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On Teissier's example of an equisingularity class that cannot be defined over the rationals

Adam Parusiński and Laurenţiu Păunescu

To the memory of Arkadiusz Płoski

Abstract. A result of Teissier says that the cone over one of classical polygon examples in the real projective space gives, by complexification, a surface singularity which is not Whitney equisingular to a singularity defined over the field $\mathbb{Q}$ of rational numbers. In this note we correct the example and give a complete proof of Teissier's result.

1. Introduction. In the Comptes Rendus note [T90], Bernard Teissier proposed an example of a surface singularity in $\mathbb{C}^3$ that is not equisingular, in the sense of Whitney, to a surface singularity in $\mathbb{C}^3$ defined by an analytic function with rational coefficients. It is known by [R] that every complex singularity is equisingular, in a way even stronger than Whitney's, to a singularity defined by polynomial equations with coefficients in the field $\overline{\mathbb{Q}}$ of algebraic numbers. Whether it is possible to deform it to a singularity defined over $\mathbb{Q}$ by a topologically equisingular deformation is still an open problem.

Teissier's example, taken from the book of Grünbaum [Gr1], is one of the classical examples in convex geometry. As B. Teissier let us know, this particular example admits a polynomial equation with rational coefficients and it has to be slightly corrected. This is explained in Section 2.

There is another slight problem in the argument of [T90]. The above mentioned property of the example is proven with the use of a theorem of [LT1] on the deformation of a surface to its tangent cone. That theorem is incorrect as stated. This is explained in Section 4. An alternative proof of the result of [T90] is given in Section 3.

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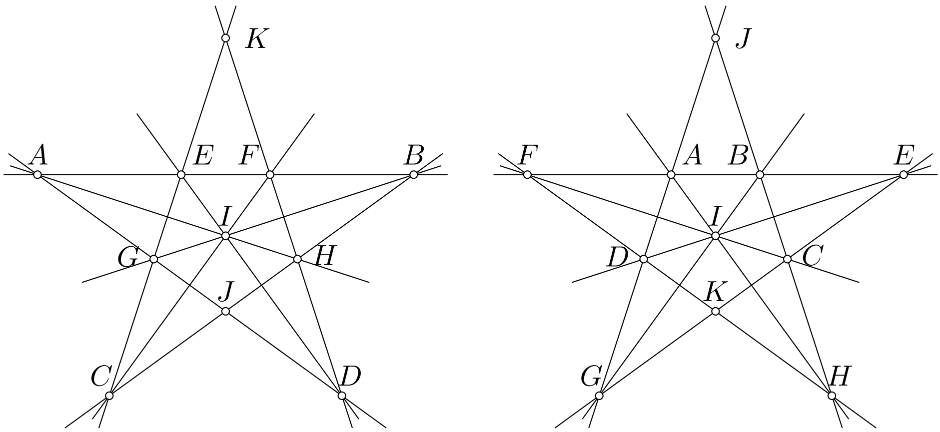
Notation. If φ and ψ are non-negative real valued functions and we write $\varphi \le C\psi$ we mean that there is a real constant C for which this inequality holds. This constant may occasionally change, but, for simplicity, it will still be denoted by C. Sometimes we even abbreviate it to $\varphi \lesssim \psi$ meaning $\varphi \le C\psi$ for a constant C.

2. Grünbaum's example. On page 34 of Grünbaum's paper [Gr2] (see also [Gr1, pp. 93–94]), one finds the following two arrangements of 9 lines in the real projective space, built in the obvious way on the regular pentagon.



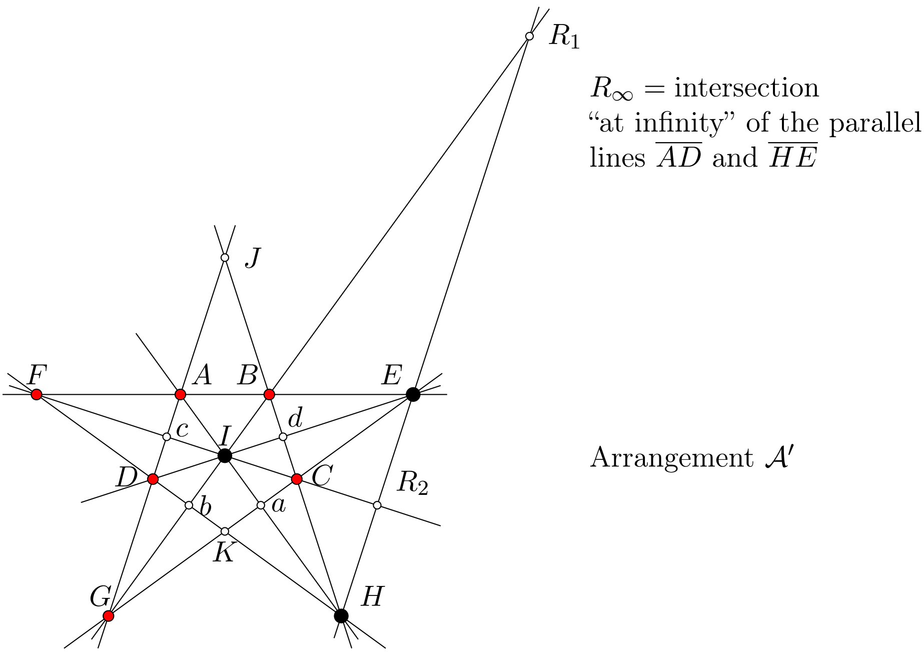
Arrangement $\mathcal{C}$ Arrangement $\mathcal{C}'$

As Grünbaum shows in [Gr2, Theorem 2.28], any arrangement equiconfigurational with $\mathcal{C}$ (i.e. corresponding by a one-to-one bijection of lines) is projectively equivalent to either $\mathcal{C}$ or $\mathcal{C}'$, by a projective isomorphism that induces exactly the same one-to-one correspondence of lines and points. Moreover, this arrangement, or any arrangement equiconfigurational with it, cannot be defined over $\mathbb{Q}$, that is, cannot be given by points with rational coordinates and lines defined by equations with rational coefficients. This follows by a direct computation as in the proof of [Gr2, Theorem 2.28], or from its statement, because the cross-ratio of the four lines at the point I (or the cross-ratio of the four points on the line AB) is irrational.

