Hopf algebraic structures of quantum field and many-body theories (II)

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Outline

- **Lecture 1**: From Hopf to Feynman diagrams
  - Coproduct
  - Hopf algebra
  - Twisted product
  - Feynman diagrams (with proofs)

- **Lecture 2**:
  - Cumulant expansion
  - A new coproduct for connected diagrams
  - Structure of Green functions
  - Renormalization

Hopf algebra definition

- A Hopf algebra is an algebra $\mathcal{A}$ with unit 1 and with
- A coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\Delta a = \sum a_{(1)} \otimes a_{(2)}$, is an algebra morphism $\Delta(ab) = (\Delta a)(\Delta b)$
- A counit $\varepsilon : \mathcal{A} \to \mathbb{C}$ such that

$$\sum \varepsilon(a_{(1)})a_{(2)} = \sum a_{(1)}\varepsilon(a_{(2)}) = a$$

- The counit is an algebra morphism

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$$

- An antipode $S : \mathcal{A} \to \mathcal{A}$

$$\sum S(a_{(1)})a_{(2)} = \sum a_{(1)}S(a_{(2)}) = \varepsilon(a)1$$

Hopf algebra of fields

- $\mathcal{A}$ is the polynomial algebra generated by the fields at distinct points.

- The coproduct is

$$\Delta\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) = \sum_{j_1=0}^{n_1}\cdots\sum_{j_k=0}^{n_k}\binom{n_1}{j_1}\cdots\binom{n_k}{j_k} \varphi^{j_1}(x_1)\cdots\varphi^{j_k}(x_k)\otimes\varphi^{n_1-j_1}(x_1)\cdots\varphi^{n_k-j_k}(x_k)$$

- The counit satisfies

$$\varepsilon\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) = \delta_{n_1,0}\ldots\delta_{n_k,0}$$

- The antipode

$$S\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) = (-1)^{n_1+\cdots+n_k}\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)$$

Laplace pairing

- A Laplace pairing is a map $(.|.) : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$
- $$\begin{aligned} (a|bc) &= \sum (a_{(1)}|b)(a_{(2)}|c) \\ (ab|c) &= \sum (a|c_{(1)})(b|c_{(2)}) \end{aligned}$$
- In particular $(a|1) = (1|a) = \varepsilon(a) \quad \big(\varphi^n(x)|1\big) = \delta_{n,0}$
- Wick's theorem $a \circ b = \sum (a_{(1)}|b_{(1)})\, a_{(2)} b_{(2)}$
- Examples
$$\begin{aligned} \varphi(x) \circ \varphi(y) &= \varphi(x)\varphi(y) + g(x,y) \\ \varphi^2(x) \circ \varphi(y) &= \varphi^2(x)\varphi(y) + 2g(x,y)\,\varphi(x) \\ \varphi^2(x) \circ \varphi^2(y) &= \varphi^2(x)\varphi^2(y) + 4g(x,y)\,\varphi(x)\varphi(y) + 4g^2(x,y) \end{aligned}$$

Twisted products

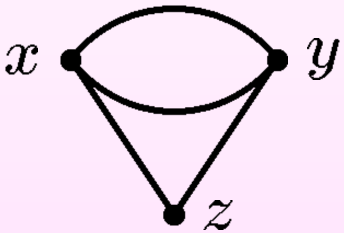
- If $g(x,y) = \langle 0|T\big(\varphi(x)\varphi(y)\big)|0\rangle$ (properly regularized), the twisted product is the (commutative) chronological (or time-ordered) product
- Previous iterated time-ordered product $T\big(\varphi^n(x)\big) = \varphi^n(x)$

  $$T\big(\varphi^{n_1}(x_1), \ldots, \varphi^{n_k}(x_k)\big) = \varphi^{n_1}(x_1) \circ \cdots \circ \varphi^{n_k}(x_k)$$
- Example: $T\big(\varphi^2(x), \varphi(y)\big) = \varphi^2(x) \circ \varphi(y)$
- New definition $T : \mathcal{A} \to \mathcal{A}$ such that

