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A Generalization of the Milnor Number

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Introduction

The Milnor number of an isolated singularity of a complex hypersurface was introduced by Milnor in [7] and afterwards widely investigated from many aspects. All these investigations have been concentrated on local properties of singularities.

The aim of this paper is to extend the notion of Milnor number to nonisolated singularities. We shall give a definition which works for any compact component of the set of singular points of a hypersurface in a complex manifold and gives the Milnor number in the case of an isolated singularity. In Sect. 1 we also study some simple properties of the defined number and give a formula for it in terms of Chern classes and the Euler characteristic.

In Sect. 2 we describe the behaviour of this number under blowing-ups. This shows the interest of our generalization, because a blowing-up of an isolated singularity can have nonisolated singularities.

1. Definition of the μ-Number

Let M be an n-dimensional connected complex manifold and let X be a hypersurface in M. This means that X is nonempty and nowhere dense in M and can be given as the zero set of a holomorphic section v of a holomorphic line bundle L over M. Fix a Hermitian metric on L. We consider the decomposition of the associated metric connection $D=D'+D''$, where

$$D' : \mathscr{A}^0(L) \to \mathscr{A}^{1,0}(L) = \mathscr{A}^0(T^{*\prime}M \otimes L),$$

$$D'' = \bar{\partial} : \mathscr{A}^0(L) \to \mathscr{A}^{0,1}(L).$$

Note that $\bar{\partial}v = 0$ and $\operatorname{Sing} X$, the set of singular points of X, is equal to $\{x \in X;\, D'v(x) = 0\}$.

Lemma 1.1. *$\operatorname{Sing} X$ is closed and open in the zero set of $D'v$.*

Proof. For $x \in \operatorname{Sing} X$ take a nonvanishing holomorphic section e of L defined in a neighbourhood of x. We have $v = we$ for some holomorphic function w. Then

$$D'v = (dw + w\theta)e,$$

where θ denotes the connection form for the frame e. By the curve selection lemma (see for instance [5])

$$|dw| \geqq 2|\theta|\,|w|$$

in a sufficiently small neighbourhood of x, so near x the zero sets of $D'v$ and dw are equal and are contained in X. This ends the proof. □

Let E be an oriented k-dimensional real vector bundle on a k-dimensional real oriented compact manifold with boundary $(B, \partial B)$. For a section s of E, which does not vanish on ∂B, we shall denote by $\operatorname{ind}_B s$ the intersection index over B of s and the zero section of E (the obstruction to extending $s|_{\partial B}$ to a nowhere vanishing section of E equals $\operatorname{ind}_B s$ times the fundamental cohomology class of $(B, \partial B)$). If $\tilde{s}$ is a small perturbation of s transversal to the zero section, then $\operatorname{ind}_B s$ equals the number of zeros of $\tilde{s}$ counted with signs (local indices). If E is trivial this definition agrees with that of the topological degree (see [1]) and it is easy to check that:

(i) $\operatorname{ind}_B s$ depends only on the homotopy class of $s|_{\partial B}$ in the space of nowhere zero sections of $E|_{\partial B}$

(ii) if s' is another section not vanishing on ∂B, then $\operatorname{ind}_B s - \operatorname{ind}_B s$ depends only on $s|_{\partial B}$ and $s'|_{\partial B}$. In fact, it is equal to $\operatorname{ind}_{\partial B \times I} f$, where $f(x,0) = s(x)$ and $f(x,1) = s'(x)$

(iii) if U is a submanifold of B such that $s^{-1}(0) \subset \operatorname{Int}(U)$, then $\operatorname{ind}_B s = \operatorname{ind}_U s$ (see also [2] for the proof using the obstruction theory).

If E (or B) has a fixed complex structure, then we treat it as canonically oriented.

Let Y be a compact connected component of $\operatorname{Sing} X$ and U a small neighbourhood of Y.