Example 2.1. Consider the points of the arrangement $\mathcal{C}'$ as points of $\mathbb{C}$ as follows: $I = 0$, $K = 1$ and A, B, C, D correspond to the primitive fifth roots of unity (i.e. we consider the picture with the x-axis directed downwards). Let G be the Galois group of the field $\mathbb{F}$, the extension of $\mathbb{Q}$ by the real coordinates of the fifth primitive roots of unity. The extension has $[\mathbb{F} : \mathbb{Q}] = 4$ and we may take $\sqrt{10 + 2\sqrt{5}}$ as its primitive element. The set of fifth roots of unity and their opposite numbers, i.e. the set of tenth roots of unity, is stable by the action of G.

Let us add to the family of lines of $\mathcal{C}'$ the 10th line $\overline{JK}$. Denote by $l_i(x,y) = x - b_i y - a_i$, $i = 1, \ldots, 9$, the equations of the 9 lines in the arrangement $\mathcal{C}'$ and take $l(x,y) = y$ as the equation of the 10th line. The group G acts on these lines by conjugating their coefficients. Since the set of points defining these lines, and hence the set of the lines, is G-invariant, we conclude that the function $l \prod_{i=1}^{9} l_i(x,y)$ is G-invariant, that is, it is a polynomial with rational coefficients. As l is fixed by G, we deduce that $\prod_{i=1}^{9} l_i(x,y)$, a polynomial defining the union of the lines in $\mathcal{C}'$, has rational coefficients.

To break the symmetry of the above arrangement we add the line $\overline{HE}$ to the arrangement $\mathcal{C}'$. The resulting arrangement consists of 10 lines intersecting in 18 points, including the point at infinity, the intersection of the lines $\overline{AD}$ and $\overline{HE}$. Denote this arrangement by $\overline{\mathcal{C}'}$.



Let $\mathcal{A}'$ be any affine line arrangement equiconfigurational with $\overline{\mathcal{C}'}$. For simplicity of notation we label the points and the lines of $\mathcal{A}'$ by the same letters as the corresponding ones of $\overline{\mathcal{C}'}$. Let $l_i(x,y)$, $i = 1, \ldots, 9$, be equations of the 9 lines of $\mathcal{A}'$ and $l_{10}(x,y)$ an equation of the line $\overline{HE}$. Consider, similarly to Example 2.1, the extension $\mathbb{F}$ of $\mathbb{Q}$ by the real coordinates of the points of $\mathcal{A}'$ and denote by G its Galois group ($\mathbb{F}$ and G depend of course on $\mathcal{A}'$ and not only on $\overline{\mathcal{C}'}$). Thus G acts on the points of $\mathbb{F}^2$ by conjugating the coordinates and on the lines in $\mathbb{F}^2$ by conjugating the coefficients of their equations.

PROPOSITION 2.2. *The product* $\varphi := \prod_{i=1}^{10} l_i(x,y)$ *is not* G*-invariant.*

Proof. Suppose for contradiction that the union of the 10 lines is G-invariant. Then so is the union of the 18 intersection points. The point at infinity cannot be moved to an affine point by an element of G. For the other points and for all the lines we show that they are G-invariant as follows.

(1) To an affine point P of $\mathcal{A}'$ we associate its weight, i.e. the number of lines intersecting at P. There are three possibilities:
   (a) 4 lines intersect at I, H, E; these points are marked in black.
   (b) 3 lines intersect at A, B, C, D, F, G; these points are marked in red.
   (c) 2 lines intersect at a, b, c, d, K, J, R_1, R_2; these points are marked in white.

   The group G preserves each of these three sets of points.
(2) The following lines are stable by the action of G: $\overline{FE}$ as the only line without points of weight 2, $\overline{HE}$ as the only line without points of weight 3, $\overline{AD}$ as the only line without points of weight 4.
(3) The intersection points of these lines, A, E (and R_∞), are stable by G.
(4) Each of the points of weight 4, I, H, E, is stable by G (the line $\overline{HE}$ and E are both stable).
(5) Lines containing two stable points are stable. Thus the lines $\overline{IE}$ and $\overline{IH}$ are stable. Hence the intersection points of these lines with the other stable lines are stable by G. Therefore D, A, and H are stable.
(6) All points of the line $\overline{AD}$ are stable (J is of weight 2 and G is of weight 3).
(7) All points of the line $\overline{HE}$ are stable (R_1 is on a stable line $\overline{IG}$).
(8) Every line of the arrangement, different from $\overline{AD}$ and $\overline{HE}$, joins a point of $\overline{AD}$ and a point of $\overline{HE}$. Therefore, by (6) and (7), all lines are stable by G.