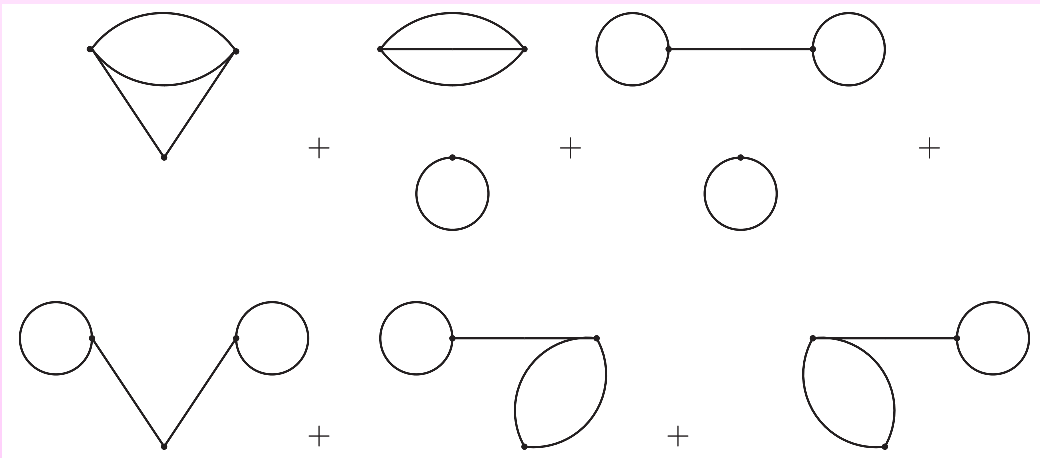
  $T(ab) = T(a) \circ T(b)$ and $T\big(\varphi(x)\big) = \varphi(x)$
- Example: $T\big(\varphi^2(x)\varphi(y)\big) = \varphi(x) \circ \varphi(x) \circ \varphi(y)$
- Feynman integral or functional derivative approaches

Examples

- For $\langle 0 | T(\phi^3(x)\phi^3(y)\phi^2(z)) | 0 \rangle$
- Previous definition

$\gamma =$ 

- New definition: add loops $g(x,x)$, etc.



Exponential form

- A *primitive element* is an element $u \in \mathcal{A}$ such that

  $$\Delta u = u \otimes 1 + 1 \otimes u$$

- The fields $\varphi(x)$ are the primitive elements
- For symmetric $g(x,y)$, define $T : \mathcal{A} \to \mathcal{A}$ by

  $$T(ab) = T(a) \circ T(b),\; T(u) = u$$

- Does not work for the operator product
- The map T is a coregular action: $T(a) = \sum t(a_{(1)})a_{(2)}$ with $t(a) = \varepsilon\big(T(a)\big)$
- Exponential form: $t(a) = e^{\star\tau}(a)$

Convolution

- If τ and σ are linear maps $\mathcal{A} \longrightarrow \mathbb{C}$, their *convolution* is

$$\sigma \star \tau(a) = \sum \sigma(a_{(1)})\tau(a_{(2)})$$

- Our coproduct is cocommutative $\sum a_{(2)} \otimes a_{(1)} = \sum a_{(1)} \otimes a_{(2)}$
- The convolution product is associative and commutative
- The convolution powers of τ are

$$\begin{aligned} \tau^{\star 0}(a) &= \varepsilon(a) \\ \tau^{\star 1}(a) &= \tau(a) \\ \tau^{\star(n+1)}(a) &= \sum \tau(a_{(1)})\tau^{\star n}(a_{(2)}) \end{aligned}$$

- The convolution exponential of τ is

$$e^{\star\tau}(a) = \sum_{n=0}^{\infty} \frac{1}{n!}\tau^{\star n}(a)$$

Convolution exponential

- Let $\tau\big(\varphi(x_1)\ldots\varphi(x_k)\big) = 0$ if $k \neq 2$, $\tau\big(\varphi(x_1)\varphi(x_2)\big) = g(x_1, x_2)$
- Then $t(a) = \langle 0|T(a)|0\rangle = e^{\star\tau}(a)$
- Examples $\langle 0|T\big(\varphi(x)\varphi(y)\big)|0\rangle = g(x,y) = e^{*\tau}\big(\varphi(x)\varphi(y)\big)$

$$\begin{aligned}\langle 0|T\big(\varphi^2(x)\varphi^2(y)\big)|0\rangle &= g(x,x)g(y,y) + 2g(x,y)^2 \\ &= e^{*\tau}\big(\varphi^2(x)\varphi^2(y)\big)\end{aligned}$$