Definition 1.2. *The intersection index $\operatorname{ind}_U D'v$* will be called the μ-number of X at Y and denoted by $\mu(X, Y)$.

Definition 1.3. Assume that X is compact. The intersection index of the zero section of $T^{*\prime}M \otimes L$ and $D'v$ over a small neighbourhood of X will be called the μ-number of X and denoted by $\mu(X)$.

Using standard homotopy arguments, it is easy to prove that $\mu(X, Y)$ ($\mu(X)$) depends only on X and Y (only on X), so the definitions are correct. It is also clear that for X compact

$$\mu(X) = \sum_{i=1}^{r} \mu(X, Y_i),$$

where $Y_1, Y_2, \ldots, Y_r$ are all connected components of $\operatorname{Sing} X$.

Let f be a holomorphic function defined in some open neighbourhood of 0 in $\mathbb{C}^n$. Suppose that 0 is an isolated singular point of the zero set Z of f. Then the vector field $(\partial f/\partial z_1, \partial f/\partial z_2, \ldots, \partial f/\partial z_n)$ has an isolated singularity at 0. The index of this vector fields at 0 is called the Milnor number of Z at 0 (the original definition is slightly different, see [7]). Assume now that x_0 is an isolated singular point of X.

Then for any system of coordinates φ at x_0, we have the Milnor number of $\varphi(X)$ at $\varphi(x_0)$ and it does not depend on φ, as is easy to check. We call this number the Milnor number of X at x_0.

Proposition 1.4. *If x_0 is an isolated singular point of X, then $\mu(X, x_0)$ equals the Milnor number of X at x_0.*

Proof. It suffices to choose a connection on L which is affine, in some system of coordinates, near x_0. □

Suppose that Y is smooth and $\dim Y = m$. To each point $x \in Y$ we attach the sequence of Teissier numbers $\mu^{n-m}(X,x), \ldots, \mu^1(X,x)$ (see [10]). In particular, when $M = \mathbb{C}^n$, $x=0$, $\mu^i(X,x)$ equals the Milnor number of $X \cap H$ at 0, where H is a generic i-dimensional linear subspace of $\mathbb{C}^n$. As was proved by Teissier [11], the pair $(X \backslash \operatorname{Sing} X, Y)$ satisfies Whitney's conditions iff the sequence of Teissier numbers is constant on Y. In such a situation we can compute $\mu(X, Y)$ in terms of Chern classes.

Proposition 1.5. *If the pair $(X \backslash \operatorname{Sing} X, Y)$ satisfies Whitney's conditions, then*

$$\mu(X, Y) = \mu \cdot \langle c_m(T^{*\prime} Y \otimes L|_Y), [Y] \rangle,$$

where $[Y]$ is the fundamental homology class of Y and $\mu = \mu^{n-m}(X,x)$ for any $x \in Y$ ($c_i(E)$ denotes, as usual, the i-th Chern class of E).

Proof. Let B^i denote the open unit ball in $\mathbb{C}^i$. Let $f: B^n \to \mathbb{C}$ be a holomorphic function and assume that $Y' = B^m \times \{0\}$ is the set of singular points of the zero set X of f. If the pair $(X' \backslash Y', Y')$ satisfies Whitney's conditions, then

$$|df| \leq C |(\partial f/\partial z_{m+1}, \ldots, \partial f/\partial z_n)| \tag{1}$$

for some positive constant C in some neighbourhood of 0 (see [11]). Furthermore, for any $y \in B^m$ $(y, 0)$ is a singular point of index μ of the vector field $(\partial f/\partial z_{m+1}, \ldots, \partial f/\partial z_n)$ on $\{y\} \times \mathbb{C}^{n-m}$.

