Dynamics hedging of CDO tranches in Markovian set-ups

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Presentation related to the papers:

- *Hedging default risks of CDOs in Markovian contagion models* (2008), to appear in Quantitative Finance, with Jean-Paul Laurent and Jean-David Fermanian
- *Hedging CDO tranches in a Markovian environment* (2009), book chapter with Monique Jeanblanc and Jean-Paul Laurent

Introduction

- In this presentation, we address the hedging issue of CDO tranches in a market model where pricing is connected to the cost of the hedge
- In credit risk market, models that connect pricing to the cost of the hedge have been studied quite lately
- Discrepancies with the interest rate or the equity derivative market
- Model to be presented is not new, require some stringent assumptions, but the hedging can be fully described in a dynamical way

Introduction

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Contents

Introduction

- Compared with previous presentations of this research
  - The theoretical framework about replication of loss derivatives is presented in more details
  - We provide a comparison analysis of hedging ratios computed in alternative models and using different methods
  - We propose a natural extension of the model where individual deltas can be discriminated by the level of CDS spreads
- The replication of CDO tranches has also been investigated in a similar framework by Bielecki, Jeanblanc and Rutkowski (2007), Frey and Backhaus (2007, 2008)

Default times

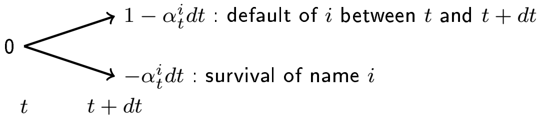
- n credit references
- $\tau_1,\ldots,\tau_n$: default times defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$
- $N_t^i = 1_{\{\tau_i \leq t\}}$, $i = 1, \ldots, n$: default indicator processes
- $\mathbb{H}^i = (\mathcal{H}_t^i)_{t \geq 0}$, $\mathcal{H}_t^i = \sigma(N_s^i,\ s \leq t)$, $i = 1, \ldots, n$: natural filtration of N^i
- $\mathbb{H} = \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$: global filtration of default times

Default times

- No simultaneous defaults : $\mathbb{P}(\tau_i = \tau_j) = 0, \forall i \neq j$
- Default times admit $\mathbb{H}$-adapted default intensities
  - For any $i = 1, \ldots, n$, there exists a non-negative $\mathbb{H}$-adapted process $\alpha^{i,\mathbb{P}}$ such that
  $$M_t^{i,\mathbb{P}} := N_t^i - \int_0^t \alpha_s^{i,\mathbb{P}} ds$$
  is a $(\mathbb{P}, \mathbb{H})$-martingale.
  - $\alpha_t^{i,\mathbb{P}} = 0$ on the set $\{t > \tau_i\}$
  - $M^{i,\mathbb{P}}$, $i = 1, \ldots, n$ will be referred to as the fundamental martingales

Market Assumption

- Instantaneous digital CDS are traded on the names $i = 1, \ldots, n$
- Instantaneous digital CDS on name i at time t is a stylized bilateral agreement
  - Offer credit protection on name i over the short period $[t, t + dt]$
  - Buyer of protection receives 1 monetary unit at default of name i
  - In exchange for a fee equal to $\alpha_t^i dt$



- Cash-flow at time $t + dt$ (buy protection position) : $dN_t^i - \alpha_t^i dt$
- $\alpha_t^i = 0$ on the set $\{t > \tau_i\}$ (Contrat is worthless)

Market Assumption

- Credit spreads are driven by defaults : $\alpha^1, \ldots, \alpha^n$ are $\mathbb{H}$-adapted processes
- Payoff of a self-financed strategy

$$V_0 e^{rT} + \sum_{i=1}^{n} \int_0^T \delta_s^i e^{r(T-s)} \underbrace{\left(dN_s^i - \alpha_s^i ds\right)}_{\text{CDS cash-flow}}.$$

- r : default-free interest rate
- V_0 : initial investment
- δ^i, $i = 1, \ldots, n$, $\mathbb{H}$-predictable process

Hedging and martingale representation theorem

Theorem (Predictable representation theorem)