- Equation $T(a) = \sum e^{\star\tau}(a_{(1)})a_{(2)}$ is equivalent to

$$T(a) = e^{\int dxdy\,\frac{1}{2}g(x,y)\,\frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)}}(a)$$

Anderson (1954), Brunetti, Dütsch, Fredenhagen (2009)

Proof (I)

- For any $\tau : \mathcal{A} \to \mathbb{C}$, any $a \in \mathcal{A}$ and any primitive u

$$e^{\star\tau}(au) = \sum e^{\star\tau}(a_{(1)})\tau(a_{(2)}u)$$

- First, prove $\tau^{\star(n+1)}(au) = (n+1)\sum \tau^{\star n}(a_{(1)})\tau(a_{(2)}u)$
  - ⋆ This is true for $n = 0$ because $\tau(au) = \sum \varepsilon(a_{(1)})\tau(a_{(2)}u)$
  - ⋆ If this is true up to n, then

$$\begin{aligned}
\tau^{\star(n+1)}(au) &= \sum \tau^{\star n}((au)_{(1)})\tau((au)_{(2)}) \\
&= \sum \tau^{\star n}(a_{(1)}u)\tau(a_{(2)}) + \sum \tau^{\star n}(a_{(1)})\tau(a_{(2)}u) \\
&= n\sum \tau^{\star(n-1)}(a_{(1)})\tau(a_{(2)}u)\tau(a_{(3)}) + \sum \tau^{\star n}(a_{(1)})\tau(a_{(2)}u)
\end{aligned}$$

- Conclude by

$$e^{\star\tau}(au) = \sum_{n=0}^{\infty} \frac{1}{n!}\tau^{\star n}(au)$$

Proof (II)

- If the only non-zero value of τ is $\tau\big(\varphi(x_1)\varphi(x_2)\big) = g(x_1, x_2)$

  then $t(a) = e^{\star\tau}(a)$

- Induction on the degree of a
  - ⋆ True for $a = 1$, $a = \varphi(x)$ and $a = \varphi(x_1)\varphi(x_2)$
  - ⋆ If this is true up to degree n, take a of degree n

$$\begin{aligned} t(au) &= \varepsilon\big(T(au)\big) = \varepsilon\big(T(a) \circ u\big) \\ &= \sum t(a_{(1)})\varepsilon(a_{(2)} \circ u) = \sum t(a_{(1)})(a_{(2)}|u) \end{aligned}$$

  - ⋆ u is of degree 1, so that $a_{(2)}$ is of degree 1 and $(a_{(2)}|u) = \tau(a_{(2)}u)$
  - ⋆ The recursion hypothesis gives us

$$t(au) = \sum e^{\star\tau}(a_{(1)})\tau(a_{(2)}u) = \sum e^{\star\tau}(au)$$

Convolution inverse

- If $t : \mathcal{A} \to \mathbb{C}$ satisfies $t(1) = 1$, there is a τ such that

  $$t(a) = e^{\star\tau}(a)$$
- The convolution inverse of t satisfies $t \star t^{-1} = \varepsilon$
- We have $t^{-1}(a) = e^{\star(-\tau)}(a)$
- Let $T, T^{-1} : \mathcal{A} \to \mathcal{A}$ with $T(a) = \sum t(a_{(1)})a_{(2)}$ and $T^{-1}(a) = \sum t^{-1}(a_{(1)})a_{(2)}$, then $T^{-1}\big(T(a)\big) = a$
- Recover the product of normal products by

  $$a \circ b = T\big(T^{-1}(a)T^{-1}(b)\big)$$

  Brunetti, Dütsch, Fredenhagen (2009)

Many-body physics

- Hamiltonian $H = H_0 + V$
- Initial state: eigenstate $|\Phi_0\rangle$ of H_0
- When the initial state is non-degenerate, it can be considered as a new vacuum (occupied and unoccupied orbitals)
- In the presence of symmetry, the initial state is often degenerate
- QFT (Wightman and Challifour)
- Degeneracy has very important technological consequences (giant magnetoresistance)