Having described the local situation, we return to our original problem. Let $p: N \to Y$ be the normal bundle of Y in M and $\varphi: N \to M$ a diffeomorphism onto a neighbourhood of Y. Such a diffeomorphism is not necessarily holomorphic but we can assume that the differential of φ on Y is the identity. The bundle $\varphi^*(T^{*\prime} M \otimes L)$ is isomorphic to $p^*(T^{*\prime} Y \otimes L|_Y) \oplus p^*(N^* \otimes L|_Y)$. Consider the image of $\varphi^*(D'v)$ under this isomorphism. Call this image s; then s and $D'v$ have the same infinitesimal properties along Y. In particular, Y is the zero set of the second component s' of s (s' is a section of $p^*(N^* \otimes L|_Y)$ and zero is a singular point of index μ of s' restricted to any fibre of N. Take a section q of $T^{*\prime} Y \otimes L|_Y$ transversal to the zero section and a smooth function $\varrho: N \to [0,1]$ such that ϱ equals 0 outside a small neighbourhood of Y and 1 in a smaller neighbourhood. Then the indices of s and $\varrho\, p^*(q) \oplus s'$ are equal. By (1), it is easy to check that the latter index is the right side of the desired formula, which ends the proof. □

In particular, we can use Proposition 1.5 for computing $\mu(X, X)$ for $X = dZ$ and Z nonsingular ($d > 1$) or for computing $\mu(X_1 \cup X_2, X_1 \cap X_2)$ for X_1, X_2 nonsingular and intersecting transversally.

In the remainder of this section we assume M to be compact. The following proposition generalizes the well-known formula for the Euler characteristic of a submanifold (see for instance [3]).

Proposition 1.6. *$\mu(X)=(-1)^n\chi(X)+\langle c_n(T^{*\prime}M\otimes L),[M]\rangle-(-1)^n\chi(M)$, where $[M]$ denotes the fundamental homology class of M.*

Proof. Fix $x_0\in X$ and take a nonvanishing holomorphic section e of L defined near x_0. We have $v=we$ for some holomorphic function w. Then

$$d|v|^2=(d|w|^2+(\theta+\bar{\theta})|w|^2)|e|^2,$$

where θ is the connection form of the frame e. By the curve selection lemma [5]

$$|d|w|^2|\geqq 2|\theta+\bar{\theta}|\,|w|^2$$

in a small neighbourhood of x_0, so if $d|v|^2$ vanishes, so does w. Hence zero is an isolated critical value of $|v|^2$. Take $U=\{x\in M; |v(x)|\leqq\varepsilon\}$ for a sufficiently small positive ε. We claim that X is a strong deformation retract of U. Indeed, this follows from the triangulability of the pair (M,X) (see for instance [6]). In particular

$$\operatorname{ind}_U d|v|^2=\chi(U)=\chi(X).$$

The canonical isomorphism between $T^{*\prime}M$ and the real cotangent bundle $T^*_{\mathbb{R}}M$ given by $\varphi\to\frac{1}{2}(\varphi+\bar{\varphi})$ changes orientation iff n is odd. This isomorphism sends $\langle D'v,v\rangle$ to $\frac{1}{2}d|v|^2$. As v does not vanish on $M\backslash X$ we have

$$\begin{aligned}\mu(X)&=\operatorname{ind}_M D'v-\operatorname{ind}_{M\backslash U}D'v\\&=\operatorname{ind}_M D'v-\operatorname{ind}_{M\backslash U}\langle D'v,v\rangle\\&=\operatorname{ind}_M D'v-(-1)^n\operatorname{ind}_{M\backslash U}d|v|^2\\&=\langle c_n(T^{*\prime}M\otimes l),[M]\rangle+(-1)^n(\chi(X)-\chi(M)).\end{aligned}$$