Let $A \in \mathcal{H}_T$ be a $\mathbb{P}$-integrable random variable. Then, there exists $\mathbb{H}$-predictable processes $\theta^i, i = 1, \ldots, n$ such that

$$
\begin{aligned}
A &= \mathbb{E}_{\mathbb{P}}[A] + \sum_{i=1}^{n} \int_0^T \theta_s^i \left(dN_s^i - \alpha_s^{i,\mathbb{P}} ds \right) \\
&= \mathbb{E}_{\mathbb{P}}[A] + \sum_{i=1}^{n} \int_0^T \theta_s^i dM_s^{i,\mathbb{P}}
\end{aligned}
$$

and $\mathbb{E}_{\mathbb{P}} \left(\int_0^T |\theta_s^i| \alpha_s^{i,\mathbb{P}} ds \right) < \infty$.

Hedging and martingale representation theorem

Theorem (Predictable representation theorem)

Let $A \in \mathcal{H}_T$ *be a* $\mathbb{Q}$*-integrable random variable. Then, there exists* $\mathbb{H}$*-predictable processes* $\hat{\theta}^i, i = 1, \ldots, n$ *such that*

$$
\begin{aligned}
A &= \mathbb{E}_{\mathbb{Q}}[A] + \sum_{i=1}^{n} \int_0^T \hat{\theta}_s^i \underbrace{\left(dN_s^i - \alpha_s^i ds\right)}_{\text{CDS cash-flow}} \\
&= \mathbb{E}_{\mathbb{Q}}[A] + \sum_{i=1}^{n} \int_0^T \hat{\theta}_s^i dM_s^i
\end{aligned}
$$

and $\mathbb{E}_{\mathbb{Q}} \left(\int_0^T |\theta_s^i| \alpha_s^{i,\mathbb{P}} ds \right) < \infty.$

Hedging and martingale representation theorem

- **Building a change of probability measure**
- Describe what happens to default intensities when the original probability is changed to an equivalent one
- From the PRT, any Radon-Nikodym density ζ (strictly positive $(\mathbb{P}, \mathbb{H})$-martingale with expectation equal to 1) can be written as

$$d\zeta_t = \zeta_{t-} \sum_{i=1}^{n} \pi_t^i dM_t^{i,\mathbb{P}}, \ \zeta_0 = 1$$

where π^i, $i = 1, \ldots, n$ are $\mathbb{H}$-predictable processes

Hedging and martingale representation theorem

- Conversely, the (unique) solution of the latter SDE is a local martingale (Doléans-Dade exponential)

$$\zeta_t = \exp\left(-\sum_{i=1}^{n}\int_0^t \pi_s^i \alpha_s^{i,\mathbb{P}} ds\right)\prod_{i=1}^{n}(1+\pi_{\tau_i}^i)^{N_t^i}$$

- The process ζ is non-negative if $\pi^i > -1$, for $i = 1, \ldots, n$
- The process ζ is a true martingale if $\mathbb{E}_{\mathbb{P}}[\zeta_t] = 1$ for any t or if π^i is bounded, for $i = 1, \ldots, n$

Hedging and martingale representation theorem

Theorem (Change of probability measure)

Define the probability measure $\mathbb{Q}$ *as*

$$d\mathbb{Q}|_{\mathcal{H}_t} = \zeta_t d\mathbb{P}|_{\mathcal{H}_t} .$$

where

$$\zeta_t = \exp\left(-\sum_{i=1}^{n} \int_0^t \pi_s^i \alpha_s^{i,\mathbb{P}} ds \right) \prod_{i=1}^{n} (1 + \pi_{\tau_i}^i)^{N_t^i}$$

Then, for any $i = 1, \ldots, n$, *the process*

$$M_t^i := M_t^{i,\mathbb{P}} - \int_0^t \pi_s^i \alpha_s^{i,\mathbb{P}} ds = N_t^i - \int_0^t (1 + \pi_s^i) \alpha_s^{i,\mathbb{P}} ds$$

is a $\mathbb{Q}$*-martingale. In particular, the* $(\mathbb{Q}, \mathbb{H})$*-intensity of* τ_i *is* $(1 + \pi_t^i)\alpha_t^{i,\mathbb{P}}$.

