all $W \in \mathfrak{g}_0$, we have $B(X,W) = B(Y,W) + iB(Z,W) = 0 \Rightarrow B_0(Y,W) = B_0(Z,W) = 0$ for all $W \in \mathfrak{g}_0$, and then $Y = Z = 0$ proving that B is non-degenerate. Consequently, if $\mathfrak{g}_0$ is semi-simple if and only if $\mathfrak{g} = \mathfrak{g}_0^{\mathbf{C}}$ is semi-simple by Cartan's criterion of semi-simplicity.

Let $\mathfrak{g}$ be a complex Lie algebra and $X, Y \in \mathfrak{g}$. Then, if $M = M_1 + iM_2$ is the matrix of $\operatorname{ad}(X)\operatorname{ad}(Y)$ in a $\mathbf{C}$-basis $(X_1, \ldots, X_m)$ of $\mathfrak{g}$, then the matrix of $\operatorname{ad}(X)\operatorname{ad}(Y)$, seen as an element of $\operatorname{End}_{\mathbf{R}}(\mathfrak{g})$, in the $\mathbf{R}$-basis $(X_1, \ldots, X_m, iX_1, \ldots, iX_m)$ of $\mathfrak{g}^{\mathbf{R}}$ is

$$\begin{pmatrix} M_1 & -M_2 \\ M_2 & M_1 \end{pmatrix}.$$

It follows that $B^{\mathbf{R}}(X,Y) = 2\operatorname{Tr}(M_1) = 2\operatorname{Re}\operatorname{Tr}(M) = 2\operatorname{Re} B(X,Y)$ as announced. Clearly, $\operatorname{Ker} B \subset \operatorname{Ker} B^{\mathbf{R}}$, and if $X \in \operatorname{Ker} B^{\mathbf{R}}$, then for all $Y \in \mathfrak{g}$, $0 = \operatorname{Re} B(X,Y) = \operatorname{Re} B(X, iY) = -\operatorname{Im} B(X,Y)$, so $X \in \operatorname{Ker} B$. □

If semi-simplicity is preserved, it does not work the same way with simplicity:

Example 5.1.4. If $\mathfrak{g}_0 = \mathfrak{so}(3,1)$, then $\mathfrak{g}_0$ is a simple real Lie algebra. However, its complexification is $\mathfrak{g} = \mathfrak{so}_4(\mathbf{C}) \simeq \mathfrak{so}_3(\mathbf{C}) \oplus \mathfrak{so}_3(\mathbf{C})$, which is semi-simple but not simple.

Nonetheless, realification preserves simplicity.

Proposition 5.1.10. *Let $\mathfrak{g}$ be a complex simple Lie algebra. Then, $\mathfrak{g}^{\mathbf{R}}$ is simple.*

Proof. Let $\mathfrak{h} \lhd \mathfrak{g}^{\mathbf{R}}$ be an ideal, *i.e.* a real vector subspace of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. We just have to prove that $i\mathfrak{h} = \mathfrak{h}$. To do so, consider $\mathfrak{a} = \mathfrak{h} \cap i\mathfrak{h}$. Then, $\mathfrak{a}$ is a (complex) ideal of $\mathfrak{g}$. Hence $\mathfrak{a} = \{0\}$ or $\mathfrak{a} = \mathfrak{g}$. The second case means that $\mathfrak{h} = \mathfrak{g}$, so let us assume $\mathfrak{h} \cap i\mathfrak{h} = \{0\}$. Then, for all $X, Y \in \mathfrak{h}$, we have $[X, iY] \in \mathfrak{h} \cap i\mathfrak{h}$ because $\mathfrak{h}$ is an ideal. Hence, $[\mathfrak{h}, \mathfrak{h}] = \{0\}$ proving that $\mathfrak{h}$ is an abelian ideal of $\mathfrak{g}^{\mathbf{R}}$, hence $\mathfrak{h} = \{0\}$ because $\mathfrak{g}^{\mathbf{R}}$ is semi-simple. □

Exercise 5.1.3. Show that $\mathfrak{sl}_2(\mathbf{C})^{\mathbf{R}} \simeq \mathfrak{so}(3,1)$.

Remark 5.1.5. In the light of a previous remark, we see that $(\mathfrak{sl}_2(\mathbf{C})^{\mathbf{R}})^{\mathbf{C}} \simeq \mathfrak{sl}_2(\mathbf{C}) \oplus \mathfrak{sl}_2(\mathbf{C})$. More generally, if $\mathfrak{g}$ is complex, simple, then $(\mathfrak{g}^{\mathbf{R}})^{\mathbf{C}} \simeq \mathfrak{g} \oplus \mathfrak{g}$.

Remark 5.1.6. If $\mathfrak{g}_0^{\mathbf{C}}$ is simple, then $\mathfrak{g}_0$ is simple. Indeed, if $\mathfrak{h}_0 \lhd \mathfrak{g}_0$ is an ideal, then $\mathfrak{h} := \mathfrak{h}_0 \oplus i\mathfrak{h}_0$ is an ideal of $\mathfrak{g}_0^{\mathbf{C}}$.

So, we have two distinct ways of producing *all* simple real Lie algebras from complex simple Lie algebras: the first is to realify it, the second is to consider its real forms. The first way produces those with a non-simple complexification, the second way produces those with a simple complexification.

Hence, all the work is to find a procedure that classifies the real forms of a complex simple Lie algebra. Let us emphasize that two real forms of a same Lie algebra may have radically different nature:

Exercise 5.1.4. Show that $\mathfrak{su}(2)$ and $\mathfrak{sl}_2(\mathbf{R})$ are real forms of $\mathfrak{sl}_2(\mathbf{C})$. Show that they are the only ones up to isomorphism. Consider their respective Killing form to deduce that they are not isomorphic.

Theorem 19. *Up to isomorphism, any simple, real, finite-dimensional Lie algebra* $\mathfrak{g}_0$ *is isomorphic to one of the following:*

1. $\mathfrak{g}^{\mathbf{R}}$*, for* $\mathfrak{g}$ *a complex, simple Lie algebra.*
2. *The compact form of a complex, simple Lie algebra.*
3. *A classical Lie algebra of matrices:*

$$\begin{array}{ll}
\mathfrak{su}(p,q), p \geqslant q \geqslant 1, p+q \geqslant 2, & \mathfrak{g}_0^{\mathbf{C}} = \mathfrak{sl}_{p+q}(\mathbf{C}) \\
\mathfrak{so}(p,q), p \geqslant q \geqslant 1, p+2 \geqslant 3, (p,q) \neq (2,2) & \mathfrak{g}_0^{\mathbf{C}} = \mathfrak{so}_{p+q}(\mathbf{C}) \\
\mathfrak{sp}(p,q), p \geqslant q \geqslant 1, p+q \geqslant 2 & \mathfrak{g}_0^{\mathbf{C}} = \mathfrak{sp}_{2(p+q)}(\mathbf{C}) \\
\mathfrak{sp}_{2n}(\mathbf{R}), n \geqslant 3 & \mathfrak{g}_0^{\mathbf{C}} = \mathfrak{sp}_{2n}(\mathbf{C}) \\
\mathfrak{so}^*(2n), n \geqslant 4 & \mathfrak{g}_0^{\mathbf{C}} = \mathfrak{so}_{2n}(\mathbf{C}) \\
\mathfrak{sl}_n(\mathbf{R}), n \geqslant 3 & \mathfrak{g}_0^{\mathbf{C}} = \mathfrak{sl}_n(\mathbf{C}) \\
\mathfrak{sl}_n(\mathbf{H}), n \geqslant 2 & \mathfrak{g}_0^{\mathbf{C}} = \mathfrak{sl}_{2n}(\mathbf{C})
\end{array}$$

4. *Twelve exceptional (non-complex, non-compact) simple Lie algebras (see [1], Ch. IV. 10, Figure 6.2, 6.3). We count: 4 real forms of* E_6*, 3 real forms of* E_7*, 2 real forms of* E_8*, 2 real forms of* F_4*, 1 real form of* G_2*.*

Chapter 6

Symmetric spaces of non-compact type

We fix $\mathbf{X} = (M, g)$ a symmetric space of non-compact type, $G = \mathrm{Iso}(\mathbf{X})_0$, an origin $o \in \mathbf{X}$, $K = \mathrm{Stab}_G(o)$, θ the Cartan involution at o and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan involution. We start with a proposition in fact valid for general symmetric spaces.