This completes the proof. □

Corollary 1.7. *If hypersurfaces X,Z are equivalent as divisors (this means that they are the zero sets of sections of the same line bundle), then*

$$\mu(X)-\mu(Z)=(-1)^n(\chi(X)-\chi(Z)).$$

2. μ-Number and Blowing-Ups

Let Q be an m-dimensional connected compact submanifold of M and let $\sigma:M'\to M$ be the blowing-up along Q. Assume that the multiplicity k of X at a generic point of Q is greater than 1. This means that $Q\subset\operatorname{Sing}X$. Assume that the connected component Y of $\operatorname{Sing}X$ which contains Q is compact. Consider the proper inverse image X' of X. It is a hypersurface in M' defined by a section v' of $\sigma^*L\otimes E^{-k}$. Here and subsequently, E denotes the line bundle on M' associated with $Q'=\sigma^{-1}(Q)$. So, there exists a section e of E such that Q' is the zero set of e and v' is equal to $\sigma^*v\otimes e^{-k}$. Let us denote $\sigma^{-1}(Y)\cap X'$ by Y' and $\sum_{i=1}^{d}\mu(X',Y'_i)$ by

$\mu(X', Y')$, where $Y'_1, \ldots, Y'_d$ are all connected components of $\operatorname{Sing} X'$ which are contained in Y'. It is natural to try to relate $\mu(X, Y)$ with $\mu(X', Y')$.

Proposition 2.1. *For any natural numbers n, m $(n > m)$ there exists a polynomial W such that for any M, X, L, Q as above*

$$\mu(X', Y') = \mu(X, Y) + (-1)^n(\chi(X') - \chi(X)) + \langle W(c_1, \ldots, c_{n-m}, d_1, \ldots, d_m, \gamma, k), [Q] \rangle,$$

where $c_1, \ldots, c_{n-m}; d_1, \ldots, d_m$ are Chern classes of the conormal and cotangent bundles of Q, γ is the first Chern class of $L|_Q$, k is the multiplicity of X at a generic point of Q, and $[Q]$ denotes the fundamental homology class of Q.

First, we shall prove an analogous result for the inverse image of X.

Lemma 2.2. $\mu(\sigma^{-1}(X), \sigma^{-1}(Y)) = \mu(X, Y) + (-1)^n(n-m-1)\langle c_m(TQ \otimes L|_Q), [Q] \rangle$.

Proof. Fix a Hermitian metric on L and the induced metric on σ^*L. Let D and $\tilde{D}$ be the associated metric connections. We denote σ^*v by $\tilde{v}$. The blowing-up σ restricted to $M' \setminus Q'$ is a biholomorphic diffeomorphism onto $M \setminus Q$ and gives an isomorphism between $T^{*\prime}(M \setminus Q) \otimes L$ and $T^{*\prime}(M' \setminus Q') \otimes \sigma^*L$. Call it σ^*: it sends $D'v$ to $\tilde{D}'\tilde{v}$. Take a small neighbourhood U of Y and put $\tilde{U} = \sigma^{-1}(U)$. In order to compute $\operatorname{ind}_U D'v - \operatorname{ind}_{\tilde{U}} \tilde{D}'\tilde{v}$ it is sufficient to find $\operatorname{ind}_U f - \operatorname{ind}_{\tilde{U}} \sigma^* f$ for some section f of $T^{*\prime}M \otimes L$ defined and nonvanishing on ∂U. In fact, we only need U to be a neighbourhood of Q.

It will be more convenient to deal with TM instead of $T^{*\prime}M$ (these two bundles are isomorphic up to orientation). Note that it is sufficient to work in the smooth category, so we can use the tubular neighbourhood theorem.

First, we describe a geometry of σ. Let $q: N \to Q$ be the normal bundle of Q in M. Then Q' is the projectivisation of N. For simplicity of notation we shall write E instead of $E|_{Q'}$ and L instead of $L|_Q$. Note that $H^*(Q')$ is a free module over $H^*(Q)$ generated by $1, \delta, \delta^2, \ldots, \delta^{n-m-1}$, where δ denotes the first Chern class of E. The tangent bundle of Q' splits into the direct sum of σ^*TQ and F – "the tangent bundle to the fibres". We can describe F using the same arguments as in the description of the tangent bundle of the projective space. First, note that E is a subbundle of σ^*N. Take a Hermitian scalar product on N. Then

$$F \cong E^* \otimes E^\perp$$

and consequently

$$F \oplus \theta \cong E^* \otimes \sigma^* N, \tag{2}$$

where θ denotes the trivial line bundle. In particular

$$0 = c_{n-m}(F \oplus \theta) = \sum_{i=0}^{n-m} (-\delta)^{n-m-i} c_i(\sigma^* N). \tag{3}$$