Hence all points of $\mathcal{A}'$ have rational coefficients. Consequently, all points of $\mathcal{C}'$ have rational coefficients. This contradicts the conclusion of [Gr2, Theorem 2.28 and discussion on p. 34]. ■

REMARK 2.3. Let $\mathcal{B}$ be a projective line arrangement equiconfigurational with $\mathcal{A}'$. Then $\mathcal{B}$ is projectively isomorphic to $\mathcal{A}'$ or to the arrangement $\mathcal{A}$ obtained from $\mathcal{C}$ by adding the line $\overline{HE}$. Indeed, suppose that $\mathcal{B}$ and $\mathcal{A}'$ are equiconfigurational by a bijection of lines denoted by Φ. Let $\tilde{\mathcal{B}}$ denote the arrangement obtained from $\mathcal{B}$ by removing the line $\Phi^{-1}(\overline{HE})$. Then $\tilde{\mathcal{B}}$ is equiconfigurational to $\mathcal{C}'$, hence, by [Gr2, Theorem 2.28], projectively isomorphic to either $\mathcal{C}'$ or $\mathcal{C}$. On these 9 lines this isomorphism induces exactly the same bijection as Φ and therefore it sends the line $\overline{HE}$ to the corresponding line.

3. Proof of Teissier's result. Let $l'_i(x,y) - a_i$, $i = 1, \dots, 9$, l'_i being a linear form, be the equations of the 9 lines defining the arrangement $\mathcal{C}'$ of the previous section. Then $f_0(x,y,z) := \prod_i \tilde{l}_i(x,y,z)$, where $\tilde{l}_i(x,y,z) =: l'_i(x,y) - a_i z$, defines the cone over the union of these lines. Consider this cone as a complex singularity in $\mathbb{C}^3$. In [T90] Teissier claims the following result.

PROPOSITION 3.1 ([T90]). *Let $f(x,y) = 0$, with $x \in \mathbb{C}^3$, $y \in \mathbb{C}^p$, be an analytic deformation of f_0 that is Whitney equisingular along $\{0\} \times \mathbb{C}^p$. Then the family of tangent cones to the fibres $y \mapsto X_y := \{(x,y);\ f(x,y) = 0\}$ is also Whitney equisingular.*

As a corollary, one finds that in Teissier's example, each of these tangent cones $C_{X_y,y}$ is the union of 9 *planes that is equivalent to X_0 by a projective transformation, hence cannot be defined over $\mathbb{Q}$.*

Besides the fact that $\mathcal{C}'$ has to be replaced by the arrangement $\overline{\mathcal{C}'}$, Teissier's proof of the first claim uses Théorème (2.1.1) of [LT1], which is incorrect as stated. We give below an alternative proof of this first claim, for the arrangement $\overline{\mathcal{C}'}$ or any arrangement of lines, that avoids using Théorème (2.1.1) of [LT1]; see Theorem 3.5. In Section 4 we provide a counterexample to Théorème (2.1.1) of [LT1].

Consider a family of surface singularities of constant multiplicity d,

$$f(x,y) = f_d(x,y) + f_{d+1}(x,y) + \cdots, \quad f(0,y) \equiv 0,$$

where $x = (x_1, x_2, x_3) \in (\mathbb{C}^3, 0), y \in (\mathbb{C}^p, 0)$, and the f_i are homogeneous in x of degree i. Let us write $X_y := \{(x,y);\ f(x,y) = 0\}$ for $y \in \mathbb{C}^p$.

We denote $X = \{f = 0\}$, $\Sigma = X_{\mathrm{sing}}$, $Y = \{x = 0\}$, and the regular part of X by X_{reg}. Let $C_{X,Y}$ denote the normal cone of X along Y, that is, $C_{X,Y} = f_d^{-1}(0)$. Similarly, $C_{\Sigma,Y}$ will represent the normal cone of Σ along Y.

3.1. Exceptional tangents. First we recall the notion of exceptional tangents of a surface singularity $S \subset (\mathbb{C}^3, 0)$. It was introduced in [HL] where the following result was shown; see also the survey paper [LS].

PROPOSITION 3.2 ([HL, Théorème 3.1]). *Let $S \subset (\mathbb{C}^3, 0)$ be an isolated surface singularity. Then the set of the limits of planes tangent to S at* 0 *is the union of the set of hyperplanes tangent to the tangent cone $C_{S,0}$ and the set of hyperplanes of a finite number of line-pencils whose axes are lines of $C_{S,0}$, called the exceptional tangents.*

Moreover, among these lines we find all the lines of the singular set of the reduced tangent cone of S at 0.

This proposition follows from a theorem of Teissier [T73, Remarque 1.6] that describes the limit of tangent hyperplanes to an isolated hypersurface singularity $(S,0) \subset (\mathbb{C}^n,0)$, of arbitrary dimension, in terms of the following criterion:

Teissier's Criterion. *A hyperplane H is not a limit of tangent hyperplanes if and only if $H \cap S$ is an isolated singularity with the minimum Milnor number among all the Milnor numbers of the intersections of S with hyperplanes.*

An analog of the main statement of Proposition 3.2 in the non-isolated case was showed by Lê [L]. We will also need a version of the last part of Proposition 3.2 which is valid for non-isolated singularities. This we show in Proposition 3.4 below.