Proposition 6.0.1. *Complete, totally geodesic submanifolds of* $\mathbf{X}$ *passing through* o *are in* $1-1$ *correspondence with vector subspaces* $V \subset \mathfrak{p}$ *such that* $[V, [V, V]] \subset V$*, the correspondence being* $V \mapsto N := \exp_{x_0}(V)$.

Proof. This condition is clearly an obstruction, as for any $X, Y, Z \in V \simeq T_o N$, we must have $R_o(X, Y, Z) = -[[X, Y], Z] \in V$ since N is assumed totally geodesic. To see that it is sufficient, consider $\mathfrak{h} := V \oplus [V, V]$. Then, the condition gives that $\mathfrak{h} < \mathfrak{g}$ is a Lie subalgebra. If $H < G$ is the corresponding connected Lie subgroup, then it is easy to see that V coincides with the tangent space at o of the H-orbit $H.o$. If $x \in N$ and $v \in T_x N$, there is $h \in H$ and $v_o \in V = T_o N$ such that $h.o = x$ and $h_* v_o = v$. Because H acts isometrically it sends $\gamma_o(t) = \exp_o(t v_o)$ to $\gamma(t) = \exp_x(tv)$ and because $N = G.o$ and $\gamma_o \subset N$ we deduce $\gamma \subset N$. ☐

In particular, maximal, totally geodesic *flats* in $\mathbf{X}$ passing through the origin are parametrized by maximal abelian subspaces $\mathfrak{a} \subset \mathfrak{p}$. We take the following definition.

Definition 6.0.1. Given a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, we call **$\mathbf{R}$-split Cartan subalgebra** of $\mathfrak{g}$ any maximal abelian subspace of $\mathfrak{p}$.

We call **Cartan subalgebra** of $\mathfrak{g}$ an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that for all $X \in \mathfrak{h}$, $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is a semi-simple endomorphism, and such that $\mathfrak{h}$ is maximal with these properties.

Remark 6.0.1. An $\mathbf{R}$-split Cartan subalgebra of $\mathfrak{g}$ can be equivalently defined as an abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that for all $X \in \mathfrak{a}$, $\mathrm{ad}(X)$ is $\mathbf{R}$-diagonalizable, and such that $\mathfrak{a}$ is maximal with these properties.

Remark 6.0.2. Note that the second definition is coherent with the definition of Cartan subalgebras of a complex semi-simple Lie algebra.

Remark 6.0.3. Equivalently, $\mathfrak{h}$ is a Cartan subalgebra of a real semi-simple Lie algebra $\mathfrak{g}$ if and only if $\mathfrak{h} + i\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$.

See Section 7.1 for examples of **R**-split Cartan subalgebras.

Fact 6.0.1. *If $\mathfrak{a}$ is an* **R***-split Cartan subalgebra, then the subspace* $\{\mathrm{ad}(X),\ X \in \mathfrak{a}\} \subset \mathrm{End}(\mathfrak{g})$ *acts codiagonally on* $\mathfrak{g}$.

Proof. Define $B_\theta(X,Y) := -B(\theta(X),Y)$ for all $X,Y \in \mathfrak{g}$. It is a symmetric bilinear form on $\mathfrak{g}$ since θ is an involutive automorphism. According to **ref**, B_θ is positive definite on $\mathfrak{g}$, hence defines a scalar product. Now, for all $X \in \mathfrak{p}$, we have $\theta(X) = -X$ by definition, and then for all $X_1, X_2 \in \mathfrak{g}$

$$\begin{aligned} B_\theta(\mathrm{ad}(X)X_1, X_2) &= -B([\theta(X),\theta(X_1)], X_2) \\ &= -B(\theta(X_1), [X, X_2]) \\ &= B_\theta(X_1, \mathrm{ad}(X)X_2). \end{aligned}$$

So, for all $X \in \mathfrak{p}$, $\mathrm{ad}(X)$ is symmetric with respect to B_θ, and the fact follows from the fact that $\mathrm{ad}(\mathfrak{a})$ is abelian. □

Thus, similarly to complex roots of a complex semi-simple Lie algebra, we get a finite family of linear forms $\Delta \subset \mathfrak{a}^* \smallsetminus \{0\}$ such that, if for all $\lambda \in \mathfrak{a}^*$ we define

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a},\ [H,X] = \alpha(H)X\},$$

then $\forall \lambda \neq 0$, we have $\mathfrak{g}_\lambda \neq 0 \iff \lambda \in \Delta$ and a (vector space) decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

called the **restricted root-space decomposition**. Elements of Δ are called **restricted roots** and Δ the **restricted root-system** of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Note that $\mathfrak{a} \subset \mathfrak{g}_0$, but a priori the inclusion is strict. In fact, $\mathfrak{g}_0$ is θ-invariant, so $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{p}) \oplus (\mathfrak{g}_0 \cap \mathfrak{k})$. By maximality of $\mathfrak{a}$, we get $\mathfrak{a} = \mathfrak{g}_0 \cap \mathfrak{p}$. It is quite standard to denote by $\mathfrak{m}$ the compact part $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = \{X \in \mathfrak{k}\ :\ \forall H \in \mathfrak{a},\ [H,X] = 0\}$ of the centralizer of $\mathfrak{a}$, so that $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$. It is also straightforward to verify that for all $\alpha \in \Delta$, $-\alpha \in \Delta$ and $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$.

6.1 Regular tangent directions

Given an **R**-split Cartan subalgebra, the presence of restricted roots indicates some anisotropy inside any maximal flat: belonging or not to the kernel of a restricted root distinguish singular/regular directions. This idea of singularity/regularity of tangent vectors is in fact an *intrinsic* property of tangent directions, as the following shows.

Proposition 6.1.1. *Let $X \in \mathfrak{p}$. Then, the following assertions are equivalent.*

1. *$\mathfrak{z}(X) \cap \mathfrak{p}$ is abelian.*

2. *There exists an $\mathbf{R}$-split Cartan subalgebra $\mathfrak{a}$ such that $X \in \mathfrak{a}$ and for all restricted root α (relative to $\mathfrak{a}$), $\alpha(X) \neq 0$.*

3. *For all $\mathbf{R}$-split Cartan subalgebra $\mathfrak{a}$ containing X and for all restricted root α (relative to $\mathfrak{a}$), $\alpha(X) \neq 0$.*

4. *The geodesic $\gamma(t) = \exp_o(tX) \in \mathbf{X}$ is contained in a unique maximal flat.*

Remark 6.1.1. If these equivalent conditions are satisfied, then there exists a unique $\mathbf{R}$-split Cartan subalgebra containing X, namely $\mathfrak{a} = \mathfrak{z}(X) \cap \mathfrak{p}$.

Definition 6.1.1. An element satisfying these equivalent conditions is said to be **regular**.

Proof. In any event, any $\mathbf{R}$-split Cartan subalgebra containing X must be included in $\mathfrak{z}(X) \cap \mathfrak{p}$. If the latter is abelian, then by maximality there is exactly one $\mathbf{R}$-split Cartan subalgebra containing X, namely $\mathfrak{z}(X) \cap \mathfrak{p}$.

If (1) is true, then let $\mathfrak{a} = \mathfrak{z}(X) \cap \mathfrak{p}$ and let α be a restricted root relatively to $\mathfrak{a}$. Then, $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{a} = 0$ by definition. For all $Y_\alpha \in \mathfrak{g}_\alpha$, we have $Y_\alpha - \theta Y_\alpha \in \mathfrak{p}$, implying $[X, Y_\alpha - \theta Y_\alpha] = \alpha(X) Y_\alpha + \alpha(X) \theta Y_\alpha \neq 0$ for all $Y_\alpha \in \mathfrak{g}_\alpha \smallsetminus \{0\}$. Necessarily, $\alpha(X) \neq 0$ and we have proved (1) ⇒(2) and (1)⇒(3).

