

Regular projections for sub-analytic sets

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Abstract — A theorem is proved which is the sub-analytic version of Mostowski's theorem on regular projections for complex analytic hypersurfaces. This theorem leads to some corollaries concerning the distance to sub-analytic sets and the metric properties of these sets. The method of proof used gives also a stronger theorem of regular projections in the complex case.

Projection régulière des ensembles sous-analytiques

Résumé — On prouve un théorème qui est la version sous-analytique du théorème de Mostowski sur les projections régulières d'hypersurfaces analytiques complexes. On en tire des corollaires pour les fonctions distances à des ensembles sous-analytiques et les propriétés métriques de ceux-ci. La méthode de la preuve donne aussi un théorème fort de projection régulière dans le cas complexe.

Version française abrégée — 1. DÉFINITIONS. — \mathbf{K} désigne le corps \mathbf{R} des nombres réels ou le corps \mathbf{C} des nombres complexes. Soit X un sous-ensemble sous-analytique de \mathbf{K}^n . Pour $\xi \in \mathbf{K}^{n-1}$, notons π_ξ la projection linéaire $\mathbf{K}^n \rightarrow \mathbf{K}^{n-1}$ parallèle au vecteur $(\xi, 1) \in \mathbf{K}^{n-1} \times \mathbf{K}$. Pour $(C, \varepsilon) \in \mathbf{R}_+^2$, nous dirons que la projection π_ξ est (C, ε) -régulière en $x \in \mathbf{K}^n$ relativement à X si :

- (a) La restriction $\pi_\xi|_X$ est un morphisme fini.
- (b) L'intersection avec X du cône ouvert

$$S_\varepsilon(x, \xi) = \{x + \lambda(\eta, 1) \mid |\eta - \xi| < \varepsilon, \lambda \in \mathbf{K} \setminus \{0\}\}$$

est réunion finie disjointe d'ensembles de la forme $\{x + \lambda_i(\eta)(\eta, 1) \mid |\eta - \xi| < \varepsilon\}$, où les $\lambda_i(\eta)$ sont des fonctions \mathbf{K} -analytiques définies sur l'ouvert $\{|\eta - \xi| < \varepsilon\}$ de \mathbf{K}^{n-1} .

- (c) Les fonctions λ_i du (b) satisfont les inégalités $\|\text{grad } \lambda_i\| < C|\lambda_i|$.

Remarques. — (i) L'intersection $S_\varepsilon(x, \xi) \cap X$ peut être vide.

(ii) Si $\pi = \pi_0$ est (C, ε) -régulière en x relativement à X , d'après le théorème des fonctions implicites, il existe des constantes positives ε', δ' et M ne dépendant que de ε, C et n et telles que $S_{\varepsilon'}(x, 0) \cap X$ soit contenu dans la réunion disjointe des graphes de fonctions \mathbf{K} -analytiques $\varphi_i: \mathbf{B}(\pi(x), |\lambda_i(0)|\delta') \rightarrow \mathbf{K}$ vérifiant les inégalités $\|\text{grad } \varphi_i\| < M$, où $\mathbf{B}(a, r)$ désigne la boule ouverte de centre a et de rayon r dans \mathbf{K}^{n-1} .

Soit q un germe en 0 de courbe \mathbf{K} -analytique. Nous dirons que la projection π_ξ est (C, ε) -régulière en q si elle l'est en tout point de $q(t)$, pour $t \neq 0$ assez petit.

Nous dirons que $E \subset \mathbf{K}^{n-1}$ est un ensemble de bonnes projections pour X s'il existe $C, \varepsilon > 0$ tels que pour tout germe en 0 de courbe analytique il existe $\xi \in E$ tel que π_ξ soit (C, ε) -régulière en q relativement à X .

2. RÉSULTATS.

THÉORÈME 1. — Soit X un sous-ensemble sous-analytique compact de \mathbf{R}^n , de dimension $< n$. Il existe un ouvert dense Ω de $(\mathbf{R}^{n-1})^{n+1}$ tel que si $(\xi_1, \dots, \xi_{n+1}) \in \Omega$, $\{\xi_1, \dots, \xi_{n+1}\}$ est un ensemble de projections régulières pour X .

Le théorème 1 est l'analogie sous-analytique d'un théorème de Mostowski [4]. L'idée de la démonstration est analogue à celle de Mostowski et s'appuie sur le théorème d'aplatissement local (voir [1]).

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THÉORÈME 2. — Soit X un germe d'hypersurface complexe de \mathbf{C}^n . Il existe un ouvert dense Ω de $(\mathbf{C}^{n-1})^n$ tel que si $(\xi_1, \dots, \xi_n) \in \Omega$, $\{\xi_1, \dots, \xi_n\}$ est un ensemble de projections régulières pour X .