The normal bundle of Q' in M' is isomorphic to $p: E \to Q'$ and the tangent bundle of E can be described as follows

$$TE = p^*\sigma^*TQ \oplus p^*E \oplus p^*F.$$

Analogously, the tangent bundle of N is isomorphic to

$$q^*TQ \oplus q^*N.$$

The manifolds $N \backslash Q$ and $E \backslash Q'$ are canonically diffeomorphic and the diffeomorphism induces an isomorphism between $TN|_{N \backslash Q}$ and $TE|_{E \backslash Q'}$.

Take a smooth section h of $TQ \otimes L$ transversal to the zero section. Let s be a section of L such that if $h(x)=0$, then $s(x) \neq 0$. Let g be the section of q^*N defined by $g(x,w)=w$. Then the section $f=q^*h \oplus g \otimes q^*s$ of $TN \otimes L$ is transversal to the zero section and its zeros are contained in Q. If U is a small neighbourhood of Q, then

$$\operatorname{ind}_U f = \langle c_m(TQ \otimes L), [Q] \rangle.$$

We can lift f to a section $\tilde{f}$ of $TE \otimes L$. We first do this outside Q and then extend uniquely onto E. We can write $\tilde{f}$ as follows

$$\tilde{f} = p^*\sigma^*h \oplus \tilde{g} \otimes p^*\sigma^*s \oplus 0,$$

where $\tilde{g}$ is the section of $p^*(E)$ defined by $g(z,w)=w$. The zero set of $\tilde{f}$ consists of a finite union of $\mathbb{C}P(n-m-1)$ and σ^*L is trivial on it. Adding to $\tilde{f}$ a suitable section of p^*F, we see that

$$\operatorname{ind}_{\tilde{U}} \tilde{f} = (n-m)\langle c_m(TQ \otimes L), [Q] \rangle,$$

where $\tilde{U} = \sigma^{-1}(U)$. This ends the proof. □

Proof of Proposition 2.1. We shall first prove the proposition for M compact. According to Proposition 1.6, we have

$$\begin{aligned}\mu(\sigma^{-1}(X), \sigma^{-1}(Y)) - \mu(X', Y') &= \mu(\sigma^{-1}(X)) - \mu(X') \\ &= (-1)^n(\chi(\sigma^{-1}(X)) - \chi(X')) \\ &\quad + \langle c_n(T^{*\prime}M' \otimes \sigma^*L), [M'] \rangle \\ &\quad - \langle c_n(T^{*\prime}M' \otimes \sigma^*L \otimes E^{-k}), M' \rangle.\end{aligned}$$

By the result of Sullivan from [9] in the form given in [4], we have

$$\begin{aligned}\chi(\sigma^{-1}(X)) - \chi(X') &= \chi(Q') - \chi(Q' \cap X') \\ &= (n-m)\chi(Q) + \chi(X) - \chi(X') - \chi(Q).\end{aligned}$$

Thus the only remaining point concerns

$$\begin{aligned}&\langle c_n(T^{*\prime}M' \otimes \sigma^*L \otimes E^{-k}), [M'] \rangle - \langle c_n(T^{*\prime}M' \otimes \sigma^*L), [M'] \rangle \\ &= \left\langle \sum_{i=1}^{n} (-k\delta)^i c_{n-i}(T^{*\prime}M' \otimes \sigma^*L), [M'] \right\rangle \\ &= -k \left\langle \sum_{i=0}^{n-1} (-k\delta)^i c_{n-i-1}(T^{*\prime}M' \otimes \sigma^*L), [Q'] \right\rangle\end{aligned}$$

(for simplicity of notation we use the same letters for elements of $H^*(M')$ and their restrictions to Q'). The last equality holds because of the fact that δ is associated with Q' by Poincaré duality.