Gemstones

- Chromium in an oxygen octahedron (ruby and emerald)

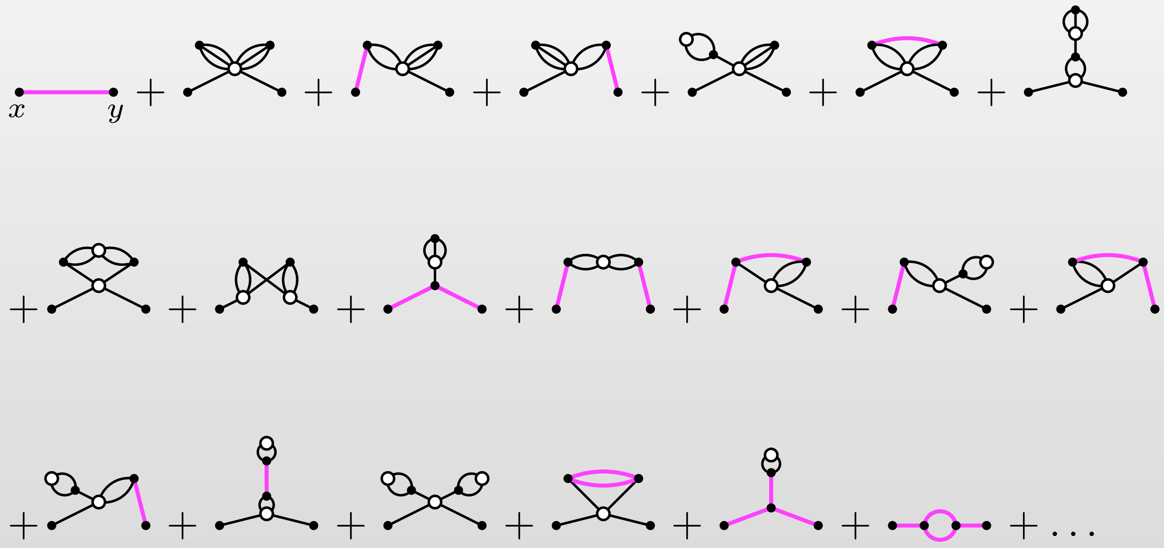
- Fe in beryl (heliodor and aquamarine)

Many-body physics

- Calculate $\langle\Phi_0|T(a)|\Phi_0\rangle$ and the Green functions for a general initial state $|\Phi_0\rangle$
- Many QFT tools break down
  - ★ Wick's theorem fails (pairings are not enough)
  - ★ Green functions cannot be written in terms of Feynman diagrams
  - ★ The Gell-Mann and Low formula does not converge
  - ★ The Dyson equation and the Bethe-Salpeter equation do not hold
- Slow reconstruction of these tools
- Combinatorics is tough

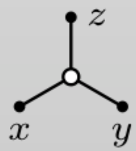
Example

- Perturbative expansion for $\varphi^3(x)$

$G(x,y) =$ 

- Cumulants $D_n(x_1,\ldots,x_n)$ of the initial state

$$\langle\Phi_0|e^{\int dx\varphi(x)j(x)}|\Phi_0\rangle = e^{\sum_{n=1}^{\infty}\frac{1}{n!}\int dx_1\ldots dx_n D_n(x_1,\ldots,x_n)j(x_1)\ldots j(x_n)}$$

- Example $D_3(x,y,z) =$ 

Hopf algebra

- In terms of Hopf algebra, we have simply

$$\langle\Phi_0|T(a)|\Phi_0\rangle = e^{\star\tau}(a)$$

- The only change is

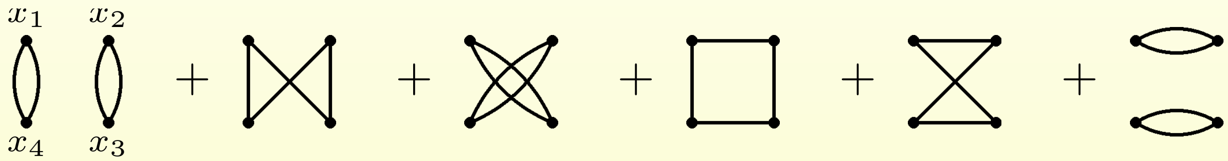
$$\tau\big(\varphi(x_1)\ldots\varphi(x_n)\big) = D_n(x_1,\ldots,x_n)$$