Hedging and martingale representation theorem

- From the absence of arbitrage opportunity

$$\{\alpha_t^i > 0\} \stackrel{\mathbb{P}-a.s.}{=} \left\{\alpha_t^{i,\mathbb{P}} > 0\right\}$$

- For any $i = 1, \ldots, n$, the process $\hat{\pi}^i$ defined by :

$$\hat{\pi}_t^i = \left(\frac{\alpha_t^i}{\alpha_t^{i,\mathbb{P}}} - 1\right)\left(1 - N_{t-}^i\right)$$

  is an $\mathbb{H}$-predictable process such that $\hat{\pi}^i > -1$
- The process ζ defined with $\pi^1 = \hat{\pi}^1, \ldots, \pi^n = \hat{\pi}^n$ is an admissible Radon-Nikodym density
- Under $\mathbb{Q}$, credit spreads $\alpha^1, \ldots, \alpha^n$ are exactly the intensities of default times

Hedging and martingale representation theorem

- The predictable representation theorem also holds under $\mathbb{Q}$
- In particular, if A is an $\mathcal{H}_T$ measurable payoff, then there exists $\mathbb{H}$-predictable processes $\hat{\theta}^i, i = 1, \dots, n$ such that

$$A = \mathbb{E}_{\mathbb{Q}}\left[A \mid \mathcal{H}_t\right] + \sum_{i=1}^{n} \int_t^T \hat{\theta}^i_s dM^i_s.$$

- Starting from t the claim A can be replicated using the self-financed strategy with
  - the initial investment $V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} A \mid \mathcal{H}_t\right]$ in the savings account
  - the holding of $\delta^i_s = \hat{\theta}^i_s e^{-r(T-s)}$ for $t \leq s \leq T$ and $i = 1, \dots, n$ in the instantaneous CDS
- As there is no charge to enter a CDS, the replication price of A at time t is $V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} A \mid \mathcal{H}_t\right]$

Hedging and martingale representation theorem

- A depends on the default indicators of the names up to time T
  - includes the cash-flows of CDO tranches or basket credit default swaps, given deterministic recovery rates
- In principle, the dynamics of a traditional CDS can also be described in terms of the dynamics of instantaneous CDS
- Can be used to replicate a CDO tranche with traditional CDS
  - Involve the inversion of a linear system

Hedging and martingale representation theorem

- Risk-neutral measure can be explicitly constructed
  - We exhibit a continuous change of probability measure
- Predictable representation theorem implies completeness of the credit market
  - Perfect replication of claims which depend only upon the default history with CDS on underlying names and default-free asset
  - Provide the replication price at time t
- But does not provide any practical way of constructing hedging strategies
- Need of a Markovian assumption to effectively compute hedging strategies

Markovian contagion model

- Pre-default intensities only depend on the **current status of defaults**

$$\alpha_t^i = \tilde{\alpha}^i \left(t, N_t^1, \ldots, N_t^n\right) 1_{t<\tau_i}, \; i = 1, \ldots, n$$

- Ex : Herbertsson - Rootzén (2006)

$$\tilde{\alpha}^i \left(t, N_t^1, \ldots, N_t^n\right) = a_i + \sum_{j \neq i} b_{i,j} N_t^j$$

- Ex : Lopatin (2008)

$$\tilde{\alpha}^i \left(t, N_t\right) = a_i(t) + b_i(t) f(t, N_t)$$

- Connection with continuous-time Markov chains
    - $\left(N_t^1, \ldots, N_t^n\right)$ Markov chain with possibly 2^n states
    - Default times follow a multivariate phase-type distribution

Homogeneous Markovian contagion model

- Pre-default intensities only depend on the current number of defaults
- All names have the same pre-default intensities

$$\alpha_t^i = \tilde{\alpha}(t, N_t) 1_{t<\tau_i}, \; i = 1, \ldots, n$$

where

$$N_t = \sum_{i=1}^{n} N_t^i$$

- The model is also referred to as the **local intensity model**

Homogeneous Markovian contagion model

- No simultaneous default, the intensity of N_t is equal to

$$\lambda(t, N_t) = (n - N_t)\tilde{\alpha}(t, N_t)$$

- N_t is a continuous-time Markov chain (pure birth process) with generator matrix :

