

Batalin–Vilkovisky Integrals in Finite Dimensions

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Abstract

The Batalin-Vilkovisky method (*BV*) is the most powerful method to analyze functional integrals with (infinite-dimensional) gauge symmetries presently known. It has been invented to fix gauges associated with symmetries that do not close off-shell. Homological Perturbation Theory is introduced and used to develop the integration theory behind *BV* and to describe the *BV* quantization of a Lagrangian system with symmetries. Localization (illustrated in terms of Duistermaat-Heckman localization) as well as anomalous symmetries are discussed in the framework of *BV*.

1 Introduction

In the Lagrangian approach to (quantum) physics, we are given a *space of fields*, M , and an *action*, S_0 , thereon. Typically, the space of fields is an infinite-dimensional space of sections of some field bundle over space-time. For the following to hold not only at formal level, we assume M to be a finite-dimensional smooth manifold without boundary. Since, in this paper, we focus our attention on various *cohomology* theories connected with gauge-fixing, we should mention that aspects of *local* cohomology are absent if we assume M to be finite-dimensional. We assume the action S_0 to be smooth and complex-valued, with a non-negative imaginary part.

The equations $S_{0,i} = 0$, where $S_{0,i}$ denotes the derivative of S_0 w.r.t. the i 'th coordinate function on M , are called *equations of motion*, and their set of solutions, $\Sigma \subseteq M$, is called the *shell*¹. A major difficulty in the calculation of path-integrals in field theory originates in the presence of *local gauge symmetries*. In general, infinitesimal local gauge symmetries do not form a Lie algebra; however, *on-shell*, they do. In our finite-dimensional setting, a symmetry is given by a linear subspace, $P \subseteq \Gamma(TM)$, of sections of the tangent bundle with the following properties:

1. For all $\mathbf{X} \in P$,

$$\mathbf{X}(S_0) = 0. \tag{1}$$

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¹This name comes from field theory jargon, where one speaks of the "mass shell" and of "on-shell" conditions etc.

2. There are tensors $T \in \Lambda^2 P^* \otimes P$ and $E \in \Lambda^2 P^* \otimes \Gamma(\Lambda^2 TM)$, with

$$[\mathbf{X}, \mathbf{Y}] = T(\mathbf{X}, \mathbf{Y}) + dS_0 \lrcorner E(\mathbf{X}, \mathbf{Y}), \quad \text{for all } \mathbf{X}, \mathbf{Y} \in P. \quad (2)$$

Furthermore, we assume that P is a finitely generated $C^\infty(M)$ -module. Since, on-shell, P is closed w.r.t. the Lie bracket, it determines a foliation of the shell. In infinite dimensions, the leaf space of the foliation, if a manifold, carries a natural symplectic structure and is the *classical phase space* of the theory (see [19]).

In field theories without anomalies, M is equipped with a formal measure, Ω_0 , that respects the symmetry, i.e., $\text{div}_{\Omega_0} \mathbf{X} = 0$, for all $\mathbf{X} \in P$. *Quantizing* these theories means calculating *vacuum expectation values* of gauge-invariant functions,

$$f \in C^\infty(M)^P := \{f \in C^\infty(M) \mid \mathbf{X}(f) = 0, \mathbf{X} \in P\},$$

by means of the following formal path-integral:

$$\langle f \rangle = \frac{\int_M f e^{iS_0/\hbar} \Omega_0}{\int_M e^{iS_0/\hbar} \Omega_0}. \quad (3)$$

Both numerator and denominator in (3) tend to diverge, due to the presence of the symmetry generated by P , and thus expression (3) needs to be replaced by a *gauge-fixed* version. We must do this in such a way that the result is manifestly independent of the gauge chosen. If the symmetry *closes* off-shell, i.e., if $E = 0$ in (2), this problem has been solved with the help of the so-called *BRST*-method (going back to Becchi, Rouet, Stora [2] and Tyutin [17]). Here we provide a brief description of this method and illustrate it with an example at the end of this introduction. The *BRST*-method extends M to a graded manifold $\mathcal{M}$, by adding auxiliary even and odd fields (*Lagrange multipliers and ghosts*), in such a way that all symmetry vector fields $\mathbf{X} \in P$ are encoded in *one odd vector field* $\mathcal{X}$ on $\mathcal{M}$, which, as an operator on functions on $\mathcal{M}$ (*BRST-operator*), squares to zero: $\mathcal{X}^2 = 0$. For this latter property to hold, ghosts must be introduced. Then, in (3), S_0 is replaced by $S_0 + \mathcal{X}(\Psi)$, for a suitably chosen *odd* function, Ψ , on $\mathcal{M}$, and integration is extended to $\mathcal{M}$. The so-called *gauge-fixing fermion* Ψ is a function of fields, Lagrange multipliers and ghosts, and we call it "suitable" if integration w.r.t. the Lagrange multipliers and ghosts (the latter being a Berezin integral) yields a *finite* measure on the space of orbits of the symmetry P acting on M . Due to the fact that S_0 and Ω_0 are invariant under the symmetry encoded in $\mathcal{X}$, (3) then turns into a functional on cohomology-classes of $\mathcal{X}$, which implies that a variation of the gauge-fixing fermion Ψ does not alter the expectation value.

The *BRST*-method can also be formulated using *anti-fields*, which were originally introduced by Zinn-Justin [18]: Each field, ghost and Lagrange multiplier is paired with a field of opposite statistics (its anti-field). Mathematically, this means that $\mathcal{M}$ is extended to its odd cotangent bundle

$$\mathcal{E} = \Pi T^* \mathcal{M}. \quad (4)$$

The action S_0 is now replaced by $S := S_0 + S_1$ with

$$S_1 = \sum_i z_i^\dagger \mathcal{X}(z^i), \quad (5)$$

where the z^i are coordinate-functions on $\mathcal{M}$ and the $z_i^\dagger$ are corresponding anti-fields. (In infinite dimensions, the sum has to be replaced by an integral.) The gauge-fixing is then encoded in the choice of a submanifold in $\mathcal{E}$ given by the equations

$$z_i^\dagger = \frac{\partial \Psi(z)}{\partial z^i}, \quad (6)$$

where $\Psi = \Psi(z)$ is the gauge-fixing fermion introduced above. As an odd cotangent bundle, $\mathcal{E}$ is equipped with the natural odd symplectic structure

$$\omega = dz^i \wedge dz_i^\dagger,$$

and eqs (6) define a *Lagrangian submanifold*², $\mathcal{L}$, in $\mathcal{E}$. We obtain

$$S|_{\mathcal{L}} = S_0 + \mathcal{X}(\Psi)$$

and the vacuum expectation value (3) is replaced by

$$\langle f \rangle = \frac{\int_{\mathcal{L}} f e^{iS/\hbar} \Omega_{\mathcal{L}}}{\int_{\mathcal{L}} e^{iS/\hbar} \Omega_{\mathcal{L}}}, \quad (7)$$

where $\mathcal{L}$ is suitably chosen; "suitable" again in the sense that it yields a finite measure on the space of orbits of the symmetry. Note that the measure $\Omega_{\mathcal{L}}$ in (7) must be well-defined for an *arbitrary* choice of a Lagrangian submanifold $\mathcal{L}$. Khudaverdian [12] calls such measures *semidensities*, and Severa [15] discovered their cohomological nature. We will give a definition of semidensities in Section 2. For the time being, we just mention that, in order to define a semidensity, we need a measure, Ω , on the even submanifold, or *body*, of $\mathcal{E}$, and Ω and hence the semidensity depend on the path-integral measure Ω_0 . Note that the *BRST*-operator (or rather: an extension of the *BRST*-operator to functions on $\mathcal{E}$) is now given by $\{S_1, \cdot\}$, where $\{\cdot, \cdot\}$ denotes the odd Poisson bracket corresponding to the odd symplectic structure on $\mathcal{E}$, and $\mathcal{X}^2 = 0$ is equivalent to $\{S_1, S_1\} = 0$. Since S_0 does not depend on any anti-fields, we also have that $\{S_0, S_0\} = 0$. Furthermore, (1) implies that $\{S_0, S_1\} = 0$ and thus

$$\{S, S\} = 0, \quad (8)$$

which is called *Classical Master Equation (CME)*.

This reformulation of the *BRST*-formalism allowed Batalin and Vilkovisky [3] to treat path-integrals with *open* symmetries (i.e., symmetries with a non vanishing E in (2)), which is why the use of anti-fields in gauge-fixing path-integrals is nowadays called the *BV-method*. In Section 3 we shall see that the cohomology of the co-boundary operator $\{S_0, \cdot\}$ is the restriction of functions (of fields and anti-fields) to the shell. This is the reason why open symmetries can potentially be gauge-fixed with the *BV-method*. While it is not possible to encode an open symmetry in a co-boundary operator $\mathcal{X}$ on functions on $\mathcal{M}$, it is often possible to do so on $\mathcal{E}$. That is, it is often possible to find an extension of the action S_0 by anti-field terms to an action $S = S_0 + S_1$ that satisfies the *CME* (8) in such a way that $\delta = \{S, \cdot\}$ encodes the symmetry, i.e., for all gauge-invariant functions $f \in C^\infty(M)^P$,

$$\delta(\pi_0^*(f)) = 0, \quad (9)$$

²A submanifold of half the dimension of $\mathcal{E}$ where ω vanishes.

where $\pi_0 : \mathcal{E} \rightarrow M$ is a projection. If the symmetry closes off-shell, the term S_1 is given by the *BRST*-operator according to (5). If it is open, S_1 contains terms of higher order in the anti-fields and $\{S_1, .\}$ does *not* square to zero.

In Section 3 we will see that, under a certain regularity condition on S_0 , gauge-invariant functions whose (closed) support lies outside the shell are δ -exact:

$$\delta(\pi_0^*(f)) = 0, \quad \overline{\text{supp } f} \cap \Sigma = \emptyset \Rightarrow \pi_0^*(f) \text{ is } \delta\text{-exact} . \quad (10)$$

Unless our system is *semi-classically exact*, (7) does *not* define a functional on δ -cohomology classes. Along with the measure Ω on the body of $\mathcal{E}$ comes a *third co-boundary operator*. In *BV*-language, the invariance of Ω under the symmetry reads

$$\Delta_\Omega S_1 = 0 ,$$

where Δ_Ω is (up to signs) the second-order differential co-boundary operator $\partial^2 / \partial z^i \partial z_i^\dagger$ depending on Ω . In Section 3 we shall see that (7) defines a functional on cohomology classes of the operator

$$\delta_{BV} := \delta + i\hbar \Delta_\Omega , \quad (11)$$

which squares to zero if the so-called *Quantum Master Equation (QME)*,

$$\frac{1}{2}\{S, S\} - i\hbar \Delta_\Omega S = 0 , \quad (12)$$

is satisfied. Furthermore, we shall see that this implies that the expectation value (7) is invariant under Hamiltonian variations of the Lagrangian submanifold $\mathcal{L}$ if f is gauge-invariant. Since Hamiltonian variations of $\mathcal{L}$ are associated with variations of the gauge-fixing, classical symmetries survive quantization if the *QME* has a solution. If the measure Ω fails to be invariant under the symmetry-flow generated by S , that is, if $\Delta_\Omega S \neq 0$, we must add higher order $\hbar$ -terms ("*counter terms*") to S to obtain a solution of the *QME*. In Section 3, we will find cohomological obstructions to solving both the *CME* and the *QME*. Obstructions to solving the *QME* are called *anomalies* since they prevent symmetries of the classical theory from surviving quantization.

From (10) and (11) we derive the statement

$$\delta(\pi_0^*(f)) = 0, \quad \overline{\text{supp } f} \cap \Sigma = \emptyset \Rightarrow \pi_0^*(f) = \delta_{BV}\text{-exact} + \mathcal{O}(\hbar) , \quad (13)$$

which gives rise to the *perturbative expansion* (in powers of $\hbar$) of a path-integral around solutions of the equations of motion. In general, such expansions do not converge. If they terminate at finite order, we say that the path-integral *localizes* on the shell. In Section 3, we will rederive Duistermaat-Heckman localization in the *BV*-formalism.

In order to illustrate some of the abstract concepts described above, we conclude this Introduction with a concrete example, pure (non-abelian) Yang-Mills theory. Consider a trivial principal G -bundle, Q , over a 4-dimensional space-time, say Minkowski space $\mathbb{M}^4$, and, as a space of fields, $\mathcal{A}$, the connections thereon. Since Q is assumed to be trivial, these are globally defined one-forms on $\mathbb{M}^4$ with values in the Lie algebra, $\mathfrak{g}$, of G :

$$\mathcal{A} := \Omega^1(\mathbb{M}^4, \mathfrak{g}) .$$

On $\mathcal{A}$ we define the pure Yang-Mills action

$$S_0(A) := \int_{\mathbb{M}^4} \text{tr}(F(A) \wedge *F(A)), \quad F(A) := dA + \frac{1}{2}[A \wedge A],$$

where $*$ denotes the Hodge-dual w.r.t. the Minkowski metric. The infinitesimal gauge symmetries,

$$A \mapsto A + \mathbf{X}_\epsilon(A),$$

are given by vector fields on $\mathcal{A}$ of the form

$$\mathbf{X}_\epsilon(A) = D_A \epsilon = d\epsilon + [A, \epsilon], \quad \epsilon \in \Omega^0(\mathbb{M}^4, \mathfrak{g}). \quad (14)$$

Corresponding to these symmetries we introduce *ghost fields*

$$\beta \in \Omega^0(\mathbb{M}^4, \mathfrak{g}[1]),$$

where the integer in brackets denotes the *ghost degree*. (Fields of even/odd ghost degree are called *bosonic/fermionic*, respectively.) Furthermore, we introduce *antighosts* (not to be confused with the anti-fields of the ghosts!) and *Lagrange multipliers*

$$\bar{\beta} \in \Omega^4(\mathbb{M}^4, \mathfrak{g}^*[-1]), \quad \lambda \in \Omega^4(\mathbb{M}^4, \mathfrak{g}^*).$$

The fields, ghosts, anti-ghosts and Lagrange multipliers determine an infinite-dimensional graded manifold, $\mathcal{M}$. Functions on $\mathcal{M}$ are local functionals of sections of a trivial graded vector bundle over $\mathbb{M}^4$. The symmetry (14) gives rise to the *BRST*-operator

$$\begin{aligned} \mathcal{X}(A) &= D_A \beta, \\ \mathcal{X}(\beta) &= -\frac{1}{2}[\beta, \beta], \end{aligned}$$

which is a co-boundary operator as the Lie bracket satisfies the Leibniz and Jacobi identities. On the anti-ghosts and Lagrange multipliers we define

$$\begin{aligned} \mathcal{X}(\bar{\beta}) &= \lambda, \\ \mathcal{X}(\lambda) &= 0. \end{aligned}$$

Finally, every field in $\mathcal{M}$ is paired with an *anti-field*, i.e., we introduce

$$\begin{aligned} A^\dagger &\in \Omega^3(\mathbb{M}^4, \mathfrak{g}^*[-1]), \\ \beta^\dagger &\in \Omega^4(\mathbb{M}^4, \mathfrak{g}^*[-2]), \\ \bar{\beta}^\dagger &\in \Omega^0(\mathbb{M}^4, \mathfrak{g}), \\ \lambda^\dagger &\in \Omega^0(\mathbb{M}^4, \mathfrak{g}[-1]). \end{aligned}$$

The fields and anti-fields together form the odd symplectic space $\mathcal{E} = \Pi T^* \mathcal{M}$; the gradings are chosen such that its natural odd symplectic structure has ghost degree -1 .

