Hopf algebraic structures of quantum field and many-body theories (I)

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Outline

- **Lecture 1**: From Hopf to Feynman diagrams
  - Coproduct
  - Hopf algebra
  - Twisted product
  - Feynman diagrams (with proofs)

- **Lecture 2**:
  - Cumulant expansion
  - A new coproduct for connected diagrams
  - Structure of Green functions
  - Renormalization

Outline of this lecture

- **Hopf algebras**
  - Simple examples **PROVIDES ALGORITHMS**
  - Coproduct
  - Convolution

- **Quantum fields**
  - Classical fields
  - Normal products
  - Hopf algebra

- **Twisted products**
  - Laplace pairing
  - Wick theorem
  - Operator and chronological products

- **Feynman diagrams**
  - Hopf algebra formula
  - Diagrammatic interpretation
  - Exponential formula

Introduction

★ **Mathematical framework for the combinatorial structure of QFT**

- **Combinatorics of QFT and statistical physics**
  - Operator, chronological and normal products
  - Wick theorem
  - Feynman diagrams
- **Natural Hopf-like constructions**
  - Vacuum expectation values
  - Ruelle product (1969)
  - Wightman, Challifour (1970) and Stora (1973)
- **Hopfish intepretation**
  - Simplifies derivations
  - Removes combinatorics
  - Used for introducing QFT (Gracia-Bondia)

What is a coproduct?

- Leibniz rule: Let $\partial = \frac{d}{dx}$

$$\begin{aligned}\partial(fg) &= (\partial f)g + f(\partial g)\\ \partial^2(fg) &= (\partial^2 f)g + 2(\partial f)(\partial g) + f(\partial^2\end{aligned}$$

- How to remove f and g?

$$\begin{aligned}\partial(fg) &= (\partial \otimes Id + Id \otimes \partial)(f \otimes g)\\ \partial^2(fg) &= (\partial \otimes Id + Id \otimes \partial)^2(f \otimes g)\end{aligned}$$

- The rule is: $(a \otimes b)(c \otimes d) = (ac) \otimes (bd)$
- Coproduct

$$\begin{aligned}\Delta\partial &= \partial \otimes Id + Id \otimes \partial\\ \Delta\partial^2 &= (\partial \otimes Id + Id \otimes \partial)^2\\ &= \partial^2 \otimes Id + 2\partial \otimes \partial + Id \otimes \partial^2\end{aligned}$$

Action on 2-body wavefunctions

- On one-body wavefunctions $\varphi_n(\mathbf{r})$

$$L_z\,\varphi_n(\mathbf{r})$$

- Two-body wavefunctions

$$\Phi_{nm}(\mathbf{r}_1,\mathbf{r}_2) = \frac{\varphi_n(\mathbf{r}_1)\otimes\varphi_m(\mathbf{r}_2) - \varphi_m(\mathbf{r}_1)\otimes\varphi_n(\mathbf{r}_2)}{\sqrt{2}}$$

- Angular momentum

$$(L_z \otimes Id + Id \otimes L_z)\Phi_{nm}(\mathbf{r}_1,\mathbf{r}_2) = \Delta L_z \Phi_{nm}(\mathbf{r}_1,\mathbf{r}_2)$$

- For $L^2 = L_x^2 + L_y^2 + L_z^2$ we have $\Delta L^2\,\Phi_{nm}(\mathbf{r}_1,\mathbf{r}_2)$

$$\Delta L_z^2 = L_z^2 \otimes Id + 2L_z \otimes L_z + Id \otimes L_z^2$$

Ruelle's product

- Families of symmetric functions $f_n(x_1, \ldots, x_n)$
- Set $X = \{x_1, \ldots, x_n\}$
- Ruelle's product

$$(f \star g)(X) = \sum_{I \subset X} f_{|I|}(I) g_{|X/I|}(X/I)$$

- Example

$$\begin{aligned}(f \star g)(x_1, x_2) &= f_2(x_1, x_2) g_0 + f_1(x_1) g_1(x_2) \\ &\quad + f_1(x_2) g_1(x_1) + f_0\, g_2(x_1, x_2)\end{aligned}$$