Lemma 3.3. *Let $S \subset (\mathbb{C}^n,0)$ be a hypersurface, $S = f^{-1}(0)$ with f reduced, and let ℓ be a line through the origin in $\mathbb{C}^n$ that is not tangent to S_{sing}. Then, for a generic linear form $\varphi : \mathbb{C}^n \to \mathbb{C}$ and for N sufficiently large, $g(x) := f(x) + \varphi^N(x)$ satisfies*

(i) $(g^{-1}(0))_{\mathrm{sing}} = S_{\mathrm{sing}} \cap \varphi^{-1}(0)$;
(ii) *for a sufficiently small conical open neighbourhood $U \subset \mathbb{C}^n$ of $\ell^* := \ell \setminus \{0\}$, the limits as $x \to 0$, $x \in U$, of hyperplanes tangent at x to the levels of f and those to the levels of g coincide.*

We say that a subset of $\mathbb{C}^n$ is *conical* at the origin if it is the intersection of a $\mathbb{C}^*$-homogeneous set, with respect to the standard action of $\mathbb{C}^*$ on $\mathbb{C}^n$: $\mathbb{C}^* \times \mathbb{C}^n \ni (s,x) \mapsto sx \in \mathbb{C}^n$, and an open neighbourhood of the origin.

Proof of Lemma 3.3. Let the system of coordinates $x = (x_1, \dots, x_n)$ be such that

$$\left\{ \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_{n-1}} = 0 \right\} = \Gamma \cup S_{\mathrm{sing}}, \tag{3.1}$$

with Γ being of dimension 1 (or empty). Then Γ, a generic relative polar curve, depends only on the projection $(x_1, \dots, x_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$. Therefore, after changing x_n if necessary, we may assume moreover that $\{x_n = 0\} \cap (\Gamma \cup \ell) = \{0\}$ and denote $H := \{x_n = 0\}$. Let U be a conical neighbourhood of ℓ^* such that $\overline{U} \cap (H \cup C_{S_{\mathrm{sing}},0}) = \{0\}$, $0 \notin U$. Then, by the Łojasiewicz inequality, there exist a positive integer N and a constant $c > 0$ such that

$$\|\mathrm{grad}\, f(x)\| \geq c|x_n|^{N-2} \qquad \text{for } x \in U. \tag{3.2}$$

Let $g(x) := f(x) + x_n^N$. Then

$$\frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} \text{ for } i = 1, \dots, n-1 \quad \text{and} \quad \frac{\partial g}{\partial x_n} = \frac{\partial f}{\partial x_n} + N x_n^{N-1}. \tag{3.3}$$

Therefore, because $H \cap \Gamma = \{0\}$, we have $(g^{-1}(0))_{\mathrm{sing}} = S_{\mathrm{sing}} \cap \varphi^{-1}(0)$ as needed. By (3.2) and (3.3), g on U satisfies the Łojasiewicz inequality similar to (3.2) and therefore

$$\left| \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|} - \frac{\operatorname{grad} g}{\|\operatorname{grad} g\|} \right| \le C|x_n|.$$

Then $\varphi(x) = x_n$ and N given by (3.2) guarantee the conclusions of the lemma. ∎

COROLLARY 3.4. *Let $S \subset (\mathbb{C}^3, 0)$ be a surface singularity. Suppose that a line ℓ through the origin in $\mathbb{C}^3$ is not tangent to S_{sing} but it is included in the singular set of the reduced tangent cone of S at 0. Then the planes containing ℓ form a pencil of limits of tangent planes to S_{reg}, that is, these planes are exceptional tangents.*

Proof. Let $S = f^{-1}(0)$, with f reduced, and suppose $\dim S_{\mathrm{sing}} = 1$. Assume that $g(x) := f(x) + \varphi^N(x)$ satisfies the conclusion of Lemma 3.3, and moreover, by (i) of this lemma, we may choose φ such that $\tilde{S} = g^{-1}(0)$ has isolated singularity at the origin. We may also choose N sufficiently large so that the multiplicities at the origin satisfy $\mathrm{mult}_0 f = \mathrm{mult}_0 g < N$. Then the normal cones $C_{S,0}$ and $C_{\tilde{S},0}$ are equal and given by the same equation.

By Proposition 3.2, the planes containing ℓ form a pencil Π of limits of tangent planes to $\tilde{S}_{\mathrm{reg}}$. To see that they form a pencil of limits of tangent planes to S_{reg}, it is sufficient to note that Π can be obtained by limits of tangent planes to $\tilde{S}_{\mathrm{reg}}$ on sequences of points $x \to 0$ with the secant lines $\overline{x0} \to \ell$, that is, the points from the set U of conclusion (ii) of Lemma 3.3. The latter claim comes from the description of the common limits of secants and tangent hyperplanes to any local singularity, as the union of dual correspondences, i.e. subsets of $\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$, $\mathbb{P}^2 \times \check{\mathbb{P}}^2$ in our case, that are projectivised conormal spaces to algebraic subsets of $\mathbb{P}^{n-1}$ (or to $\check{\mathbb{P}}^{n-1}$ by duality). This follows, for instance, from [LT2, Théorème 2.1.1], in the case where the small stratum Y is just the origin. ∎

3.2. Main Theorem

MAIN THEOREM 3.5. *Let $f(x,y)$ with $x \in \mathbb{C}^3$, $y \in \mathbb{C}^p$, be a reduced analytic function germ of multiplicity d at the origin. Denote $X = f^{-1}(0)$, $\Sigma = X_{\mathrm{sing}}$, $Y = \{0\} \times \mathbb{C}^p$. Assume that $f(x,0) = f_d(x,0)$ and that $(X_{\mathrm{reg}}, \Sigma \setminus Y, Y)$ is a Whitney stratification of X. Then the following hold:*

(i) *for all y, $X_y = \{x \in \mathbb{C}^3;\ f(x,y) = 0\}$ has no exceptional tangents at $x = 0$;*
(ii) *$((C_{X,Y})_{\mathrm{reg}}, Y)$ satisfies Whitney's conditions;*
(iii) *$(C_{X,Y})_{\mathrm{sing}} = C_{\Sigma,Y}$;*
(iv) *$C_{\Sigma,Y} \setminus Y$ is smooth;*

(v) $(C_{\Sigma,Y}\setminus Y, Y)$ *satisfies Whitney's conditions;*
(vi) $((C_{X,Y})_{\rm reg}, C_{\Sigma,Y}\setminus Y, Y)$ *is a Whitney stratification of the normal cone* $C_{X,Y}$.

