Generalized Bollobás-Riordan polynomials and partial duality

Fabien Vignes-Tourneret

Universität Wien

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Outline of the talk

Introduction

Topological graphs

A topological graph (or ribbon graph, or fatgraph) is a surface (not necessarily orientable) with boundary represented by the union of two sets of topological discs called vertices and edges.

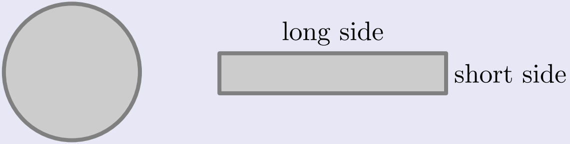
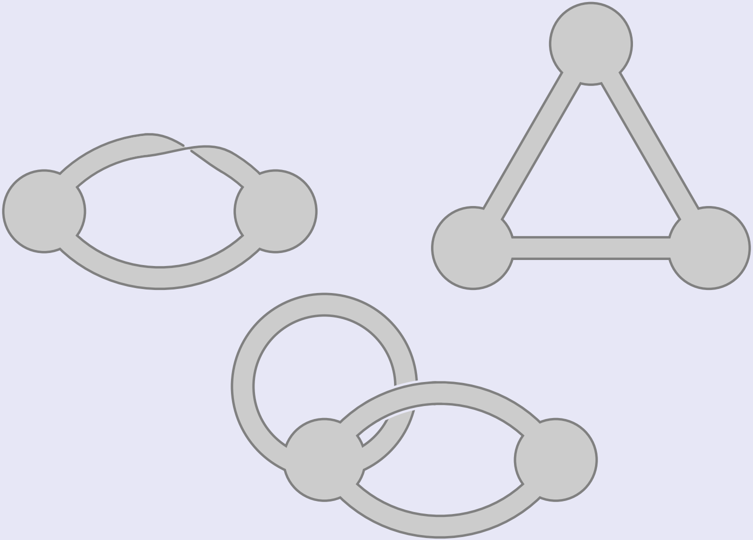


Figure: Vertex and edge of topological graphs

These two sets obey the following rules:

- vertices and edges intersect along line segments,
- these segments lie on the boundary of exactly one vertex and one edge,
- each edge contains exactly two such line segments.

Examples of topological graphs



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- A graph is signed if an element of $\{1, -1\}$ is attached to each edge. One defines $\varepsilon_G : E(G) \to \{1, -1\}$.

From Tutte to Bollobás and Riordan

In 1947, William Tutte defined a polynomial invariant of graphs. One of its definitions is:

$$T(G;x,y) = \sum_{F\subset G}(x-1)^{r(G)-r(F)}(y-1)^{n(F)}$$

$T($$;x,y) = (x-1)+2+(y-1)$

$= x + y$

In 2000, B. Bollobás and O. Riordan defined a new invariant of topological graphs:

$$R(G;x+1,y,z) = \sum_{F\subset G} x^{r(G)-r(F)} y^{n(F)} z^{k(F)-f(F)+n(F)}$$

<u>Properties</u>: $R(G;x+1,y,1) = T(G;x+1,y+1)$.
If $g(G)=0$, $R(G)$ is independent of z.

Motivations

Why a multivariate polynomial?

1. **Quantum field theory** : certain graph invariants appear in the parametric representation.
   - Symanzik = Tutte :

$$U(G;\boldsymbol{\alpha}) = \Big(\prod_{e\in E(G)} \alpha_e\Big) \lim_{\lambda\to 0} \lambda^{1-v(G)} Z_T(G;0,\frac{\lambda}{\boldsymbol{\alpha}}),$$

   - "Non-commutative" Symanzik = Bollobas-Riordan :

$$U^\star(G;\boldsymbol{\alpha},\theta) = \Big(\frac{\theta}{2}\Big)^{e(G)-v(G)+1} \Big(\prod_{e\in E(G)} \alpha_e\Big) \lim_{c\to 0} c^{-1} Z(G;1,\frac{\theta}{2\boldsymbol{\alpha}},c).$$

➡ Algebraic methods (Grassmann-Berezin) for combinatorics and graph theory.