- Coproduct $\Delta X = \sum_{I \subset X} I \otimes X/I$

$$\Delta\{x_1, x_2\} = \{x_1, x_2\} \otimes \emptyset + \{x_1\} \otimes \{x_2\} + \{x_2\} \otimes \{x_1\} + \emptyset \otimes \{x_1, x_2\}$$

The coproduct (I)

- The coproduct of an element is the sum of all the ways to split this element into two parts

$$\Delta\{x_1,x_2\} = \{x_1,x_2\}\otimes\emptyset + \{x_1\}\otimes\{x_2\} + \{x_2\}\otimes\{x_1\} + \emptyset\otimes\{x_1,x_2\}$$

- Sweedler's notation

$$\Delta a = \sum a_{(1)} \otimes a_{(2)}$$

- Split into three parts (coassociativity)

$$\Delta^2 a = \sum (\Delta a_{(1)}) \otimes a_{(2)} = \sum a_{(1)} \otimes (\Delta a_{(2)})$$

- Sweedler's notation

$$\begin{aligned}\Delta^2 a &= \sum a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = \sum a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} \\ &= \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}\end{aligned}$$

The coproduct (II)

- Derivative of a product of three functions

$$D(fgh) = \sum (D_{(1)}f)\,(D_{(2)}g)\,(D_{(3)}h)$$

- Example

$$\Delta^2\partial = \partial \otimes Id \otimes Id + Id \otimes \partial \otimes Id + Id \otimes Id \otimes \partial$$

$$\partial(fgh) = (\partial f)gh + f(\partial g)h + fg(\partial h)$$

- Action of a one-body operator on a three-body wavefunction

$$\sum (O_{(1)} \otimes O_{(2)} \otimes O_{(3)})\Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$$

- Examples

$$\Delta^2 L_z = L_z \otimes Id \otimes Id + Id \otimes L_z \otimes Id + Id \otimes Id \otimes L_z$$

$$\begin{aligned}\Delta^2 L_z^2 \quad &= \quad L_z^2 \otimes Id \otimes Id + Id \otimes L_z^2 \otimes Id + Id \otimes Id \otimes L_z^2 \\ &\quad +2Id \otimes L_z \otimes L_z + 2L_z \otimes Id \otimes L_z + 2L_z \otimes L_z \otimes Id\end{aligned}$$

The coproduct (III)

- An algebra $\mathcal{A}$
- A coassociative linear map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$
- Polynomial algebra in the variable ∂ with unit Id

$$\Delta\partial^n = (\Delta\partial)^n = \sum_{k=0}^{n} \binom{n}{k} \partial^k \otimes \partial^{n-k}$$

- Polynomial algebra in the variables L_x, L_y and L_z
- The algebra of sets is generated as a vector space by the sets $X = \{x_1, \ldots, x_n\}$ with the disjoint union as a product and the empty space as a unit
- For a group algebra $\Delta g = g \otimes g$ (diagonal map)

Algebra morphism

- For any $a, b \in \mathcal{A}$

  $\Delta(ab) = (\Delta a)(\Delta b)$
- Sweedler's notation

$$\sum (ab)_{(1)} \otimes (ab)_{(2)} = \sum \sum a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}$$

- Example

$$\begin{aligned} \Delta \partial^2 &= (\partial \otimes Id + Id \otimes \partial)(\partial \otimes Id + Id \otimes \partial) \\ &= \partial^2 \otimes Id + 2\partial \otimes \partial + Id \otimes \partial^2 \end{aligned}$$

$$\Delta \partial^n = (\Delta \partial)^n = (\partial \otimes Id + Id \otimes \partial)^n = \sum_{k=0}^{n} \binom{n}{k} \partial^k \otimes \partial^{n-k}$$

$(\partial^0 = Id)$

QFT and Hopf algebra

- ★ Anything that you can do with Hopf algebra, you can also do without it, and the reverse is not true
- Ruelle's product (1969)
- Wightman and Challifour (1970): triple dots for QFT with a general vacuum
- Stora (Les Houches, 1971)
- Connes et Kreimer (1998 – 2002): renormalization
- Fauser (2001): Wick's theorem
- B., Fauser, Frabetti, Oeckl (2004): chronological product