Pairing each anti-field with the *BRST*-transformation of the corresponding field according to (5), we extend the action S_0 to

$$S = S_0 + \int_{\mathbb{M}^4} \langle A^\dagger \wedge D_A \beta \rangle - \langle \beta^\dagger, \frac{1}{2}[\beta, \beta] \rangle + \langle \bar{\beta}^\dagger, \lambda \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing of elements in $\mathfrak{g}$ with elements in $\mathfrak{g}^*$. The *CME* $\{S, S\} = 0$ follows from the facts that $\mathcal{X}$ squares to zero and S_0 is gauge-invariant. Furthermore, gauge-invariant functionals on $\mathcal{A}$ have vanishing odd Poisson bracket with S , so that condition (9) is satisfied.

The *BV* Laplacian Δ_Ω appearing in the *QME* is given by the operator

$$\Delta_\Omega = \int_{\mathbb{M}^4} \left\{ \frac{\delta^2}{\delta A \delta A^\dagger} + \frac{\delta^2}{\delta \beta \delta \beta^\dagger} + \frac{\delta^2}{\delta \bar{\beta} \delta \bar{\beta}^\dagger} + \frac{\delta^2}{\delta \lambda \delta \lambda^\dagger} \right\},$$

and we find that $\Delta_\Omega S = 0$. Hence S is a solution of the *QME*. Notice, however, that Δ_Ω is singular. We refer the reader to Costello [7] for a *renormalized* version of the *QME* and its solutions for the example considered here.

Next, we impose a general gauge-fixing condition

$$G(A) = 0, \tag{15}$$

where $G : \Omega^1(\mathbb{M}^4, \mathfrak{g}) \rightarrow \Omega^0(\mathbb{M}^4, \mathfrak{g})$ is a local map, i.e., $(G(A))(x)$ depends only on a finite number of derivatives of A at x . The Lorentz-gauge, $G(A) = \partial^\mu A_\mu$, provides an example. Condition (15) is implemented via the gauge-fixing fermion

$$\Psi := \int_{\mathbb{M}^4} \langle \bar{\beta}, G(A) \rangle + i \int_{\mathbb{M}^4} \alpha(\bar{\beta}, * \lambda), \quad \alpha : \text{metric on } \mathfrak{g}^*,$$

where the second term is added in order to achieve a Gaussian average around the gauge-fixing (15). The associated Lagrangian submanifold, $\mathcal{L}_\Psi$, is determined by the equations

$$A^\dagger = \frac{\delta \Psi}{\delta A} = \delta_A G^*(\bar{\beta}), \quad \bar{\beta}^\dagger = \frac{\delta \Psi}{\delta \bar{\beta}} = G(A) + i\alpha(*\lambda), \quad \beta^\dagger = \lambda^\dagger = 0,$$

where $\delta_A G^* : \Omega^4(\mathbb{M}^4, \mathfrak{g}^*) \rightarrow \Omega^3(\mathbb{M}^4, \mathfrak{g}^*)$ is the dual of the derivative, $\delta_A G : \Omega^1(\mathbb{M}^4, \mathfrak{g}) \rightarrow \Omega^0(\mathbb{M}^4, \mathfrak{g})$, of G at A . The restriction of the action S to $\mathcal{L}_\Psi$ reads

$$S|_{\mathcal{L}_\Psi} = S_0 + \int_{\mathbb{M}^4} \langle \bar{\beta}, \delta_A G(D_A \beta) \rangle + \langle \lambda, G(A) \rangle + i\alpha(\lambda, * \lambda). \tag{16}$$

The first term under the integral yields the usual Faddeev-Popov determinant after Berezin integration, while the last two terms implement the gauge-fixing.

Note that if $\alpha = 0$ we actually do not need to introduce Lagrange multipliers and antighosts in order to implement the gauge-fixing (15). Instead, we can use the Lagrangian submanifold, $\mathcal{L}$, defined by the equations

$$G(A) = 0, \quad \beta^\dagger = 0, \quad A^\dagger = \delta_A G^*(\bar{\beta}) \quad \text{for } \bar{\beta} \in \Omega^4(\mathbb{M}^4, \mathfrak{g}^*[-1]).$$

The organization of this paper is as follows: In Section 2 we first introduce general notions of graded differential geometry. Then we review homological perturbation theory and use it to define semidensities and develop their integration theory. In Section 3 we construct an action satisfying the *CME* (8) and property (9), for the finite dimensional Lagrangian systems introduced above. Homological Perturbation Theory will be the essential tool for the construction of a solution of the *CME* for an open symmetry. We discuss the role of the operator Δ_Ω in connection with localization and illustrate our formalism with Duistermaat-Heckman localization. We conclude with a comment on anomalies, i.e., on obstructions to solving the *QME*.

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2 Mathematical Formalism

2.1 Elements of Graded Algebra

The notion of a *grading* for an algebraic object is defined over any monoid G . For us, $G = \mathbb{N}, \mathbb{Z}$ or $\mathbb{Z}_2 = \{0, 1\}$. Instead of using the term $\mathbb{Z}_2$ -graded we also use the prefix "*super*".

A G -*graded ring* is a ring, $R = \bigoplus_{i \in G} R_i$, such that $R_i R_j \subseteq R_{ij}$, where ij is the product of $i, j \in G$. Given a G -graded ring R , a G -*graded R -module* is an R -module $M = \bigoplus_{i \in G} M_i$ such that $R_i M_j \subseteq M_{ij}$. A G -*graded R -algebra* is an R -algebra $A = \bigoplus_{i \in G} A_i$ such that $A_i A_j \subseteq A_{ij}$ and $R_i A_j \subseteq A_{ij}$. An element $x \in M_i$ (or A_i or R_i) is called *homogeneous of degree i* , and we write $|x| = i$.

Let M, N be graded R -modules. Then we define

$$\text{Hom}(M, N)^j = \{ \varphi : M \rightarrow N \mid \varphi \text{ is } R\text{-linear and } \varphi(M_i) \subseteq N_{i+j} \}, \quad (17)$$

and

$$\text{Hom}(M, N) = \bigoplus_{j \in G} \text{Hom}(M, N)^j.$$

The *tensor algebra* of M over R ,

$$T_R(M) = \bigoplus_{n=0}^{\infty} M^{\otimes n}, \quad (18)$$

where $M^0 = R$ and the tensor product is taken w.r.t. R , is a bi-graded associative algebra with multiplication $M^{\otimes n} \otimes M^{\otimes m} \rightarrow M^{\otimes(n+m)}$ given by $(m_1, m_2) \mapsto m_1 \otimes m_2$. Let J_S be the ideal of $T_R(M)$ generated by *graded commutators*,

$$[[m_1, m_2]] = m_1 \otimes m_2 - (-1)^{|m_1||m_2|} m_2 \otimes m_1,$$

with m_1, m_2 homogeneous in M . Then

$$S_R(M) = \bigoplus_{n=0}^{\infty} S_R^n(M) = T_R(M)/J_S \quad (19)$$

is the *graded commutative algebra* generated by M . Similarly, the *graded anti-commutative algebra* generated by M is given by

$$\bigwedge_R(M) = T_R(M)/J_{\wedge}, \quad (20)$$

where $J_{\wedge}$ is the ideal of $T_R(M)$ generated by $m_1 \otimes m_2 + (-1)^{|m_1||m_2|} m_2 \otimes m_1$, with m_1, m_2 homogeneous in M .

2.2 Graded Manifolds

Consider a bundle, E , of graded vector spaces over a smooth manifold M . By this we mean that the local sections, $\Gamma(U, E)$, for any local neighbourhood $U \subseteq M$, are $C^\infty(M)$ -modules freely generated by homogeneous sections. To each open subset, $U \subseteq M$, we assign the graded commutative $C^\infty(U)$ -algebra

$$C^\infty(\mathcal{U}) := S_{C^\infty(U)}(\Gamma(U, E^*)) = \Gamma(U, S_{\mathbb{R}}(E^*)) \quad (21)$$

as defined in (19). In (21), E^* denotes the bundle dual to E , and the degree of the dual of a homogeneous section is the negative of the degree of the original section. Just as the assignment $U \mapsto C^\infty(U)$ may be viewed as defining a smooth structure on a topological manifold, we shall use the assignment $U \mapsto C^\infty(\mathcal{U})$ to equip M with the structure of a graded manifold. We denote the latter by $\mathcal{M}$ and write $C^\infty(\mathcal{M})$ if $U = M$ in (21). Elements of $C^\infty(\mathcal{M})$ are called *graded functions*, their degree is called *ghost degree*.

This subsection is devoted to an abstract definition of a graded manifold inspired by the properties of the assignment (21). We will see that the isomorphism classes of graded manifolds are in bijection with the isomorphism classes of graded vector bundles. However, the category of graded vector bundles has fewer morphisms than the category of graded manifolds, since morphisms of graded manifolds do not need to be linear in the generators of the fibres of the associated graded vector bundle.

That the assignment (21) behaves nicely under restrictions to smaller open sets is encoded in the definition of a *sheaf*. In general, a sheaf assigns to every open subset, U , of a topological space M some algebraic structure $A(U)$ (in (21), $A(U)$ is a graded commutative $C^\infty(U)$ -algebra), such that there is a restriction map

$$r_V^U : A(U) \longrightarrow A(V), \quad V \subseteq U \subseteq M,$$

satisfying the following natural axioms: For open subsets $W \subseteq V \subseteq U \subseteq M$:

$$(S1) \quad r_W^U = r_W^V \circ r_V^U;$$

$$(S2) \quad r_U^U = \text{id}_{A(U)}.$$

Furthermore, given any collection $U = \bigcup_i U_i$ of open subsets in M ,

$$(S3) \quad \text{for } s, t \in A(U), \text{ the conditions } r_{U_i}^U(s) = r_{U_i}^U(t), \text{ for all } i, \text{ imply } s = t;$$

$$(S4) \quad \text{for } s_i \in A(U_i), \text{ the conditions } r_{U_i \cap U_j}^{U_i}(s_i) = r_{U_i \cap U_j}^{U_j}(s_j), \text{ for all } i, j, \text{ imply that there is an } s \in A(U) \text{ such that } r_{U_i}^U(s) = s_i, \text{ for all } i.$$

The sheaf (21) has an additional property, which must be satisfied by any graded manifold: it is *locally trivial*, which means that for any local neighbourhood (coordinate patch) $U \subseteq M$ with local coordinates $\{x^1, \dots, x^{(m-m')}\}$ there are graded functions

$$\{y^1, \dots, y^{m'}, \beta^1, \dots, \beta^n\}, \quad (22)$$

such that $C^\infty(\mathcal{U})$ is generated - as a $C^\infty(U)$ -algebra - by the even functions $\{y^1, \dots, y^{m'}\}$ and the odd functions $\{\beta^1, \dots, \beta^n\}$ and such that the requirements of graded commutativity yield the only relations among them. We say that $C^\infty(\mathcal{U})$ is freely generated by m' even and

n odd generators. We then say that the graded manifold $\mathcal{M}$ over a manifold M of dimension $(m - m')$ has dimension (m, n) and call $\{x^1, \dots, x^{(m-m')}, y^1, \dots, y^{m'}, \beta^1, \dots, \beta^n\}$ a set of *local coordinates* on $\mathcal{U}$. Hence, in local coordinates, any $f \in C^\infty(\mathcal{U})$ can be written as

$$f = \sum_{i_1, \dots, i_{m'}, \alpha_1, \dots, \alpha_n} f_{i_1, \dots, i_{m'}, \alpha_1, \dots, \alpha_n}(x)(y^1)^{i_1} \dots (y^{m'})^{i_{m'}} (\beta^1)^{\alpha_1} \dots (\beta^n)^{\alpha_n},$$

where the i 's are in $\mathbb{N}_0$, the α 's in $\mathbb{Z}_2$ and $f_{i_1, \dots, i_{m'}, \alpha_1, \dots, \alpha_n} \in C^\infty(U)$.

Definition 2.1. A graded manifold of dimension (m, n) over a smooth, $(m - m')$ -dimensional manifold M is a sheaf of graded commutative $C^\infty(U)$ -algebras

$$M \supseteq U \mapsto C^\infty(\mathcal{U})$$

that are locally freely generated by m' even and n odd generators.

A morphism $(\phi, \phi^*) : (M, C^\infty(\mathcal{M})) \rightarrow (N, C^\infty(\mathcal{N}))$ of graded algebras consists of a smooth map $\phi : M \rightarrow N$ and, for each open set $U \subseteq N$ and $V := \phi^{-1}(U)$, a morphism of graded commutative algebras $\phi^* : C^\infty(\mathcal{U}) \rightarrow C^\infty(\mathcal{V})$, i.e.,

$$\phi^*(f) := f \circ \phi \quad \text{and} \quad \phi^*(fab) = \phi^*(f)\phi^*(a)\phi^*(b), \quad \text{with} \quad |\phi^*(a)| = |a|, \quad |\phi^*(b)| = |b|,$$

for all $f \in C^\infty(U)$ and all homogeneous $a, b \in C^\infty(\mathcal{U})$.