- Decomposition of Green function in terms of connected Green functions
- One-particle irreducible structure of Green functions
- Dyson equation
- Bethe-Salpeter equation
- Effective Hamiltonians

Connected graphs

- How to describe a connected graph ?

$$\sum_\gamma \gamma =$$



- Define a second coproduct

$$\delta\varphi^n(x) = \varphi^n(x) \otimes 1 + 1 \otimes \varphi^n(x)$$

$$\begin{aligned}
\delta\big(\varphi^n(x)\varphi^m(y)\big) &= \big(\delta\varphi^n(x)\big)\big(\delta\varphi^m(y)\big) \\
&= \varphi^n(x)\varphi^m(y) \otimes 1 + \varphi^n(x) \otimes \varphi^m(y) \\
&\quad +\varphi^m(y) \otimes \varphi^n(x) 1 \otimes \varphi^n(x)\varphi^m(y)
\end{aligned}$$

- The convolution product for this coproduct is Ruelle's product

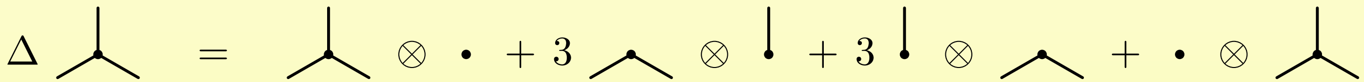
Connecting coproduct

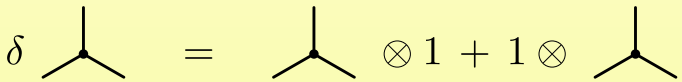
- Distinguish the two coproducts

$$\Delta\varphi^n(x) = \sum_{k=0}^{n} \binom{n}{k} \varphi^k(x) \otimes \varphi^{n-k}(x)$$

$$\delta\varphi^n(x) = \varphi^n(x) \otimes 1 + 1 \otimes \varphi^n(x)$$

- Graphical representation





- Sweedler's notation

$$\Delta a = \sum a_{(1)} \otimes a_{(2)}$$

$$\delta a = \sum a_{[1]} \otimes a_{[2]}$$

A bit of Hopf theory

- $(\mathcal{A},\delta)$ is a coalgebra with the counit defined by

$$\varepsilon_\delta(1)=1,\ \varepsilon_\delta\big(\varphi^n(x)\big)=0 \text{ for all } n$$

- The map $\beta=\Delta$ is a right coaction of $(\mathcal{A},\Delta)$ on $(\mathcal{A},\delta)$

$$\begin{aligned}(\beta\otimes Id)\beta &= (Id\otimes\Delta)\beta\\ (Id\otimes\varepsilon)\beta &= Id\end{aligned}$$

- $(\mathcal{A},\delta)$ is a $(\mathcal{A},\Delta)$-comodule coalgebra

$$\begin{aligned}(\varepsilon_\delta\otimes Id)\beta &= 1\varepsilon_\delta\\ (\delta\otimes Id)\beta &= (Id\otimes Id\otimes\mu)(Id\otimes\tau\otimes Id)(\beta\otimes\beta)\delta\end{aligned}$$

- Sweedler's notation

$$\begin{aligned}\sum\varepsilon_\delta(a_{(1)})a_{(2)} &= \varepsilon_\delta(a)1\\ \sum a_{(1)[1]}\otimes a_{(1)[2]}\otimes a_{(2)} &= \sum a_{[1](1)}\otimes a_{[2](1)}\otimes a_{[1](2)}a_{[2](2)}\end{aligned}$$