If (2) is true, we prove $\mathfrak{a} = \mathfrak{z}(X) \cap \mathfrak{p}$ showing that the latter is abelian. Let $Y \in \mathfrak{z}(X) \cap \mathfrak{p}$. We decompose it with respect to the restricted root-spaces relative to $\mathfrak{a}$: $Y = Y_\mathfrak{a} + Y_\mathfrak{m} + \sum_{\alpha \in \Delta} Y_\alpha$. By hypothesis,

$$0 = [X, Y] = 0 + 0 + \sum_{\alpha \in \Delta} \alpha(X) Y_\alpha.$$

Because all $\alpha(X)$ are non-zero, we obtain $Y \in \mathfrak{a} \oplus \mathfrak{m}$. And since $Y \in \mathfrak{p}$, we get $Y \in \mathfrak{a}$. Thus, $\mathfrak{z}(X) \cap \mathfrak{p} = \mathfrak{a}$ is abelian. This proves (2)⇒(1), and the three first assertions are equivalent. The fourth is a geometric translation of these properties. □

As a consequence, we obtain the following.

Proposition 6.1.2. *Any two $\mathbf{R}$-split Cartan subalgebras are conjugate by an element of K. Otherwise stated, for any fixed $\mathbf{R}$-split Cartan subalgebra $\mathfrak{a}$, we have*

$$\mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}.$$

Proof. Let $\mathfrak{a}, \mathfrak{a}'$ be two $\mathbf{R}$-split Cartan subalgebras of $\mathfrak{g}$ and let $X \in \mathfrak{a}$ and $X' \in \mathfrak{a}'$ be regular elements, so that $\mathfrak{a} = \mathfrak{z}(X) \cap \mathfrak{p}$ and $\mathfrak{a}' = \mathfrak{z}(X') \cap \mathfrak{p}$. Consider now the map

$$f : k \in K \mapsto B(\mathrm{Ad}(k)X, X').$$

By compactness, f reaches its maximum at some point $k_0 \in K$. Expressing that k_0 is a critical point of f, we derive:

$$\forall Y \in \mathfrak{k},\ B([Y, \mathrm{Ad}(k_0)X], X') = 0.$$

Therefore, by ad-skew-symmetry of B, we obtain $B([\mathrm{Ad}(k_0)X, X'], Y) = 0$ for all $Y \in \mathfrak{k}$, hence $[\mathrm{Ad}(k_0)X, X'] = 0$ because B is negative definite in restriction to $\mathfrak{k}$. Thus, $\mathrm{Ad}(k_0)X \in \mathfrak{z}(X') \cap \mathfrak{p} = \mathfrak{a}'$, and then $\mathfrak{a}' \subset \mathfrak{z}(\mathrm{Ad}(k_0)X) \cap \mathfrak{p} = \mathrm{Ad}(k_0)(\mathfrak{z}(X) \cap \mathfrak{p}) = \mathrm{Ad}(k_0)\mathfrak{a}$. A symmetric argument gives $\mathfrak{a} \subset \mathrm{Ad}(k_0^{-1})\mathfrak{a}'$, and we get the announced conjugacy. □

Remark 6.1.2. If $\mathfrak{a}' = \mathrm{Ad}(k)\mathfrak{a}$ is another $\mathbf{R}$-split Cartan subalgebra, then the roots of $\mathfrak{a}'$ are deduced from those of $\mathfrak{a}$ by precomposition by $\mathrm{Ad}(k)$. More pompously, ${}^t\mathrm{Ad}(k) : (\mathfrak{a}')^* \to \mathfrak{a}^*$ is an isometry sending restricted roots to restricted roots. Modulo such identifications, the restricted roots are defined independently of $\mathfrak{a}$.

The structure of the set of restricted roots is central in the geometric structure of the visual boundary $\partial_\infty \mathbf{X}$, so let's look a bit further to them.

Fact 6.1.1. *The space $\mathfrak{a}^*$ comes with a natural Euclidean structure, provided by the Killing form B of $\mathfrak{g}$.*

Indeed, since $\mathbf{X}$ is of non-compact type, $B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite. So, it induces a natural identification $\mathfrak{a} \simeq \mathfrak{a}^*$. Notably, if $H_\alpha \in \mathfrak{a}$ represents the linear form α, then for all $\alpha, \beta \in \mathfrak{a}^*$, $<\alpha, \beta>= \alpha(H_\beta) = \beta(H_\alpha)$ is the Euclidean scalar product we are talking about.

Fact 6.1.2. *The restricted-roots satisfy the following properties:*

1. *$0 \notin \Delta$ and Δ spans linearly $\mathfrak{a}^*$;*
2. *For all $\alpha \in \Delta$, the orthogonal symmetry s_α with respect to $\alpha^\perp$ preserves Δ;*
3. *For all $\alpha, \beta \in \Delta$, we have $2\frac{<\alpha,\beta>}{|\alpha|^2} \in \mathbf{Z}$.*

Note that s_α is given by $s_\alpha(v) = v - 2\frac{<\alpha,v>}{|\alpha|^2}\alpha$. So, for all $\alpha, \beta \in \Delta$, $s_\alpha(\beta)$ is another root γ of the form $\gamma = \beta + n\alpha$, with $n \in \mathbf{Z}$. Using Cauchy-Schwarz's inequality, we can see that if α is not colinear to β, this n belongs to $\{0, \pm1, \pm2, \pm3\}$.

Besides, if $\alpha \in \Delta$, then $-\alpha = s_\alpha(\alpha) \in \Delta$. However, if $t\alpha \in \Delta$, we can only deduce $t \in \{\pm1, \pm2, \pm\frac{1}{2}\}$. Indeed, by (3), $2t \in \mathbf{Z}$ and $\frac{2}{t} \in \mathbf{Z}$.

6.2 Abstract root-systems

Definition 6.2.1. Given a Euclidean vector space V of dimension n and a finite set $\Delta \subset V$, we say that Δ is an abstract **root system of rank** n if it satisfies properties (1) to (3) of Fact 6.1.2.

Remark 6.2.1. In the light of a previous remark, we say that two root systems (V, Δ) and (V', Δ') are isomorphic when there is a similarity $f : V \to V'$ sending Δ to Δ'.

Remark 6.2.2. The difference with *reduced* root systems is that we do **not** assume $t\alpha \Rightarrow t = \pm 1$.

Definition 6.2.2. A root system $\Delta \subset V$ is **irreducible** if there does not exist non-trivial orthogonal decomposition $V = V_1 \oplus V_2$ and disjoint union $\Delta = \Delta_1 \sqcup \Delta_2$ with $\Delta_1 \subset V_1$ and $\Delta_2 \subset V_2$.

Theorem 20. *Irreducible root systems are classified. Apart from reduced root-systems already classified in* ***ref****, only one infinite family of non-reduced root system appears:* $(BC)_n$*,* $n \geqslant 1$.

Hence, they are classified into five infinite families and five additional exceptions:

"Regular" infinite families	Exceptional root systems
$A_n,\ B_n,\ C_n,\ D_n,\ (BC)_n,\ n \geqslant 1$	$E_6,\ E_7,\ E_8,\ F_4,\ G_2$

As before, the index indicates the rank of the root-system, for instance E_7 is living in a 7-dimensional Euclidean space.

Going back to semi-simple Lie algebras without compact factor, we stress that contrarily to Theorem **ref** for the root-system of a complex simple Lie algebra, restricted root-system is **not** a classifying notion: several real simple Lie algebras may have the same restricted root-system. For instance, we will see that A_n is the restricted root-system of $\mathfrak{sl}_{n+1}(\mathbf{R})$, $\mathfrak{sl}_{n+1}(\mathbf{C})$ (seen as a real Lie algebra) and $\mathfrak{sl}_{n+1}(\mathbf{H})$, which are of course non-isomorphic (for dimensional reason for instance).

Explicit instances of reduced root-systems were given in **ref**. As for our new character $(BC)_n$, a root-system of type $(BC)_n$ is a root-system isomorphic to

$$\Delta = \{\pm e_i \pm e_j,\ i \neq j\} \cup \{\pm e_i, \pm 2e_i\} \subset V = \mathbf{R}^n.$$

6.3 Cartan's fixed point theorem and maximal compact subgroups

A beautiful consequence of the Lie theoretic interpretation is the application of a geometric result of non-positive curvature to obtain existence and uniqueness up to conjugacy of the maximal compact subgroup of a semi-simple Lie group without compact factor.

