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Algebraic Equisingularity and Whitney's Conditions

by

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Summary. We present an example of complex analytic hypersurfaces X in some open set $U \subset \mathbf{C}^n$ such that: Sing X is nonsingular, the pair of manifolds X-Sing X, Sing X satisfies Whitney's conditions but X is not algebraically equisingular along Sing X.

0. Introduction. Let X be an analytic (complex) hypersurface in some open subset U of $\mathbf{C}^n$, and let Y be a nonsingular analytic subset of X. Then the following holds: There exists a proper analytic subset Y_1 of Y such that X is topologically equisingular along $Y - Y_1$ (in the sense of Zariski [12]), particularly for all $y \in Y - Y_1$ germs (X, y) have the same topological type. There are two different methods of proving this theorem. The first method given by Varchenko in [8], is based on properties of algebraic equisingularity (see O. Zariski [11] and [12]). The second method follows from the existence of Whitney's stratification for analytic sets (see Whitney [9]) and its topological properties (see Mather [4] and Thom [7]). We are going to discuss "wrong sets" Y_1 which can be obtained by these two methods. By the example of Briançon and Speder [1], the "wrong set" which is eliminated by the first method need not contain the set which is eliminated by the second method. In this article we shall construct an example which shows the converse. Our example will develop the example given by Briançon and Speder in [2].

1. Some facts about algebraic equisingularity. At first we recall the definition and some properties of algebraic equisingularity. Let X be an analytic hypersurface in some open neighbourhood U of the origin 0 in $\mathbf{C}^n$. Suppose that Y is a nonsingular analytic subset of X and $0 \in Y$. An analytic mapping $\pi = (\pi_1, \ldots, \pi_{n-1}): V \to C^{n-1}$ is called a *system of local parameters for* X if

(i) V is an open neighbourhood of 0 and $\pi(0)=0$;

(ii) π is a submersion;

(iii) $\pi^{-1}(0)\cap X$ is a discrete set.

If additionally $\ker d\pi(0)$ is transversal to a tangent cone $C(X,0)$ to X at 0 we call this system *transversal.*

DEFINITION 1. (see [12]) The definition is inductive for $\dim X-\dim Y$. We say that X *is algebraically equisingular along* Y *at the origin* 0 if there is a system of local parameters for X $\pi: V\to \mathbf{C}^{n-1}$ such that:

(a) $\pi|_{Y\cap V}: Y\cap V\to \pi(Y\cap V)$ is an analytic isomorphism;

(b) a branch locus of $\pi|_X$ B_π is algebraically equisingular along $\pi(Y\cap V)$ at the origin 0. (We know that $B_\pi=\{\text{critical values of } \pi|_{X\cap V-\mathrm{Sing}\,X}\}\cup\cup\pi(\mathrm{Sing}\,X\cap V)$ is an analytic hypersurface in some open neighbourhood of 0 in $\mathbf{C}^{n-1}$).

In the case if $\dim X=\dim Y$, we say that X *is algebraically equisingular along* Y *at* 0 if $0\notin \mathrm{Sing}\,X$.

In papers [10], [11] O. Zariski proved the following facts about algebraic equisingularity in codimension 1 ($\dim X-\dim Y=1$), which we shall need in the sequel. Now we assume additionally that $\dim X-\dim Y=1$.

Fact 1. If X is algebraically equisingular along Y at 0 then this equisingularity occurs for all transversal systems of local parameters for X.

Fact 2. Let $\sigma:\tilde{X}\to X$ be a blowing up of X along Y. We denote $\sigma^{-1}(Y)$ by $\tilde{Y}$. Then X is algebraically equisingular along Y at 0 if and only if $\sigma^{-1}(0)$ is a finite set, say $\sigma^{-1}(0)=\{s_1,\dots,s_l\}$, $\tilde{Y}$ is nonsingular at $s_1,\dots,s_l$, and $\tilde{X}$ is algebraically equisingular along $\tilde{Y}$ at $s_1,\dots,s_l$. (We know that $\tilde{X}$ is contained in some n-dimensional complex manifold).

Fact 3. If X is algebraically equisingular along Y at 0 for a system of local parameters $\pi: V\to C^{n-1}$, then intersection multiplicites $i(X\cdot\pi^{-1}(z))$ are constant for a sufficiently small neighbourhood $W\subset\pi(V)$ of 0. Particularly the multiplicity of X along Y is constant (in a sufficiently small neighbourhood of 0).