If the grading under consideration is a $\mathbb{Z}_2$ -grading, graded manifolds are called *super-manifolds*. Batchelor [4] showed that, up to isomorphism, every super-manifold arises from a unique graded vector bundle as in (21). Her arguments extend to general gradings, and hence, up to isomorphism, every graded manifold arises from a unique graded vector bundle, which we call the *graded vector bundle associated with the graded manifold*. However, since the linear structure of the latter is not remembered by the graded manifold, not every morphism of a graded manifold arises from a morphism of the associated graded vector bundle.

Note that $C^\infty(\mathcal{M})$ has a distinguished ideal, I_0 , generated by its odd functions (it is clear that this notion is independent of the choice of coordinates). The ideal I_0 defines a distinguished even submanifold $\iota : M_0 \hookrightarrow \mathcal{M}$, called the *body* and defined by the natural projection

$$\iota^* : C^\infty(\mathcal{M}) \longrightarrow C^\infty(M_0) = C^\infty(\mathcal{M})/I_0.$$

While there is no canonical projection $\pi : \mathcal{M} \rightarrow M_0$, the existence of a graded vector bundle associated with $\mathcal{M}$ guarantees the existence of a projection: The body M_0 is the graded manifold arising from the even subbundle of the vector bundle associated with $\mathcal{M}$, and the projection of this bundle onto its even subbundle yields a projection $\pi : \mathcal{M} \rightarrow M_0$.

2.3 Graded Differential Geometry

In this subsection, we introduce graded versions of (multi-) vector fields and differential forms, together with the operations of exterior differentiation, interior multiplication and Lie derivation and establish the commutation relations among these operations.

We define a homogeneous (*left-*) *vector field* of ghost degree q , denoted by $\vec{\mathbf{X}} \in \mathcal{X}^{1,q}(\mathcal{M})$, as a linear map of degree q , $\vec{\mathbf{X}} \in \text{Hom}(C^\infty(\mathcal{M}), C^\infty(\mathcal{M}))^q$ (see eq. (17)), satisfying the *graded Leibniz rule*

$$\vec{\mathbf{X}}(f \cdot g) = \vec{\mathbf{X}}(f) \cdot g + (-1)^{(|\vec{\mathbf{X}}|+1)|f|} f \cdot \vec{\mathbf{X}}(g), \quad (23)$$

where we define the *total degree* of the vector field $\vec{\mathbf{X}}$ to be

$$|\vec{\mathbf{X}}| := q - 1.$$

The graded commutator,

$$[[\vec{\mathbf{X}}, \vec{\mathbf{Y}}]] = \vec{\mathbf{X}}\vec{\mathbf{Y}} - (-1)^{(|\vec{\mathbf{X}}|+1)(|\vec{\mathbf{Y}}|+1)}\vec{\mathbf{Y}}\vec{\mathbf{X}}, \quad (24)$$

equips $\mathcal{X}^{1,\bullet}(\mathcal{M})$ with the structure of a graded Lie algebra, i.e., it satisfies the *graded Jacobi identity*

$$(-1)^{(|\vec{\mathbf{X}}|+1)(|\vec{\mathbf{Z}}|+1)}[[\vec{\mathbf{X}}, [[\vec{\mathbf{Y}}, \vec{\mathbf{Z}}]]]] + \text{cyclic permutations} = 0. \quad (25)$$

Sometimes matters simplify by using *right vector fields*, which are defined as follows:

$$(f)\overleftarrow{\mathbf{X}} := (-1)^{(|\vec{\mathbf{X}}|+1)(|f|+1)}\vec{\mathbf{X}}(f). \quad (26)$$

From (23) and (26) we deduce the Leibniz rule for right vector fields

$$(f \cdot g)\overleftarrow{\mathbf{X}} = f \cdot (g)\overleftarrow{\mathbf{X}} + (-1)^{(|\mathbf{X}|+1)|g|}(f)\overleftarrow{\mathbf{X}} \cdot g. \quad (27)$$

In local coordinates $\{z^1, \dots, z^{m+n}\}$, the left vector fields $\{\frac{\vec{\partial}}{\partial z^i}\}_{i=1, \dots, m+n}$, determined by

$$\frac{\vec{\partial}}{\partial z^i}(z^j) = \delta_i^j, \quad (28)$$

form a local basis of $\mathcal{X}^{1,\bullet}(\mathcal{U})$ as a $C^\infty(\mathcal{U})$ -module. We then obtain

$$(z^j)\frac{\overleftarrow{\partial}}{\partial z^i} = \delta_i^j,$$

for the associated right vector fields, and the graded commutation relations

$$\left[\left[\frac{\vec{\partial}}{\partial z^i}, \frac{\vec{\partial}}{\partial z^j} \right] \right] = \left[\left[\frac{\overleftarrow{\partial}}{\partial z^i}, \frac{\overleftarrow{\partial}}{\partial z^j} \right] \right] = 0 \quad (29)$$

hold. By definition,

$$\left| \frac{\overleftarrow{\partial}}{\partial z^i} \right| = \left| \frac{\vec{\partial}}{\partial z^i} \right| = -|z_i| - 1.$$

Next, we define (see (19))

$$\mathcal{X}^{p,\bullet}(\mathcal{M}) := S_{C^\infty(\mathcal{M})}^p(\mathcal{X}^{1,\bullet}(\mathcal{M})),$$

and the graded commutative algebra

$$\mathcal{X}(\mathcal{M}) := \bigoplus_{p,q} \mathcal{X}^{p,q}(\mathcal{M}),$$

where "graded commutative" is to be understood w.r.t. the total degree: An element, χ , of $\mathcal{X}^{p,q}(\mathcal{M})$ is called a *multi-vector field* and has total degree $|\chi| = q - p$. We use the symbol $\wedge$ for the associative product in $\mathcal{X}^{p,q}(\mathcal{M})$, i.e., for $\chi_1 \in \mathcal{X}^{p_1,q_1}(\mathcal{M})$ and $\chi_2 \in \mathcal{X}^{p_2,q_2}(\mathcal{M})$ we have

$$\chi_1 \wedge \chi_2 = (-1)^{|\chi_1||\chi_2|} \chi_2 \wedge \chi_1 \in \mathcal{X}^{p_1+p_2,q_1+q_2}(\mathcal{M}). \quad (30)$$

We introduce *graded differential forms* starting with one-forms, which are defined to be dual to the (left-) vector fields:

$$\Omega^{1,q}(\mathcal{M}) := \text{Hom}(\mathcal{X}^{1,\bullet}(\mathcal{M}), C^\infty(\mathcal{M}))^q. \quad (31)$$

Differential forms of higher form degree are elements of the graded commutative algebra

$$\Omega(\mathcal{M}) := \bigoplus_{p,q} \Omega^{p,q}(\mathcal{M}),$$

with

$$\Omega^{p,\bullet}(\mathcal{M}) := S_{C^\infty(\mathcal{M})}^p(\Omega^{1,\bullet}(\mathcal{M})).$$

Again, "graded commutative" is understood w.r.t. the total degree, $|\eta| := p + q$, for $\eta \in \Omega^{p,q}(\mathcal{M})$. The associative product in $\Omega^{p,q}(\mathcal{M})$ is denoted by $\wedge$, and

$$\eta_1 \wedge \eta_2 = (-1)^{|\eta_1||\eta_2|} \eta_2 \wedge \eta_1. \quad (32)$$

According to (31), vector fields pair with one-forms. We write

$$\iota_{\vec{\mathbf{X}}} \eta := \eta(\vec{\mathbf{X}})$$

and extend $\iota_{\vec{\mathbf{X}}}$ to an *interior multiplication* imposing the graded Leibniz rule

$$\iota_{\vec{\mathbf{X}}}(\eta_1 \wedge \eta_2) = \iota_{\vec{\mathbf{X}}} \eta_1 \wedge \eta_2 + (-1)^{|\eta_1||\vec{\mathbf{X}}|} \eta_1 \wedge \iota_{\vec{\mathbf{X}}} \eta_2,$$

for $\eta_1, \eta_2 \in \Omega(\mathcal{M})$, as well as the rule

$$\iota_{\vec{\mathbf{X}} \wedge \vec{\mathbf{Y}}} := \iota_{\vec{\mathbf{X}}} \iota_{\vec{\mathbf{Y}}}, \quad (33)$$

for $\vec{\mathbf{X}}, \vec{\mathbf{Y}} \in \mathcal{X}^{1,\bullet}(\mathcal{M})$. For $\chi \in \mathcal{X}^{p,q}(\mathcal{M})$, the operator ι_χ has degree

$$|\iota_\chi| = |\chi| = q - p,$$

and it follows immediately from (30) and (33) that

$$[[\iota_{\vec{\mathbf{X}}}, \iota_{\vec{\mathbf{Y}}}] = 0, \quad \vec{\mathbf{X}}, \vec{\mathbf{Y}} \in \mathcal{X}^{1,\bullet}(\mathcal{M}).$$

Next, we show that there is a unique *exterior differential*, d , of degree $|d| = 1$, satisfying the following equalities:

$$d(\eta_1 \wedge \eta_2) = d\eta_1 \wedge \eta_2 + (-1)^{|\eta_1|} \eta_1 \wedge d\eta_2 \quad (\text{graded Leibniz rule}), \quad (34)$$

$$\iota_{\vec{\mathbf{X}}} df = \vec{\mathbf{X}}(f), \quad (35)$$

$$\frac{1}{2} [[d, d]] = d^2 = 0. \quad (36)$$

From (34) to (36) it is clear that we only need to verify condition (36) on functions. We introduce local coordinates $\{z^1, \dots, z^{m+n}\}$ and calculate, using (35),

$$df = dz^i \frac{\vec{\partial} f}{\partial z^i}.$$

Hence

$$d^2 f = (-1)^{|z^i|+1} dz^i \wedge dz^j \frac{\vec{\partial}^2 f}{\partial z^j \partial z^i},$$

which vanishes due to (24), (29) and (32).

Finally, the *Lie-derivative* of a differential form, η , along a vector field, $\vec{\mathbf{X}}$, is given by

$$L_{\vec{\mathbf{X}}} \eta := [[\iota_{\vec{\mathbf{X}}}, d]] \eta = \iota_{\vec{\mathbf{X}}} (d\eta) - (-1)^{|\vec{\mathbf{X}}|} d(\iota_{\vec{\mathbf{X}}} \eta). \quad (37)$$

It has degree $|L_{\vec{\mathbf{X}}}| = |\vec{\mathbf{X}}| + 1$ and obeys the commutation relations

$$[[L_{\vec{\mathbf{X}}}, L_{\vec{\mathbf{Y}}}]] = L_{[[\vec{\mathbf{X}}, \vec{\mathbf{Y}}]]}, \quad (38)$$

$$[[L_{\vec{\mathbf{X}}}, \iota_{\vec{\mathbf{Y}}}]] = \iota_{[[\vec{\mathbf{X}}, \vec{\mathbf{Y}}]]}, \quad (39)$$

with $[[\vec{\mathbf{X}}, \vec{\mathbf{Y}}]]$ as defined in (24).

2.4 Odd Symplectic Manifolds

An *odd symplectic manifold*, $(\mathcal{E}, \omega)$, is a graded manifold $\mathcal{E}$ equipped with an *odd symplectic structure*, more precisely, with a closed, non-degenerate two-form ω of ghost degree -1 , i.e., $\omega \in \Omega^{2,-1}(\mathcal{E})$.

To each function $f \in C^\infty(\mathcal{E})$ we associate a *Hamiltonian vector field*, $\vec{\mathbf{X}}_f$, defined by

$$\iota_{\vec{\mathbf{X}}_f} \omega := (-1)^{|f|} df, \quad (40)$$

of degree

$$|\vec{\mathbf{X}}_f| = |f|. \quad (41)$$

The odd symplectic form ω yields the *odd Poisson structure* defined by

$$\{f, g\} := \vec{\mathbf{X}}_f(g) = \iota_{\vec{\mathbf{X}}_f} dg = (-1)^{|g|} \iota_{\vec{\mathbf{X}}_f \wedge \vec{\mathbf{X}}_g} \omega, \quad f, g \in C^\infty(\mathcal{E}), \quad (42)$$

where the second equality follows from (35) and the third from (40) and (33). From (39), (40) and (42) we derive the formula

$$\vec{\mathbf{X}}_{\{f, g\}} = [[\vec{\mathbf{X}}_f, \vec{\mathbf{X}}_g]]. \quad (43)$$

From these formulae together with (23) to (25) it is straightforward to derive graded versions of the usual properties of a Poisson bracket.

Lemma 2.2. *For an odd symplectic graded manifold $(\mathcal{E}, \omega)$, the bracket defined in (42) satisfies the following properties, for all homogeneous $f, g, h \in C^\infty(\mathcal{E})$:*

1. *Graded Commutation Relation:* $\{f, g\} = -(-1)^{(|f|+1)(|g|+1)}\{g, f\}$,
2. *Graded Jacobi Identity:* $(-1)^{(|h|+1)(|g|+1)}\{h, \{f, g\}\} + \text{cyclic permutations} = 0$,
3. *Graded Leibniz Rule:* $\{f, gh\} = \{f, g\}h + (-1)^{|g|(|f|+1)}g\{f, h\}$.

In the mathematics literature, a graded commutative associative algebra together with a bracket satisfying the equations in Lemma 2.2 is called a *Gerstenhaber algebra*.