Proof. Step 1. We use [LT2, Théorème 2.1.1] to show (i), (ii), and (v), the latter under the assumtpion that (iii) and (iv) hold. (It is possible to reduce to the case $\dim Y = 1$ using the argument of the proof of [T82, Proposition 1.2.1], but this is not necessary in our case.)

By the assumption that $(X_{\rm reg}, Y)$ satisfies Whitney's conditions, the family of limits of secants and tangent hyperplanes to the fibers over Y is equidimensional over Y. It contains the (projective) conormal to the tangent cone, that is,

$$\overline{\{(x,H,y)\in \mathbb{C}^3\times \check{\mathbb{P}}^2\times \mathbb{C}^p;\ (x,y)\in (C_{X,Y})_{\rm reg},\ H=T_x((C_{X_y,y})_{\rm reg})\}},$$

and the conormals to the subcones of $C_{X,Y}$, denoted by $Y(V_\alpha)$ in [LT2] and called the exceptional cones (see Définition 2.1.4 ibid.). By Whitney's conditions the conormal spaces to the exceptional cones are also equidimensional over Y, and therefore by the homogeneity assumption for $y=0$, i.e. $X_0 = C_{X_0,0}$, they are empty. This shows that for all y, X_y has no exceptional tangents, which is (i).

Then, the reciprocal argument (see [LT2, Théorème 2.1.1]) for $V_\alpha = C_{X,Y}$ shows that the pair of strata $((C_{X,Y})_{\rm reg}, Y)$ satisfies Whitney's conditions, proving (ii). (This also follows by an elementary computation: see Lemma 3.7 below.)

A similar argument shows (v), provided that (iii) and (iv) hold. Because $(\Sigma\setminus Y, Y)$ satisfies Whitney's conditions, by [LT2, Theorem 2.1.1] the set

$$W=\overline{\{([x],H,y);\ x\in (C_{\Sigma,Y})_{\rm reg},\ T_x((C_{\Sigma,Y})_{\rm reg})\subset H\}}\subset \mathbb{P}^2\times\check{\mathbb{P}}^2\times\mathbb{C}^p$$

is equidimensional over Y, and $((C_{\Sigma,Y})_{\rm reg}, Y)$ satisfies Whitney's conditions.

Step 2. We show the third and the fourth claims of Theorem 3.5.

By definition, for an ideal I the normal cone of $V(I)$ along Y is $C_{V(I),Y} = V(I_{\rm in})$, where $I_{\rm in} := (g_{\rm in}, g\in I)$. Here $g_{\rm in}$ denotes the homogeneous initial form of g with respect to the variable x. Thus $C_{X,Y}=V(f_d)$ and

$$(C_{X,Y})_{\rm sing} = V\biggl(\biggl(\frac{\partial f_d}{\partial x_1},\dots,\frac{\partial f_d}{\partial x_n},\frac{\partial f_d}{\partial y_1},\dots,\frac{\partial f_d}{\partial y_p}\biggr)\biggr).$$

The singular locus of X is defined as $\Sigma = V(J_f)$, where

$$J_f = \biggl(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n},\frac{\partial f}{\partial y_1},\dots,\frac{\partial f}{\partial y_p}\biggr).$$

Its normal cone along Y is $C_{\Sigma,Y} = V((J_f)_{\rm in})$. Clearly $\frac{\partial f_d}{\partial x_i}$ is either zero or the initial part of $\frac{\partial f}{\partial x_i}$, and similarly for $\frac{\partial f_d}{\partial y_j}$. This shows the inclusion $C_{\Sigma,Y}\subset (C_{X,Y})_{\rm sing}$.

Note that X_0 is reduced because so is X, and both the regular and the singular parts of X are Whitney equisingular along Y. Therefore, since X_0 is reduced and homogeneous, its singular part Σ_0 is a finite union of lines in $C_{X_0,0}$. Then, by [H], because $(\Sigma \setminus Y, Y)$ satisfies Whitney's conditions, Σ is normally flat (i.e. equimultiple) along Y, and Σ is topologically trivial along Y. This shows that $\Sigma_0 = \Sigma \cap \{y = 0\}$ and that Σ, with reduced structure, is a finite union of mutually transverse smooth subvarieties intersecting along Y. Therefore $C_{\Sigma,Y}$ is a finite union of smooth families of lines intersecting only along Y. In particular, $C_{\Sigma,Y} \setminus Y$ is smooth, and this shows (iv).

As shown before, $C_{\Sigma,Y} \subset (C_{X,Y})_{\mathrm{sing}}$, and hence, for every y,

$$(C_{\Sigma,Y})_y \subset ((C_{X,Y})_{\mathrm{sing}})_y \subset (C_{X_y,y})_{\mathrm{sing}}.$$

By Corollary 3.4, $(C_{X_y,y})_{\mathrm{sing}} \setminus C_{\Sigma_y,y}$ is contained in the set of exceptional tangents, empty in our case. This shows that $(C_{X,Y})_{\mathrm{sing}} = C_{\Sigma,Y}$, i.e. the third claim of Theorem 3.5. This, as explained before, completes the proof of (v).