Motivations

Why studying dualities?

2. **The role of duality**
   - in quantum field theory:

$$U^\star(G;\boldsymbol{\alpha},\theta) = \left(\frac{\theta}{2}\right)^{e(G)-v(G)+1} \sum_{\text{quasi-trees } \mathcal{T}^\star} \prod_{e \notin \mathcal{T}^\star} \frac{2\alpha_e}{\theta},$$

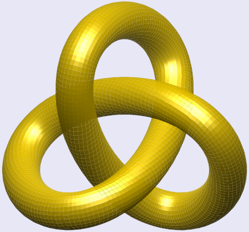
   - in combinatorics/graph theory:

$$R(G;x,y,z) = \sum_{\text{quasi-trees } \mathcal{T}^\star} y^{n(\mathcal{D}(\mathcal{T}^\star))} z^{2g(\mathcal{D}(\mathcal{T}^\star))} (1+y)^{|\varepsilon(\mathcal{T}^\star)|}\, T(G_{\mathcal{T}^\star};x,1+yz^2)$$

   - But also in...

Motivations

3. **Knot theory**
   - A *knot* is an embedding of the circle into $\mathbb{R}^2 \times \mathrm{I}$ upto ambient isotopy.



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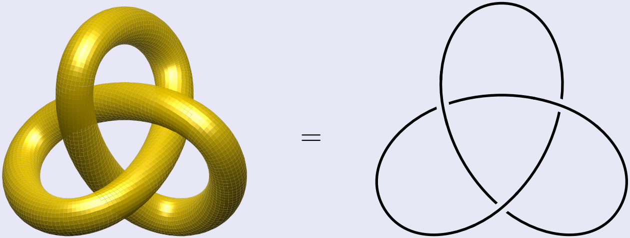

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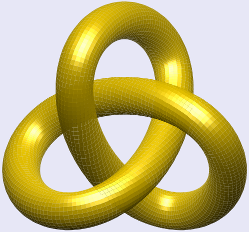

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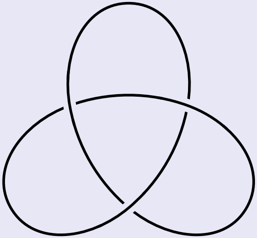

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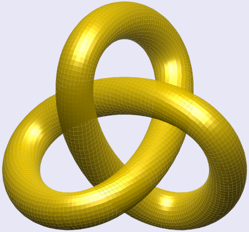

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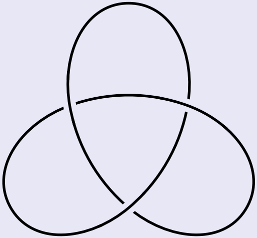

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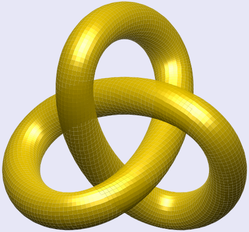

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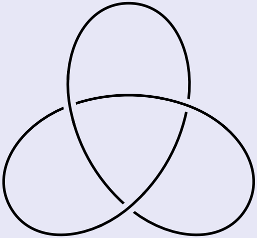

- There exist several link invariants: the Jones polynomial, the Kauffman bracket...
- One can also associate polynomials to a link diagram (e.g. the Kauffman polynomial $[L](A, B, d)$):

$$V_L(t) = (-t)^{3w(L)/4} \langle L \rangle (t^{-1/4}) = (-t)^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{-1/2} - t^{1/2}).$$

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3. **Knot theory**
   - There exist relations between the knot polynomials and the graph invariants:

Theorem (Kauffman 1989)

For any link L, $[L](A, 1, d) = d^{-v(G_L)-1-k(G_L)} A^{k(G_L)} Z^s_T(G_L; d^2, Ad)$.