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & & 0 \\ & & \ddots & \ddots & \\ 0 & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Model involves as many parameters as the number of names

Homogeneous Markovian contagion model

- **Replication price of a European type payoff**

$$V(t,k) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\Phi(N_T) \mid N_t = k\right]$$

- $V(t,k),\ k = 0,\dots,n-1$ solve the backward Kolmogorov differential equations :

$$\frac{\delta V(t,k)}{\delta t} = rV(t,k) - \lambda(t,k)\left(V(t,k+1) - V(t,k)\right)$$

- Approach also puts in practice by van der Voort (2006), Schönbucher (2006), Herbersson (2007), Arnsdorf and Halperin (2007), Lopatin and Misirpashaev (2007), Cont and Minca (2008), Cont and Kan (2008), Cont, Deguest and Kan (2009)

Homogeneous Markovian contagion model

- **Computation of credit deltas...**
- $V(t, N_t)$, price of a CDO tranche (European type payoff)
- $V^I(t, N_t)$, price of the CDS index (European type payoff)

$$V(t, N_t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\Phi(N_T) \mid N_t\right]$$

$$V^I(t, N_t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\Phi^I(N_T) \mid N_t\right]$$

- Using standard Itô's calculus

$$dV(t, N_t) = \left(V(t, N_t) - \delta^I(t, N_t)V^I(t, N_t)\right) rdt + \delta^I(t, N_t)dV^I(t, N_t)$$

where

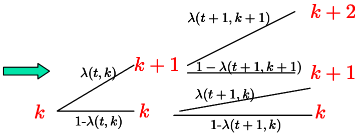
$$\delta^I(t, N_t) = \frac{V(t, N_t + 1) - V(t, N_t)}{V^I(t, N_t + 1) - V^I(t, N_t)}.$$

- Perfect replication with the index and the risk-free asset

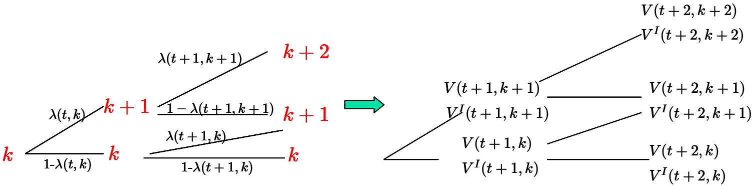
Pricing and hedging in a binomial tree

- Binomial tree : discrete version of the homogeneous contagion model

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & & 0 \\ & & \ddots & \ddots & \\ 0 & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



- Calibration of loss intensities $\lambda(t,k)$ on a loss surface by forward induction



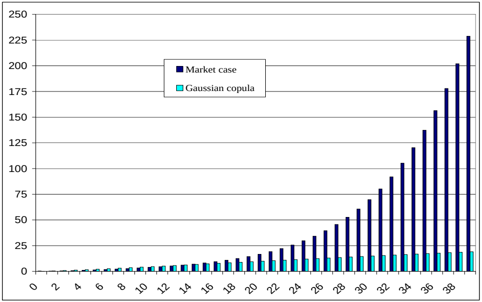
- CDO tranches and index price computed by backward induction

Empirical results

- Calibration of loss intensities : Number of names : 125, risk-free interest rate : $r = 3\%$, recovery rate : $R = 40\%$, 5Y credit spreads : 20bps

[0-3%]	[0-6%]	[0-9%]	[0-12%]	[0-22%]
18%	28%	36%	42%	58%

- Time-homogeneous intensities : $\lambda(t,k) = \lambda(k)$, $k = 0, \ldots, 125$