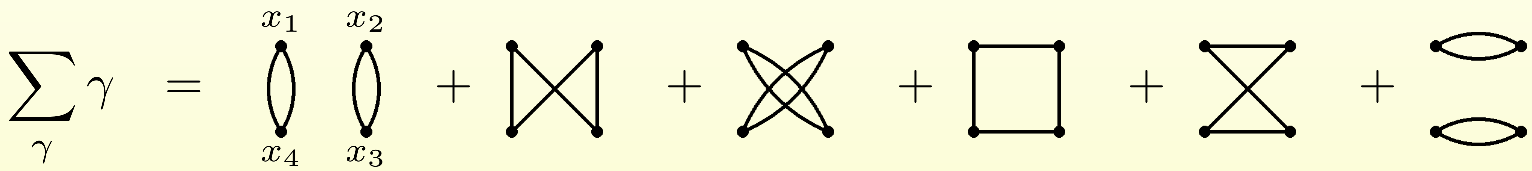
Connected graphs

- For $a = \varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)$, $t(a) = \langle 0|T(a)|0\rangle$ is the sum over all Feynman diagrams with k vertices, with n_i edges attached to vertex x_i
- The sum over connected diagrams is $t_c(a)$
- The relation is
$$t_c(a) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sum t(a_{[\underline{1}]})\ldots t(a_{[\underline{n}]})$$

  where $\sum a_{[\underline{1}]}\otimes a_{[\underline{2}]} = \underline{\Delta} a = \Delta a - a\otimes 1 - 1\otimes a$
- Examples
$$\begin{aligned} t_c(u) &= t(u) \\ t_c(uv) &= t(uv) - t(u)t(v) \\ t_c(uvw) &= t(uvw) - t(u)t(vw) - t(v)t(uw) - t(w)t(uv) + 2t(u)t(v)t(w) \end{aligned}$$

Connected graphs

- For $a = \varphi^2(x_1)\varphi^2(x_2)\varphi^2(x_3)\varphi^2(x_4)$, the graphs are



- The sum over connected graphs is indeed

$$
\begin{aligned}
t_c(a) \quad = \quad & t(a) - t\big(\varphi^2(x_1)\varphi^2(x_2)\big)t\big(\varphi^2(x_3)\varphi^2(x_4)\big) \\
& -t\big(\varphi^2(x_1)\varphi^2(x_3)\big)t\big(\varphi^2(x_2)\varphi^2(x_4)\big) \\
& -t\big(\varphi^2(x_1)\varphi^2(x_4)\big)t\big(\varphi^2(x_2)\varphi^2(x_3)\big)
\end{aligned}
$$

- Well known: Haag (1958), Epstein and Glaser (1973),Fredenhagen et al. (1997,2001), Mestre and Oeckl (2006)

Connected time-ordering

- For $a = \varphi(x_1)\ldots\varphi(x_k)$, $T(a) = \varphi(x_1) \circ \cdots \circ \varphi(x_k)$
- We have $T(a) = \sum t(a_{(1)})a_{(2)}$, with $t(a) = \varepsilon\big(T(a)\big)$
- We defined the connected time-ordering

$$T_c(a) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum T(a_{[\underline{1}]})\ldots T(a_{[\underline{n}]})$$

- Using the comodule coalgebraic structure,

$$T_c(a) = \sum t_c(a_{(1)})a_{(2)}$$

with

$$t_c(a) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum t(a_{[\underline{1}]})\ldots t(a_{[\underline{n}]})$$

Exponential form

- The Ruelle product is

$$f \diamond g(a) = \sum f(a_{[1]}) g(a_{[2]})$$

- If we put $t_c(1) = 0$, then $t(a) = e^{\diamond t_c}(a)$
- A local Lagrangian is primitive: if $\mathcal{L}(x) = \lambda\varphi^4(x)$

$$\delta\mathcal{L}(x) = \mathcal{L}(x) \otimes 1 + 1 \otimes \mathcal{L}(x)$$

- If $u = \int dx \mathcal{L}(x)$, then $\delta e^u = e^u \otimes e^u$
- Thus, $t(e^u) = e^{t_c(e^u)}$. We recover $Z = e^W$
- How about Green functions?