Theorem 21 (Cartan's fixed point theorem)**.** *Let* $\mathbf{X}$ *be a symmetric space of non-compact type. Let* $G = \mathrm{Iso}(\mathbf{X})_0$*,* $x_0 \in \mathbf{X}$.

If $H < G$ *is a subgroup such that the orbit closure* $\overline{H.x_0}$ *is compact, then there exists* $y \in \mathbf{X}$ *fixed by* H*, i.e. such that* $H.y = \{y\}$.

The proof only uses the fact that $\mathbf{X}$ is simply-connected of non-positive sectional curvature, $A := \overline{H.x}$ is compact and H-invariant and that H acts by isometry of $\mathbf{X}$ (this is the classic of setting of Cartan's fixed point theorem). The naive idea is to take for y a "barycenter" of A, notion which does not exist in this setting..

Proof. Let $A = \overline{H.x_0}$ and denote by d the length distance on $\mathbf{X}$. Define $r : \mathbf{X} \to \mathbf{R}_+$ by $r(x) = \sup_{y \in A} d(x,y)$. Then, r is continuous since $|r(x_1) - r(x_2)| \leqslant d(x_1, x_2)$. Since $r(x) \geqslant d(x, x_0)$, and closed balls are compact, there exists a compact subset $K \subset M$ such that $r \geqslant r(x_0)$ on $M \smallsetminus K$, hence r admits a global minimum. Now the key fact is:

Lemma 6.3.1. *This minimum is reached at a unique point.*

Proof. This is based on the so-called *law of cosines* valid in any complete, simply-connected Riemannian manifold of non-positive sectional curvature (Hadamard manifold). If $\{p,q,r\}$ is a triangle in $\mathbf{X}$, with length $a = d(p,q)$, $b = d(p,r)$ and $c = d(q,r)$, and if α, β, γ are the opposite corresponding angles, then

$$c^2 \geqslant a^2 + b^2 - 2ab\cos(\gamma),$$

and of course all permutations of the letters.

Now assume that r reaches its minimum at two distinct points $x_1, x_2 \in \mathbf{X}$. Let x^* be the middle of the geodesic segment $[x_1, x_2]$. Let $a \in A$ be such that $r(x^*) = d(x^*, a)$. Then, because their sum is π, either $\alpha_1 \geqslant \pi/2$ or $\alpha_2 \geqslant \pi/2$ where α_1 is the angle at x^* of $\{x^*, x_1, a\}$ and α_2 is the angle at x^* of $\{x^*, x_2, a\}$. Say $\alpha_1 \geqslant \pi/2$. Then, the law of cosines gives

$$d(x_1,a)^2 \geqslant d(x^*,a)^2 + d(x_1,x^*)^2$$

so $x_1 \neq x_2 \Rightarrow d(x_1,a) > d(x^*,a) = r(x^*)$, contradicting $r(x_1) \leqslant r(x^*)$. ☐

Let x^* be the unique point of $\mathbf{X}$ minimizing r. Because $A = \overline{H.x_0}$ is H-invariant, we get $r(h.x) = r(x)$ for all $h \in H$. Hence, $h.x^* = x^*$ for all $h \in H$, proving the theorem. ☐

Corollary 6.3.1. *Let now $x_0 \in \mathbf{X}$ be an origin in our symmetric space of non-compact type. Then, G_{x_0} is a maximal compact subgroup of G, and any maximal compact subgroup of G is a conjugate of G_{x_0}: the family of maximal compact subgroups is $\{G_x,\ x \in \mathbf{X}\}$.*

Remark 6.3.1. Remark that it's not clear whether or not a compact subgroup is contained in a maximal compact subgroup in general.

Proof. Let $K < G$ be a compact subgroup such that $G_{x_0} < K$. The previous theorem gives a point $x \in \mathbf{X}$ fixed by K. By homogeneity of $\mathbf{X}$, we have $g \in G$ such that $x = g.x_0$. Therefore, $K < G_x = gG_{x_0}g^{-1} < gKg^{-1}$. Although it is not true in general[1] that $gHg^{-1} \subset H$ for a given g implies that H is normalized by g, here we can use that K is a compact Lie subgroup of G. Let $f = i_{g^{-1}}$ be the inner automorphism associated to g^{-1}. By assumption $f(K) \subset K$, and then $\mathrm{d}_e f(\mathfrak{k}) \subset \mathfrak{k}$, and we get $\mathrm{d}_e f(\mathfrak{k}) = \mathfrak{k}$ because f is an automorphism of G. So $f(\exp(\mathfrak{k})) = \exp(\mathfrak{k})$ and then $f(K_0) = K_0$. Consequently, f induces an injective group homomorphism $\overline{f} : K/K_0 \to K/K_0$, which must be onto because K/K_0 is finite (discrete and compact). So, for every $k \in K$, there exists $k' \in K$

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[1] In $G = \mathrm{SL}_2(\mathbf{R})$, consider $H = U^+ \cap \mathrm{SL}_2(\mathbf{Z})$, where U^+ is the unipotent one-parameter subgroup of upper triangular matrices, and $g = \mathrm{diag}(2, \frac{1}{2})$. Then, $gHg^{-1} \subsetneq H$.

and $k_0 \in K_0$ such that $k = f(k')k_0$, and there exists $k_0' \in K_0$ such that $k_0 = f(k_0')$, so $k = f(k'k_0')$, proving that $f(K) = K$. Hence, g normalizes K as expected.

So, all inclusions in the chain $K < G_x = gG_{x_0}g^{-1} < gKg^{-1}$ are equalities, and in particular $G_{x_0} = K$ as expected.

Let now K be a maximal compact subgroup of G. Always by Cartan's fixed point theorem, there exists $x \in \mathbf{X}$ such that $K < G_x$. Since G_x is compact, $K = G_x$. Conversely, if $x = g.x_0 \in \mathbf{X}$, and $K \supset G_x$ is a compact subgroup, then $g^{-1}Kg$ is a compact subgroup containing G_{x_0}. Hence, $G_{x_0} = g^{-1}Kg$, and $K = gG_{x_0}g^{-1}$. □

6.4 Symmetric space associated to a semi-simple Lie group

Conversely, we have the following:

Proposition 6.4.1. *Let G be a center-free, real semi-simple Lie group without compact factor. Then, there exists a unique conjugacy class of maximal compact subgroups of G. If $K < G$ is maximal compact, then there exists a -Riemannian metric g on $M = G/K$ such that (M, g) is symmetric, of non-compact type and* $\mathrm{Iso}(M, g)_0 = G$.

If G is simple, then this metric is unique up to a positive constant. In the general case, it is unique up to a multi-homothetie constant on each irreducible factor.

This section is devoted to the proof of this proposition. The second part of the statement implies the first, so let us see how to produce such a Riemannian metric on G/K. We take the following

Definition 6.4.1. A Cartan involution of $\mathfrak{g}$ is a Lie algebra automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$ such that, if we denote by B_θ the bilinear form $B_\theta(X, Y) = -B(\theta X, Y)$, then B_θ is positive definite on $\mathfrak{g}$.

Cartan involutions appeared when G was exhibited as the isometry group of a symmetric space of non-compact type. Their existence is in fact a purely algebraic result.

Theorem 22. *If $\mathfrak{g}$ is a semi-simple Lie algebra, then it admits Cartan involutions. Moreover, Cartan involutions are pairwise conjugate.*

Proof. We just give the plan of the proof and refer to [1], Ch. VI, §2 for a complete, self-contained and relatively short proof.

The first step is to prove that any complex semi-simple Lie algebra $\mathfrak{g}$ admits a *compact* real form $\mathfrak{u}_0$ (*i.e.* a real-form with negative definite Killing form). Then, it is not very long to observe that the conjugation $\sigma : \mathfrak{g} \to \mathfrak{g}$ with respect to $\mathfrak{u}_0$ is a Cartan involution of $\mathfrak{g}^{\mathbf{R}}$.

This already gives a proof for complex semi-simple Lie algebras. If $\mathfrak{g}_0$ is now a general real semi-simple Lie algebra, then let $\mathfrak{u}_0$ be a compact form of $\mathfrak{g} = \mathfrak{g}_0^{\mathbf{C}}$. Let $\sigma \in \mathrm{Aut}(\mathfrak{g}^{\mathbf{R}})$ be the conjugation with respect to $\mathfrak{g}_0$ and let τ that with respect to $\mathfrak{u}_0$. Then, the final step is to prove that there exists an inner automorphism φ such that σ and $\tau' := \varphi\tau\varphi^{-1}$ commute. Then, τ' preserves $\mathfrak{g}_0$ which is the subspace of fixed points of σ and the restriction $\tau'|_{\mathfrak{g}_0}$ is a Cartan involution.