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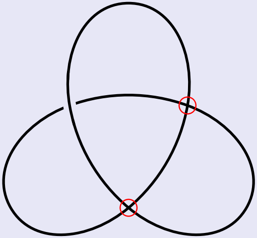

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   - One can generalize knots to *virtual knots*, considered as embeddings of the circle into $S_g \times I$.
   - One can relate the Kauffman polynomial of a virtual link to the Bollobás-Riordan polynomial:

Theorem (Chmutov 2007)

Let L be a virtual link diagram and s one of its states,

$$[L](A,B,d) = A^{e(G_s)}\Big(x^k y^v z^{v+1}\, R_s(G_s; x,y,z)|_{x=Ad/B, y=Bd/A, z=1/d}\Big).$$

Partial duality

Combinatorial representation

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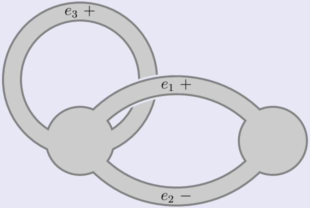

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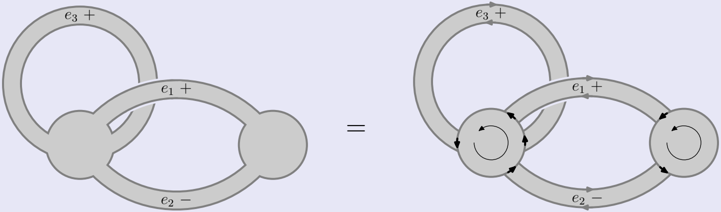

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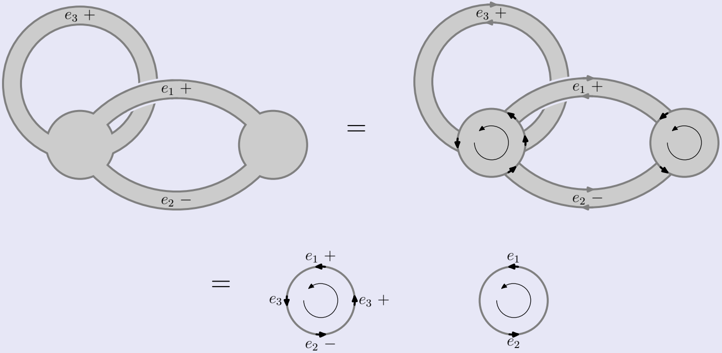

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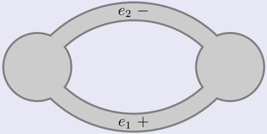
3. $\forall e \in E(G) - E'$, $\varepsilon_{G^{E'}} := \varepsilon_G(e)$ and $\forall e \in E'$, $\varepsilon_{G^{E'}} := -\varepsilon_G(e)$.

Example of partial duality 1



A ribbon graph G

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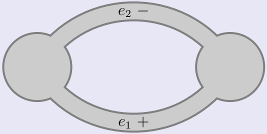

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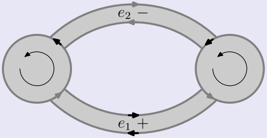

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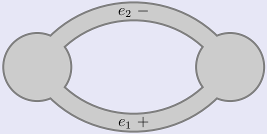

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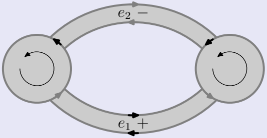

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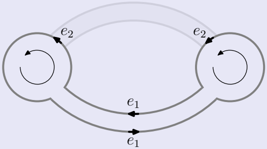

The boundary of $F_{E'}$

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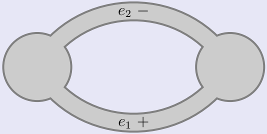

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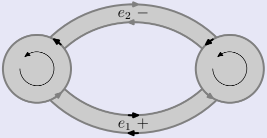