Step 3. To complete the proof of Theorem 3.5 we show that

$$((C_{X,Y})_{\mathrm{reg}}, C_{\Sigma,Y} \setminus Y) \tag{3.4}$$

satisfies Whitney's conditions. In the proof we will use the standard action of $\mathbb{C}^*$ on $\mathbb{C}^3$.

Let ℓ be a line in $(C_{X_0,0})_{\mathrm{sing}}$, say given by $x_1 = x_2 = 0$. Denote by L the component of $C_{\Sigma,Y}$ containing it and let $N = \{x_3 = c\}$ be a transverse slice of ℓ. Then, to show that (3.4) is Whitney, it is equivalent to show that

$$(N \cap (C_{X,Y})_{\mathrm{reg}}, N \cap C_{\Sigma,Y} \setminus Y) \tag{3.5}$$

is Whitney. Indeed, it follows from the fact that $\ell^* = \ell \setminus \{0\}$ is an orbit of the $\mathbb{C}^*$-action.

Note that $N \cap C_{X,Y}$ is a family of plane curve singularities along $N \cap L$. Therefore, to show that it is Whitney it suffices to show that this family is μ constant (see [LR] or [T76]), or equivalently that

$$\left|\frac{\partial f_d}{\partial y}\right| \le C \left|\frac{\partial f_d}{\partial x_1}, \frac{\partial f_d}{\partial x_2}\right| \tag{3.6}$$

in a neighbourhood of $\ell \cap N$ in N.

Note also that (3.6) has to hold in a whole neighbourhood of ℓ in N, not merely on the zero set of f_d. Therefore to show it we consider the strict Thom stratifications of f and of f_d. A strict Thom stratification of f is a Whitney stratification of X satisfying the strict Thom condition w_f along each stratum (see [HMS]). In particular, along the smallest stratum Y this condition reads

$$\left|\frac{\partial f}{\partial y}\right| \le C|x|\left|\frac{\partial f}{\partial x}\right|. \tag{3.7}$$

By [P] or [BMM], every Whitney stratification of X satisfies the w_f condition along every stratum. In particular, we may assume that (3.7) holds.

Recall that we say that a subset of $\mathbb{C}^3$ is *conical* if it is the intersection of a $\mathbb{C}^*$-homogeneous set and an open neighbourhood of the origin.

LEMMA 3.6. *There is an open conical neighbourhood U of $\ell^* := \ell \setminus \{0\}$ in $\mathbb{C}^3$ such that*

$$\left|\frac{\partial f}{\partial x_3}\right| \le C\left|\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right| \tag{3.8}$$

on $U \times \mathbb{C}$.

Proof. First note that, for $y = 0$, (3.8) holds on an open conical neighbourhood of ℓ^* in $\mathbb{C}^3$. Indeed, it follows from the homogeneity of f_d that there is a Whitney stratification of X_0 with ℓ^* as a stratum, and both sides of (3.8) are homogeneous of the same degree.

To show that (3.8) holds also for $y \neq 0$ and small, maybe with a slightly different constant C, we may use the fact that if a stratification satisfies the w_f condition then the exceptional divisor $E \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2 \times \mathbb{C}$ of the blowing-up of the ideal $(x_1, x_2, x_3)\big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\big)$ is equidimensional (see [HMS, Proposition 3.3.1, Théorème 6.1, and Remarque 6.2.1]. Recall that the line ℓ is given by $x_1 = x_2 = 0$. Thus the conclusion of Lemma 3.6 means that $([0:0:1],[0:0:1]) \notin E$ for $y = 0$. But, if this is the case for $y = 0$, then it holds as well for y close to 0. ∎

LEMMA 3.7. *Let*

$$F(x, y, t) = t^{-d} f(tx, y) = f_d(x, y) + t f_{d+1}(x, y) + \cdots$$

be the deformation to the normal cone $C_{X,Y}$ induced by f. If f satisfies (3.7)*, then the induced deformation to the normal cone satisfies*

$$\left|\frac{\partial F}{\partial y}\right| \le C|x|\left|\frac{\partial F}{\partial x}\right|. \tag{3.9}$$

Similarly, if for a conical set $U \subset \mathbb{C}^3$ the inequality

$$\left|\frac{\partial f}{\partial y}\right| \le C|x|\left|\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right| \tag{3.10}$$

holds on U, then the analogous inequality holds on $U \times \mathbb{C} \times \overline{D}_1$ for F,

$$\left|\frac{\partial F}{\partial y}\right| \le C|x|\left|\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right|. \tag{3.11}$$

(*Here $\overline{D}_1$ denotes* $\{|t| \le 1\}$.)