2. Construction and properties of the example. Now we construct our example. Let X be a hypersurface in $U=\{(x,y,z,t)\in C^4: |t|<3\}$ defined by

$$f(x,y,z,t)=x^9+y^{12}+z^{15}+tx^3y^4z^5=0$$

Let $Y=\mathrm{Sing}\,X=\{(x,y,z,t)\in U;\ x=y=z=0\}$ and let $X(t_0)$ denote a set $\{(x,y,z)\in C^3;\ f(x,y,z,t_0)=0\}$. It is easy to check that for each t_0 ($|t_0|<3$), $X(t_0)$ has a single singular point at the origin, and the polynomial

$f(\cdot,\cdot,\cdot,t_0)$ is square-free (in other words, $f(\cdot,\cdot,\cdot,t_0)$ should be either irreducible or the product of distinct irreducible polynomials). Now we prove that this example has all properties required above.

Property 1. The pair of manifolds $X-Y$, Y satisfies Whitney's conditions.

Proof. Due to [6], it sufficies to show that the numbers $\mu^*(X(t),0)$ are independent of t. We can compute μ^* using [5] and [3], and so we obtain

$$\mu^3(X(t),0) = (15-1)(12-1)(9-1)$$

$$\mu^2(X(t),0) = (12-1)(9-1)$$

$$\mu^1(X(t),0) = (9-1)$$

and consequently we see that μ^* is independent of t.

Next we shall prove that the pair X, Y does not satisfy conditions of algebraic equisingularity at the origin. Before that we shall prove some technical lemmas.

LEMMA 1. *Let an analytic function F be of the from*

$$F(x, y_1, \ldots, y_k) = x^s + p_1(y_1, \ldots, y_k)\, x^{s-1} + \ldots + p_s(y_1, \ldots, y_k).$$

We assign to variables $x, y_1, \ldots, y_k$ weights $f, e_1, \ldots, e_k$, respectively. Suppose that x^s is contained in terms of the smallest weight. We denote these terms by $F_{\min}$. If the discriminant $\Delta(F_{\min})$ of $F_{\min}$ is not equal to 0, then

$$\Delta(F_{\min}) = (\Delta(F))_{\min},$$

where we assign to variables $y_1, \ldots, y_k$ in $\Delta(F)$ weights $e_1, \ldots, e_k$.

Proof. The discriminant of a polynomial $\sigma = x^s + p_1 x^{s-1} + \ldots + p_s = (x-\xi_1)(x-\xi_2)\ldots(x-\xi_s)$ can be written as

$$\Delta(\sigma) = \prod_{i<j} (\xi_i - \xi_j)^2.$$

By the fundamental theorem on symetric functions $\Delta(\sigma)(p_1, \ldots, p_s)$ is a quasi-homogeneous polynomial in $p_1, \ldots, p_s$, with weights $1, \ldots, s$, respectively.

A term $x^{s-i} y_1^{\alpha_1} \ldots y_k^{\alpha_k}$ of F is contained in $F_{\min}$ if and only if $\alpha_1 e_1 + \ldots + \alpha_k e_k = i \cdot s$, thus $(\Delta(F))_{\min}$ depends only on $F_{\min}$. This ends the proof.

LEMMA 2. *Let T be the zero set of $g \in \mathbf{C}\{x, y, t\}$ in some open nieghbourhood V of the origin 0. We assign to variables x, y, t weights $r, s, 0$ $(r, s > 0)$. Let $g_{\min}$ be of a weight $r \cdot s$ and coefficients at y^r and x^s are non-zero. Then if T is algebraically equisingular along $S = \{(x, y, t) \in U: x = y = 0\}$ at 0, then the zero set $T_{\min}$ of $g_{\min}$ is algebraically equisingular along S at 0, too.*

Proof. The proof will be by induction on $\max\{s, r\}$. For $r = s$ the lemma reduces to the well-known fact (see [11]) about tangent cones in algebraically equisingular families of curves, which can be deduced from Fact 2.