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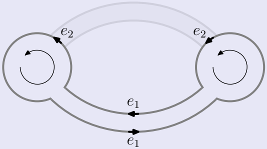

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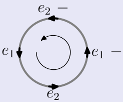

The combinatorial representation of $G^{E'}$

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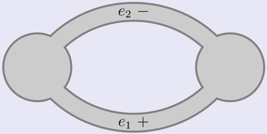

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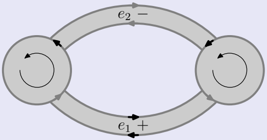

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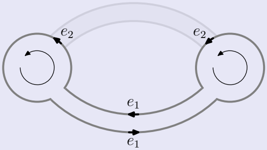

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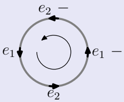

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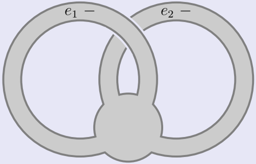

The dual graph $G^{E'}$

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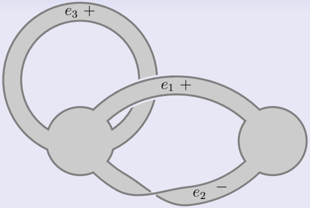

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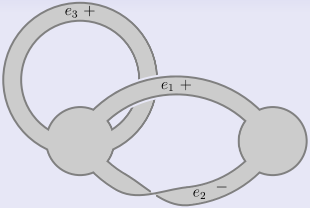

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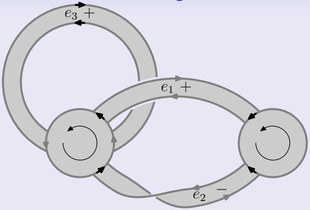

Orientation of the edges

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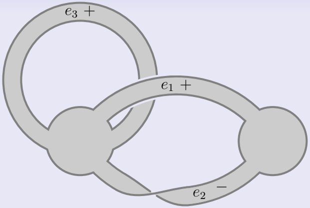

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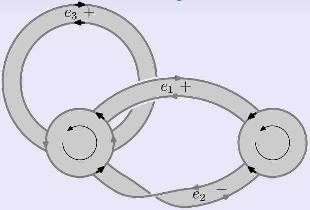

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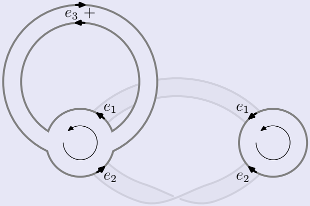

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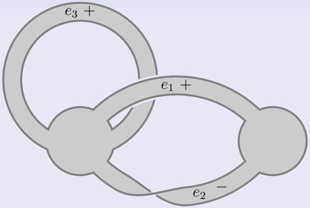

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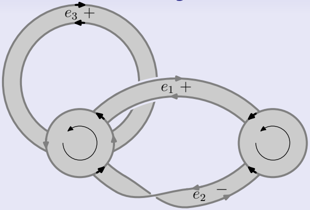
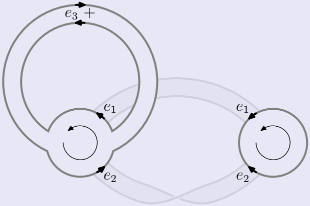
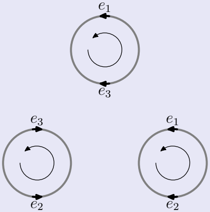