As we have seen in Subsection 2.2, the graded manifold $\mathcal{E}$ has a distinguished even submanifold, namely its body, E_0 . Hence $\mathcal{E}$ is associated with some odd vector bundle, E , over E_0 . Since ω is of ghost degree -1 it induces a symplectic structure on E , such that the fibres and the zero section of E are Lagrangian. It is then a standard result from symplectic geometry that E is isomorphic to the cotangent bundle over E_0 . Since ω has ghost degree -1 , we need to increase the degree of the fibres of this co-tangent bundle by one. We denote this by $E = \Pi T^*E_0$. We have proven the following version of the Darboux theorem for graded odd symplectic manifolds:

Lemma 2.3. *Any graded manifold $\mathcal{E}$, equipped with a non-degenerate closed two-form ω of ghost degree -1 , is associated with the graded vector bundle ΠT^*E_0 over its body, where the functor Π increases the ghost degree by one.*

Choose a projection $\pi : \mathcal{E} \rightarrow E_0$ and a set of local³ coordinates, $\{\hat{x}^i\}$, on E_0 . Together with the corresponding local coordinate vector fields $\{\partial/\partial\hat{x}^i\}$, they define a set of local graded coordinate functions, $\{x^i, x_i^\dagger\}$ (with $|x_i^\dagger| = -|x^i| - 1$), on $\mathcal{E}$ given by the equations

$$x^i := \pi^*(\hat{x}^i), \quad \{\pi^*(\cdot), x_i^\dagger\} := \pi^*\left(\frac{\partial}{\partial\hat{x}^i}(\cdot)\right). \quad (44)$$

We call $x_i^\dagger$ the *anti-coordinate* corresponding to the coordinate x^i , and the coordinates together with their anti-coordinates form a system of local *Darboux coordinates*. With respect to these Darboux coordinates, the odd symplectic two-form ω is given by

$$\omega = dx^i \wedge dx_i^\dagger. \quad (45)$$

In order to see this, we derive from (26), (40), (42) and (45) the local expression for the odd Poisson structure

$$\{f, g\} = \frac{\overleftarrow{f} \overrightarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial} g}{\partial x_i^\dagger} - \frac{\overleftarrow{f} \overrightarrow{\partial}}{\partial x_i^\dagger} \frac{\overrightarrow{\partial} g}{\partial x^i}, \quad \text{for } f, g \in C^\infty(\mathcal{E}).$$

For $f = x^i, g = x_j^\dagger$, we find agreement with (44).

The identification of graded functions on $\mathcal{E}$ with multi-vector fields on E_0 depends on the choice of projection $\pi : \mathcal{E} \rightarrow E_0$, and, by abuse of notation, we write ΠT^*E_0 for $\mathcal{E}$ together

³Remember that E_0 is an even manifold associated with some even vector bundle over a manifold M . Locality is to be understood w.r.t. M .

with a choice of projection π . Analogously, we define the odd cotangent bundle, $\Pi T^* \mathcal{M}$, of an arbitrary graded manifold, $\mathcal{M}$. If $\{z^i\}$ are local coordinates on $\mathcal{M}$, corresponding anti-coordinates, $\{z_i^\dagger\}$, have degree $|z_i^\dagger| = -|z^i| - 1$ and the two-form ω , which, in local coordinates, is given by

$$\omega = dz^i \wedge dz_i^\dagger$$

defines a natural odd symplectic structure on $\mathcal{E}$. The corresponding odd Poisson structure is then given by

$$\{f, g\} = (-1)^{|z^i|} \left(\frac{\overleftarrow{f} \overrightarrow{\partial}}{\partial z^i} \frac{\overrightarrow{\partial} g}{\partial z_i^\dagger} - \frac{\overleftarrow{f} \overrightarrow{\partial}}{\partial z_i^\dagger} \frac{\overrightarrow{\partial} g}{\partial z^i} \right), \quad \text{for } f, g \in C^\infty(\mathcal{E}).$$

Obviously, odd symplectic manifolds are of dimension (m, m) , and, being odd, the symplectic structure vanishes on its $(m, 0)$ -dimensional body. Hence the body of an odd symplectic manifold is a *Lagrangian submanifold*, locally given by the ideal

$$I_0 := \langle x_i^\dagger \rangle_{i=1, \dots, m},$$

for any choice of local Darboux coordinates. In the *BV*-framework, we consider more general $(k, m - k)$ -dimensional Lagrangian submanifolds given by a sheaf of ideals,

$$I(U) = \langle f^i, \zeta_\alpha \rangle_{\alpha=1, \dots, k}^{i=k+1, \dots, m}, \quad (46)$$

where the f 's (ζ 's) are homogeneous even (odd) functions in $C^\infty(\mathcal{U})$ (also called *constraints*) that are in involution, i.e.,

$$\{f^i, f^j\} = \{f^i, \zeta_\alpha\} = \{\zeta_\alpha, \zeta_\beta\} = 0, \quad k+1 \leq i, j \leq m, \quad 1 \leq \alpha, \beta \leq k. \quad (47)$$

Consider a Lagrangian submanifold, $\mathcal{L}$, of $\mathcal{E}$, given by the sheaf of ideals (46). The body, $L \subseteq E_0$, of $\mathcal{L}$ is defined as

$$L := \mathcal{L} \cap E_0,$$

i.e., $C^\infty(L) := C^\infty(\mathcal{E})/(I \cup I_0)$. A projection, $\pi : \mathcal{E} \rightarrow E_0$, of $\mathcal{E}$ onto its body is said to be *adapted* to $\mathcal{L}$, write $\pi = \pi_{\mathcal{L}}$, if

$$\pi_{\mathcal{L}}(\mathcal{L}) = L,$$

i.e., if $\pi_{\mathcal{L}}^*(f + I_0) \in I$, for all $f \in I$.

Lemma 2.4. *For any Lagrangian submanifold $\mathcal{L}$ of the odd symplectic manifold $\mathcal{E}$ there is a projection $\pi_{\mathcal{L}} : \mathcal{E} \rightarrow E_0$ adapted to $\mathcal{L}$.*

Proof. Let $\mathcal{L}$ be defined by the constraints (46). Since the Hamiltonian vector fields $\overrightarrow{\mathbf{X}}_{f^i}, \overrightarrow{\mathbf{X}}_{\zeta_\alpha}$ are in involution because of (43) and (47), there are local Darboux coordinates $\{x^1, \dots, x^m, x_1^\dagger, \dots, x_m^\dagger\}$ such that

$$x^i := f^i, \quad k+1 \leq i \leq m, \quad x_\alpha^\dagger := \zeta_\alpha, \quad 1 \leq \alpha \leq k$$

and

$$\frac{\overrightarrow{\partial}}{\partial x^\alpha} = \overrightarrow{\mathbf{X}}_{\zeta_\alpha}, \quad 1 \leq \alpha \leq k, \quad \frac{\overrightarrow{\partial}}{\partial x_i^\dagger} = \overrightarrow{\mathbf{X}}_{f^i}, \quad k+1 \leq i \leq m.$$

Locally, we define $\pi_{\mathcal{L}}^*(\hat{x}^i) := x^i$, for $\hat{x}^i := x^i + I_0$. Then the lemma follows from a standard partition of unity argument. $\square$

2.5 Semidensities and the BV Operator

Let $(\mathcal{E}, \omega)$ be an odd symplectic graded manifold. Clearly, $\omega \wedge \omega = 0$, and hence $\omega \wedge (\cdot)$ yields a co-boundary operator on $\Omega(\mathcal{E})$. Severa [15] recognized the corresponding cohomology classes to be the semidensities, which Khudaverdian [12] had previously introduced as the natural objects to be integrated over Lagrangian submanifolds in $\mathcal{E}$. Furthermore, he gave a cohomological definition of the so-called BV -operator, a co-boundary operator acting on semidensities. In this subsection, we introduce the complex given by semidensities and the BV -operator and establish an isomorphism with the de Rham complex of the body of $\mathcal{E}$. Instead of the spectral sequences used by Severa, we shall use the language of Homological Perturbation Theory (HPT), which will also be useful in Section 3.

The basic object in HPT is a *contraction*, as introduced by Eilenberg and McLane in the fifties. It "contracts some big object onto some small object" without loss of cohomology. In the easiest case, these objects are differential graded modules:

Definition 2.5. A contraction consists of two differential graded modules (M, d_M) and (N, d_N) over some ring, together with chain maps $\iota : N \rightarrow M$, $p : M \rightarrow N$, i.e.,

$$d_M \circ \iota = \iota \circ d_N, \quad d_N \circ p = p \circ d_M,$$

and a morphism, $h : M \rightarrow M$, of degree -1 such that

1. $p \circ \iota = id_N$,
2. $\iota \circ p - id_M = hd_M + d_M h$,
3. $h^2 = h \circ \iota = p \circ h = 0$.

Then p is a surjection called the projection, ι is an injection called the inclusion and h is called the homotopy operator. We write

$$(N, d_N) \begin{matrix} \xrightarrow{\iota} \\ \xleftarrow[p]{} \end{matrix} (M, d_M) \curvearrowright h.$$

Condition 2. implies that the cohomologies of M and N are isomorphic, since it implies that the kernel of p has trivial cohomology:

$$p\alpha = 0, \quad d_M \alpha = 0 \quad \Rightarrow \quad -\alpha = d_M \circ h(\alpha).$$

Conditions 3. are also called *side conditions* and can always be satisfied (see, e.g., [14]). Lambe and Stasheff [14] adapted this definition to the situation where the objects M and N carry, in addition, an algebra or coalgebra structure. We shall come back to this special case in Section 3.

The basic theorem of HPT is the so-called *Perturbation Lemma* and goes back to Brown [6] and Shih [16]. A *perturbation* of the differential d_M is a morphism $\delta : M \rightarrow M$ of degree $+1$, such that $(d_M + \delta)^2 = 0$.

Theorem 2.6 (Perturbation Lemma). *Given a contraction*

$$(N, d_N) \xrightleftharpoons[p]{\iota} (M, d_M) \curvearrowright h$$

and a perturbation, δ , of d_M , with the property that the series below converge, then there is a contraction

$$(N, d_N + \tilde{\delta}) \xrightleftharpoons[\tilde{p}]{\tilde{\iota}} (M, d_M + \delta) \curvearrowright \tilde{h},$$

where

$$\begin{aligned} \tilde{\delta} &= \sum_{n \geq 0} p \delta (h \delta)^n \iota, \\ \tilde{\iota} &= \sum_{n \geq 0} (h \delta)^n \iota, \\ \tilde{p} &= \sum_{n \geq 0} p (\delta h)^n, \\ \tilde{h} &= \sum_{n \geq 0} (h \delta)^n h. \end{aligned}$$

Proof. It is straightforward to verify that, formally, conditions 1. - 3. of Definition 2.5 hold for $\tilde{p}$, $\tilde{\iota}$, $d_M + \delta$, $d_N + \tilde{\delta}$ and $\tilde{h}$. $\square$

Next, let $(\mathcal{E}, \omega)$ be an (m, m) -dimensional odd symplectic manifold. Since ω is non-degenerate, there is a two-vector-field $\chi \in \mathcal{X}^{2,1}(\mathcal{E})$ with $\iota_\chi \omega = m$. In local Darboux coordinates (45) χ is given by

$$\chi = \frac{\vec{\partial}}{\partial x^i} \wedge \frac{\vec{\partial}}{\partial x_i^\dagger}.$$

For $\eta = f_{i_1, \dots, i_r}^{j_1, \dots, j_s}(x, x^\dagger) dx^{i_1} \wedge \dots \wedge dx^{i_r} \wedge dx_{j_1}^\dagger \wedge \dots \wedge dx_{j_s}^\dagger \in \Omega(\mathcal{E})$, we find that

$$(i_\chi \circ (\omega \wedge) + (\omega \wedge) \circ i_\chi) \eta = k(\eta) \eta, \quad (48)$$

where $k(\eta) = m - r + s \in \mathbb{N}_0$, and we define the homomorphism $h : \Omega(\mathcal{E}) \rightarrow \Omega(\mathcal{E})$ of $C^\infty(\mathcal{E})$ -modules by

$$h(\eta) := \begin{cases} -\frac{1}{k(\eta)} \iota_\chi(\eta) & \text{if } k(\eta) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (49)$$

Depending on the choice of a projection $\pi : \mathcal{E} \rightarrow E_0$, k defines a grading on $\Omega(\mathcal{E})$,

$$\Omega(\mathcal{E}) = \bigoplus_{k=0}^{2m} \{\eta \in \Omega(\mathcal{E}) | k(\eta) = k\},$$

and $k(\eta) = 0$ if and only if η is in the $C^\infty(\mathcal{E})$ -module $\pi^*(\Omega^m(E_0))$. Hence there is a contraction

$$(\pi^*(\Omega^m(E_0)), 0) \xrightleftharpoons[p]{\iota'} (\Omega(\mathcal{E}), \omega \wedge) \curvearrowright h, \quad (50)$$

and the differential on the l.h.s. of (50) is zero since $\omega \wedge \eta = 0$ for $\eta \in \pi^*(\Omega^m(E_0))$. Therefore the l.h.s. of (50) coincides with the cohomology of the r.h.s.:

$$\pi^*(\Omega^m(E_0)) \xrightarrow{\cong} H(\Omega(\mathcal{E}), \omega \wedge), \quad s \longmapsto [s]. \quad (51)$$

We write $\mathcal{S}(\mathcal{E}) := H(\Omega(\mathcal{E}), \omega \wedge)$ and call classes of forms in $\mathcal{S}(\mathcal{E})$ *semidensities*. According to (51), each semidensity, $[s] \in \mathcal{S}(\mathcal{E})$, has a representative of the form $f\pi^*\hat{\Omega}$ with $f \in C^\infty(\mathcal{E})$ and $\hat{\Omega} \in \Omega^m(E_0)$, called the *normal form* of the semidensity $[s]$.

Since $|\omega| = 1$ and $d\omega = 0$, the de Rham differential d is a perturbation of $\omega \wedge$, i.e., $(\omega \wedge (\cdot) + d)^2 = 0$, and the Perturbation Lemma yields the contraction

$$(\mathcal{S}(\mathcal{E}), \Delta) \cong (\pi^*(\Omega^m(E)), \Delta) \xrightleftharpoons[p]{\iota} (\Omega(\mathcal{E}), \omega \wedge (\cdot) + d) \xrightarrow{\sim} \tilde{h}, \quad (52)$$

with the differential Δ given by

$$\Delta = p \sum_{n \geq 0} (dh)^n dt = pdhdt, \quad (53)$$

because the summand corresponding to $n = 1$ is the only one that preserves $k(\eta)$ and thus the only one that is not annihilated by p . The differential Δ on $\pi^*(\Omega^m(E))$ determines the *BV-operator* on $\mathcal{S}(\mathcal{E})$, via the isomorphism (51). To compute the BV-operator on $\mathcal{S}(\mathcal{E})$, also denoted by Δ , take $[s] \in \mathcal{S}(\mathcal{E})$ with $s \in \pi^*(\Omega^m(E))$. Then $k(s) = 0$ and $k(ds) = 1$, so that (48) implies

$$ds = \iota_\chi(\omega \wedge ds) + \omega \wedge (\iota_\chi ds) = \omega \wedge t, \quad (54)$$

where $t = \iota_\chi ds$. Noting that $k(t) = 1$, we obtain

$$\begin{aligned} \Delta[s] &= [dhds] \\ &= [dh(\omega \wedge t)] \quad \text{by (54)} \\ &= -[d(\iota_\chi(\omega \wedge t))] \quad \text{by (49)} \\ &= -[d(t - \omega \wedge (\iota_\chi t))] \quad \text{by (48)} \\ &= -[dt] \\ &= -[d(\omega \wedge)^{-1} ds] \quad \text{by (54)}, \end{aligned} \quad (55)$$

which coincides (up to a minus sign) with the definition of the BV-operator given in [15].