3. COROLLAIRES ET REMARQUES. — Remarque 1. — Soient X un sous-ensemble sous-analytique compact de \mathbf{R}^n et E un ensemble de projections régulières pour $X \setminus \dot{X}$. Il existe $C > 0$ et un sous-ensemble sous-analytique Y de X de dimension $< n-1$ tel que pour tout $x \in \mathbf{R}^n$ on ait :

$$\min_{\xi \in E} (\min \text{dist}_{\xi}(x, X), \text{dist}(x, Y)) \leq C \cdot \text{dist}(x, X), \quad \text{où } \text{dist}_{\xi}(x, X) = \text{dist}(x, \pi_{\xi}^{-1} \pi_{\xi}(x) \cap X).$$

Remarque 2. — Soit X un sous-ensemble sous-analytique compact de \mathbf{R}^n de dimension $< n-1$. Il existe un sous-ensemble fini E de \mathbf{R}^{n-1} et une constante $C > 0$ tels que, pour tout $x \in \mathbf{R}^n$, on ait

$$\text{dist}(x, X) \leq C \cdot \max_{\xi \in E} \{ \text{dist}(\pi_{\xi}(x), \pi_{\xi}(X)) \}.$$

Un sous-ensemble sous-analytique compact X de \mathbf{R}^n est dit L -régulier si :

- (i) Si $\dim X = 0$, X est un point.
- (ii) Si $\dim X = n$, X devient, dans un système convenable de coordonnées, de la forme $\{(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}; f(x') \leq x_n \leq g(x'), x' \in Y\}$, où Y est L -régulier, f et g sont des fonctions continues sous-analytiques qui sur \dot{Y} sont analytiques et à dérivées bornées et vérifient $f < g$.
- (iii) Si $\dim X = k < n$, X devient, dans un système convenable de coordonnées, le graphe d'une application sous-analytique continue $\varphi: Y \rightarrow \mathbf{R}^{n-k}$, où Y est L -régulier dans \mathbf{R}^k et φ est analytique et à dérivées bornées dans \dot{Y} .

COROLLAIRE. — Tout sous-ensemble sous-analytique compact de \mathbf{R}^n peut être recouvert par une famille finie de sous-ensembles sous-analytiques L -réguliers (non nécessairement disjoints).

La preuve est la même que dans le cas semi-analytique (voir [6]).

1. DEFINITIONS. — Let X be a sub-analytic subset of \mathbf{K}^n ($\mathbf{K} = \mathbf{R}$ or \mathbf{C}). For $\xi \in \mathbf{K}^{n-1}$, we denote by π_{ξ} the linear projection $\mathbf{K}^n \rightarrow \mathbf{K}^{n-1}$ parallel to the vector $(\xi, 1) \in \mathbf{K}^{n-1} \times \mathbf{K}$. For $(C, \varepsilon) \in \mathbf{R}_+^2$, we say that π_{ξ} is (C, ε) -regular at $x \in \mathbf{K}^n$ (with respect to X) if:

- (a) the restriction $\pi_{\xi}|_X$ is finite;
- (b) the intersection of X with the open cone $S_{\varepsilon}(x, \xi) = \{x + \lambda(\eta, 1) \mid |\eta - \xi| < \varepsilon, \lambda \in \mathbf{K} - \{0\}\}$ is a finite disjoint union of sets of the form $\{x + \lambda_i(\eta)(\eta, 1) \mid |\eta - \xi| < \varepsilon\}$, where λ_i are \mathbf{K} -analytic functions for $|\eta - \xi| < \varepsilon$ [in particular $S_{\varepsilon}(x, \xi) \cap X$ can be empty];
- (c) the functions λ_i from (b) satisfy $\|\text{grad } \lambda_i(\eta)\| \leq C |\lambda_i(\eta)|$ for $|\eta - \xi| < \varepsilon$.

Note that if $\pi = \pi_0$ is (C, ε) -regular at x then, by the implicit function theorem, there exist positive constants ε', δ', M depending only on ε, C, n and such that $S_{\varepsilon'}(x, 0) \cap X$ is contained in the disjoint finite union of graphs of \mathbf{K} -analytic functions $\varphi_i: \mathbf{B}(\pi(x), |\lambda_i(0)|\delta') \rightarrow \mathbf{K}$, satisfying $\|\text{grad } \varphi_i\| < M, (\mathbf{B}(y, R) = \{x \in \mathbf{K}^{n-1} \mid \|x - y\| < R\})$, and these graphs are contained in $S_{\varepsilon}(x, 0) \cap X$.