- Comparison with loss intensities calibrated on a flat correlation structure

Empirical results

- Dynamics of CDS index spreads in the Markov chain

Nb Defaults	Weeks			
	0	**14**	**56**	**84**
0	20	19	17	16
1	0	31	23	20
2	0	95	57	43
3	0	269	150	98
4	0	592	361	228
5	0	1022	723	490
6	0	1466	1193	905
7	0	1870	1680	1420
8	0	2243	2126	2423
9	0	2623	2534	2423
10	0	3035	2939	2859

- Explosive behavior associated with upward base correlation curve

Empirical results

- Dynamics of credit deltas equity tranche $[0, 3\%]$

Nb Defaults	Outstanding Nominal	Weeks			
		0	14	56	84
0	3.00%	0.541	0.617	0.823	0.910
1	2.52%	0	0.279	0.510	0.690
2	2.04%	0	0.072	0.166	0.304
3	1.56%	0	0.016	0.034	0.072
4	1.08%	0	0.004	0.006	0.012
5	0.60%	0	0.002	0.002	0.002
6	0.12%	0	0.001	0.000	0.000
7	0.00%	0	0	0	0

- Deltas are between 0 and 1
- Gradually decrease with the number of defaults (concave payoff)
- Increase with time (consistent with a decrease of time value)

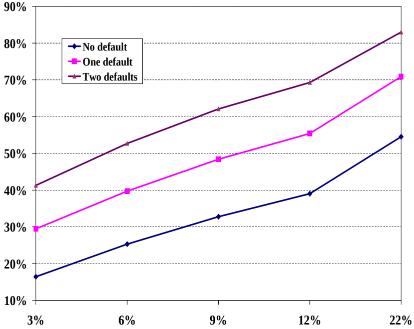
Empirical results

- **Market and theoretical deltas at inception**
- Market deltas computed under the Gaussian copula model
  - Uniform bump of index spreads
  - Market delta = Change in PV of the tranche/ Change in PV of the CDS index
  - Base correlation is unchanged when shifting spreads
  - Standard way of computing CDS index hedges in trading desks

Tranches	[0-3%]	[3-6%]	[3-9%]	[9-12%]	[12-22%]
Market deltas	27	4.5	1.25	0.6	0.25
Model deltas	21.5	4.63	1.63	0.9	0.6

Empirical results

- **Smaller equity tranche deltas in the Markov chain model**
  - How would we explain this ?
- **Contagion effect** : default is associated with a dynamic increase in dependence



- Increasing correlation leads to a decrease in the PV of the equity tranche

Empirical results

- Comparison with results provided by Arnsdorf and Halperin (2007) : *BSLP : Markovian bivariate spread-loss model for portfolio credit derivatives*

Tranches	[0-3%]	[3-6%]	[3-9%]	[9-12%]	[12-22%]
Market deltas	26.5	4.5	1.25	0.65	0.25
BSLP model deltas	21.9	4.81	1.64	0.79	0.38

- Computed in March 2007 on the iTraxx tranche
- Two dimensional Markov chain, shift in credit spreads, deltas not related to replication strategies

Tranches	[0-3%]	[3-6%]	[3-9%]	[9-12%]	[12-22%]
Market deltas	27	4.5	1.25	0.6	0.25
Model deltas	21.5	4.63	1.63	0.9	0.6

- Note that our results are quite similar
- Equity tranche deltas are smaller in contagion models

Empirical results

- Consistent with results provided by Frey and Backhaus (2007) : *Dynamic hedging of synthetic CDO tranches with spread risk and default contagion*

Tranche	[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
Spread	26 %	84 bp	24 bp	14 bp	11 bp
Tranche Correlation	17.30 %	3.22 %	9.93 %	15.81 %	27.46 %
Gauss Cop. Δ	0.61	0.23	0.06	0.03	0.07

- VOD : Value-on-default

	VOD in the Markov model	VOD in the Copula model
$[0,3]$	0.344	1.002
$[3,6]$	0.138	0.171
$[6,9]$	0.058	0.023
$[9,12]$	0.039	0.008
$[12,22]$	0.107	0.010