QFT based on normal product

- A very old and natural idea
  - Gupta, *Phys. Rev.* **107** (1957) 1722
- Quantum fields as deformations of classical fields
  - Dito *J. Math. Phys.* **33** (1992) 791
  - Hirshfelder & Henselder *Ann. Phys.* **308** (2003) 311
- Nonlinear evaluation functionals (Dütsch and Fredenhagen)
  - D & F *Commun. Math. Phys.* **219** (2001) 5
  - D & F *Commun. Math. Phys.* **243** (2003) 275
  - D & F *Rev. Math. Phys.* **16** (2004) 1291

Hopf algebra of classical fields

- Product of fields at a point

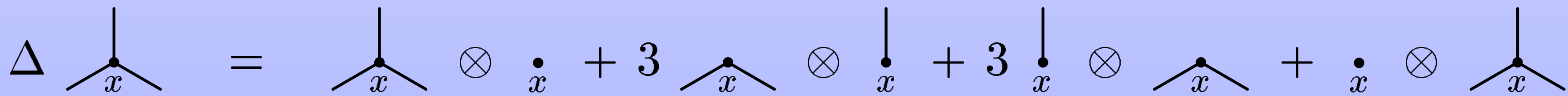
$$\varphi^n(x)\varphi^m(x) = \varphi^{n+m}(x)$$

- Coproduct of fields at a point:

$$\Delta\varphi(x) = \varphi(x) \otimes 1 + 1 \otimes \varphi(x)$$

$$\Delta\varphi^n(x) = \sum_{k=0}^{n} \binom{n}{k} \varphi^k(x) \otimes \varphi^{n-k}(x)$$

- Graphical notation for $\varphi^3(x)$



Counit = VEV

- For normal products of fields the vacuum expectation

$$\langle 0|\varphi^n(x)|0\rangle = \langle 0|{:}\varphi^n(x){:}|0\rangle = \delta_{n,0}$$

- The counit is defined by

$$\varepsilon\big(\varphi^n(x)\big) = \delta_{n,0}$$

- The counit ε of a Hopf algebra $\mathcal{A}$ is a linear map from $\mathcal{A}$ to $\mathbb{C}$ such that

$$\sum \varepsilon(a_{(1)})a_{(2)} = \sum a_{(1)}\varepsilon\big(a_{(2)}\big) = a$$

- Example

$$\sum_{k=0}^{n} \binom{n}{k} \varepsilon\big(\varphi^k(x)\big)\varphi^{n-k}(x) = \varphi^n(x)$$

Hopf algebra definition

- A Hopf algebra is an algebra $\mathcal{A}$ with unit 1 and with
- A coassociative coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ which is an algebra morphism
- A counit $\varepsilon : \mathcal{A} \to \mathbb{C}$ such that

$$\sum \varepsilon(a_{(1)})a_{(2)} = \sum a_{(1)}\varepsilon(a_{(2)}) = a$$

- The counit is an algebra morphism

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$$

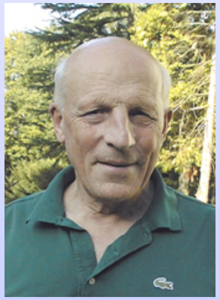
- An antipode $S : \mathcal{A} \to \mathcal{A}$

$$\sum S(a_{(1)})a_{(2)} = \sum a_{(1)}S(a_{(2)}) = \varepsilon(a)1$$

Hopf algebra

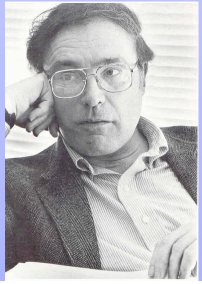
- **Heinz Hopf (1894 - 1971)**
  constructed the first one (1941)

- **Armand Borel (1923 - 2003)**
  coined the name (1952)



- **Pierre Cartier**
  connected cocommutative bialgebra (1955-6)
- **John W. Milnor and John C. Moore**
  connected bialgebra (1959-65)

- **Bertram Kostant**
  modern sense (1966)



- **Giancarlo Rota (1932 - 1999)**
  applied it to combinatorics (1979)

Hopf algebra of fields

- $\mathcal{A}$ is the polynomial algebra generated by the fields at a finite set of distinct points