Proof. Using the action of $s \in \mathbb{C}^*$ we have

$$\frac{\partial F}{\partial x}(sx,y,t) = s^{d-1}\frac{\partial F}{\partial x}(x,y,st), \qquad \frac{\partial F}{\partial y}(sx,y,t) = s^{d}\frac{\partial F}{\partial y}(x,y,st).$$

Then, by (3.7),

$$\begin{aligned}\left|\frac{\partial F}{\partial y}(x,y,s)\right| &= |s^{-d}|\left|\frac{\partial F}{\partial y}(sx,y,1)\right| \\ &\lesssim |s^{-d}|\,|sx|\left|\frac{\partial F}{\partial x}(sx,y,1)\right| = |x|\left|\frac{\partial F}{\partial x}(x,y,s)\right|.\end{aligned}$$

The proof of (3.11) is similar. ∎

Note that (3.7) and (3.8) give (3.10), and hence, by Lemma 3.7, (3.11). The latter gives

$$\left|\frac{\partial f_d}{\partial y}\right| \le C|x|\left|\frac{\partial f_d}{\partial x_1}, \frac{\partial f_d}{\partial x_2}\right|,$$

which implies (3.6). This shows (vi) of Theorem 3.5 and completes the proof. ∎

4. When is the deformation to the tangent cone equisingular? Let $X := \{f(x,y,z) = 0\} \subset (\mathbb{C}^3, 0)$ be a surface singularity. Suppose that the tangent cone $C_{X,0}$ of X at the origin is reduced. Let $F(x,y,z,t)$ denote the deformation of X to the tangent cone $C_{X,0}$ and denote $F^{-1}(0)$ by $\mathfrak{X}$. Then Théorème (2.1.1) of [LT1] claims that the following two conditions are equivalent:

(1) X has no exceptional tangents.
(2) The deformation of X to the tangent cone $C_{X,0}$ is equisingular.

There are various notions of equisingularity stated in this theorem and claimed equivalent in this case. One of them, condition (b) of Théorème (2.1.1) of [LT1], says that the partition of $\mathfrak{X}$: $(\mathfrak{X} - \Sigma, \Sigma \setminus \{0\} \times \mathbb{C}, \{0\} \times \mathbb{C})$, where $\Sigma := \mathfrak{X}_{\mathrm{sing}}$, is a Whitney stratification of $\mathfrak{X}$.

Suppose that X has no exceptional tangents. Then $(\mathfrak{X} - \Sigma, \Sigma \setminus \Sigma_{\mathrm{sing}})$ satisfies Whitney's conditions, i.e. condition (a) of loc. cit. This part of the proof is correct and was generalized to the hypersurfaces of arbitrary dimensions in [F]; see also [FT].

Both the counterexample below and our proof of the previous section suggest that Théorème (2.1.1) of [LT1] may be true under some additional assumptions.

Question. *Suppose moreover that the (reduced) singular locus of X is a finite union of smooth curves mutually not tangent. Is then $(\mathfrak{X} - \Sigma, \Sigma \setminus \{0\} \times \mathbb{C}, \{0\} \times \mathbb{C})$ a Whitney stratification of $\mathfrak{X}$?*

Note that Théorème (2.1.1) of [LT1] holds true if X is an isolated singularity.

4.1. Example of a singularity that is not equisingular to its tangent cone. Let

$$f(x,y,z) = z(zx - y^2) + zx^3 = z(zx - y^2 + x^3), \qquad (x,y,z) \in \mathbb{C}^3.$$

Then $X = \{f = 0\}$ is the union of the plane $X_1 = \{z = 0\}$ and the Morse singularity (ordinary double point) $X_2 = \{zx - y^2 + x^3 = 0\}$ that is analytically isomorphic to $zx - y^2 = 0$ (by the change $(x,y,z) \mapsto (x,y,\tilde{z} = z + x^2)$, $zx - y^2 + x^3 = (z + x^2)x - y^2 = \tilde{z}x - y^2$.) We note that

(1) neither X_1 nor X_2 has exceptional tangents and hence neither has X;
(2) the singular locus of X is $X_1 \cap X_2 = \{z = zx - y^2 + x^3 = 0\}$, i.e. the cusp $y^2 = x^3, z = 0$.

Deformation to the normal cone. Consider the deformation to the normal cone of f,

$$F(x,y,z,t) = z(zx - y^2) + tzx^3 = z(zx - y^2 + tx^3). \tag{4.1}$$

Let $\mathfrak{X} := \{F = 0\} = \mathfrak{X}_1 \cup \mathfrak{X}_2$, where $\mathfrak{X}_1 = \{z = 0\}$ and $\mathfrak{X}_2 = \{zx - y^2 + tx^3 = 0\}$. Then $\mathfrak{X}$ has the following properties:

(1) Both $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are analytically trivial along $\{0\} \times \mathbb{C}$.
(2) $\mathfrak{X}_{\text{sing}} = \mathfrak{X}_1 \cap \mathfrak{X}_2 = \{z = y^2 - tx^3 = 0\}$ is the deformation of the cusp to a double line.
(3) $(\mathfrak{X}_{\text{reg}}, \{0\} \times \mathbb{C})$ satisfies Whitney's conditions (because the components are analytically trivial along $\{0\} \times \mathbb{C}$).

The deformation $t \mapsto \mathfrak{X}_t$ is not equisingular. The singular locus of $\mathfrak{X}$, considered as a function of t, splits by (2). It is not topologically trivial either. If $t \mapsto \mathfrak{X}_t$ were topologically trivial, then the induced deformation on the normal section by $N = \{x = 1\}$, at a singular point of $C_{X,0}$,

$$h(y,z,t) = z(z - y^2 + t), \tag{4.2}$$

would be topologically trivial as a deformation of plane curves, which is not the case.