Let $s \neq r$, then we can assume that $s < r$. Due to the Weierstrass preparation theorem, g is of the form

$$g(x, y, t) = x^s + p_1(y, t)\, x^{s-1} + \ldots + p_s(y, t).$$

Then $\tilde{T}$ a blowing-up T along S is defined by

$$0 = \tilde{g}(x, y, t) = x^s + \tilde{p}_1(y, t)\, x^{s-1} + \ldots + \tilde{p}_s(y, t),$$

where $\tilde{p}_i(y, t) = p_i(y, t)/y^i$. We define analogously $(\widetilde{T_{\min}})$ a blowing-up $T_{\min}$ along S, as the zero set of $(\widetilde{g_{\min}})$. Using a proof of Lemma 1 it is easy to prove that

$$(\widetilde{g_{\min}}) = (\tilde{g})_{\min},$$

where we assign to variables x, y, t in $\tilde{g}$ weights $r-s, s, 0$, respectively.

Now $\tilde{T}, \tilde{g}, S$ satisfy the conditions of the lemma for weights $r-s, s, 0$ and the lemma follows by an induction argument and fact 2.

Now we can prove:

Property 2. X is not algebraically equisingular along Y at 0.

Proof. We have to consider all systems of local parameters for X

$$\pi: (C^4, 0) \to (C^3, 0).$$

There are only three possible cases.

Case 1. *A transversal system of local parameters.* First we change the local coordinate system in $(C^3, 0)$ to obtain a more useful form of π. Since π is a transversal system, we can assume that $d\pi_1(0) = y - ax$, $d\pi_2(0) = z - bx$, $d\pi_3(0) = t - cx$. Hence we can write

$$y = \pi_1 + ax + A_1(x, \pi_1, \pi_2, \pi_3),$$

$$z = \pi_2 + bx + A_2(x, \pi_1, \pi_2, \pi_3),$$

$$t = \pi_3 + cx + A_3(x, \pi_1, \pi_2, \pi_3),$$

where $A_1, A_2, A_3 \in m^2$ (by m denote a maximal ideal in $\mathbf{C}\{x, y, z, t\}$). Then we put

$$\hat{y} = \pi_1 + A_1(0, \pi_1, \pi_2, \pi_3),$$

$$\hat{z} = \pi_2 + A_2(0, \pi_1, \pi_2, \pi_3),$$

$$\hat{t} = \pi_3 + A_3(0, \pi_1, \pi_2, \pi_3).$$

And now

$$y = \hat{y}+ax+B_1(x,\hat{y},\hat{z},\hat{t}),$$
$$z = \hat{z}+bx+B_2(x,\hat{y},\hat{z},\hat{t}),$$
$$t = \hat{t}+cx+B_3(x,\hat{y},\hat{z},\hat{t}),$$

where $B_i \in m^2$ and $B_i(0,\hat{y},\hat{z},\hat{t}) \equiv 0$.

In the coordinate system $(x,\hat{y},\hat{z},\hat{t})$, X is defined by

$$F(x,\hat{y},\hat{z},\hat{t}) = x^9+(\hat{y}+ax+B_1)^{12}+(\hat{z}+bx+B_2)^{15}+ \\ +(\hat{t}+cx+B_3)\,x^3(\hat{y}+ax+B_1)^4(\hat{z}+bx+B_2)^5 = 0$$

and Y is defined by $Y = \{(x,y,z,t)\in C^4;\ x=\hat{y}=\hat{z}=0\}$.
By the Weierstrass preparation theorem we can write

$$F(x,\hat{y},\hat{z},\hat{t}) = w(x,\hat{y},\hat{z},\hat{t})\cdot q(x,\hat{y},\hat{z},\hat{t}),$$

where $q(0)=1$ and $w(x,\hat{y},\hat{z},\hat{t}) = x^9+p_1(\hat{y},\hat{z},\hat{t})\,x^8+\ldots+p_9(\hat{y},\hat{z},\hat{t})$.
A branch locus B_π is the zero set of $\Delta(w)$. We assign to variables $x,\hat{y},\hat{z},\hat{t}$ weights 20, 15, 12, 0, respectively and then we obtain

$$F_{\min} = w_{\min} = x^9+\hat{y}^{12}+\hat{z}^{15}+\hat{t}x^3\hat{y}^4\hat{z}^5.$$

Lemma 1 gives

$$(\Delta(w))_{\min} = \Delta(w_{\min}) = d\cdot(\hat{y}^{12}+\hat{z}^{15})^2\{27(\hat{y}^{12}+\hat{z}^{15})^2+4\hat{t}^3\hat{y}^{12}\hat{z}^{15}\}^3,$$

where $d\in\mathbf{C}$ is non zero. From fact 2 one can derive that the zero set of $(\Delta(w))_{\min}$ is not algebraically equisingular along $\pi(Y) = \{(\hat{y},\hat{z},\hat{t})\in\mathbf{C}^3;\ \hat{y}=\hat{z}=0\}$ at 0 (the zero set of $(\Delta(w))_{\min}(\cdot,\cdot,\hat{t}_0)$ has three irreducible components at 0 for $\hat{t}_0=0$ and six components for $\hat{t}\neq 0$). From Lemma 2 we deduce that the zero set of $\Delta(w)$ is not algebraically equisingular along $\pi(Y)$ at 0. This completes the proof of Case 1.