- The coproduct is

$$\Delta\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) = \sum_{j_1=0}^{n_1}\cdots\sum_{j_k=0}^{n_k}\binom{n_1}{j_1}\cdots\binom{n_k}{j_k} \varphi^{j_1}(x_1)\cdots\varphi^{j_k}(x_k)\otimes\varphi^{n_1-j_1}(x_1)\cdots\varphi^{n_k-j_k}(x_k)$$

- The counit satisfies

$$\varepsilon\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) = \delta_{n_1,0}\ldots\delta_{n_k,0}$$

- The antipode

$$S\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) = (-1)^{n_1+\cdots+n_k}\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)$$

Dual Hopf algebra (I)

- The dual $\mathcal{A}^*$ of a Hopf algebra $\mathcal{A}$ is a Hopf algebra
- The coproduct in $\mathcal{A}^*$ is defined from the product in $\mathcal{A}$

  $$\langle \Delta a^*, b \otimes c \rangle = \langle a^*, bc \rangle$$

- Exemple: dual of the product of integers
  - ★ What is $\Delta 6^*$?
  - ★ For any integers i, j, $\langle \Delta 6^*, i \otimes j \rangle = \langle 6^*, ij \rangle$
  - ★ We need $ij = 6$
  - ★ Thus, $\Delta 6^* = 1^* \otimes 6^* + 2^* \otimes 3^* + 3^* \otimes 2^* + 6^* \otimes 1^*$

Dual Hopf algebra (II)

- The product in $\mathcal{A}^*$ is defined from the coproduct in $\mathcal{A}$

$$\langle a^* b^*, c\rangle = \langle a^* \otimes b^*, \Delta c\rangle = \sum \langle a^*, c_{(1)}\rangle \langle b^*, c_{(2)}\rangle$$

- Example $\Delta \partial^n = (\Delta \partial)^n = \sum_{k=0}^{n} \binom{n}{k} \partial^k \otimes \partial^{n-k}$

$$\langle \partial^{*i} \partial^{*j}, \partial^n\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle \partial^{*i} \otimes \partial^{*j}, \partial^k \otimes \partial^{n-k}\rangle = \delta_{n,i+j} \binom{n}{i}$$

- Thus, $\partial^{*i} \partial^{*j} = \binom{i+j}{i} \partial^{*(i+j)}$
- The unit is $\langle 1^*, a\rangle = \varepsilon(a)$
- The counit is $\varepsilon^*(a^*) = \langle a^*, 1\rangle$
- The antipode is $\langle S^*(a^*), b\rangle = \langle a^*, S(b)\rangle$

Laplace pairing

- For the kind of Hopf algebra that we consider, a Laplace pairing is a bilinear map $(.|.) : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$ such that, for $a, b, c \in \mathcal{A}$

$$\begin{aligned} (ab|c) &= \sum (a|c_{(1)})(b|c_{(2)}) \\ (a|bc) &= \sum (a_{(1)}|b)(a_{(2)}|c) \end{aligned}$$

- In particular $(a|1) = (1|a) = \varepsilon(a)$ $\qquad \big(\varphi^n(x)|1\big) = \delta_{n,0}$

- Very natural object: bicharacter (Borcherds), coquasitriangular structure (Drinfeld), a braiding (Hodges), an R-form (Jacobs), a universal r-form (Schmugden), a duality pairing (Majid), a Hopf pairing (Brown Goodearl)

- For fermionic variables, these relations are equivalent to the Laplace identity of determinants

Examples

- The Laplace coupling can be written in terms of

$$\big(\varphi(x)|\varphi(y)\big) = g(x,y)$$

- We use the rule $(a|bc) = \sum (a_{(1)}|b)(a_{(2)}|c)$

$$\Delta\varphi(x) = 1\otimes\varphi(x) + \varphi(x)\otimes 1$$

$$\big(\varphi(x)|\varphi(y)\varphi(z)\big) = \big(1|\varphi(y)\big)\big(\varphi(x)|\varphi(z)\big) + \big(\varphi(x)|\varphi(y)\big)\big(1|\varphi(z)\big)$$

$$\big(\varphi(x)|\varphi(y)\varphi(z)\big) = 0$$

- Similarly $\Delta\varphi^2(x) = 1\otimes\varphi^2(x) + 2\varphi(x)\otimes\varphi(x) + \varphi^2(x)\otimes 1$

$$\big(\varphi^2(x)|\varphi(y)\varphi(z)\big) = 2\big(\varphi(x)|\varphi(y)\big)\big(\varphi(x)|\varphi(z)\big) = 2g(x,y)g(x,z)$$