Note that the discriminant of $(y,z,t) \mapsto (t, y - bz)$ restricted to the zero set of h equals $(y^2 - t)^2$ (up to a non-zero unit). Thus the sum of the multiplicities of its zero set is independent of t and equals 4. Recall from [T76, Chap. II, Prop. 1.2] the formula for the multiplicity of the discriminant: $\operatorname{mult} \Delta = \sum_i (\mu_i + (m_i - 1))$, where μ_i denote the Milnor numbers and m_i the multiplicities of the singular points of h. Then, for $t = 0$, the fiber is $z(z - y^2) = 0$, with a single singular point $z = y = 0$, its multiplicity and its Milnor number are $m = 2, \mu = 3$, respectively, whilst for $t \neq 0$ and small, $0 = z(z - y^2 + t) = 0$ with two singular points with multiplicity and Milnor number $m = 2, \mu = 1$ each. Thus, the sum of the multiplicities at the zero set is

- for $t = 0, \operatorname{mult} \Delta = 3 + (2-1) = 4$,
- for $t \neq 0, \operatorname{mult} \Delta^1 + \operatorname{mult} \Delta^2 = 1 + (2-1) + 1 + (2-1) = 4$.

Thus, the constancy (with respect to the parameter t) of the sum of the multiplicities of the discriminant does not guarantee the non-splitting of the singular locus, nor the constancy of the sum of the Milnor numbers.

Some other properties:

(1) The limits of $\left[\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z} : \frac{\partial F}{\partial t}\right]_{|\mathfrak{X}} \to [\eta_1 : \eta_2 : \eta_3 : \eta_4] \in \check{\mathbb{P}}^3$ at the origin are of the following form. The last component η_4 is 0 and $[\eta_1 : \eta_2 : \eta_3]$ belongs to the dual of $C_{X_1,0}$, i.e. $[0:0:1]$, or to the dual of $C_{X_2,0}$, i.e. it satisfies $4\eta_1\eta_3 - \eta_2^2 = 0$.

(2) The relative polar curve $\overline{\left\{\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0, F \neq 0\right\}}$ is empty.

(3) The inequality $\left|\frac{\partial F}{\partial t}\right| \leq C\left|\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right|$ fails.

(4) Every plane of $\check{\mathbb{P}}^3$ is the limit of the (projectivized) gradient

$$\left[\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z} : \frac{\partial F}{\partial t}\right].$$

That is, $0 \times \check{\mathbb{P}}^3$ is a component of the exceptional divisor of the blowing-up of the jacobian ideal $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \frac{\partial F}{\partial t}\right)$.

Proof. (1) For $g = zx - y^2, \eta_1 = \frac{\partial g}{\partial x} = z, \eta_2 = \frac{\partial g}{\partial y} = -2y, \eta_3 = \frac{\partial g}{\partial z} = x$ the limits satisfy the equation of the dual curve $4\eta_1\eta_3 - \eta_2^2 = 4zx - 4y^2 = 0$.

(2) We show that $\{\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0\} \subset F = 0$. If $\frac{\partial F}{\partial y} = -2yz = 0$, then either $z = 0$ or $y = 0$. If $z = 0$ then $F = 0$.

Suppose $y = 0$. Then $x\frac{\partial F}{\partial x} = z^2x + 3tzx^3 = 0, z\frac{\partial F}{\partial z} = 2z^2x + tzx^3 = 0$. This gives $z^2x = 0$. If $z = 0$ then $F = 0$. If $x = 0$, then $\frac{\partial F}{\partial x} = z^2 = 0$ and hence again $z = 0$.

(3) It fails on the curve $y = 0, 2z + tx^2 = 0, z = x^4, t = -2x^2$. Indeed,

$$\begin{aligned}\left|\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right| &= |z^2 + 3tzx^2, 0, 0| = |x^8 - 2x^8| \\ &= o\left(\left|\frac{\partial F}{\partial t}\right| = |zx^3| = |x^7|\right).\end{aligned}$$

(4) On the curve $y = \gamma x^3 + \cdots, z = \alpha_1 x^3 + \alpha_2 x^4 + \alpha_3 x^5 + \cdots, t = \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \cdots$ parameterized by x, if we assume $2\alpha_1 + \beta_1 = 2\alpha_2 + \beta_2 = 0$, we have

$$
\begin{aligned}
\frac{\partial F}{\partial x} &= z^2 + 3tzx^2 = (\alpha_1^2 + 3\alpha_1\beta_1)x^6 + \cdots = -5\alpha_1^2 x^6 + \cdots ,\\
\frac{\partial F}{\partial y} &= -2yz = -2\alpha_1\gamma x^6 + \cdots ,\\
\frac{\partial F}{\partial z} &= 2zx - y^2 + tx^3\\
&= (2\alpha_1 + \beta_1)x^4 + (2\alpha_2 + \beta_2)x^5 + (2\alpha_3 + \beta_3 - \gamma^2)x^6 + \cdots\\
&= (2\alpha_3 + \beta_3 - \gamma^2)x^6 + \cdots ,\\
\frac{\partial F}{\partial t} &= zx^3 = \alpha_1 x^6 + \cdots .
\end{aligned}
$$

Then, $\left[\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z} : \frac{\partial F}{\partial t}\right] \to [-5\alpha_1 : -2\gamma : \alpha_1^{-1}(2\alpha_3 + \beta_3 - \gamma^2) : 1]$. Choosing $\alpha_1, \gamma, \alpha_3, \beta_3$ we can get as limits all points of a dense set of $\mathbb{P}^3$, and of course, the set of limits is closed. ∎

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Adam Parusiński
Université Côte d'Azur
CNRS, LJAD, UMR 7351
06108 Nice, France
E-mail: adam.parusinski@univ-cotedazur.fr

Laurenţiu Păunescu
School of Mathematics and Statistics
The University of Sydney
Sydney, NSW, 2006, Australia
E-mail: laurentiu.paunescu@sydney.edu.au