- Much smaller delta in the contagion model than in the Gaussian copula model

Empirical results

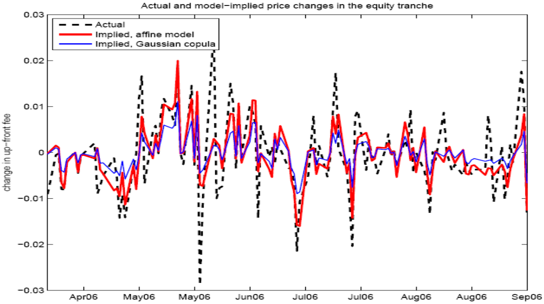
- Comparison with results provided by Eckner (2007)
- Deltas computed in a Duffie and Garleanu (2001) reduced-form model
  - Model calibrated on December 2005 CDX data
  - Spread sensitivity deltas

Tranches	**[0-3%]**	**[3-7%]**	**[7-10%]**	**[10-15%]**	**[15-30%]**
AJD model deltas	21.7	6.0	1.1	0.4	0.1
Market deltas	18.5	5.5	1.5	0.8	0.4
Contagion model deltas	17.9	6.3	2.5	1.3	0.8

- Deltas go in opposite direction when comparing with the contagion model

Empirical results

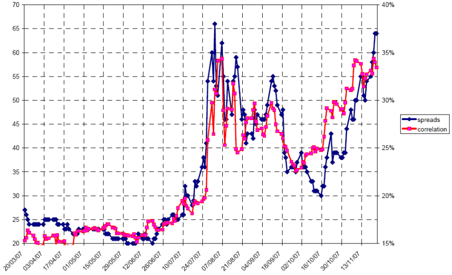
- Consistent with Feldhütter (2008) empirical study of the affine intensity model
- Comparison of hedging performance with the Gaussian copula model
- Back-test study : Use information at time $t+1$ to compute hedge ratios at time t
- Higher deltas for the equity tranche in the affine model compared with the 1F Gaussian copula



- Prediction of (equity tranche) MTM change are better in the intensity model
- But results are pre-crisis...

Empirical results

- The recent crisis is associated with joint upward shifts in credit spreads
- And an increase in base correlations
- Dependence parameters and credit spreads may be highly correlated



- Should go in favour of the contagion model but...

Empirical results

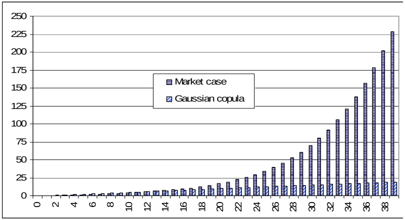
- Cont and Kan (2008) perform a similar study for various hedging strategies
  - Comparison of spread-sensitivity deltas and jump-to-default deltas
  - Computed using several market models calibrated to the same data set
  - Back-test the strategies before and during the crisis
- Spread-deltas are very similar across models (5Y Europe iTraxx on 20 September 2006)

- Gaussian copula model
- Local intensity (contagion model)
- BSLP (Arnsdorf and Halperin (2007))
- GPL : generalized Poisson loss model (Brigo et al. (2006))

Tranche	Gauss	Local	BSLP	GPL
0 - 3	24.48	24.52	24.79	24.48
3 - 6	5.54	5.45	5.30	5.54
6 - 9	1.79	1.80	1.80	1.79
9 - 12	0.87	0.85	0.88	0.87
12 - 22	0.35	0.35	0.32	0.35
22 - 100	0.08	0.08	0.09	0.08

Empirical results

- Cont and Kan (2008) show rather poor performance of the contagion model even during the crisis period
- However, hedging performance may significantly depend on the calibration method
- Identification problem : several specification of loss intensities may be compatible with the same set of market data



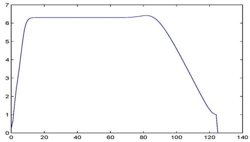
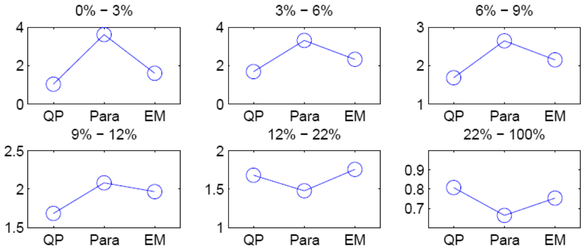


Figure 2: Dependence of default intensity on number of defaults for $t = 1$ year: ITRAXX Europe Series 6, March 15 2007..