The *BV-Laplacian*, Δ_Ω , is a second order graded differential operator on $C^\infty(\mathcal{E})$ that depends on the choice of pull-back $\Omega = \pi^*(\hat{\Omega})$ of a volume form $\hat{\Omega}$ on the body E_0 , and is determined by

$$\Delta[f\Omega] =: (\Delta_\Omega f)[\Omega]. \quad (56)$$

In local Darboux coordinates, with $\Omega = dx^1 \wedge \dots \wedge dx^m$, a direct calculation yields

$$\Delta[f\Omega] = -\frac{\overset{\rightarrow}{\partial} f}{\partial x^i \partial x_i^\dagger} [\Omega],$$

and thus

$$\Delta_\Omega = -\frac{\overset{\rightarrow}{\partial}}{\partial x^i \partial x_i^\dagger}. \quad (57)$$

What prevents the BV -Laplacian from being a derivation on the associative algebra $C^\infty(\mathcal{E})$ is the odd Poisson bracket:

$$\Delta_\Omega(f \cdot g) = (\Delta_\Omega f) \cdot g + (-1)^{|f|} \{f, g\} + (-1)^{|f|} f \cdot (\Delta_\Omega g). \quad (58)$$

From (58) we deduce, however, that the BV -Laplacian is a derivation for the Poisson bracket:

$$\Delta_\Omega\{f, g\} = \{\Delta_\Omega f, g\} + (-1)^{|f|+1} \{f, \Delta_\Omega g\}. \quad (59)$$

In the mathematics literature, a Gerstenhaber algebra whose bracket is derived from a differential according to (58) is called a *BV algebra*.

Given an odd symplectic manifold $(\mathcal{E}, \omega)$, we may also treat $\omega \wedge$ as a perturbation of the de Rham differential d on $\mathcal{E}$. Since ω is exact, there is a one-form, θ , s.t. $\omega = d\theta$ and since ω is non-degenerate, we may associate to θ a vector field $\vec{\mathbf{X}}$, which, in local Darboux coordinates, takes the form $\vec{\mathbf{X}} = x_i^\dagger \frac{\partial}{\partial x_i^\dagger}$. For $\eta = f_K^{i_1, \dots, i_\ell, j_1, \dots, j_n}(x) x_{i_1}^\dagger \dots x_{i_\ell}^\dagger dx^K \wedge dx_{j_1}^\dagger \wedge \dots \wedge dx_{j_n}^\dagger$, with $dx^K = dx^{k_1} \wedge \dots \wedge dx^{k_q}$, we find that

$$(d\iota_{\vec{\mathbf{X}}} + \iota_{\vec{\mathbf{X}}}d)\eta = k'(\eta)\eta, \quad (60)$$

where $k'(\eta) = (n + \ell)$. We define a homomorphism $h' : \Omega(\mathcal{E}) \rightarrow \Omega(\mathcal{E})$ of $C^\infty(\mathcal{E})$ -modules by

$$h'(\eta) = \begin{cases} -\frac{1}{k'(\eta)} \iota_{\vec{\mathbf{X}}}(\eta) & \text{if } k'(\eta) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The choice of a projection, $\pi : \mathcal{E} \rightarrow E_0$, together with the natural inclusion $\iota : E_0 \rightarrow \mathcal{E}$ induces a contraction

$$(\Omega(E_0), d) \xrightleftharpoons[\iota^*]{\pi^*} (\Omega(\mathcal{E}), d) \bigcirc h', \quad (61)$$

since $k'(\eta) = 0$ if and only if $\pi^* \circ \iota^*(\eta) = \eta$. (Note that d on the l.h.s. of (61) denotes the de Rham differential on the body.)

We then consider $\omega \wedge$ as a perturbation of d and apply the Perturbation Lemma. Since $\iota^* \circ \omega \wedge = 0$ neither the projection ι^* nor the differential on the l.h.s. of (61) get modified and we obtain the contraction

$$(\Omega(E_0), d) \xrightleftharpoons[\iota^*]{\widetilde{\pi^*}} (\Omega(\mathcal{E}), d + \omega \wedge) \bigcirc \tilde{h}'. \quad (62)$$

Hence, composing the contractions (52) and (62), we obtain an isomorphism of differential graded modules,

$$F^\pi : (\mathcal{S}(\mathcal{E}), \Delta) \xrightarrow{\cong} (\Omega(E_0), d), \quad \text{with } F^\pi := \iota^* \circ \tilde{\iota}'. \quad (63)$$

This isomorphism depends on the choice of a projection $\pi : \mathcal{E} \rightarrow E_0$, a fact one is advised to remember. Note that $(F^\pi)^{-1} = \tilde{p} \circ \widetilde{\pi^*}$, since

$$\begin{aligned} \tilde{p} \circ \widetilde{\pi^*} \circ \iota^* \circ \tilde{\iota}' &= \tilde{p}(id + \tilde{h}'(d + \omega \wedge) + (d + \omega \wedge)\tilde{h}')\tilde{\iota}' \\ &= id + \sum_{n \geq 0} p(dh)^n \sum_{n \geq 0} (h'\omega \wedge)^n h'(d + \omega \wedge) \sum_{n \geq 0} (hd)^n \iota' \\ &\quad + \sum_{n \geq 0} p(dh)^n (d + \omega \wedge) \sum_{n \geq 0} (h'\omega \wedge)^n h' \sum_{n \geq 0} (hd)^n \iota' = id, \end{aligned}$$

where the last equality is most easily seen using local Darboux coordinates and keeping track of the number of $dx^\dagger$'s that appear. We also use the isomorphism (63) in order to define a grading on semidensities:

$$||s|| := |F^\pi[s]|. \quad (64)$$

Then $|\Delta| = 1$.

For $f = g^{i_1, \dots, i_\ell}(x) x_{i_1}^\dagger \dots x_{i_\ell}^\dagger$ and $s = f\Omega$, we compute

$$F^\pi([s]) = \iota^* \circ \tilde{\iota}'(s) = \iota^* \circ \sum_{n \geq 0} (hd)^n \iota'(s) = \iota_{\chi(f)} \Omega, \quad (65)$$

where $\chi(f) := g^{i_1, \dots, i_\ell}(x) \frac{\vec{\partial}}{\partial x^{i_1}} \wedge \dots \wedge \frac{\vec{\partial}}{\partial x^{i_\ell}}$. From (65) we get an isomorphism from the graded functions on $\mathcal{E}$ to the multi-vector fields on the body E_0 :

$$\chi : C^\infty(\mathcal{E}) \xrightarrow{\cong} \mathcal{X}^\bullet(E_0). \quad (66)$$

It is then clear that χ - and hence F^π - only depend on the choice of a projection $\pi : \mathcal{E} \rightarrow E_0$, since such a choice allows for an identification of $\mathcal{E}$ with ΠT^*E_0 and, therefore, for an identification of graded functions on $\mathcal{E}$ with multi-vector fields on E_0 . The map χ allows us to identify multiplication with graded functions, Q , on the l.h.s. of (63) with interior multiplication, $\iota_{\chi(Q)}$, on the r.h.s. of (63)

$$Q \cdot \mapsto \iota_{\chi(Q)}.$$

Furthermore, we may use it to define the so-called *Schouten-Nijenhuis bracket* on multi-vector fields:

$$[\chi(f), \chi(g)] := -\chi(\{f, g\}). \quad (67)$$

If we define the *Lie derivative* for semidensities as

$$L_Q := [[Q, \Delta]] = Q\Delta - (-1)^{|Q|} \Delta Q, \quad (68)$$

we deduce from the well-known relations between the operators d , ι_χ and L_χ acting on differential forms corresponding relations between the operators Δ , $Q \cdot$ and L_Q acting on semidensities, which we state in the following proposition.

Proposition 2.7. *For all Q, Q_1 and Q_2 in $C^\infty(\mathcal{E})$, the following identities hold:*

1. $[[\Delta, \Delta]] = 2\Delta^2 = 0$,
2. $[[Q_1, Q_2]] = 0$,
3. $[[L_{Q_1}, L_{Q_2}]] = -L_{\{Q_1, Q_2\}}$,
4. $[[L_{Q_1}, Q_2]] = -\{Q_1, Q_2\}$ and
5. $[[\Delta, L_Q]] = 0$.

2.6 BV-Integrals

Thanks to the isomorphism (63) there is a natural pairing between semidensities and smooth submanifolds $L \subseteq E_0$, once a projection $\pi : \mathcal{E} \rightarrow E_0$ is chosen. In other words, there is a natural pairing between semidensities $[s]$ and pairs (L, π) :

$$\int_{(L, \pi)} [s] := \int_L F^\pi[s]. \quad (69)$$

Each pair (L, π) defines a Lagrangian submanifold, $\mathcal{L} \subset \mathcal{E}$, whose body is L and such that π is adapted to $\mathcal{L}$. We show that the r.h.s. of (69) only depends on the Lagrangian submanifold $\mathcal{L}$ and not on the particular choice of π . Consider an infinitesimal isomorphism of the odd symplectic manifold $(\mathcal{E}, \omega)$ that preserves $\mathcal{L}$ (but not necessarily π). Inserting a partition of unity under the integral (69) we may work locally and restrict to Hamiltonian variations δ_Q . Let $\mathcal{L}$ be defined by a sheaf of ideals I . Then

$$\delta_Q \mathcal{L} = 0 \quad \Leftrightarrow \quad Q \in I.$$

Under such a transformation, the semidensity transforms as $\delta_Q[s] = [L_{\vec{\mathbf{X}}_Q} s]$, where $\vec{\mathbf{X}}_Q$ is the Hamiltonian vector field associated with Q , as defined in (40), and $L_{\vec{\mathbf{X}}_Q}$ denotes the Lie derivative (37). Since ω is invariant under Hamiltonian flows, $[L_{\vec{\mathbf{X}}_Q} s]$ is well-defined. We need to show that, for $Q \in I$,

$$\int_{(L, \pi)} [L_{\vec{\mathbf{X}}_Q} s] = 0, \quad (70)$$

which follows from the following lemma.

Lemma 2.8. *For $Q \in C^\infty(\mathcal{E})$ and $[s] \in \mathcal{S}(\mathcal{E})$, the Lie-derivative L_Q defined in (68) satisfies the equation*

$$L_Q[s] = [L_{\vec{\mathbf{X}}_Q} s].$$

Proof. Consider a semidensity $[s]$ with representative $s \in \Omega(\mathcal{E})$. Then, as we have seen in (54) and (55), $ds = \omega \wedge t$, for some $t \in \Omega(\mathcal{E})$, and $\Delta[s] = -[dt]$. By (37) and (40)

$$\begin{aligned} [L_{\vec{\mathbf{X}}_Q} s] &= [\iota_{\vec{\mathbf{X}}_Q} ds - (-1)^{|Q|} d\iota_{\vec{\mathbf{X}}_Q} s] \\ &= [\iota_{\vec{\mathbf{X}}_Q} (\omega \wedge t) - (-1)^{|Q|} d\iota_{\vec{\mathbf{X}}_Q} s] \\ &= [(-1)^{|Q|} (dQ) \wedge t - (-1)^{|Q|} d\iota_{\vec{\mathbf{X}}_Q} s]. \end{aligned}$$

On the other hand, by (68), we find that

$$\begin{aligned} L_Q[s] &= (Q\Delta - (-1)^{|Q|}\Delta Q)[s] \\ &= [-Qdt + (-1)^{|Q|}d(\omega \wedge)^{-1}d(Qs)] \\ &= [-Qdt + (-1)^{|Q|}d(\omega \wedge)^{-1}(dQ) \wedge s + d(\omega \wedge)^{-1}Q\omega \wedge t] \\ &= [-Qdt + (-1)^{|Q|}d(\omega \wedge)^{-1}(dQ) \wedge s + (-1)^{|Q|}d(Qt)] \\ &= [(-1)^{|Q|}d(\omega \wedge)^{-1}(dQ) \wedge s + (-1)^{|Q|}(dQ) \wedge t], \end{aligned}$$

and the lemma follows by noticing that

$$0 = \iota_{\vec{\mathbf{x}}_Q}^{\rightarrow} (\omega \wedge s) = (-1)^{|Q|} (dQ) \wedge s + (-1)^{|Q|} \omega \wedge \iota_{\vec{\mathbf{x}}_Q}^{\rightarrow} s$$

and hence that

$$(\omega \wedge)^{-1} (dQ) \wedge s = -\iota_{\vec{\mathbf{x}}_Q}^{\rightarrow} s.$$

□

Lemma 2.8 implies

$$\int_{(L,\pi)} [L_{\vec{\mathbf{x}}_Q}^{\rightarrow} s] = \int_{(L,\pi)} Q \Delta[s] - (-1)^{|Q|} \int_{(L,\pi)} \Delta Q[s] = \int_{(L,\pi)} Q \Delta[s], \quad (71)$$

where the last eq. follows from (63), (69) and the standard Stokes theorem. Claim (70) follows from the observation that, for any semidensity $[s]$,

$$Q \in I \quad \Rightarrow \quad \int_{(L,\pi)} Q[s] = 0,$$

if I defines the submanifold with body L to which π is adapted. Moreover, if $\Delta[s] = 0$, we deduce from (71) that the integral (69) is preserved under *any* Hamiltonian variation of $\mathcal{L}$. Hence we have proven the following result.