Green functions

- For $a = \varphi(x_1)\ldots\varphi(x_n)$, the Green function is

$$G(a) = G(x_1, \ldots, x_n) = \frac{t(ae^u)}{t(e^u)}$$

- The relations $t(a) = e^{\diamond t_c}(a)$, $\delta e^u = e^u \otimes e^u$ and $e^{\diamond t_c}(va) = \sum t_c(va_{[1]})e^{\diamond t_c}(a_{[2]})$ imply

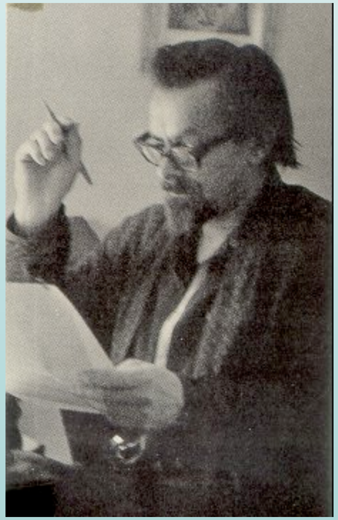
$$t(\varphi(x)e^u) = t_c(\varphi(x)e^u)t(e^u)$$

- In other words $G(x) = G_c(x)$ where $G_c(a) = t_c(ae^u)$
- More generally, assuming $G_c(1) = 0$, the relation between standard and connected Green functions is

$$G(a) = e^{\diamond G_c}(a)$$

Renormalization

- Johann Melchior Ernst Karl Gerlach Stückelberg von Breidenbach zu Breidenstein und Melsbach (1905-1984)
- Nikolay Nikolaevich Bogoliubov (1909-1992)
- Henri Epstein
- Vladimir Jurko Glaser (1924-1984)
- Gudrun Pinter



Transformation of T-products

- For a linear map $\Lambda : \mathcal{A} \to \mathcal{A}$, we define

$$T_\Lambda(a) = \sum_{n=1}^{\infty} \frac{1}{n!} T\big(\Lambda(a_{[\underline{1}]}) \ldots \Lambda(a_{[\underline{n}]})\big)$$

- The reduced coproduct is $\underline{\delta} a = \delta a - a \otimes 1 - 1 \otimes a$
- Its iteration is

$$\begin{aligned} \underline{\delta}^0 &= Id \\ \underline{\delta}^1 &= \underline{\delta} \\ \underline{\delta}^{n+1} &= (\underline{\delta} \otimes Id^{\otimes n})\underline{\delta}^n \end{aligned}$$

- Notation $\underline{\delta}^{n-1} a = \sum a_{[\underline{1}]} \otimes \cdots \otimes a_{[\underline{n}]}$
- If $\Lambda(1) = 0$, then $T_\Lambda(a) = T\big(e^{\diamond \Lambda}(a)\big)$

Transformation of T-products

- Example $a = \varphi^3(x)\varphi^3(y)$
- Reduced coproduct

$$\begin{aligned} \underline{\delta}^0 a &= \varphi^3(x)\varphi^3(y) \\ \underline{\delta}^1 a &= \varphi^3(x) \otimes \varphi^3(y) + \varphi^3(y) \otimes \varphi^3(x) \\ \underline{\delta}^n a &= 0, \quad n > 1 \end{aligned}$$

- Transformed T-product

$$T_\Lambda(a) = T\Big(\Lambda(\varphi^3(x)\varphi^3(y))\Big) + T\Big(\Lambda(\varphi^3(x))\Lambda(\varphi^3(y))\Big)$$

- Λ is called a generalized vertex by Bogoliubov

Resummation

- Bogoliubov-Pinter $T_\Lambda(a) = \sum_{n=1}^{\infty} \frac{1}{n!} T\big(\Lambda(a_{[\underline{1}]}) \ldots \Lambda(a_{[\underline{n}]})\big)$
- For a local Lagrangian $u = \int \varphi^n(x)dx$

$$\underline{\delta}(e^u - 1) = (e^u - 1) \otimes (e^u - 1)$$

- Thus, $\begin{aligned} T_\Lambda(e^u) &= T_\Lambda(1) + T_\Lambda(e^u - 1) \\ &= T_\Lambda(1) + T\big(e^{\Lambda(e^u - 1)} - 1\big) \end{aligned}$
- If we assume $T_\Lambda(1) = T(1) = 1$, then

$$T_\Lambda(e^u) = T\big(e^{\diamond\Lambda}(e^u)\big) = T(e^{u_\Lambda})$$

- The renormalized Lagrangian is $u_\Lambda = \Lambda(e^u - 1)$

Renormalized Lagrangian

- The renormalized Lagrangian is $u_\Lambda = \Lambda(e^u - 1)$
- The image of Λ is local (i.e. primitive)
- For example

$$\Lambda\Big(\varphi^4(x_1)\ldots\varphi^4(x_n)\Big) = \int dx\delta(x_1 - x)\ldots\delta(x_n - x)$$
$$\Big(C_{n1}\varphi^2(x) + C_{n2}\varphi^4(x) + C_{n3}\varphi(x)\Box\varphi(x)\Big)$$