Let q be a germ at 0 of a \mathbf{K} -analytic curve. We say that π_{ξ} is (C, ε) -regular at q if it is so at every $q(t)$ for $t \neq 0$ and sufficiently small.

We will say that $E \subset \mathbf{K}^{n-1}$ is a set of regular projections for X if there exist $C, \varepsilon > 0$ such that for every germ q at 0 of a \mathbf{K} -analytic curve we can find $\xi \in E$ such that π_{ξ} is (C, ε) -regular at q with respect to X .

2. RESULTS. — Regular projections were introduced by Mostowski in [4] and they play an important role in the proof of existence of Lipschitz stratification of complex analytic and real semi-analytic sets (see [4], [5] and [6]). In this Note we shall prove the sub-analytic version of Mostowski's theorem of regular projections (Proposition 4.2 from [4]).

THEOREM 1. — *Let X be a compact sub-analytic subset of \mathbf{R}^n and $\dim X < n$. Then there exists an open dense subset Ω of $(\mathbf{R}^{n-1})^{n+1}$ such that if $(\xi_1, \xi_2, \dots, \xi_{n+1}) \in \Omega$, then $E = \{\xi_1, \xi_2, \dots, \xi_{n+1}\}$ is a set of regular projections for X .*

Proof. — The idea of proof is similar to that of Mostowski's theorem of regular projections. Instead of blowing up some ideal (which is in fact the flattener of some morphism, see [4]) we use the local flattening theorem (see e. g. [1]). By the Koopmann-Brown Lemma ([2], [3]), there is no problem with finiteness of projections.

To make the proof simpler we consider only ξ from the unit ball D in \mathbf{R}^{n-1} and q contained in the unit ball B in \mathbf{R}^n .

First we prove the statement for q not contained in some sub-analytic nowhere dense subset of B .

Step 1. — Take $\tilde{X} = p^{-1}(X)$, where $p: B \times D \times \mathbf{R} \rightarrow \mathbf{R}^n$ is defined by $p(x, \xi, \lambda) = x + \lambda(\xi, 1)$. There exists a compact sub-analytic nowhere dense subset Y of $B \times D$ such that $\tilde{X} \cap p^{-1}((B \times D) - Y)$ is a finite disjoint union of graphs of analytic functions $\lambda_i(p_1: B \times D \times \mathbf{R} \rightarrow B \times D$ is the standard projection). Consider the projection $\pi: Y \rightarrow B$. It can be deduced easily from the local flattening theorem that there exists a finite collection of maps $\pi_j: B_j \rightarrow B$ (each π_j is a finite sequence of local blowing-ups) and compact subsets K_j of B_j such that $\bigcup_j \pi_j(K_j) = B$ and for each j .

(*) the strict transform $\tilde{\pi}_j: Y_j \rightarrow B_j$ of π by π_j has the fibres of dimension strictly less than $n-1$.

Fix j for a moment and consider the family of compact sub-analytic subsets $(B_j)_\xi = \tilde{\pi}_j(\tilde{p}_2^{-1}(\xi)) \cap K_j$ of K_j parametrized by $\xi \in D$, where $p_2: Y_j \rightarrow D$ is the standard projection ($Y_j \subset B_j \times D$). By (*), we have for generic $\xi_1, \xi_2, \dots, \xi_{n+1}$

$$(**) \quad \bigcap_k (B_j)_{\xi_k} = \emptyset,$$

so for each $\tilde{x} \in K_j$, we can find $\xi(\tilde{x}) \in \{\xi_1, \xi_2, \dots, \xi_{n+1}\}$ such that $(\tilde{x}, \eta) \notin Y_j$ for $|\eta - \xi(\tilde{x})| < \varepsilon$ (for some universal ε).

Each π_j is an isomorphism outside some proper analytic subset E_j of Y_j . Let $V = \bigcup_j \pi_j(K_j \cap E_j)$.

Let q be a germ at 0 of an \mathbf{R} -analytic curve not contained in V . Take q_j the strict transform of q by π_j (it exists at least for one j). Then $\xi(q_j(0))$ is ε -regular [without property (c)] for q .