Orientation of the edges

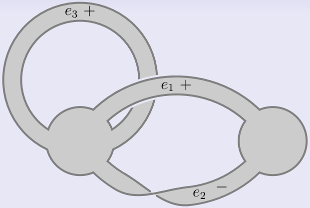


The boundary of $F_{E'}$

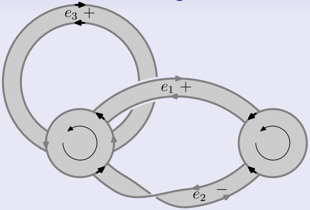


Combinatorial representation of $G^{E'}$

Example of partial duality 2



A ribbon graph G
$E' = \{e_3\}$



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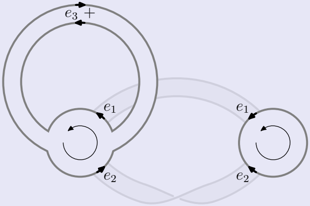

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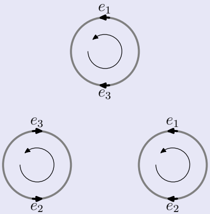
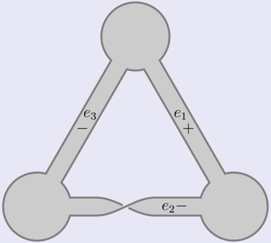


Combinatorial representation of $G^{E'}$



The dual graph $G^{E'}$

Properties

Lemma (S. Chmutov)

Let G be any ribbon graph and $E' \subset E(G)$ any subset of edges. Then we have:

- *if $e \notin E'$ then $G^{E' \cup \{e\}} = (G^{E'})^{\{e\}}$.*
- $(G^{E'})^{E'} = G$.
- *The partial duality preserves the number of connected components.*

Definition

Let G be a ribbon graph. For any edge $e \in E(G)$, one defines

$$G/e := G^{\{e\}} - e.$$

Multivariate Bollobás-Riordan polynomial

Generalization

Method

Let G be a signed ribbon graph.
Let $E(G) =: \boldsymbol{E_+} \cup \boldsymbol{E_-}$ where $E_\pm$ is the set of positive (resp. negative) edges, with cardinality $e_\pm$. For any spanning subgraph $F = (V(G), E(F))$ of G, let $\overline{\boldsymbol{F}}$ be the spanning subgraph of G, the edge set of which is $E(G) - E(F)$ and let $\boldsymbol{s(F)} := \frac{1}{2}(e_-(F) - e_-(\overline{F}))$.

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The polynomial $R_s(G; x+1, y, z)$ defined by S. Chmutov and I. Pak (2006) is:

$$R_s(G; x+1, y, z) = \sum_{F \subseteq G} x^{r(G)-r(F)+s(F)} y^{n(F)-s(F)} z^{k(F)-f(F)+n(F)}$$

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$$Z(G; xyz^2, yz, z) = \sum_{F \subseteq G} (xyz^2)^{k(F)} (yz)^{e(F)} z^{-f(F)} x^{s(F)} y^{-s(F)}.$$

With new variables $q := xyz^2$, $\alpha := yz$, $c := z^{-1}$:

$$Z(G; q, \alpha, c) = \sum_{F \subseteq G} q^{k(F)+s(F)} \alpha^{e(F)-2s(F)} c^{f(F)}.$$

Generalization

Definition and properties

The multivariate signed Bollobás-Riordan polynomial is defined as:

$$Z(G;q,\boldsymbol{\alpha},c) \coloneqq \sum_{F\subseteq G} q^{k(F)+s(F)}\Big(\prod_{\substack{e\in E_+(F)\\ \cup E_-(\overline{F})}} \alpha_e\Big)c^{f(F)}.$$

Proposition (Disjoint union, one-point join)

Let $G_1 \cup G_2$ be the disjoint union of G_1 and G_2. Then

$$Z(G_1 \cup G_2;q,\boldsymbol{\alpha},c) = Z(G_1;q,\boldsymbol{\alpha_1},c)Z(G_2;q,\boldsymbol{\alpha_2},c)$$

where $\boldsymbol{\alpha} = \boldsymbol{\alpha_1} \cup \boldsymbol{\alpha_2}$.
Let $G_1 \cdot G_2$ the one-point join of G_1 and G_2. Then

$$Z(G_1 \cdot G_2;q,\boldsymbol{\alpha},c) = \frac{1}{qc}Z(G_1;q,\boldsymbol{\alpha_1},c)Z(G_2;q,\boldsymbol{\alpha_2},c).$$