As we have seen $TM'|_{\tilde{Q}}=\sigma^*TQ\oplus E\oplus F$, so from (2), we conclude that each $c_j(T^{*\prime}M'|_{Q'}\otimes\sigma^*L|_{Q'})$ is a polynomial in $\sigma^*(c_1),\ldots,\sigma^*(c_{n-m})$; $\sigma^*(d_1),\ldots,\sigma^*(d_m)$; $\sigma^*(\gamma)$ and δ.

Assume that we have an element $\sum_i \delta^i a_i=A$ of $H^*(Q')$, where $a_i\in H^*(Q)$. Then we can compute $\langle A,[Q']\rangle$ in the following way. First, using (3) we have

$$A=\sum_{j=0}^{n-m-1}\delta^j b_j,$$

where b_j are polynomials in a_i. Next, it is easy to prove that $\langle A,[Q']\rangle=\langle b_{n-m-1},[Q]\rangle$. Combining all these formulas we can construct the required polynomial.

We now turn to the general case. Let $\tilde{U}\subset M'$ be a compact manifold with boundary such that $\sigma^{-1}(Y)\subset\operatorname{Int}(\tilde{U})$. Taking, if necessary, the double of $\tilde{U}$ instead of M', we assume that M' is an oriented compact smooth manifold (not necessarily holomorphic) and the complex line bundles σ^*L, E are defined on it. Because E restricted to $\tilde{U}\setminus Q'$ is trivial and e is a trivialisation, we can assume that e is a section of E defined on the whole M' and vanishing only on Q'. Extend $\tilde{v}$ restricted to $\tilde{U}$ to a section of σ^*L defined on the whole M' and transversal to the zero section outside $\tilde{U}$. Take any Hermitian metric on σ^*L and consider the section $d|\tilde{v}|^2$ of T^*M'. Note that $d|\tilde{v}|^2\otimes\bar{\tilde{v}}^{-1}$ extends uniquely to a continuous section w of $T^*M'\otimes\sigma^*L^{-1}$. Let V be a neighbourhood of the zero set of $\tilde{v}$. If V is sufficiently small, then w does not vanish on $V\setminus\tilde{U}$. So

$$\begin{aligned}\operatorname{ind}_{\tilde{U}}w&=\operatorname{ind}_V w=\operatorname{ind}_{M'}w-\operatorname{ind}_{M'\setminus V}w\\&=\operatorname{ind}_{M'}w-\operatorname{ind}_{M'\setminus V}d|\tilde{v}|^2\\&=\operatorname{ind}_{M'}w-\operatorname{ind}_{M'}d|\tilde{v}|^2+\chi(V).\end{aligned}$$

By similar arguments as in the proof of Proposition 1.6 we have $\chi(V)=\chi(\tilde{v}^{-1}(0))$ and

$$\operatorname{ind}_{\tilde{U}}w=(-1)^n\mu(\sigma^{-1}(X),\sigma^{-1}(Y)).$$

We now apply these arguments again for $\tilde{v}\otimes e^{-k}$. The rest of proof is the same as in the special case. □

Remarks. The proof above gives not only the existence of W, but also a method for computing it.

If Q is a point, then all bundles are trivial on Q and it can be computed that

$$W_n(k)=-(k-1)^n+(-1)^n-(-1)^n l(n-1,k),$$

where $l(n-1,k)=n+\dfrac{(1-k)^n-1}{k}$ is the Euler characteristic of a nonsingular hypersurface of degree k in $\mathbb{C}P(n-1)$. Thus Proposition 2.1 generalizes the well-known formula for the behaviour of the Milnor number of singularities of curves in $\mathbb{C}^2$ under blowing-ups (see for example [8]).

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