As for the uniqueness, the approach is similar to the last step. The same argument shows that if θ, θ' are two Cartan involutions, then there exists an inner automorphism φ such that θ' commutes to $\theta'' := \varphi\theta\varphi^{-1}$, which is still a Cartan involution. It is then easy to check that two commuting Cartan involutions are equal. □

Let now θ be a Cartan involution of $\mathfrak{g}$. Recall that G is connected and center-free. Note that it implies that G is isomorphic to $\tilde{G}/\mathcal{Z}$, where $\tilde{G}$ refers to its universal cover and $\mathcal{Z}$ to the center of $\tilde{G}$. Now, define $\tilde{\sigma} : \tilde{G} \to G$ the Lie group homomorphism defined by $\theta : \mathfrak{g} \to \mathfrak{g}$. Let $\gamma \in \mathcal{Z}$. Then, $\tilde{\sigma}$ being a local diffeomorphism, $\tilde{\sigma}(\tilde{G}) = G$ because it is an open subgroup of G, which is connected. Therefore, $\tilde{\sigma}(\gamma)$ is central in G, hence $\mathcal{Z} \subset \operatorname{Ker} \tilde{\sigma}$, so that σ factorizes into $\sigma : G \to G$ satisfying $\mathrm{d}_e\sigma = \theta$. Note that σ is an involution.

Definition 6.4.2. We say that σ is the Cartan involution of G associated to θ.

In order to build our symmetric space, we consider $K = G^\sigma = \{g \in \sigma \mid \sigma(g) = g\}$.

Lemma 6.4.1. *K is a compact Lie subgroup of G.*

Proof. K can be easily seen to be a closed (Lie) subgroup with Lie algebra $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta X = X\}$. Hence, by definition of θ, the Killing form B is negative definite in restriction to $\mathfrak{k}$. □

Define now $M = G/K$ and $x_0 = eK$. Define $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$ the other eigenspace of θ. If $k \in K$ and $X \in \mathfrak{p}$, we have $\sigma(e^{tX}) = e^{-tX}$, and then $ke^{-tX}k^{-1} = \sigma(ke^{tX}k^{-1})$. Taking derivative at $t = 0$, we obtain $-\operatorname{Ad}(k)X = \theta(\operatorname{Ad}(k)X)$, showing $\operatorname{Ad}(k)\mathfrak{p} = \mathfrak{p}$. The derivative of the canonical projection $G \to G/K$ yields a natural identification $\varphi : \mathfrak{p} \to T_{x_0}M$. As in **ref**, this map conjugates for all $k \in K$ the action of $\operatorname{Ad}(k)$ on $\mathfrak{p}$ to the action of $\mathrm{d}_{x_0}k$ on $T_{x_0}M$. By definition of θ, the Killing form B is positive definite in restriction to $\mathfrak{p}$, so it induces an $\operatorname{Ad}(K)$-invariant scalar product. Define now a Riemannian metric g on M by assigning to $x = gK$ and $u, v \in T_xM$, the well-defined $g_x(u, v) = B(\varphi^{-1}(\mathrm{d}L_{g^{-1}}u), \varphi^{-1}(\mathrm{d}L_{g^{-1}}v))$.

Fact 6.4.1. *The Riemannian manifold (M, g) is symmetric.*

Indeed, by homogeneity it is enough to verify it at $x_0 = eK$. By definition, $\sigma(K) = K$, so the involution $\sigma \in \operatorname{Aut}(G)$ induces a diffeomorphism $s : G/K \to G/K$. We claim that it is a geodesic involution at x_0.

Fact 6.4.2. *M is diffeomorphic to $\mathbf{R}^{\dim M}$, hence simply connected.*

We use the *polar decomposition* of G. If $\mathfrak{p}$ still denote the -1-eigenspace of θ, then $(k, X) \in K \times \mathfrak{p} \mapsto k \exp(X) \in G$ is a diffeomorphism. Hence, $M = G/K$ is diffeomorphic to $\mathfrak{p}$.

Fact 6.4.3. *(M, g) is symmetric of non-compact type.*

Indeed, by definition of θ, the Killing form B is positive definite in restriction to $\mathfrak{p}$.

Fact 6.4.4. $G = \mathrm{Iso}(M,g)_0$.

Proof. Let $G' = \mathrm{Iso}(M,g)_0$. Let $K' = G'_{x_0}$ and let $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ be the Cartan decomposition of $\mathfrak{g}'$ at x_0. Then, $\mathfrak{g} \subset \mathfrak{g}'$, $\mathfrak{k} \subset \mathfrak{k}'$ and $\theta = \theta'|_{\mathfrak{g}}$. Since $M = G/K = G'/K'$, we have $\mathfrak{p} = \mathfrak{p}'$. Let B, B' be the Killing forms of $\mathfrak{g}, \mathfrak{g}'$ respectively. Then, B' is negative definite in restriction to $\mathfrak{k}'$, positive definite in restriction to $\mathfrak{p}' = \mathfrak{p}$ and $\mathfrak{k}' \perp \mathfrak{p}'$. Because B' is still negative definite on $\mathfrak{k}$, we deduce that B' is non-degenerate in restriction to $\mathfrak{g}$. Let $\mathfrak{s} = \mathfrak{g}^{\perp}$ be the orthogonal in $\mathfrak{g}'$ with respect to B'. It is a negative definite subspace of $\mathfrak{g}'$, which is $\mathrm{Ad}(G)$-invariant. Since G is semi-simple without compact factor, $\mathrm{Ad}(.)|_{\mathfrak{s}} : G \to O(\mathfrak{s})$ must be trivial and it follows that $\mathfrak{g}$ centralizes $\mathfrak{s}$. In fact, $\mathfrak{s} = \mathfrak{z}(\mathfrak{g})$ is the centralizer because if $X = X_1 + X_2$ with $X_1 \in \mathfrak{g}$ and $X_2 \in \mathfrak{s}$ centralizes $\mathfrak{g}$, then for all $Y \in \mathfrak{g}$, $0 = [X_1, Y]$ and X_1 is central in $\mathfrak{g}$ so $X_1 = 0$ by semi-simplicity. Therefore, $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}'$ because for all $X_1, X_2 \in \mathfrak{s}$ and $Y \in \mathfrak{g}$, we have $[Y,[X_1,X_2]] = [[Y,X_1],X_2] + [X_1,[Y,X_2]] = 0$. Finally, $\mathfrak{s}$ is an ideal of $\mathfrak{g}'$ with negative-definite Killing form. Hence, if non-zero, it would be a semi-simple factor of compact type of $\mathfrak{g}'$, a contradiction. So, $\mathfrak{s} = 0$, $\mathfrak{g} = \mathfrak{g}'$ and $G = G'$ by connectedness. □

Consequently, K, defined as $K = G^{\sigma}$, is the isotropy of the simply-connected symmetric space (M,g) of non-compact type. Hence, it is a maximal compact subgroup of G and every maximal compact subgroup of G is conjugate of K.

6.5 A couple examples in rank one

We can now illustrate the interplay between the algebraic and geometric description of symmetric spaces. We start with the real and complex hyperbolic spaces which are Riemannian manifolds which are usually defined geometrically as the projectivization $P(Q<0)$ of the negative locus of a quadratic/hermitian form of signature $(1,n)$. We see in this section that how they can be recovered via Lie theory.