- More general example

$$\big(\varphi^m(x)|\varphi^n(y)\big) = \delta_{m,n} n!\, g^n(x,y)$$

Proof

- We want to prove $\big(\varphi^m(x)|\varphi^n(y)\big) = \delta_{m,n} n!\, g^n(x,y)$ by induction
- This is true for $n = m = 1$
- This is true for $m = 1$ and all $n > 1$

$$\begin{aligned} \big(\varphi(x)|\varphi^{n+1}(y)\big) &= \big(\varphi(x)|\varphi^n(y)\big)\big(1|\varphi(y)\big) + \big(1|\varphi^n(y)\big)\big(\varphi(x)|\varphi(y)\big) \\ &= 0 \end{aligned}$$

- This is true for all m and n

$$\begin{aligned} \big(\varphi^{1+m}(x)|\varphi^n(y)\big) &= \sum_{k=0}^{n} \binom{n}{k} \big(\varphi(x)|\varphi^k(y)\big)\big(\varphi^m(x)|\varphi^{n-k}(y)\big) \\ &= n\big(\varphi(x)|\varphi(y)\big)\big(\varphi^m(x)|\varphi^{n-1}(y)\big) \\ &= ng(x,y)\delta_{n,m+1}(n-1)!g^{n-1}(x,y) \end{aligned}$$

Wick's theorem

- For $a, b \in \mathcal{A}$ we define the twisted (or smash or circle) product by
$$a \circ b = \sum (a_{(1)}|b_{(1)})\, a_{(2)} b_{(2)}$$
- Example
$$\Delta\varphi(x) = 1 \otimes \varphi(x) + \varphi(x) \otimes 1, \Delta\varphi(y) = 1 \otimes \varphi(y) + \varphi(y) \otimes 1$$
$$\begin{aligned} \varphi(x) \circ \varphi(y) &= \big(1|1\big)\varphi(x)\varphi(y) + \big(\varphi(x)|1\big)\varphi(y) \\ &\quad + \big(1|\varphi(y)\big)\varphi(x) + \big(\varphi(x)|\varphi(y)\big)1 \end{aligned}$$
$$\varphi(x) \circ \varphi(y) = \varphi(x)\varphi(y) + g(x,y)$$
- Other examples
$$\varphi^2(x) \circ \varphi(y) = \varphi^2(x)\varphi(y) + 2g(x,y)\,\varphi(x)$$
$$\varphi^2(x) \circ \varphi^2(y) = \varphi^2(x)\varphi^2(y) + 4g(x,y)\,\varphi(x)\varphi(y) + 4g^2(x,y)$$

Twisted products

- If $g(x,y) = \langle 0|T\big(\varphi(x)\varphi(y)\big)|0\rangle$ (properly regularized), the twisted product is the (commutative) chronological (or time-ordered) product
- If $g(x,y) = \langle 0|\varphi(x)\cdot\varphi(y)|0\rangle$ (properly regularized), the twisted product is the (noncommutative) operator product
- The twisted product is associative. Therefore, the (unrenormalized) time-ordered product is associative
- Iterated chronological product $T : \mathcal{A} \to \mathcal{A}$

$$T\big(\varphi^n(x)\big) = \varphi^n(x)$$

$$T\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) = \varphi^{n_1}(x_1)\circ\cdots\circ\varphi^{n_k}(x_k)$$

Applications

- The S-matrix

$$S = T\left(e^{-i\lambda \int dx \varphi^4(x)}\right)$$

- Perturbative expansion of the S-matrix

$$S = 1 + \sum_{k=1}^{\infty} \frac{(-i\lambda)^k}{k!} \int dx_1 \ldots dx_k T\big(\varphi^4(x_1) \ldots \varphi^4(x_k)\big)$$

- Green functions

$$G(x,y) = \frac{\langle 0|T\left(\varphi(x)\varphi(y)e^{-i\lambda \int dz \varphi^4(z)}\right)|0\rangle}{\langle 0|S|0\rangle}$$