Theorem 2.9. *Let $[s] \in \mathcal{S}(\mathcal{E})$ be a semidensity and $\mathcal{L} \subset \mathcal{E}$ a Lagrangian submanifold with body L (without boundary). Then the following integral is well defined*

$$\int_{\mathcal{L}} [s] := \int_L F^{\pi_{\mathcal{L}}} [s], \quad \pi_{\mathcal{L}} \text{ adapted to } \mathcal{L}. \quad (72)$$

Furthermore, if $\Delta[s] = 0$ then the integral (72) is preserved under Hamiltonian variations of $\mathcal{L}$.

We conclude this mathematical section with a remark on *Berezin integration*. Let $\mathcal{M}$ be a graded manifold. W.r.t. local (even/odd) coordinate functions, $\{x^i, \beta^\alpha\}$ ($i = 1, \dots, m$, $\alpha = 1, \dots, n$), we define a so-called *Berezinian*

$$\Omega_B := f(x, \beta) d\beta^1 \wedge \dots \wedge d\beta^n \wedge dx^1 \wedge \dots \wedge dx^m, \quad (73)$$

where f may be expressed in local coordinates as

$$f(x, \beta) = f_0(x) + f_\alpha(x) \beta^\alpha + \dots + f_{top}(x) \beta^n \dots \beta^1.$$

In the literature, the Berezin-integral of Ω_B over $\mathcal{M}$ is then defined to be

$$\int_{\mathcal{M}} \Omega_B := \int_{M_0} f_{top}(x) dx^1 \wedge \dots \wedge dx^m, \quad (74)$$

where M_0 is the body of $\mathcal{M}$. There is, however, no natural (coordinate-independent) way of characterizing differential forms of the kind (73) on a graded manifold without introducing

additional structure. A natural way of defining "Berezinians" on a graded manifold $\mathcal{M}$ is via semidensities on $\Pi T^*\mathcal{M}$. Then (73) should be replaced by the semidensity

$$f(x, \beta)[d\beta_1^\dagger \wedge \cdots \wedge d\beta_n^\dagger \wedge dx^1 \wedge \cdots \wedge dx^m]$$

and (74) by the BV -integral

$$\int_{\mathcal{M} \subset \Pi T^*\mathcal{M}} f(x, \beta)[d\beta_1^\dagger \wedge \cdots \wedge d\beta_n^\dagger \wedge dx^1 \wedge \cdots \wedge dx^m],$$

which coincides with the r.h.s. of (74) due to (65) and (69).

3 BV -quantization

In this section, we give a general description of how to solve the problem stated in the Introduction using the formalism introduced in Section 2: For a given Lagrangian system with symmetry, (M, S_0, P) , we describe how to construct the BV -space, $\mathcal{E}$, (see (4)) as well as a solution of the QME (12), for a given pull-back, Ω , of a measure on the body of $\mathcal{E}$, such that condition (9) is satisfied.

The idea is to first extend the original space of fields, M , by ghosts - as dictated by the symmetry P - to a graded manifold $\mathcal{M}$ and to pair each field and ghost with an anti-field so as to obtain the odd cotangent bundle⁴

$$\begin{array}{c} \Pi T^*\mathcal{M} = \mathcal{E} \\ \downarrow \pi_0 \\ M \end{array}$$

and, second, to find a suitable Δ -closed semidensity, $[s]$, on $\mathcal{E}$, in order to define an expectation, $\langle f \rangle$, for functions $f \in C^\infty(M)$, by

$$\langle f \rangle := Z^{-1} \int_{\mathcal{L}} \pi_0^* f \cdot [s], \quad \text{with} \quad Z := \int_{\mathcal{L}} [s], \quad (75)$$

for a suitable Lagrangian submanifold $\mathcal{L} \subset \mathcal{E}$. The choice of $\mathcal{L}$ encodes the choice of a gauge. The semidensity $[s]$ is chosen to be of the normal form

$$[s] = \left[e^{\frac{i}{\hbar} S} \Omega \right], \quad (76)$$

where Ω is the pull-back (w.r.t. some projection) of a measure, $\hat{\Omega}$, on the body of $\mathcal{E}$, and S is an extension of S_0 by ghost- and higher $\hbar$ -terms:

$$S = S_0 + \hbar \times \text{ghost terms} + \mathcal{O}(\hbar^2). \quad (77)$$

The condition $\Delta[s] = 0$ is equivalent to

$$\frac{1}{2} \{S, S\} - i\hbar \Delta_\Omega S = 0, \quad (78)$$

⁴Note that M is *not* the body of $\mathcal{E}$.

which is the *QME* (12). According to Theorem 2.9, the first integral in (75) is *gauge-invariant* (i.e., invariant under Hamiltonian variations of $\mathcal{L}$) if

$$\Delta(\pi_0^* f[s]) = 0 \quad \Leftrightarrow \quad \delta_{BV}(\pi_0^* f) = 0, \quad (79)$$

where

$$\delta_{BV} := \{S, \cdot\} - i\hbar \Delta_\Omega. \quad (80)$$

The *QME* implies that $\delta_{BV}^2 = 0$, and hence (75) defines a measure on the cohomology classes of δ_{BV} . Note that the δ_{BV} -cohomology is isomorphic to the Δ -cohomology and hence to the deRham cohomology of the body of $\mathcal{E}$. Since the body of $\mathcal{E}$ is a vector bundle over M , the δ_{BV} -cohomology is nothing but the *deRham cohomology of M* .

The information encoded in the action S_0 , the symmetry P and the measure $\hat{\Omega}$ on the body enables us to split δ_{BV} according to (80) and (77) and to reinterpret the δ_{BV} -cohomology by using Homological Perturbation Theory (*HPT*): Under a regularity condition (see Sect. 3.1), the cohomology of the differential $\delta_0 = \{S_0, \cdot\}$ on $C^\infty(\mathcal{E})$, given by the first term in (77), coincides with the *restriction of graded functions to the shell*. A solution of the classical master equation (*CME*) $\{S, S\} = 0$, given by adding the ghost terms in (77) to S_0 , can be seen as a perturbation of δ_0 that induces the *BRST-differential on the shell*. We will see that the *BRST*-cohomology at ghost degree zero contains the *classical observables*, i.e., the gauge-invariant functions on the shell, and that the *BRST*-cohomology at ghost degree one contains the *anomalies*, i.e., obstructions to solving the *QME*.

Perturbation of the *BRST*-cohomology by Δ_Ω in (80) then enables us to construct an *invariant effective measure on the shell* that contains the off-shell contributions to (75) as a formal power series in $\hbar$. In general, this power series in $\hbar$ does not converge. If it terminates, we say that the integral *localizes* on the shell. As an example, we will rederive Duistermaat-Heckman localization in the *BV*-formalism. If the measure $\hat{\Omega}$ is not invariant under the symmetry, we need to add higher order $\hbar$ -terms to S - as indicated in (77) - in order to obtain a solution of the *QME*. We will find obstructions to solving the *QME* (anomalies) as well as obstructions to solving the *CME* for the case of an open symmetry in the *BRST*-cohomology at ghost degree one.

3.1 The shell

We start with a discussion of the cohomology generated by the first term, S_0 , in (77). The ghosts being spectators, we may restrict our attention to the space $\mathcal{E}_0 := \Pi T^*M$, where the fibres carry ghost degree one, and consider the differential

$$\delta_0 := \{S_0, \cdot\} \quad (81)$$

thereon. The set of critical points of S_0 is denoted by Σ and is called the *shell*. We impose the following *regularity condition* on S_0 : On the shell, the Hessian of S_0 is supposed to be of constant rank $m - k$, where m is the dimension of M and k is the dimension of the shell:

$$rk(S_{0,ij})|_\Sigma = m - k. \quad (82)$$

This is to say that any function $f \in C^\infty(M)$ that vanishes on-shell can be written as a linear combination of equations of motion $S_{0,i}$ (with coefficients in $C^\infty(M)$), which, in turn, amounts to saying that f is δ_0 -exact:

$$f = \sum_i S_{0,i} g^i = \{S_0, x_i^\dagger g^i\}.$$

Hence, at ghost degree 0, the cohomology of δ_0 is the restriction of graded functions to the shell. At negative ghost degree $\delta_0(f) = 0$ implies that the multi-vector field associated with f is tangential to the shell. Hence

$$H(C^\infty(\mathcal{E}_0), \delta_0) = C^\infty(\Pi T^*\Sigma). \quad (83)$$

3.2 Closed irreducible symmetries

Next, we introduce ghosts, as dictated by the symmetry P , and perturb the δ_0 -cohomology by adding ghost terms to S_0 . In this subsection we consider *closed symmetries*, i.e., symmetries where $E = 0$ in (2). Due to the assumption that P is finitely generated, there is a vector bundle E_1 and a bundle map ρ_1

$$\begin{array}{ccc} E_1 & \xrightarrow{\rho_1} & TM \\ & \searrow & \swarrow \\ & M & \end{array} \quad (84)$$

such that

$$\rho_1(\Gamma(E_1)) = P.$$

We call the bundle E_1 a *ghost bundle*. Assigning ghost degree -1 to the fibres of E_1 yields the graded vector bundle denoted by $E_1[-1]$. We denote the associated graded manifold by $\mathcal{M}$. According to (21), local graded functions on $\mathcal{M}$ are generated by local sections, β^α , of $E_1^*[1]$ - the *ghosts* that carry degree one. The symmetry P is said to be *irreducible* if there is a ghost bundle (84) such that $\rho_1 : \Gamma(E_1) \rightarrow P$ is injective. Then the tensor T in (2) equips E_1 with the structure of a *Lie algebroid*. Pairing each field (coordinate function on M) and ghost with an anti-field extends $\mathcal{M}$ to the graded manifold $\mathcal{E} := \Pi T^*\mathcal{M}$, which is associated with the graded vector bundle $T^*[1]M \oplus E_1[-1] \oplus E_1^*[2]$. Graded functions on $\mathcal{E}$ are then generated, locally, by the coordinates $\{x_i^\dagger, \beta^\alpha, \beta_\alpha^\dagger\}$ of degrees -1 , 1 and -2 , respectively, with coefficients in $C^\infty(M)$. Then the natural odd symplectic structure on $\mathcal{E}$, $\omega = dx^i \wedge dx_i^\dagger + d\beta^\alpha \wedge d\beta_\alpha^\dagger$, has ghost degree -1 . The body of $\mathcal{E}$ is $E_1^*[2]$. From the theory of Lie algebroids we know that the dual of a Lie algebroid is equipped with a natural Poisson bracket, i.e., with a two-vector field $P_1 \in \Gamma \Lambda^2 T E_1^*$ with the property that $[P_1, P_1] = 0$, where $[\cdot, \cdot]$ denotes the Schouten-bracket. Due to the isomorphism (66) and eq. (67), P_1 corresponds to a graded function, S_1 , on $\mathcal{E}$ with $\{S_1, S_1\} = 0$. The section of $S_{\mathbb{R}}(T[-1]M \oplus E_1^*[1] \oplus E_1[-2])$ associated with S_1 is given by the sum of the two terms $\rho_1 \in E_1^* \otimes TM$ and $-\rho_1^{-1}[\rho_1 \wedge \rho_1]/2 \in E_1^* \wedge E_1^* \otimes E_1$. What this means becomes clear in local coordinates, in which the graded function S_1 reads

$$S_1 = x_i^\dagger \rho_{1\alpha}^i(x) \beta^\alpha - \beta_\alpha^\dagger T_{\beta\gamma}^\alpha(x) \beta^\beta \beta^\gamma, \quad T_{\beta\gamma}^\alpha(x) := (\rho_1^{-1})_i^\alpha(x) \rho_{1\beta}^j(x) \rho_{1\gamma,j}^i(x). \quad (85)$$

The restriction of the differential

$$\delta_1 := \{S_1, \cdot\} \quad (86)$$

to graded functions on $\mathcal{M}$ is usually called *BRST-differential*. If the symmetry arises from a Lie group acting on M , the *BRST*-cohomology coincides with the corresponding Lie algebra cohomology with coefficients in $C^\infty(M)$. In the open case (see Sect. 3.4), δ_1 is not a differential but still a perturbation of δ_0 , i.e., the *BRST*-differential is only defined on the shell. The *BRST*-differential on the shell contains interesting physical information: the *classical observables* (gauge-invariant functions on the shell) at ghost degree zero and the *anomalies* at ghost degree 1 (see Subsect. 3.6).

3.3 Closed reducible symmetries

In the *reducible* case, replacing ρ_1^{-1} in (85) by an arbitrary injection

$$\iota : P \hookrightarrow \Gamma(E_1), \quad \text{with} \quad \rho_1 \circ \iota = id, \quad (87)$$

we again obtain a coboundary operator δ_1 , which produces, however, extra cycles, corresponding to δ_1 -closed graded functions that are associated with one-forms $\eta \in \Gamma E_1^*$ that vanish on the image of ι . (If the one-form associated with the graded function $\delta_1(f)$, for any $f \in C^\infty(M)$, were to vanish on the image of ι , $\delta_1(f)$ would have to vanish as well.) In order to kill these cycles, we need to introduce second-order ghosts. Iterating this process, we end up with a sequence of ghost bundles

$$\begin{array}{ccccc} \dots & E_2 & \xrightarrow{\rho_2} & E_1 & \xrightarrow{\rho_1} & TM \\ & & \searrow & \downarrow & \swarrow & \\ & & & M & & \end{array} \quad (88)$$

such that

$$\rho_i(\Gamma(E_i)) = \text{Ker}(\rho_{i-1}).$$

The graded bundle $E_1[-1] \oplus E_2[-2] \oplus \dots$ defines a graded manifold $\mathcal{M}$. We then find a natural graded function S_1 , with $\{S_1, S_1\} = 0$, on $\mathcal{E} = \Pi T^* \mathcal{M}$. Detailed descriptions of how to construct S_1 in the presence of higher order ghosts can be found, e.g., in [9].