Real hyperbolic space Recall that the real hyperbolic space is by definition the unique (up to isometry) complete, simply-connected Riemannian manifold of sectional curvature constant equal to -1. There are several geometric definitions, for instance the following three manifolds are pairwise isometric and give models of the n-dimensional real hyperbolic space:

- The ball model, *i.e.* $(\mathbf{B}^n, 4\frac{\mathrm{d}x_1^2+\cdots+\mathrm{d}x_n^2}{(1-|x|^2)^2})$, where $\mathbf{B}^n = \{x \in \mathbf{R}^n \mid |x| < 1\}$.
- The upper half-space model, *i.e.* $(\mathbf{H}^n, \frac{\mathrm{d}x_1^2+\cdots+\mathrm{d}x_n^2}{x_n^2}$ where $\mathbf{H}^n = \{x = (x_1,\dots,x_n) \in \mathbf{R}^n \mid x_n > 0\}$.
- The hyperboloid model, *i.e.* in the Lorentzian Minkowski space $(\mathbf{R}^{n+1}, -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2)$, let $\mathcal{H} = \{x \in \mathbf{R}^{n+1} \mid -x_0^2 + x_1^2 + \cdots + x_n^2 = -1 \text{ and } x_0 > 0\}$ endowed with the restriction of the Lorentzian metric $-\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2$.

The advantage of the last characterization is that it exhibits the isometry group of $\mathbf{H}^n_{\mathbf{R}}$. It can be shown to be exactly $\mathrm{PO}(1,n)$, but using **ref**, we can directly see that it's identity component is $O_0(1,n)$. Let us algebraically that the symmetric space of $G := O_0(1,n)$ given by Proposition 6.4.1 is indeed isometric, up to a constant scaling, to $\mathbf{H}^n_{\mathbf{R}}$. First, we easily verify that G has trivial center (to show that $-I_{n+1} \notin G$, observe that G preserves $\mathcal{H}$ defined above by connectedness).

Let

$$I_{1,n} = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$$

so that $O(1,n) = \{g \in \mathrm{GL}_{n+1}(\mathbf{R}) \mid g I_{1,n} {}^t g = I_{1,n}\}$ and $\mathfrak{so}(1,n) = \{X \in \mathfrak{gl}_{n+1}(\mathbf{R}) \mid X I_{1,n} + I_{1,n} {}^t X = 0\}$. Explicitly,

$$\mathfrak{so}(1,n) = \left\{ \begin{pmatrix} 0 & {}^t u \\ u & A \end{pmatrix},\ u \in \mathbf{R}^n,\ A \in \mathfrak{so}(n) \right\}$$

According to **ref**, the Killing form is given by $B(X,Y) = 2(n+1)\operatorname{Tr} XY$. So it is immediate that $\theta(X) = -{}^t X$ is a Cartan involution of $\mathfrak{g}$. We then have

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix},\ A \in \mathfrak{so}(n) \right\} \text{ and } \mathfrak{p} = \left\{ \begin{pmatrix} 0 & {}^t u \\ u & 0 \end{pmatrix},\ u \in \mathbf{R}^n \right\}.$$

As for the Cartan involution at the Lie group level, it is given by $\sigma(g) = {}^t(g^{-1})$. Hence, $K := G^\sigma = O(1,n)_0 \cap O(n+1)$ which can be easily shown to be

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},\ A \in \mathrm{SO}(n) \right\} \simeq \mathrm{SO}(n).$$

What we need to do is to see that G/K, endowed with symmetric Riemannian metric given by the Killing form, has constant negative sectional curvature. To do so, we simply use **ref** to compute that $K_{x_0}(P) < 0$ is independent of the 2-plane $P \subset T_{x_0} M$. The result follows by homogeneity. Elementary computations give for $u, v \in \mathbf{R}^n$

$$\left[\begin{pmatrix} 0 & {}^t u \\ u & 0 \end{pmatrix}, \begin{pmatrix} 0 & {}^t v \\ v & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ 0 & u{}^t v - v{}^t u \end{pmatrix} \in \mathfrak{k},$$

and

$$B\left(\begin{pmatrix} 0 & {}^t u \\ u & 0 \end{pmatrix}, \begin{pmatrix} 0 & {}^t v \\ v & 0 \end{pmatrix} \right) = 4(n+1)\, {}^t u\, v = 4(n+1) \sum u_i v_i.$$

The expression of the bracket implies that two commuting elements $X, Y \in \mathfrak{p}$ are colinear (since $u{}^t v - v{}^t u = 0$ implies $|v|^2 u - \langle u, v\rangle v = 0$). Hence, $\mathbf{R}$-split Cartan subalgebras are 1-dimensional showing that $\mathfrak{g} = \mathfrak{so}(1,n)$ has real-rank 1.

Recall that we identify $T_{x_0}G/K \simeq \mathfrak{p}$ and that by construction, the metric at x_0 is $B|_{\mathfrak{p}\times\mathfrak{p}}$. So, if $u, v \in \mathbf{R}^n$ form an orthonormal pair for the standard scalar product $\sum u_i v_i$, and if

$$X = \begin{pmatrix} 0 & {}^t u \\ u & 0 \end{pmatrix} \in \mathfrak{p} \text{ and } Y = \begin{pmatrix} 0 & {}^t v \\ v & 0 \end{pmatrix} \in \mathfrak{p}$$

then

$$\begin{aligned} \frac{R_{x_0}(X,Y,Y,X)}{g_{x_0}(X,X)g_{x_0}(Y,Y)} &= \frac{B\left(\begin{pmatrix} 0 & 0 \\ 0 & u\,{}^t v - v\,{}^t u \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & u\,{}^t v - v\,{}^t u \end{pmatrix}\right)}{16(n+1)^2} \\ &= \frac{1}{8(n+1)} \operatorname{Tr}((u\,{}^t v - v\,{}^t u)^2) \\ &= -\frac{1}{8(n+1)} (\operatorname{Tr}(u\,{}^t v v\,{}^t u) + \operatorname{Tr}(v\,{}^t u u\,{}^t v)) \\ &= -\frac{1}{4(n+1)}. \end{aligned}$$

So, for this normalization, the symmetric space $O(1,n)_0/\operatorname{SO}(n)$ has constant sectional curvature $-\frac{1}{4(n+1)}$. It means that up to a scaling, it is isometric to the real hyperbolic space.

Complex hyperbolic space Let now $G = \operatorname{PU}(1,n) = U(1,n)/\mathcal{Z}(U(1,n))$. It is a connected, simple Lie group without center. It has Lie algebra

$$\begin{aligned} \mathfrak{su}(1,n) &= \{X \in \mathfrak{gl}_{n+1}(\mathbf{C}) \mid {}^t\overline{X} I_{1,n} + I_{1,n} X = 0 \text{ and } \operatorname{Tr} X = 0\} \\ &= \left\{ \begin{pmatrix} -\operatorname{Tr} A & {}^t\overline{u} \\ u & A \end{pmatrix},\ u \in \mathbf{C}^n,\ A \in \mathfrak{u}(n) \right\}. \end{aligned}$$

Then, $\theta(X) = -{}^t\overline{X}$ is a Cartan involution of $\mathfrak{su}(1,n)$, with Cartan decomposition $\mathfrak{su}(1,n) = \mathfrak{k} \oplus \mathfrak{p}$ given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} -\operatorname{Tr} A & 0 \\ 0 & A \end{pmatrix},\ A \in \mathfrak{u}(n) \right\} \text{ and } \mathfrak{p} = \left\{ \begin{pmatrix} 0 & {}^t\overline{u} \\ u & 0 \end{pmatrix},\ u \in \mathbf{C}^n \right\}.$$

The corresponding Cartan involution $\sigma : \operatorname{PU}(1,n) \to \operatorname{PU}(1,n)$ at the Lie group level is the involution induced by $\widehat{\sigma} : U(1,n) \to U(1,n)$ given by $\widehat{\sigma}(g) =^t \overline{g}^{-1}$. The corresponding compact subgroup $K = G^\sigma$ is the projection in $\operatorname{PU}(1,n)$ of

$$U(1,n) \cap U(n+1) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & A \end{pmatrix},\ |\lambda| = 1,\ A \in U(n) \right\}.$$

We endow $M = G/K$ with the symmetric Riemannian metric of non-compact type such that on $T_{x_0}M \simeq \mathfrak{p}$, the metric is given by $B_{\mathfrak{p}\times\mathfrak{p}}$ where B is the Killing form of $\mathfrak{g}$. Here,

$B(X,Y) = 4(n+1)\operatorname{Re}(\operatorname{Tr}(XY))$. If, for $u, v \in \mathbf{C}^n$

$$X = \begin{pmatrix} 0 & {}^t\overline{u} \\ u & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & {}^t\overline{v} \\ v & 0 \end{pmatrix}$$

then $B(X,Y) = 4(n+1)\operatorname{Re}({}^t\overline{u}v + \operatorname{Tr}(u{}^t\overline{v})) = 8(n+1)\operatorname{Re}({}^t\overline{u}v)$.