QFT basic formulas

- In the QFT notation (Epstein-Glaser)

$$T\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) \quad = \quad \sum_{p_1=0}^{n_1}\cdots\sum_{p_k=0}^{n_k}\binom{n_1}{p_1}\cdots\binom{n_k}{p_k}\langle 0\big|T\big(\varphi^{p_1}(x_1)\ldots\varphi^{p_k}(x_k)\big)\big|0\rangle$$
$$\varphi^{n_1-p_1}(x_1)\ldots\varphi^{n_k-p_k}(x_k)$$

- Hopf calculation:

$$\langle 0\big|T\big(\varphi^{p_1}(x_1)\ldots\varphi^{p_k}(x_k)\big)\big|0\rangle = p_1!\ldots p_k!\sum_M\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{g(x_i,x_j)^{m_{ij}}}{m_{ij}!}$$

where $M=\{m_{ij}\}$ runs over all symmetric nonnegative integer matrices with zero diagonal such that $\sum_j m_{ij}=p_i$

The first equation (I)

- The Hopf algebraic translation of

$$T\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) \quad = \quad \sum_{p_1=0}^{n_1}\cdots\sum_{p_k=0}^{n_k}\binom{n_1}{p_1}\ldots\binom{n_k}{p_k}\langle 0|T\big(\varphi^{p_1}(x_1)\ldots\varphi^{p_k}(x_k)\big)|0\rangle$$
$$\varphi^{n_1-p_1}(x_1)\ldots\varphi^{n_k-p_k}(x_k)$$

- We have
  - ★ $T\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) = \varphi^{n_1}(x_1)\circ\cdots\circ\varphi^{n_k}(x_k)$
  - ★ $\langle 0|a|0\rangle = \varepsilon(a)$
  - ★ $\Delta\big(\varphi^{n_1}(x_1)\ldots\varphi^{n_k}(x_k)\big) \quad = \quad \sum_{j_1=0}^{n_1}\cdots\sum_{j_k=0}^{n_k}\binom{n_1}{j_1}\cdots\binom{n_k}{j_k}$
$$\varphi^{j_1}(x_1)\cdots\varphi^{j_k}(x_k)\otimes\varphi^{n_1-j_1}(x_1)\cdots\varphi^{n_k-j_k}(x_k)$$

- This gives us

$$a^1\circ\cdots\circ a^k = \sum\varepsilon(a^1_{(1)}\circ\cdots\circ a^k_{(1)})a^1_{(2)}\cdots a^k_{(2)}$$

The first equation (II)

- We prove $a^1 \circ \cdots \circ a^k = \sum \varepsilon(a^1_{(1)} \circ \cdots \circ a^k_{(1)}) a^1_{(2)} \cdots a^k_{(2)}$

- We show that, for $a, b \in \mathcal{A}$, $\varepsilon(a \circ b) = (a|b)$

$$\begin{aligned} \varepsilon(a \circ b) &= \sum (a_{(1)}|b_{(1)})\, \varepsilon(a_{(2)} b_{(2)}) = \sum (a_{(1)}|b_{(1)})\, \varepsilon(a_{(2)}) \varepsilon(b_{(2)}) \\ &= \left(\sum a_{(1)} \varepsilon(a_{(2)}) \middle| \sum b_{(1)} \varepsilon(b_{(2)}) \right) = (a|b) \end{aligned}$$

- We show that $\Delta\, a \circ b = \sum a_{(1)} \circ b_{(1)} \otimes a_{(2)} b_{(2)}$

$$\begin{aligned} \Delta\, a \circ b &= \sum (a_{(1)}|b_{(1)}) \Delta\, a_{(2)} b_{(2)} \\ &= \sum (a_{(1)}|b_{(1)})\, a_{(2)(1)} b_{(2)(1)} \otimes a_{(2)(2)} b_{(2)(2)} \\ &= \sum \left(a_{(1)(1)} \middle| b_{(1)(1)} \right) a_{(1)(2)} b_{(1)(2)} \otimes a_{(2)} b_{(2)} \\ &= \sum a_{(1)} \circ b_{(1)} \otimes a_{(2)} b_{(2)} \end{aligned}$$

The first equation (III)