Proposition (Contraction and deletion)

Let G be a signed ribbon graph. Then for any positive edge e which is not an orientable loop,

$$Z(G;q,\boldsymbol{\alpha},c) = \alpha_e Z(G/e;q,\boldsymbol{\alpha}_e,c) + Z(G-e;q,\boldsymbol{\alpha}_e,c).$$

For any trivial *positive orientable loop e,*

$$\begin{aligned} Z(G;q,\boldsymbol{\alpha},c) &= q^{-1}\alpha_e Z(G/e;q,\boldsymbol{\alpha}_e,c) + Z(G-e;q,\boldsymbol{\alpha}_e,c) \\ &= (\alpha_e c + 1) Z(G-e;q,\boldsymbol{\alpha}_e,c). \end{aligned}$$

For any negative edge e which is not an orientable loop,

$$Z(G;q,\boldsymbol{\alpha},c) = q^{1/2} Z(G/e;q,\boldsymbol{\alpha}_e,c) + q^{-1/2}\alpha_e Z(G-e;q,\boldsymbol{\alpha}_e,c).$$

For any trivial *negative orientable loop e,*

$$\begin{aligned} Z(G;q,\boldsymbol{\alpha},c) &= q^{-1/2}\big(Z(G/e;q,\boldsymbol{\alpha}_e,z) + \alpha_e Z(G-e;q,\boldsymbol{\alpha}_e,z)\big) \\ &= (q^{1/2}c + q^{-1/2}\alpha_e) Z(G-e;q,\boldsymbol{\alpha}_e,c). \end{aligned}$$

Invariance under duality

Main theorem

Theorem (F. V.-T.)

Let G be a signed ribbon graph. For any subset $E' \subset E(G)$, the multivariate signed Bollobás-Riordan polynomial at $q = 1$ is invariant under the partial duality with respect to E':

$$Z(G; 1, \alpha, c) = Z(G^{E'}; 1, \alpha, c).$$

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Proof : $Z(G; q, \alpha, c) =: \sum_{F \subseteq G} M(F)$.
There is a bijection between the spanning subgraphs of G and those of $G^{E'}$ given by: $F \subset G \mapsto F' \subset G^{E'}$, with $E(F') := (E' \cup E(F)) - (E' \cap E(F))$.
One proves that $M(F)|_{q=1} = M(F')|_{q=1}$.

Proof

Application

Contractions and deletions

At $q = 1$, one can obtain reduction relations for the non trivial orientable loops.

Lemma

Let G be a signed ribbon graph and let $e \in E(G)$ be a non trivial orientable loop. Then

$$Z(G; 1, \boldsymbol{\alpha}, c) = \begin{cases} \alpha_e Z(G/e; 1, \boldsymbol{\alpha}_e, c) + Z(G - e; 1, \boldsymbol{\alpha}_e, c) & \text{if } e \text{ is positive,} \\ Z(G/e; 1, \boldsymbol{\alpha}_e, c) + \alpha_e Z(G - e; 1, \boldsymbol{\alpha}_e, c) & \text{if } e \text{ is negative.} \end{cases}$$

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Proof: e is ordinary in $G^{E'}$.

Natural duality

Motivations

Let G be a graph. Let $R(G; x+1, y, z, w)$ be the Bollobás-Riordan polynomial.
It has been shown [Ellis-Monaghan and Sarmiento, Moffatt]:

$$x^{g(G)} R(G; x, y, 1/\sqrt{xy}, 1) = y^{g(G)} R(G^\star; y, x, 1/\sqrt{xy}, 1).$$

This duality takes place on the surface $xyz^2 = 1$ equivalent to $q = 1$.
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This duality takes place on the surface $xyz^2 = 1$ equivalent to $q = 1$. Does the partial duality generalize this result?