Case 2. Ker $d\pi(0)\subset\{(x,y,z,t)\in C^4;\ x=0 \text{ and } y\neq 0\}$. Similarly, as in Case 1, we can find new coordinates $\hat{x},\hat{z},\hat{t}$ such that

$$x = \hat{x}+B_1(\hat{x},y,\hat{z},\hat{t}),$$
$$z = \hat{z}+by+B_2(\hat{x},y,\hat{z},\hat{t}),$$
$$t = \hat{t}+cy+B_3(\hat{x},y,\hat{z},\hat{t}),$$

where $B_i\in m^2$ and $B_i(\hat{x},0,\hat{z},\hat{t})\equiv 0$. We can write $B_1(\hat{x},y,\hat{z},\hat{t}) = y\cdot A(\hat{t})+$ $+y^2\cdot(\ldots)+y\hat{x}\cdot(\ldots)+y\hat{z}\cdot(\ldots)$. If $A(\hat{t})\equiv 0$ then $x_{\min}(\hat{x},y,\hat{z},\hat{t}) = \hat{x}$ (weights such as in Case 1) and we proceed as in Case 1.
We assume that $A(\hat{t}) = a\cdot\hat{t}^s+\hat{t}^{s+1}\cdot R(\hat{t})$ and $a\neq 0$. We assign to variables $\hat{x},y,\hat{z},\hat{t}$ weights 20, 15, 20, $5/s$, respectively. Similarly to Case 1 we obtain

F, w, q and $w(\hat{x}, y, \hat{z}, \hat{t}) = y^{12} + p_1(\hat{x}, \hat{z}, \hat{t})\, y^{11} + \ldots + p_{12}(\hat{x}, \hat{z}, \hat{t})$, $w_{\min} = F_{\min} = y^{12} + (\hat{x} + ay\hat{t}^s)^9$. From Lemma 1 follows that $(\Delta(w))_{\min} = \Delta(w_{\min}) = \sum_{20i + \frac{5}{S} j = \text{constant}} a_{ij}\, \hat{x}^i\, \hat{t}^j$ and consequently we get

(i) $\hat{t}$ does not divide $\Delta(w_{\min})$, because for $\hat{t} = 0$ and $\hat{x} \neq 0$ $w_{\min}$ has no multiply roots

(ii) a_{N0} is not the only non-zero coefficient $\Delta(w_{\min})$, because if $\Delta(w_{\min}) = a_{N0}\, x^N$ then: for a system of parameters $(\hat{x}, y, t) \to (\hat{x}, \hat{t})$, $\{(\hat{x}, y, \hat{t}) \in C^3;\ w_{\min}(\hat{x}, y, 0, \hat{t}) = 0\}$ would be algebraically equisingular along $\{(\hat{x}, y, t) \in C^3;\ \hat{x} = y = 0\}$ at 0, but this contradicts Fact 3.

(i) and (ii) show that multiplicity of the zero set of $\Delta(w)$ varies along $\pi(Y) = \{(\hat{x}, \hat{z}, \hat{t}) \in C^3;\ \hat{x} = \hat{z} = 0\}$ at the origin, so by Fact 3, the zero set of $\Delta(w)$ is not algebraically equisingular along $\pi(Y)$ at 0. This ends the proof Case 2.

Case 3. Ker $d\pi(0) \subset \{(x, y, z, t) \in C^4;\ x = y = 0 \text{ and } z \neq 0\}$. In this case we proceed similarly to the previous cases. Because ker $d\pi(0)$ cannot be contained in $\{(x, y, z, t) \in C^4;\ x = y = z = 0\}$, the proof of Property 2 is complete.

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А. Парусиньски, **Алгебраическая эквисингулярность и условия Уитни**

Строится пример гиперповерхности в открытом подмножестве $U \subset \mathbf{C}^n$ такой, что: sing X является многообразием, пара X — sing X, sing X удовлетворяет условиям Уитни, но X не эквисингулярно вдоль sing X.