- Proof of $\Delta\, a^1 \circ \cdots \circ a^k = \sum a^1_{(1)} \circ \cdots \circ a^k_{(1)} \otimes a^1_{(2)} \cdots a^k_{(2)}$
- True for $k = 1$by definition of the coproduct
- If true up to k, put $U = a^1 \circ \cdots \circ a^k$

$$\Delta\, U \circ a^{k+1} = \sum U_{(1)} \circ a^{k+1}_{(1)} \otimes U_{(2)} a^{k+1}_{(2)}$$

- Proof of $a^1 \circ \cdots \circ a^k = \sum \varepsilon(a^1_{(1)} \circ \cdots \circ a^k_{(1)}) a^1_{(2)} \cdots a^k_{(2)}$
- True for $k = 1$ by definition of the counit
- If true up to k, put $U = a^1 \circ \cdots \circ a^k$

$$\begin{aligned} U \circ a^{k+1} &= \sum (U_{(1)} | a^{k+1}_{(1)})\, U_{(2)} a^{k+1}_{(2)} = \sum \varepsilon\big(U_{(1)} \circ a^{k+1}_{(1)}\big)\, U_{(2)} a^{k+1}_{(2)} \\ &= \sum \varepsilon(a^1_{(1)} \circ \cdots \circ a^{k+1}_{(1)}) a^1_{(2)} \cdots a^{k+1}_{(2)} \end{aligned}$$

The second equation (I)

- Hopf algebraic translation of

$$\langle 0|T(\phi^{p_1}(x_1)\dots\phi^{p_k}(x_k))|0\rangle = p_1!\dots p_k!\sum_M \prod_{1\le i<j\le k} \frac{g(x_i,x_j)^{m_{ij}}}{m_{ij}!}$$

- The left hand side is $\varepsilon(a^1\circ\cdots\circ a^k)$
- Examples $\varepsilon(a\circ b) = (a|b)$

$$\varepsilon(a\circ b\circ c) = \sum (a_{(1)}|b_{(1)})(a_{(2)}|c_{(1)})(b_{(2)}|c_{(2)})$$

$$\begin{aligned}\varepsilon(a^1\circ a^2\circ a^3\circ a^4) \quad &= \quad \sum (a^1_{(1)}|a^2_{(1)})(a^1_{(2)}|a^3_{(1)})(a^1_{(3)}|a^4_{(1)})\\ &\quad (a^2_{(2)}|a^3_{(2)})(a^2_{(3)}|a^4_{(2)})(a^3_{(3)}|a^4_{(3)})\end{aligned}$$

- Iterated coproduct $\Delta^{k-2}a = \sum a_{(1)}\otimes\cdots\otimes a_{(k-1)}$
- The general case $\varepsilon(a^1\circ\cdots\circ a^k) = \sum \prod_{1\le i<j\le k} (a^i_{(j-1)}|a^j_{(i)})$

The second equation (II)

- Proof of $\varepsilon(a^1 \circ \cdots \circ a^k) = \sum \prod_{i=1}^{k-1} \prod_{j=i+1}^{k} (a^i_{(j-1)} | a^j_{(i)})$
- True for $k = 2$, let $U = a^1 \circ \cdots \circ a^k$

$$\begin{aligned}
\varepsilon(U \circ a^{k+1}) &= (U|a^{k+1}) = \sum \varepsilon(U_{(1)})(U_{(2)}|a^{k+1}) \\
&= \sum \varepsilon(a^1_{(1)} \circ \cdots \circ a^k_{(1)})(a^1_{(2)} \cdots a^k_{(2)}|a^{k+1}) \\
&= \sum \varepsilon(a^1_{(1)} \circ \cdots \circ a^k_{(1)}) \prod_{q=1}^{k} (a^q_{(2)}|a^{k+1}_{(q)}) \\
\varepsilon(U \circ a^{k+1}) &= \sum \prod_{i=1}^{k-1} \prod_{j=i+1}^{k} (a^i_{(j-1)}|a^j_{(i)}) \prod_{q=1}^{k} (a^q_{(k)}|a^{k+1}_{(q)}) \\
&= \sum \prod_{i=1}^{k-1} \prod_{j=i+1}^{k+1} (a^i_{(j-1)}|a^j_{(i)})(a^k_{(k)}|a^{k+1}_{(k)})
\end{aligned}$$

The second equation (III)