3.4 Open symmetries

In this subsection, we consider an *open* gauge algebra, i.e., a symmetry P with $E \neq 0$ in (2). As usual, we write $\delta_0 := \{S_0, \cdot\}$ and choose a contraction

$$(H(C^\infty(\mathcal{E})), 0) \xrightleftharpoons[p]{\iota} (C^\infty(\mathcal{E}), \delta_0) \xrightarrow{h} h, \quad (89)$$

where h satisfies

$$h\delta_0 + \delta_0 h = \iota p - id \quad (90)$$

as well as the side conditions

$$h^2 = h \circ \iota = p \circ h = 0. \quad (91)$$

The data (2) yield an even function $S_1 \in C^\infty(\mathcal{E})$ with the properties

$$\delta_0 S_1 = 0, \quad \{S_1, S_1\} = \delta_0\text{-exact}. \quad (92)$$

We attempt to find a solution of the *CME* $\{S, S\} = 0$ of the form

$$S = \sum_{m \geq 0} S_m.$$

The *CME* can be written in the form of the Maurer-Cartan equation

$$\delta_0 \left(\sum_{m \geq 1} S_m \right) + \frac{1}{2} \left\{ \sum_{m \geq 1} S_m, \sum_{m \geq 1} S_m \right\} = 0. \quad (93)$$

Following ideas of Kontsevich [13], we want to treat $\{.,.\}$ as a perturbation of δ_0 . Therefore, we need to extend the contraction (89) to a contraction of graded anti-commutative algebras as defined in (20):

$$(\bigwedge_{\mathbb{R}}(H(C^\infty(\mathcal{E})))[1], 0) \xrightleftharpoons[p]{\iota} (\bigwedge_{\mathbb{R}}(C^\infty(\mathcal{E}))[1], \delta_0) \xrightarrow{\quad} h. \quad (94)$$

We use the symbol $\wedge$ for the associative products in $\bigwedge_{\mathbb{R}}(C^\infty(\mathcal{E}))[1]$ and $\bigwedge_{\mathbb{R}}(H(C^\infty(\mathcal{E})))[1]$. The $[1]$ indicates that the anti-commutative product is taken w.r.t. the ghost gradings in $C^\infty(\mathcal{E})$ and $H(C^\infty(\mathcal{E}))$ shifted by one, i.e.,

$$f \wedge g = -(-1)^{(|f|+1)(|g|+1)} g \wedge f. \quad (95)$$

In order for (94) to be a contraction, we extend the definitions of ι , p , δ_0 and h as follows:

$$\begin{aligned} \iota(f_1 \wedge \cdots \wedge f_k) &= \iota(f_1) \wedge \cdots \wedge \iota(f_k), \\ p(f_1 \wedge \cdots \wedge f_k) &= p(f_1) \wedge \cdots \wedge p(f_k), \\ \delta_0(f_1 \wedge \cdots \wedge f_k) &= \sum_{i=1}^k (-1)^{|f_1|+\cdots+|f_{i-1}|+i+k} f_1 \wedge \cdots \wedge (\delta_0 f_i) \wedge \cdots \wedge f_k, \\ h(f_1 \wedge \cdots \wedge f_k) &= \sum_{i=1}^k (-1)^{|f_1|+\cdots+|f_{i-1}|+i+k} f_1 \wedge \cdots \wedge (h f_i) \wedge \iota p(f_{i+1}) \wedge \cdots \wedge \iota p(f_k). \end{aligned}$$

Eqs (90) and (91) then hold on $\bigwedge_{\mathbb{R}}(C^\infty(\mathcal{E}))[1]$, e.g.,

$$\begin{aligned} (h\delta_0 + \delta_0 h)(f_1 \wedge \cdots \wedge f_k) &= \sum_{i=1}^k f_1 \wedge \cdots \wedge (h\delta_0 + \delta_0 h) f_i \wedge \iota p f_{i+1} \wedge \cdots \wedge \iota p f_k \\ &= (\iota p - \text{id})(f_1 \wedge \cdots \wedge f_k), \end{aligned}$$

where the first equation is derived from the fact that h and δ_0 both change the ghost degree by one, and, in the second equation, (90) on $C^\infty(\mathcal{E})$ has been used. The contraction (94) is an example of a contraction of *co-algebras*, as studied by Gugenheim, Lambe and Stasheff in [10]. Now, the bracket

$$\{.,.\} : C^\infty(\mathcal{E}) \wedge C^\infty(\mathcal{E}) \longrightarrow C^\infty(\mathcal{E})$$

can be considered as a perturbation of δ_0 on $\bigwedge_{\mathbb{R}}(C^\infty(\mathcal{E}))[1]$. Therefore, we define

$$\{.,.\}(f_1 \wedge \cdots \wedge f_k) = \sum_{\sigma} (-1)^{\sigma+k} \{f_{i_1}, f_{i_2}\} \wedge f_{i_3} \wedge \cdots \wedge f_{i_k},$$

where the sum ranges over all $(2, k-2)$ shuffles, i.e., over all permutations of k letters that leave the order of the first two and the order of the last $k-2$ letters unchanged. The symbol $(-1)^\sigma$ denotes the sign that is associated with the permutation σ according to (95). For a triple, e.g., we get the formula

$$\begin{aligned} \{.,.\}(f_1 \wedge f_2 \wedge f_3) &= \\ &= \{f_1, f_2\} \wedge f_3 - (-1)^{(|f_2|+1)(|f_3|+1)+1} \{f_1, f_3\} \wedge f_2 - (-1)^{(|f_1|+1)(|f_2|+|f_3|)} \{f_2, f_3\} \wedge f_1. \end{aligned}$$

The fact that $\{.,.\}$ is a perturbation of δ_0 follows from the graded Leibniz- and Jacobi-identities of the bracket, which may be written in terms of operators on $\bigwedge_{\mathbb{R}}(C^\infty(\mathcal{E}))[1]$ as

$$\delta_0\{.,.\} + \{.,.\}\delta_0 = \{.,.\}^2 = 0.$$

Note that the sign $(-1)^k$ in the definition of δ_0 is needed for δ_0 and $\{.,.\}$ to *anti*-commute.

Applying the Perturbation Lemma yields a co-boundary operator, D , on $\bigwedge_{\mathbb{R}}(H(C^\infty(\mathcal{E})))[1]$ of the form

$$D = \sum_{n \geq 2} D_n, \quad D_n = p\{.,.\}(h\{.,.\})^{n-2} \iota : H(C^\infty(\mathcal{E}))^{\wedge n} \longrightarrow H(C^\infty(\mathcal{E})) \quad (96)$$

and a new contraction

$$(\bigwedge_{\mathbb{R}}(H(C^\infty(\mathcal{E})))[1], D) \xrightleftharpoons[\tilde{p}]{\tilde{\iota}} (\bigwedge_{\mathbb{R}}(C^\infty(\mathcal{E}))[1], \delta_0 + \{.,.\}) \bigcirc \tilde{h}. \quad (97)$$

The new inclusion $\tilde{\iota}$ reads

$$\tilde{\iota} = \sum_{n \geq 1} \tilde{\iota}_n, \quad \tilde{\iota}_n = (h\{.,.\})^{n-1} \iota \quad (98)$$

and the fact that it is a chain map is expressed by

$$(\delta_0 + \{.,.\}) \circ \tilde{\iota} = \tilde{\iota} \circ D. \quad (99)$$

We will need the formulae

$$\begin{aligned} \tilde{\iota}_k(a^{\wedge m}) &= \frac{1}{m-k+1} \sum_{i=1}^k \binom{m}{i} \tilde{\iota}_{k+1-i}(a^{\wedge(m-i)}) \wedge \tilde{\iota}_i(a^{\wedge i}) \\ &= \sum_{i_1 + \dots + i_{k'} = m} \frac{1}{(k')!} \binom{m}{i_1} \binom{m-i_1}{i_2} \dots \binom{m-i_1-i_2-\dots-i_{k-1}}{i_{k'}} \tilde{\iota}_{i_1}(a^{\wedge i_1}) \wedge \dots \wedge \tilde{\iota}_{i_{k'}}(a^{\wedge i_{k'}}), \quad (k' = m-k+1), \end{aligned} \quad (100)$$

for $k \leq m$, which hold for any $a \in C^\infty(\mathcal{E})$ of *even* ghost degree and are proven by induction. The new projection $\tilde{p}$ reads

$$\tilde{p} = \sum_{n \geq 1} \tilde{p}_n \quad \tilde{p}_n = p(\{.,.\}h)^{n-1} \quad (101)$$

and the fact that it is a chain map is expressed by

$$\tilde{p} \circ (\delta_0 + \{.,.\}) = D \circ \tilde{p}. \quad (102)$$

Eqs (92) can be expressed as

$$(\delta_0 + \{.,.\})S_1 \wedge S_1 = \delta_0\text{-exact}. \quad (103)$$

By applying $\tilde{p}$ on both sides of (103) and using (102), we get that

$$\tilde{p}(\delta_0 + \{.,.\})(S_1 \wedge S_1) = D\tilde{p}(S_1 \wedge S_1) = D_2\tilde{p}_1(S_1 \wedge S_1) = D_2(pS_1)^{\wedge 2} = 0.$$

We will show below that the condition

$$D_k(pS_1)^{\wedge k} = 0 \quad (104)$$

is satisfied, for all $k \geq 2$, if a certain cohomology class vanishes. Then we have that

$$D \left(\sum_{n \geq 1} \frac{1}{n!} (pS_1)^{\wedge n} \right) = 0. \quad (105)$$

Furthermore, we deduce from eqs (100) that

$$\tilde{t} \left(\sum_{n \geq 1} \frac{1}{n!} (pS_1)^{\wedge n} \right) = \sum_{n \geq 1} \frac{1}{n!} \left(\sum_{m \geq 1} \frac{1}{m!} \tilde{t}_m((pS_1)^{\wedge m}) \right)^{\wedge n}. \quad (106)$$

If we choose the contraction (89) in such a way that

$$h(S_1) = 0, \quad (107)$$

which, due to (90), implies that

$$\iota p(S_1) = S_1, \quad (108)$$

we may set

$$S_m = \frac{1}{m!} \tilde{t}_m(pS_1)^{\wedge m} \quad (m \geq 1). \quad (109)$$

From eqs (99), (105) and (106) we obtain

$$(\delta_0 + \{\cdot, \cdot\}) \left(\sum_{n \geq 1} \frac{1}{n!} \left(\sum_{m \geq 1} S_m \right)^{\wedge n} \right) = 0, \quad (110)$$

which implies that $S := \sum_{m \geq 0} S_m$ is a solution of the *CME*, written in the form (93):

$$\delta_0 \left(\sum_{m \geq 1} S_m \right) + \frac{1}{2} \left\{ \sum_{m \geq 1} S_m, \sum_{m \geq 1} S_m \right\} = 0.$$

In the remainder of this subsection we show that contraction (89) can be chosen such that condition (104) is satisfied, for all $k \geq 2$, if the first cohomology class of the *BRST* operator, defined on the shell, vanishes. First, use eqs (100) in order to express the l.h.s. of (104) as

$$D_k(pS_1)^{\wedge k} = p\{\cdot, \cdot\} \tilde{t}_{k-1}(pS_1)^{\wedge k} = \frac{k!}{2} p \sum_{i_1+i_2=k} \{S_{i_1}, S_{i_2}\}. \quad (111)$$

If we define

$$T_k := \frac{k!}{2} \sum_{i_1+i_2=k} \{S_{i_1}, S_{i_2}\} = \{\cdot, \cdot\} \tilde{t}_{k-1}(pS_1)^{\wedge k}, \quad (112)$$

conditions (104) thus read

$$pT_k = 0, \quad k \geq 2. \quad (113)$$

Furthermore, we deduce from (109) and (112) that

$$S_k = \frac{1}{k!} h(T_k), \quad k \geq 2. \quad (114)$$

Now assume that conditions (113) are satisfied up to order n :

$$p(T_k) = 0, \quad 2 \leq k \leq n. \quad (115)$$

We show that this implies that

$$\delta_0 T_k = 0, \quad 2 \leq k \leq n+1. \quad (116)$$

Using eqs (90), (98), (112) and the Jacobi identity $\{.,.\}^2 = 0$, we see that $\delta_0 T_k = 0$ if the following conditions are met

$$p\{.,.\} \tilde{l}_i(pS_1)^{\wedge k} = 0, \quad i = 1, \dots, k-2.$$

But these conditions follow from the conditions (115), as an application of eq. (100) shows:

$$\begin{aligned} p\{.,.\} \tilde{l}_i(pS_1)^{\wedge k} &= p\{.,.\} \frac{k!}{(k')!} \sum_{i_1 + \dots + i_{k'} = k} S_{i_1} \wedge \dots \wedge S_{i_{k'}} \quad (k' = k - i + 1) \\ &= \frac{(-1)^{k'+1} k!}{(k' - 2)!} \frac{1}{2} \sum_{i_1 + \dots + i_{k'} = k} p\{S_{i_1}, S_{i_2}\} \wedge pS_{i_3} \wedge \dots \wedge pS_{i_{k'}} \\ &= \frac{(-1)^{k'+1} k!}{(k' - 2)!} \sum_{I + i_3 + \dots + i_{k'} = k} \frac{1}{I!} pT_I \wedge pS_{i_3} \wedge \dots \wedge pS_{i_{k'}} = 0. \end{aligned}$$