Step 2. — In order to prove (c) we consider on $(B \times D) - Y$ the sub-analytic functions

$$f_{k,i}(x, \eta) = (\partial \lambda_i / \partial \eta_k(x, \eta)) \cdot \lambda_i^{-1}(x, \eta).$$

Fix $f = f_{k,i}$. Let $Z \subset B \times D \times \mathbf{P}^1$ be the closure of the graph of f . Apply the local flattening theorem to $\pi: Z \rightarrow B$ in the same way as in Step 1. The maps obtained $\tilde{\pi}_j: Z_j \rightarrow B_j$ satisfy $D \times \{\infty\} \not\subseteq \tilde{\pi}_j^{-1}(\tilde{x})$ for each $\tilde{x} \in B_j$. In fact, let q be a germ of an analytic curve in B such that the strict transform \tilde{q} of q by π_j exists and satisfies $\tilde{q}(0) = \tilde{x}$. By Pawlucki's version of the Puiseux theorem [7] applied to $\lambda_i(q(t), \eta)$ the function $f(q(t), \eta)$ is bounded near $(q(0), \xi)$ for some $\xi \in D$. So, by the curve selection

lemma, $D \times \{\infty\}$ can not be contained in $\tilde{\pi}_j^{-1}(\tilde{x})$. Now we proceed with $Z_j \cap (B \times D \times \{\infty\}) \rightarrow B_j$ as with $Y_j \rightarrow B_j$ in Step 1.

To prove the result for q from a sub-analytic nowhere dense subset W of B we repeat the above construction for $\tilde{X} \cap p_3^{-1}(W)$, where $p_3: \tilde{X} \rightarrow B$ is the standard projection.

If we want to obtain a set of regular projections consisting only of $n+1$ elements we should consider simultaneously all the maps like $Y \rightarrow B$, $Z \rightarrow B$, occurring in Step 1 and Step 2, so we obtain one family of B_j 's. This family satisfies (***) for generic $\xi_1, \xi_2, \dots, \xi_{n+1}$. This ends the proof.

In the local situation one can prove the analogous result with sets of regular projections consisting only of n elements [(***) can be obtained taking n generic ξ 's]. In particular adapting the above ideas to the proof of Mostowski [4] in the complex case, we can estimate the number of regular projections needed by n .

THEOREM 2. — *Let X be a germ of a complex hypersurface of \mathbf{C}^n . Then there exists an open dense subset Ω of $(\mathbf{C}^{n-1})^n$ such that if $(\xi_1, \xi_2, \dots, \xi_n) \in \Omega$, then $\{\xi_1, \xi_2, \dots, \xi_n\}$ is a set of regular projections for X .*

3. COROLLARIES AND REMARKS. — *Remark 1.* — Let X be a compact sub-analytic subset of \mathbf{R}^n and let E be a set of regular projections for $X - \dot{X}$. Then, there exist $C > 0$ and a sub-analytic subset Y of X such that $\dim Y < n-1$ and for every $x \in \mathbf{R}^n$

$$\min_{\xi \in E} (\min \text{dist}_{\xi}(x, X), \text{dist}(x, Y)) \leq C \text{dist}(x, X),$$

where $\text{dist}_{\xi}(x, X) = \text{dist}(x, \pi_{\xi}^{-1}(\pi_{\xi}(x)) \cap X)$.

Remark 2. — Let X be a compact sub-analytic subset of \mathbf{R}^n such that $\dim X < n-1$ and let E be a set of regular projections for X . Then, there exists $C > 0$ such that for every $x \in \mathbf{R}^n$

$$\text{dist}(x, X) \leq C \max_{\xi \in E} (\text{dist}(\pi_{\xi}(x), \pi_{\xi}(X))).$$

The proof of Remark 1 is similar to that of Proposition 2 of [5]. Remark 2 follows from the fact that if π_{ξ} is (C, ε) -regular at x and $\dim X < n-1$, then $S_{\varepsilon}(x, \xi) \cap X = \emptyset$.

We call a compact sub-analytic subset X of \mathbf{R}^n *L-regular* if:

- (i) if $\dim X = 0$, then X is a point;
- (ii) if $\dim X = n$, then, in some system of coordinates, X is of the form $\{(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}; f(x') \leq x_n \leq g(x'), x' \in Y\}$, where Y is L-regular, f and g are continuous sub-analytic functions which on \dot{Y} are analytic, satisfy $f < g$ and have bounded derivatives;
- (iii) if $\dim X = k < n$, then, in some system of coordinates, X is the graph of continuous sub-analytic map $\Phi: Y \rightarrow \mathbf{R}^{n-k}$, where Y is an L-regular subset of \mathbf{R}^k and Φ is analytic and with bounded derivatives in \dot{Y} .

Using the same methods as in the semi-analytic case [6] we obtain:

COROLLARY. — *Every compact sub-analytic subset of \mathbf{R}^n can be covered with a finite (not necessarily disjoint) family of L-regular sets.*

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