- Iterated coproduct

$$\Delta^{k-2}\varphi^p(x) = \sum_{q_1+\cdots+q_{k-1}=p} \frac{p!}{q_1!\ldots q_{k-1}!}\varphi^{q_1}(x)\otimes\cdots\otimes\varphi^{q_{k-1}}(x)$$

- Let $U = \varphi^{p_1}(x_1)\circ\cdots\circ\varphi^{p_k}(x_k)$

$$\varepsilon(U) \quad = \quad \sum_Q \Big(\prod_{i=1}^{k}\frac{p_i!}{q_{i1}!\ldots q_{ik-1}!}\Big)\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\big(\varphi^{q_{ij-1}}(x_i)|\varphi^{q_{ji}}(x_j)\big)$$

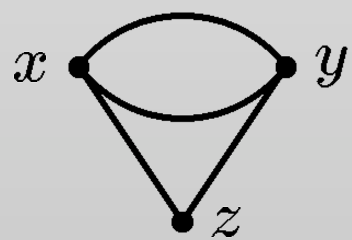
- The $k\times(k-1)$ matrix Q satisfies $q_{i1}+\cdots+q_{ik-1}=p_i$
- We use $\big(\varphi^{q_{ij-1}}(x_i)|\varphi^{q_{ji}}(x_j)\big) = \delta_{q_{ij-1},q_{ji}}q_{ji}!\,g^{q_{ji}}(x_i,x_j)$ to transform Q into the symmetric $k\times k$ matrix M

$$\begin{pmatrix} \bullet & \bullet \\ \times & \bullet \\ \times & \times \end{pmatrix} \mapsto \begin{pmatrix} 0 & \bullet & \bullet \\ \times & 0 & \bullet \\ \times & \times & 0 \end{pmatrix}$$

Examples

- For $\langle 0 \big| T\big(\phi^3(x)\phi^3(y)\phi^2(z)\big) \big| 0 \rangle$
- The matrix is $M = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
- $\langle 0 \big| T\big(\phi^3(x)\phi^3(y)\phi^2(z)\big) \big| 0 \rangle = 36 g(x,y)^2 g(x,z) g(y,z)$
- M is the adjacency matrix of the Feynman diagram

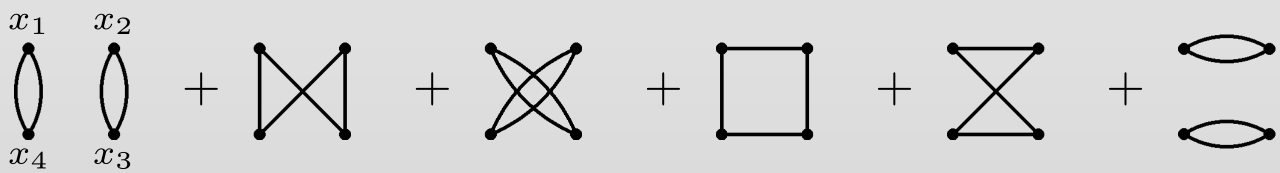
$\gamma \quad =$



Examples

- For $X = \langle 0 \big| T\big(\varphi^2(x_1)\varphi^2(x_2)\varphi^2(x_3)\varphi^2(x_4)\big)\big| 0\rangle$
- $$\begin{aligned} X \;=\; & 4g(x_1,x_4)^2 g(x_2,x_3)^2 + 16 g(x_1,x_3)g(x_1,x_4)g(x_2,x_3)g(x_2,x_4) + 4g(x_1,x_3)^2 g(x_2,x_4)^2 \\ & + 16 g(x_1,x_2)g(x_1,x_4)g(x_2,x_3)g(x_3,x_4) + 16 g(x_1,x_2)g(x_1,x_3)g(x_2,x_4)g(x_3,x_4) \\ & + 4g(x_1,x_2)^2 g(x_3,x_4)^2 \end{aligned}$$
- Feynman diagrams

$$\sum_\gamma \gamma \;=$$



- All these results are obtained without combinatorics (i.e. counting) whatsoever

Conclusion

- Get Feynman diagrams with the proper combinatorial factors in 90 minutes (including an introduction to Hopf algebra)
- Formulas valid for time-ordered as well as operator products
- Combinatorics is replaced by straightforward algebra
- Expressions are valid in any order of perturbation theory
- Noncommutative analogues