Hence, up to a constant, we get the standard Euclidean scalar product on $\mathbf{C}^n \simeq \mathbf{R}^{2n}$. Now,

$$[X,Y] = \begin{pmatrix} {}^t\overline{u}v - {}^t\overline{v}u & 0 \\ 0 & u{}^t\overline{v} - v{}^t\overline{u} \end{pmatrix},$$

so

$$\begin{aligned} \frac{1}{4(n+1)}B([X,Y],[X,Y]) &= -4(\operatorname{Im}{}^t\overline{u}v)^2 + {}^t\overline{v}u\operatorname{Tr}(u{}^t\overline{v}) + {}^t\overline{u}v\operatorname{Tr}(v{}^t\overline{u}) \\ &\quad - {}^t\overline{u}u\operatorname{Tr}(u{}^t\overline{u}) - {}^t\overline{v}v\operatorname{Tr}(v{}^t\overline{v}) \\ &= -4(\operatorname{Im}(\langle u,v\rangle))^2 + \langle u,v\rangle^2 + \langle v,u\rangle^2 - 2|u|^2|v|^2 \end{aligned}$$

where $\langle u,v\rangle = {}^t\overline{u}v$ stands for the standard Hermitian structure on $\mathbf{C}^n$ and $|u|^2 = \langle u,u\rangle$.

Consider now $u, v \in \mathbf{C}^n$ such that $|u| = |v| = 1$ and $\operatorname{Re}(\langle u,v\rangle) = 0$, so that $\langle u,v\rangle^2 + \langle v,u\rangle^2 = -2(\operatorname{Im}(\langle u,v\rangle))^2$. Then, if $X, Y \in \mathfrak{p}$ are the corresponding elements and if $P \in T_{x_0}M$ is the 2-plane they span, we have

$$\begin{aligned} K_{x_0}(P) &= \frac{B([X,Y],[X,Y])}{B(X,X)B(Y,Y)} \\ &= -\frac{1+3(\operatorname{Im}(\langle u,v\rangle))^2}{2(n+1)} \end{aligned}$$

This shows that the complex hyperbolic space has pinched sectional curvature: up to scale, its sectional curvature has range in $[-1, -\frac{1}{4}]$.

6.6 A couple examples in higher-rank

Symmetric space of $\mathrm{PSL}_n(\mathbf{R})$, $n \geqslant 3$

6.7 Weyl chambers and action of the isotropy

Definition 6.7.1. If $\mathfrak{a}$ is a $\mathbf{R}$-split Cartan subalgebra, then a **Weyl chamber** of $\mathfrak{a}$ is a connected component of the set of regular vectors contained in $\mathfrak{a}$. Equivalently, it is a connected component of $\mathfrak{a} \smallsetminus \cup_{\alpha\in\Delta} \operatorname{Ker}\alpha$.

Definition 6.7.2. If (V, Δ) is a root-system, we define its **Weyl group** $W = W(\Delta)$ as being the subgroup of $O(V)$ generated by the reflections s_α, $\alpha \in \Delta$.

It is immediate to see that W must be **finite**: by axioms, W permutes the roots, which form a finite set, and no non-trivial element of W can act trivially on Δ, since the latter generates V linearly.

Example 6.7.1. 1. In the case $\Delta = A_n$, as explicited above, Weyl chambers are connected components of the open-dense subset $U \subset V = \{x_1 + \cdots + x_{n+1} = 0\}$ formed of elements with pairwise distinct coordinates. Thus, a Weyl chamber is of the form

$$C_\sigma = \{(x_1, \dots, x_{n+1}) \ : \ \sum x_i = 0 \text{ and } x_{\sigma(1)} < \cdots < x_{\sigma(n+1)}\}$$

for some permutation $\sigma \in \mathfrak{S}_{n+1}$. The reflection defined by $e_i - e_j$ acts on V as the transposition of coordinated x_i and x_j, so the Weyl group identifies with $\mathfrak{S}_{n+1}$ via permutation of coordinates. Dessin.

2. In the case of B_n, the Weyl group action on $V = \mathbf{R}^n$ is generated by coordinate transpositions (reflections with respect to $e_i - e_j$) and the coordinate reflections $(x_1, \dots, x_i, \dots, x_n) \mapsto (x_1, \dots, -x_i, \dots, x_n)$. So, it is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n \rtimes \mathfrak{S}_n$, whose action is given by $(\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)})$, for all permutation σ and signs ε_i. A Weyl chamber is then of the form $\{x_{\sigma(1)} < \cdots < x_{\sigma(n)}\} \cap \{\varepsilon_1 x_1 > 0, \ \dots, \varepsilon_n x_n > 0\}$.

3. The Weyl group and Weyl chambers of C_n are the same as those of B_n.

4. In the case of D_n, the Weyl group is generated by coordinate transpositions and products of pairs of coordinates reflexions. So it is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{n-1} \rtimes \mathfrak{S}_n$, where the factor $(\mathbf{Z}/2\mathbf{Z})^{n-1}$ corresponds to n-tuples of signs $(\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_1 \dots \varepsilon_n = 1$.

Fact 6.7.1. *W permutes the Weyl chambers, and its action on the set of Weyl chambers is simply transitive.*

Remark 6.7.1. When Δ is irreducible, it can be proved that W is a finite Coxeter group, that is a finite group with presentation $W =< s_1, \dots, s_n \mid s_i^2, \ (s_i s_j)^{m_{ij}} >$, where the coefficients m_{ij} are integers in $\{2, 4, 6\}$.

Chapter 7

Annex

7.1 Some explicit instances of abstract root-systems as restricted root-systems of real semi-simple Lie algebras

- A_n, $n \geqslant 1$. When $n = 1$, it is the restricted root-system of $\mathfrak{sl}_2(\mathbf{R})$, but also for all $\mathfrak{so}(1,n)$, $n \geqslant 3$.

  Let's start with the simplest one. For $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{R})$ with standard presentation (H,E,F), $[H,E] = 2E$, $[H,F] = -2F$ and $[E,F] = H$, we can take $\mathfrak{a} = \mathbf{R}.H$, and then the restricted root-space decomposition with respect to $\mathfrak{a}$ is

  $$\mathfrak{sl}_2(\mathbf{R}) = \mathbf{R}.H \oplus \underbrace{0}_{\mathfrak{m}} \oplus \underbrace{\mathbf{R}.E}_{\mathfrak{g}_\alpha} \oplus \underbrace{\mathbf{R}.F}_{\mathfrak{g}_{-\alpha}},$$

  where $\alpha \in \mathfrak{a}^*$ is defined by $\alpha(H) = 2$. This proves that the restricted root-system of $\mathfrak{sl}_2(\mathbf{R})$ is $\{\pm\alpha\}$ which is indeed isomorphic to A_1. For $\mathfrak{so}(1,n)$, see below ($A_1 = B_1$).

  Now, let's consider $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{R})$, $n \geqslant 3$. With the standard Cartan involution $\theta(X) = -{}^tX$, the space of diagonal matrices $\mathfrak{a} = \{\mathrm{diag}(\lambda_1,\dots,\lambda_n),\ \sum\lambda_i = 0\}$ is an $\mathbf{R}$-split Cartan subalgebra, the roots are the

  $$\alpha_{ij} : X = \mathrm{diag}(\lambda_1,\dots,\lambda_n) \mapsto \lambda_i - \lambda_j,$$

  the restricted root-space of α_{ij} is $\mathbf{R}.E_{ij}$, the matrices with zeros everywhere except on the (i,j) entry. Here $\mathfrak{m} = 0$ again, and we see that the restricted root-system is indeed A_{n-1}.