Let (G, ε_G) be a signed ribbon graph such that $\forall e \in E(G)$, $\varepsilon_G(e) = 1$. Then $R_s(G; x, y, z) = R(G; x, y, z)$. One defines $(G^\star, \varepsilon_{G^\star}) := (G^E, -\varepsilon_{G^E})$.

Change of sign(s)

Proposition

Let (G,ε) *be a signed ribbon graph, let* $e_i \in E(G)$ *and let* G_{-e_i} *be the same ribbon graph but with a sign function* ε_{-e_i} *given by:* $\forall e \in E(G_{-e_i}) - \{e_i\}$, $\varepsilon_{-e_i}(e) = \varepsilon(e)$ *and* $\varepsilon_{-e_i}(e_i) = -\varepsilon(e_i)$. *Then*

$$Z(G_{-e_i};q,\boldsymbol{\alpha},c) = \frac{\alpha_{e_i}}{\sqrt{q}}\, Z(G;q,\boldsymbol{\alpha}^i,c)$$

with $\boldsymbol{\alpha}^i := \{\alpha_1,\dots,\alpha_{i-1},\frac{q}{\alpha_{e_i}},\alpha_{i+1},\dots\}$.

Corollary

Let $(G_\varepsilon,\varepsilon)$ *be a signed ribbon graph and* $G_{-\varepsilon}$ *be the same graph but with the sign function* $-\varepsilon$. *Then*

$$Z(G_{-\varepsilon};q,\boldsymbol{\alpha},c) = \Big(\prod_{e\in E(G)} \frac{\alpha_e}{\sqrt{q}}\Big)\, Z(G_\varepsilon;q,\frac{q}{\boldsymbol{\alpha}},c).$$

Natural duality

Proposition

Let $G^\star$ be the natural dual of a ribbon graph G. Then

$$Z(G;1,\boldsymbol{\alpha},c) = \Big(\prod_{e\in E(G)} \alpha_e\Big) Z(G^\star;1,\boldsymbol{\alpha}^{-1},c).$$

Remark: one can show that the duality property of the multivariate Tutte polynomial is a consequence of the previous proposition.

$$\begin{aligned} Z_T(G;q,\boldsymbol{\alpha}) &\coloneqq \sum_{F\subseteq G} q^{k(F)} \prod_{e\in E(F)} \alpha_e \\ &= q^{1-v(G^\star)}\Big(\prod_{e\in E(G)} \alpha_e\Big) Z_T(G^\star;q,q/\boldsymbol{\alpha}) \end{aligned}$$

$$\Longrightarrow T(G;x,y) = T(G^\star;y,x)$$

Perspectives

- Expansion on the spanning trees of the dual graph(s)
- Physical system corresponding to the Bollobás-Riordan polynomial
- Study of its zeros
- Generalization to matroids

Main theorem

Steps of the proof

1. It is enough to consider the case where E' contains a single edge e'.

Back to thm

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4. If $\varepsilon_G(e') = 1$, $s(F) = s(F') + 1/2$, $E_+(F) = E_+(F') \cup \{e'\}$ and
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   If $\varepsilon_G(e') = -1$, $s(F) = s(F') + 1/2$, $E_+(F) = E_+(F')$ and $E_-(\overline{F}) = E_-(\overline{F'})$.

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$$\begin{aligned} M(F) :=& q^{k(F)+s(F)} \Big(\prod_{\substack{e \in E_+(F) \\ \cup E_-(\overline{F})}} \alpha_e \Big) c^{f(F)} \\ =& q^{k(F)+s(F')+1/2} \Big(\prod_{\substack{e \in E_+(F') \\ \cup E_-(\overline{F'})}} \alpha_e \Big) c^{f(F')} \\ =& q^{k(F)-k(F')+1/2} M(F'). \end{aligned}$$

Back to thm