Notice that we are free to define h and, because of (114), S_k , for $k \geq 2$, as long as eqs (90) and (91) are satisfied. Using eqs (114), (115) and (116) we see that eqs (90) and (91) lead to the following conditions for S_k :

$$\delta_0 S_k = -\frac{1}{k!} T_k, \quad 2 \leq k \leq n, \quad (117)$$

and

$$pS_k = 0, \quad 2 \leq k \leq n. \quad (118)$$

Forgetting condition (118) for a moment, we see that S_n is only determined by S_k , for $1 \leq k < n$, up to a δ_0 -closed term. We need to show that condition (115) is satisfied for $k = n+1$. Due to (116) this is the same as to show that T_{n+1} is δ_0 -exact. We show that the freedom of adding a δ_0 -closed term to S_n is enough to render T_{n+1} δ_0 -exact, if the on-shell $BRST$ -cohomology at ghost degree one vanishes. Thus, denote the $BRST$ co-boundary operator on $H(C^\infty(\mathcal{E}))$ by

$$\delta_{BRST} := p\{S_1, .\} \iota, \quad (119)$$

and check that it squares to zero. We show that pT_{n+1} is δ_{BRST} -closed if conditions (115)

hold:

$$\begin{aligned}
\delta_{BRST} p T_{n+1} &= (n+1)! p \sum_{i_1+i_2=n+1} \{ \{S_1, S_{i_1}\}, S_{i_2} \} \\
&= (n+1)! p \left(\sum_{\substack{i_1+i_2=n+1 \\ i_1, i_2 \geq 2}} \{ \{S_1, S_{i_1}\}, S_{i_2} \} + \{ \{S_1, S_1\}, S_n \} + \{ \{S_1, S_n\}, S_1 \} \right) \\
&= -(n+1)! p \left(\sum_{\substack{i_1+i_2=n+1 \\ i_1, i_2 \geq 2}} \left(\{ \delta_0 S_{i_1+1}, S_{i_2} \} + \frac{1}{2} \sum_{\substack{j_1+j_2=i_1+1 \\ j_1, j_2 \geq 2}} \{ \{S_{j_1}, S_{j_2}\}, S_{i_2} \} \right) \right. \\
&\quad \left. - \{S_n, \delta_0 S_2\} \right) \\
&= -(n+1)! p \left(\delta_0 \sum_{\substack{i_1+i_2=n+2 \\ i_1, i_2 \geq 2}} \{S_{i_1}, S_{i_2}\} + \frac{1}{2} \sum_{\substack{j_1+j_2+i_2=n+2 \\ j_1, j_2, i_2 \geq 2}} \{ \{S_{j_1}, S_{j_2}\}, S_{i_2} \} \right) = 0
\end{aligned}$$

where we used (90), (112), (116) and $p\{S_1, \cdot\}\delta_0 = 0$ in the first equation, (112) and (117) in the third and the Jacobi identity as well as $p \circ \delta_0 = 0$ in the last equation. If the δ_{BRST} -cohomology vanishes at ghost degree one we can thus write

$$T_{n+1} = \{S_1, G\} + \delta_0 - \text{exact},$$

for some δ_0 -closed term G . And, since

$$T_{n+1} = k! \{S_1, S_n\} + \text{terms independent of } S_n,$$

we may use the freedom of adding a δ_0 -closed term to S_n to render T_{n+1} δ_0 -exact. Finally, condition (118) can always be satisfied without spoiling the δ_0 -exactness of T_{n+1} by replacing S_n by $S_n - \iota \circ p S_n$. Hence conditions (104) are satisfied for all k , which was to be shown.

For a more general view on this subject we refer to [11].

3.5 The BV -Laplacian and localization

The last three subsections were devoted to finding a solution of the CME $\{S, S\} = 0$. In this subsection, we assume that there are no symmetries, i.e. $S = S_0$, and perturb the differential $\delta_0 := \{S_0, \cdot\}$ by the BV -Laplacian so as to obtain the BV -differential

$$\delta_{BV} = \delta_0 - i\hbar \Delta_\Omega;$$

see (80). We restrict our attention to $\mathcal{E}_0 := \Pi T^*M$ and the differential $\delta_0 := \{S_0, \cdot\}$ and choose a contraction

$$(H(\delta_0), 0) \xrightleftharpoons[p]{\iota} (C^\infty(\mathcal{E}_0), \delta_0) \bigcirc \quad h. \quad (120)$$

Under the regularity condition (82) we have seen in (83) that $H(\delta_0) = C^\infty(\Pi T^*\Sigma)$. Then the projection p is simply the restriction to Σ and we can choose ι to extend a super-function on $\Pi T^*\Sigma$ to a smooth super-function supported on a ϵ -neighbourhood of Σ in M . We shall

keep track of the ϵ -dependence of ι and write ι^ϵ instead. We perturb δ_0 by $-i\hbar\Delta_\Omega$ - for some pulled-back measure Ω on M - and apply the Perturbation Lemma in order to get a perturbed contraction with inclusion, $\tilde{\iota}$, projection, $\tilde{p}$, and homotopy operator, $\tilde{h}$, given by formal power series in $\hbar$. Applying equation 2. in Definition 2.5 (see Sect. 2.5) of the perturbed contraction to the constant function 1 on M gives

$$\tilde{\iota}^\epsilon \tilde{p}(1) - 1 = \frac{i}{\hbar} \delta_{BV} \tilde{h}(1).$$

Since

$$\Delta \left[\tilde{h}(1) e^{iS_0/\hbar} \Omega \right] = \frac{i}{\hbar} \delta_{BV}(\tilde{h}(1)) \left[e^{iS_0/\hbar} \Omega \right]$$

and integrals of Δ -exact semidensities vanish, we get that

$$\int_M e^{iS_0/\hbar} [\Omega] = \int_M \tilde{\iota}^\epsilon \tilde{p}(1) e^{iS_0/\hbar} [\Omega]. \quad (121)$$

Note, that $\tilde{\iota}^\epsilon \tilde{p}(1)$ is localized in an ϵ -neighbourhood of Σ , since $\Delta_\Omega(\iota^\epsilon \tilde{p}(1)) = 0$ and hence $\tilde{\iota}^\epsilon \tilde{p}(1)$ can be replaced by $\iota^\epsilon \tilde{p}(1)$. In the limit $\epsilon \rightarrow 0$ the integrand on the r.h.s. of eq. (121) tends to an effective measure on Σ in the form of a formal power series in $\hbar$. In general, the latter does not converge. As an example where it terminates, we rederive the Duistermaat-Heckman localization formula. Before we do so, we describe an equivalent way of constructing an effective measure on Σ . We remove the critical points from M : $\tilde{M} := M \setminus \Sigma$, $\tilde{\mathcal{E}}_0 := \Pi T^* \tilde{M}$ and choose a contraction

$$0 \xrightleftharpoons[p]{\iota} (C^\infty(\tilde{\mathcal{E}}_0), \delta_0) \xrightarrow{\quad} h. \quad (122)$$

The operator h is multiplication with a super-function, denoted h as well, of anti-ghost-degree -1 that satisfies

$$\delta_0(h) = 1.$$

Such a function exists on $\tilde{\mathcal{E}}_0$.

Now, we perturb the differential δ_0 by $-i\hbar\Delta_\Omega$. The Perturbation Lemma (or a direct computation) then tells us that

$$id = \tilde{h} \delta_{BV} + \delta_{BV} \tilde{h},$$

where $\tilde{h}$ is given by the formal power series

$$\tilde{h} := \sum_{n \geq 0} (-i\hbar)^n (h \Delta_\Omega)^n h. \quad (123)$$

Assuming that Σ has zero measure w.r.t. Ω , we obtain a formula of the kind (121)

$$\int_M e^{iS_0/\hbar} [\Omega] = \int_{\tilde{M}} e^{iS_0/\hbar} [\Omega] = \int_{\tilde{M}} \delta_{BV}(\tilde{h}) e^{iS_0/\hbar} [\Omega] = -i\hbar \lim_{\epsilon \rightarrow 0} \int_{\Sigma_\epsilon} \tilde{h} e^{iS_0/\hbar} [\Omega], \quad (124)$$

where Σ_ϵ denotes the boundary of a small tubular neighbourhood of Σ in M . Eq. (124) yields an effective measure on Σ in the limit $\epsilon \rightarrow 0$ as a formal power series in $\hbar$.

As an example, we study Duistermaat-Heckman localization [8]. We consider a compact $2m$ -dimensional symplectic manifold (M, ω) without boundary and a Hamiltonian $H \in$

$C^\infty(M; \mathbb{R})$ with the property that its Hamiltonian vector field integrates to a $U(1)$ -action on M whose fixed points (the critical points of H) are isolated. Then the Duistermaat-Heckman localization formula states that

$$\int_M e^{iH/\hbar} \frac{\omega^m}{m!} = (-2\pi i \hbar)^m \sum_{dH=0} \frac{e^{iH/\hbar} \sqrt{\det \omega}}{\sqrt{\det(H_{,ij})}}. \quad (125)$$

As discovered by Atiyah and Bott [1], the fact that (125) localizes at critical points can be seen with the help of *equivariant cohomology*. We give a description following Blau and Thompson [5]. Note, first, that the integrand in (125) coincides with the top degree part of the inhomogeneous form

$$e^{iH/\hbar + \omega}, \quad (126)$$

which is closed w.r.t. to the equivariant differential

$$d_{\mathbf{X}} := d - i_{\mathbf{X}}, \quad i_{\mathbf{X}} \omega = dH. \quad (127)$$

We show that, on $\tilde{M}$, the form (126) is also $d_{\mathbf{X}}$ -exact. For this purpose, we choose a metric, g , on M such that

$$e^{iH/\hbar + \omega} = d_{\mathbf{X}} \left(g(\mathbf{X}) (d_{\mathbf{X}}(g(\mathbf{X})))^{-1} e^{iH/\hbar + \omega} \right). \quad (128)$$

The one form $d_{\mathbf{X}}(g(\mathbf{X}))$ is invertible, since its zero-degree part, $-g(\mathbf{X}, \mathbf{X})$, is nowhere vanishing on $\tilde{M}$. Note that $d_{\mathbf{X}}$ does not square to zero, but

$$d_{\mathbf{X}}^2 = -L_{\mathbf{X}} = -di_{\mathbf{X}} - i_{\mathbf{X}}d. \quad (129)$$

Hence, in order for (128) to be true, we need to choose g to be invariant under the $U(1)$ -action, $L_{\mathbf{X}}g = 0$, which can always be achieved by averaging. Taking the top-degree part on both sides of eq. (128) we see that the measure in (125) becomes d -exact, if the critical points are removed.

Knowing that the integral (125) localizes, we calculate the contribution from a critical point $p \in \Sigma$, i.e., $dH(p) = 0$, to the integral (125), using formulae (123) and (124). We start by choosing radial local coordinates in a neighbourhood of p , such that

$$H(r) = H(p) + \frac{1}{2}r^2.$$

We may set

$$h := \frac{r^\dagger}{r},$$

since $\delta_0(h) = 1$. The Liouville measure, $\Omega = \omega^m/m!$, expressed in radial coordinates, then reads

$$\Omega = \frac{\sqrt{\det \omega} r^{2m-1}}{\sqrt{\det(H_{,ij})}} \Omega_{S^{2m-1}} \wedge dr,$$

where $\Omega_{S^{2m-1}}$ denotes the standard volume element on the unit sphere. The term of order $\hbar^{n-1}$ in (123) is calculated to be

$$(-i\hbar)^{n-1} (2m-2)(2m-4) \dots (2m-2(n-1)) r^{-(2n-1)} r^\dagger.$$

Its contribution to the integral (124) is thus given by

$$\lim_{\epsilon \rightarrow 0} \int_{S_r^{2m-1}} (-i\hbar)^n (2m-2)(2m-4) \dots (2m-2(n-1)) \epsilon^{2(m-n)} e^{iH/\hbar} \frac{\sqrt{\det \omega} \text{Vol}(S^{2m-1})}{\sqrt{\det(H_{ij})}}.$$

The area of the standard sphere around the critical point measured w.r.t. $\Omega_{S^{2m-1}}$, $\text{Vol}(S^{2m-1})$, is given by the formula

$$\text{Vol}(S^{2m-1}) = \frac{(2\pi)^m}{2 \cdot 4 \cdot \dots \cdot (2m-2)}.$$

Hence the terms of order $n < m$ yield contributions that vanish in the limit $\epsilon \rightarrow 0$. The term of order $n = m$ yields

$$(-2\pi i\hbar)^m \frac{e^{iH/\hbar} \sqrt{\det \omega}}{\sqrt{\det(H_{ij})}},$$

and the terms of order $n > m$ vanish.

3.6 Anomalies

If a symmetry P is present, we aim at finding a solution of the QME

$$\frac{1}{2} \{S, S\} - i\hbar \Delta_\Omega S = 0$$

in such a way that P is encoded in the corresponding co-boundary operator

$$\delta_{BV} = \{S, \cdot\} - i\hbar \Delta_\Omega;$$

see eq. (9). Since the expectation values, $\langle f \rangle$, defined in (75) are then defined on cohomology classes of δ_{BV} , they are independent of the particular gauge-fixing, provided f is gauge-invariant. This is to say, that the classical symmetry P survives quantization if a solution of the QME can be found. Classical symmetries that do not survive quantization are called *anomalous* and can be seen as obstructions to solving the QME . In Subsection 3.4 we have found obstructions to solving the CME in the first cohomology classes of the $BRST$ -operator on the shell. In this subsection, we show that the same cohomology classes contain obstructions to solving the QME .

Set $\delta := \{\tilde{S}, \cdot\}$ for a solution of the CME , $\{\tilde{S}, \tilde{S}\} = 0$, and assume that a measure $\Omega = \pi^* \hat{\Omega}$ is not invariant w.r.t. the symmetry encoded in $\tilde{S}$, i.e., $\Delta_\Omega(\tilde{S}) \neq 0$. Instead of changing $\hat{\Omega}$ or the projection π from $\mathcal{E}$ onto its body, we may attempt to add terms of higher order in $\hbar$ to $\tilde{S}$, $S = \tilde{S} + \hbar Q$, for $Q \in C^\infty(\mathcal{E})[[\hbar]]$, so as to obtain a solution of the QME : Notice that $\Delta_\Omega \tilde{S}$ is $\{\tilde{S}, \cdot\}$ -closed as a consequence of eq. (59). If the $\{\tilde{S}, \cdot\}$ -cohomology at ghost degree one vanishes, there is a $T_1 \in C^\infty(\mathcal{E})$, such that

$$\Delta_\Omega \tilde{S} = \{\tilde{S}, T_1\}. \quad (130)$$

We get

$$\frac{1}{2} \{\tilde{S} + i\hbar T_1, \tilde{S} + i\hbar T_1\} - i\hbar \Delta(\tilde{S} + i\hbar T_1) = -\hbar^2 \left(\frac{1}{2} \{T_1, T_1\} - \Delta T_1 \right).$$

With

$$\{\tilde{S}, \frac{1}{2} \{T_1, T_1\} - \Delta T_1\} = 0 \quad (131)$$

and the above assumption on the cohomology, we get the next order $\hbar$ -correction, and so on. In this way, we find a solution of the *QME* in the form of a formal power series in $\hbar$.

Note that the $\{\tilde{S}, .\}$ -differential is a perturbation of $\delta_0 = \{S_0, .\}$ and induces a differential on $H(\delta_0)$ whose first term is the *BRST*-differential (119). Hence the obstructions to solving the *QME* are in the same cohomology classes as the obstructions to solving the *CME* in the case of an open symmetry.

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