  Let us consider the case of $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{C})$, seen as a real Lie algebra (we forget the complex structure). Then we can identify a maximal abelian Lie algebra whose adjoint action is semi-simple, that is diagonalizable over $\mathbf{C}$: the set of diagonal matrices

$$\left\{\begin{pmatrix}\lambda_1+i\mu_1 & & \\ & \ddots & \\ & & \lambda_n+i\mu_n\end{pmatrix},\ \sum\lambda_i=\sum\mu_i=0\right\}\subset\mathfrak{sl}(n,\mathbf{C}).$$

Those with real-entries will form a maximal abelian space acting $\mathbf{R}$-diagonally on $\mathfrak{g}$, hence an $\mathbf{R}$-split Cartan subalgebra. Alternatively, consider the Cartan decomposition with respect to the Cartan involution $\theta(X)=-{}^t\overline{X}$. Those with purely imaginary entries will form the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$:

$$\mathfrak{m}=\left\{\begin{pmatrix}i\mu_1 & & \\ & \ddots & \\ & & i\mu_n\end{pmatrix},\ \sum\mu_i=0\right\}.$$

It is direct that we obtain the same restricted-roots $\alpha_{ij}:\mathfrak{a}\to\mathbf{R}$ as in the case of $\mathfrak{sl}(n,\mathbf{R})$, and the corresponding root spaces are the (real-spaces) $\mathbf{C}.E_{ij}$.

It is then easy to generalize this to the case of $\mathfrak{sl}(n,\mathbf{H})$.

- B_n, $n\geqslant 2$. In $\mathbf{R}^{2n+1}$, let Q be the quadratic form $2x_1x_{2n+1}+2x_2x_{2n}+\cdots+2x_nx_{n+2}+x_{n+1}^2$ and let $\mathfrak{g}=\mathfrak{so}(Q)\simeq\mathfrak{so}(n,n+1)$. A maximal $\mathbf{R}$-split diagonalizable abelian subalgebra is conjugate to the space $\mathfrak{a}$ of matrices of the form

$$\begin{pmatrix}\lambda_1 &&&&&&&& \\ &\lambda_2 &&&&&&& \\ &&\ddots &&&&&& \\ &&&\lambda_n &&&&& \\ &&&&0 &&&& \\ &&&&&-\lambda_n &&& \\ &&&&&&\ddots && \\ &&&&&&&-\lambda_2 & \\ &&&&&&&&-\lambda_1\end{pmatrix}.$$

It is immediate to see that $\mathfrak{a}$ coincides with its centralizer in $\mathfrak{so}(Q)$, proving $\mathfrak{m}=0$. If M is any element of $\mathfrak{so}(Q)$, because Me_i is always orthogonal to e_i, the ascendant diagonal is formed of zeros. It is also easy to see that M is "skew-symmetric" with respect to this ascendant diagonal, and because the Cartan involution $\theta(M)=-{}^tM$ will send positive restricted root-spaces to their opposites, we just have to consider

$\mathrm{ad}(\mathfrak{a})$ in restriction to the upper quarter defined by the two diagonals:

$$\begin{pmatrix}
\lambda_1 & & & & & & & & 0 \\
& \lambda_2 & & & & & & 0 & \\
& & \ddots & & \mathrm{ad}(\mathfrak{a})? & & \iddots & & \\
& & & \lambda_n & & 0 & & & \\
& & & & 0 & & & & \\
& & & 0 & & -\lambda_n & & & \\
& & \iddots & & & & \ddots & & \\
& 0 & & & & & & -\lambda_2 & \\
0 & & & & & & & & -\lambda_1
\end{pmatrix}.$$

Basic computations gives that on the i-th row, inside the quarter, an element $H \in \mathfrak{a}$ acts with the following eigenvalues (in order of appearance from left to right):

$$\lambda_i - \lambda_{i+1},\ \lambda_i - \lambda_{i+2}, \dots, \lambda_i - \lambda_n, \boxed{\lambda_i}, \lambda_i + \lambda_n, \dots,\ \lambda_i + \lambda_{i+2},\ \lambda_i + \lambda_{i+1}.$$

The restricted roots λ_i correspond to the entry of the matrix at the middle of the segment, so the image that we can retain is

$$\begin{pmatrix}
\lambda_1 & & & & \ast & & & & 0 \\
& \lambda_2 & & & \ast & & & 0 & \\
& & \ddots & & \vdots & & \iddots & & \\
& & & \lambda_n & \ast & 0 & & & \\
\ast & \ast & \cdots & \ast & 0 & \ast & \cdots & \ast & \ast \\
& & & 0 & \ast & -\lambda_n & & & \\
& & \iddots & & \vdots & & \ddots & & \\
& 0 & & & \ast & & & -\lambda_2 & \\
0 & & & & \ast & & & & -\lambda_1
\end{pmatrix},$$

the stars being the locus of restricted root-spaces of the $\pm\lambda_i$'s. Thus, we have obtained that the restricted roots $\{\pm\lambda_i \pm \lambda_j\} \cup \{\pm\lambda_i\}$, which is B_n.

Now, let's look at non-split cases: if $m \geqslant n+1$, we consider the quadratic form on $\mathbf{R}^{n+m}$ given by: $Q = 2x_1x_{n+m} + 2x_2x_{n+m-1} + \cdots + 2x_nx_{m+1} + x_{n+1}^2 + \cdots + x_m^2$. Then, what we obtain is the same as before except that we have "enlarged the stars": the zeros become matrices $A \in \mathfrak{so}(m-n)$ and the stars vectors in $\mathbf{R}^{m-n}$, see below where $u_i, v_i \in \mathbf{R}^{m-n}$ are row vectors.

$$\begin{pmatrix}
\lambda_1 & & & & u_1 & & & & 0 \\
& \lambda_2 & & & u_2 & & & 0 & \\
& & \ddots & & \vdots & & \iddots & & \\
& & & \lambda_n & u_n & 0 & & & \\
-{}^t v_1 & -{}^t v_2 & \cdots & -{}^t v_n & A & -{}^t u_n & \cdots & -{}^t u_2 & -{}^t u_1 \\
& & & 0 & v_n & -\lambda_n & & & \\
& & \iddots & & \vdots & & \ddots & & \\
& 0 & & & v_2 & & & -\lambda_2 & \\
0 & & & & v_1 & & & & -\lambda_1
\end{pmatrix}.$$

The centralizer of $\mathfrak{a}$ is then clear: it is

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & A & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}, \ A \in \mathfrak{so}(m-n) \right\},$$

and the restricted root-spaces remain similar, except those associated to the $\pm\lambda_i$, which have now dimension $m - n \geqslant 1$. Thus the restricted root-system of $\mathfrak{so}(n,m)$ is also B_n.

- C_n, $n \geqslant 1$. On $\mathbf{R}^{2n} = \mathbf{R}^n \oplus \mathbf{R}^n$, we note $\omega((x,y),(x',y')) = \sum x_i y_i' - x_i' y_i$ the standard symplectic form. Then, a matrix $M \in \mathfrak{gl}(2n,\mathbf{R})$ belongs to $\mathfrak{sp}(n,\mathbf{R})$ - the Lie algebra of the subgroup of $\mathrm{GL}(2n,\mathbf{R})$ of elements preserving ω - if and only if it has the $(n+n)$-block form

$$M = \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}, \text{ with } B, C \text{ symmetric.}$$

Again, $\theta(X) = -{}^tX$ is a Cartan involution of $\mathfrak{sp}(n,\mathbf{R})$, and a natural $\mathbf{R}$-split Cartan subalgebra $\mathfrak{a}$ is given by diagonal matrices:

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -\lambda_n \end{pmatrix}$$

Then, in the block A, we obtain all the restricted-roots of type A_{n-1} (the computation is the same as in $\mathfrak{sl}(n,\mathbf{R})$), that is the $\lambda_i - \lambda_j$, $i \neq j$. In the block B, we obtain the roots $\lambda_i + \lambda_j$, $i \neq j$, outside the diagonal of B, and the roots $2\lambda_i$ on the diagonal. On the block C, we get the opposite (logic since θ sends $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$). The locus of the restricted root spaces of the $\pm 2\lambda_i$'s is represented with stars

$$\left(\begin{array}{ccc|ccc} \lambda_1 & & & * & & \\ & \ddots & & & \ddots & \\ & & \lambda_n & & & * \\ \hline * & & & -\lambda_1 & & \\ & \ddots & & & \ddots & \\ & & * & & & -\lambda_n \end{array}\right)$$

- D_n, $n \geqslant 1$. We just consider $\mathfrak{so}(n,n)$. The presentation is the same as $\mathfrak{so}(n,n+1)$, except that we erase the row and colon with zeros at the middle. The result is that the roots $\pm\lambda_i$ disappear, and we get D_n.

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