- Left : Laurent *et al.* (2007), Right : Cont and Minca (2008)

Empirical results

- And computed deltas are rather sensitive to the calibration of contagion parameters on quoted CDO tranches
- Cont, Deguest and Kan (2009) : Computation of jump-to-default deltas using different calibration methods (5Y Europe iTraxx on 25 March 2008)



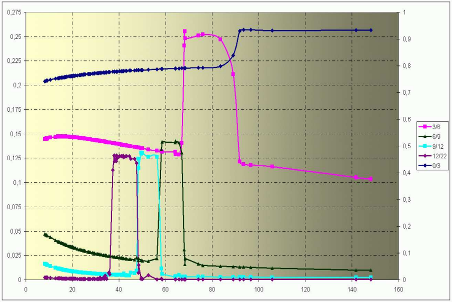
- QP : Quadratic programming method
- Para : Parametric method (piecewise constant default intensities proposed by Herbertsson (2007))
- EM : Entropy minimization method (Cont and Minca (2008))

Empirical results

- Limited hedging performance of the contagion model may be related to absence of specific spread risk
- But incorporating additional risks will create incompleteness
- Introducing some Brownian risks on top of jump-to-default risks brings unclear practical issues
    - It is not clear how defaults would drive the volatility of credit spreads
    - Regarding the hedging issue, one can think of using CDS with two different maturities for each name to cope both with default and credit spread risks
    - or using local risk minimization techniques as in Frey and Backhaus (2008)

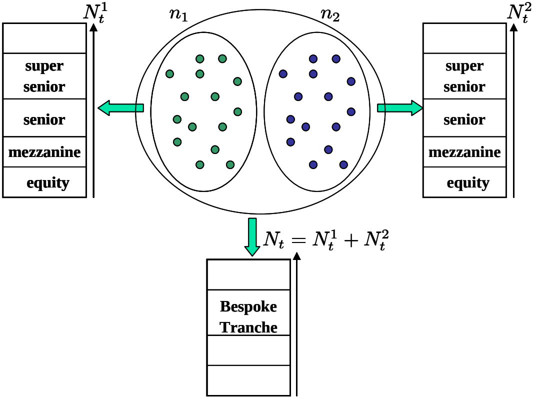
Empirical results

- Hedging with individual CDS may perform a better hedge (than hedging with the index)
    - Heterogeneous portfolio where some individual spreads are suddenly widening
    - Equity tranche very sensitive to idiosyncratic risk
- Obviously, beyond the scope of a pure top model
- Individual spread-ratios may be very different across names when computed in a bottom-up approach



Empirical results

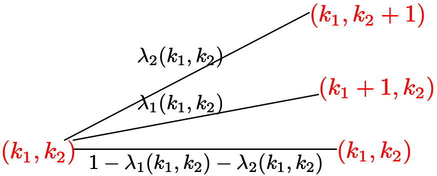
- **One natural extension of the Markovian contagion model...**
- CDO Tranches on a portfolio composed with two disjoint sub-groups



- $n_1 + n_2 = n$, $N_t = N_t^1 + N_t^2$

Empirical results

- (N^1, N^2) is a bivariate Markov chain, simultaneous defaults are precluded
  - **Markovian contagion model**



- Dynamics of sub-index loss processes can be described in a trinomial tree
- As in the previous approaches, replication is theoretically feasible
- Loss intensities $\lambda_1(k_1, k_2)$ and $\lambda_2(k_1, k_2)$:

$$\begin{cases} \lambda_1(N_t^1, N_t^2) = (n_1 - N_t^1)\alpha_1(N_t^1, N_t^2) \\ \lambda_2(N_t^1, N_t^2) = (n_2 - N_t^2)\alpha_2(N_t^1, N_t^2) \end{cases}$$

  - α_1 pre-default individual intensity of names in sub-portfolio 1
  - α_2 pre-default individual intensity of names in sub-portfolio 2