- In general $\Lambda(u) = u$ is the bare Lagrangian and $u_\Lambda - u$ are the counterterms
- Compatible with connectivity

Renormalization group (I)

- The renormalize Lagrangian u_Λ is related to the bare Lagrangian u by $e^{\diamond\Lambda}(e^u) = e^{u_\Lambda}$
- We compose two renormalizations $e^{\diamond\Lambda'}\big(e^{\diamond\Lambda}(a)\big)$
- For example

$$e^{\diamond\Lambda}(uv) = \Lambda(uv) + \Lambda(u)\Lambda(v)$$

$$e^{\diamond\Lambda'}\big(e^{\diamond\Lambda}(uv)\big) = \Lambda'\big(\Lambda(uv)\big) + \Lambda'\big(\Lambda(u)\Lambda(v)\big) + \Lambda'\big(\Lambda(u)\big)\Lambda'\big(\Lambda(v)\big)$$

- We define the product $\Lambda' \bullet \Lambda(a) = \Lambda'\big(e^{\diamond\Lambda}(a)\big)$
- We have $e^{\diamond\Lambda'}\big(e^{\diamond\Lambda}(a)\big) = e^{\diamond(\Lambda'\bullet\Lambda)}(a)$

Renormalization group (II)

- Let $\mathcal{L}$ be the set of linear maps from $\mathcal{A}$ to the primitive elements of $\mathcal{A}$
- $\mathcal{L}$ endowed with the product $\bullet$ is a unital associative algebra
- The unit of this algebra is the map Λ_0 that is the identity for primitive elements and zero otherwise
- The invertible elements of this algebra are the maps that are invertible over the primitive elements
- In the standard case $\Lambda(u) = u$ and we have a group

Conclusion

- Gauge theory: the Slavnov-Taylor identities generate a Hopf ideal of the Hopf algebra of renormalization (van Suijlekom)
- Noncommutative analogues of connectivity, one-particle irreducibility, renormalization
- Extension to non-relativistic case (same time instead of same spacetime point)
- Open problems
  - ★ Direct connection with BPHZ
  - ★ Rigorous formulation of the quantum field Hopf algebra over spacetime

Analytical questions

- Product of fields at a point

$$\varphi^n(x)\varphi^m(x) = \varphi^{n+m}(x)$$

- Coproduct of smoothed fields $\varphi(g) = \int dx \varphi(x) g(x)$

$$\Delta\varphi^n(g) = \sum_{k=0}^{n} \binom{n}{k} \int dx\, \varphi^k(x) \otimes \varphi^{n-k}(x) g(x)$$

- Product of the polynomial algebra in the variable φ by the manifold $\mathbb{R}^4$

- For a general manifold, define a Hopf algebra bundle

Analytical questions (II)

- Dütsch and Fredenhagen: Non linear evaluation functionals: $\varphi^n(x)[h] = h^n(x)$
- Smoothed fields: $\varphi^n(g)[h] = \int dx\, h^n(x)\, g(x)$
- Coproduct

$$\begin{aligned} \Delta\varphi^n(x)[h_1, h_2] &= \sum_{k=0}^{n} \binom{n}{k} \varphi^k(x)[h_1] \otimes \varphi^{n-k}(x)[h_2] \\ &= \varphi^n(x)[h_1 + h_2] \end{aligned}$$

- Smoothed coproduct

$$\begin{aligned} \Delta\varphi^n(g)[h_1, h_2] &= \sum_{k=0}^{n} \binom{n}{k} \int dx \big(h_1^k(x) \otimes h_2^{n-k}(x)\big) g(x) \\ &= \varphi^n(g)[h_1 + h_2] \end{aligned}$$