Finite Volume discretization of two phase Darcy flows with discontinuous capillary pressures

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Outline

- Two phase Darcy flow in phase pressures formulation
- Gradient scheme discretization
- Convergence analysis
- Vertex Approximate Gradient scheme
- Numerical tests

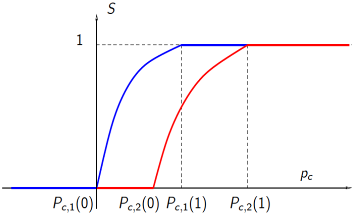
Incompressible two-phase flow in heterogeneous porous media

Equations for $(\mathbf{x}, t)$ in the space-time domain $\Omega \times (0, T)$

$$\begin{cases} \Phi(x)\partial_t S(x, p_c(\mathbf{x}, t)) - \operatorname{div}\left(M^1(x, S(x, p_c(\mathbf{x}, t)))\Lambda(x)(\nabla p^1(\mathbf{x}, t) - \rho^1 g)\right) = f^1(\mathbf{x}, t) \\ -\Phi(x)\partial_t S(x, p_c(\mathbf{x}, t)) - \operatorname{div}\left(M^2(x, S(x, p_c(\mathbf{x}, t)))\Lambda(x)(\nabla p^2(\mathbf{x}, t) - \rho^2 g)\right) = f^2(\mathbf{x}, t) \\ p_c(\mathbf{x}, t) = p^1(\mathbf{x}, t) - p^2(\mathbf{x}, t) \end{cases}$$

Formulation in phase-pressures (p^1, p^2) with

- p^1 pressure of the phase 1 (non wetting phase)
- p^2 pressure of the phase 2 (wetting phase)
- $p_c = p^1 - p^2$ is the capillary pressure
- rocktypes: $\overline{\Omega} = \bigcup_{j \in J} \overline{\Omega_j}$



- $S(x, p_c) \in [0, 1]$ is the saturation of the phase 1 with
  - $S(x, p_c) = S_j(p_c)$ for a.e. $x \in \Omega_j$ and all $p_c \in \mathbb{R}$
  - S_j is a non decreasing Lipschitz continuous function

Gradient scheme discretization [Eymard et al 2010]

$$\mathcal{D} = (X_{\mathcal{D}},\ \Pi_{\mathcal{D}},\ \nabla_{\mathcal{D}})$$

- discrete space $X_{\mathcal{D}} = \mathbb{R}^{\{d.o.f.\}}$ ($X_{\mathcal{D},0}$ with homogeneous Dirichlet BC)
- reconstruction of function $\Pi_{\mathcal{D}} \ : \ X_{\mathcal{D}} \to L^2(\Omega)$ linear mapping
- reconstruction of gradient $\nabla_{\mathcal{D}} \ : \ X_{\mathcal{D}} \to L^2(\Omega)^d$ linear mapping

such that $\| \cdot \|_{\mathcal{D}} = \|\nabla_{\mathcal{D}} \cdot \|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$

Examples of gradient schemes :

– Conforming and Mixed Finite Elements

– SUSHI and Mimetic schemes

– Symmetric MPFA O scheme on tetrahedral meshes

– VAG scheme

Discretization of the two-phase Darcy flow problem

$$p^{1,(n+1)} - \bar{p}^1_{\mathcal{D}} \in X_{\mathcal{D},0} \qquad p^{2,(n+1)} - \bar{p}^2_{\mathcal{D}} \in X_{\mathcal{D},0} \qquad p_c^{(n+1)} = p^{1,(n+1)} - p^{2,(n+1)} \in X_{\mathcal{D}}$$

$$s_{\mathcal{D}}^{(n+1)}(\mathbf{x}) = S(\mathbf{x}, \Pi_{\mathcal{D}} p_c^{(n+1)}(\mathbf{x}))$$

$$\int_\Omega \Phi(\mathbf{x}) \frac{s_{\mathcal{D}}^{(n+1)}(\mathbf{x}) - s_{\mathcal{D}}^{(n)}(\mathbf{x})}{t^{(n+1)} - t^{(n)}} \Pi_{\mathcal{D}} w(\mathbf{x}) \mathrm{d}\mathbf{x}$$

$$+ \int_\Omega M^1(\mathbf{x}, s_{\mathcal{D}}^{(n+1)}(\mathbf{x})) \Lambda(\mathbf{x}) (\nabla_{\mathcal{D}} p^{1,(n+1)}(\mathbf{x}) - \rho^1 g) \cdot \nabla_{\mathcal{D}} w(\mathbf{x}) \mathrm{d}\mathbf{x}$$

$$= \frac{1}{t^{(n+1)} - t^{(n)}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f^1(\mathbf{x}, t) \Pi_{\mathcal{D}} w(\mathbf{x}) \mathrm{d}\mathbf{x} \mathrm{d}t \quad \forall w \in X_{\mathcal{D},0}, \ \ \forall n = 0, \ldots, N-1$$

+ equation for phase 2

Convergence analysis for two phase flow models

- Global pressure formulation:
  - MFE-FE [Ewing et al 2001]
  - Finite Volume TPFA (Michel et al 2003)
  - Finite Volume Sushi, Mimetic [Brenner 2011], VAG [Brenner et al 2012],
  - Gradient schemes [Eymard et al 2013]
- Phase by phase upwind scheme [Michel et al 2003]: only TPFA
- Discontinuous capillary pressures [Cances et al 2012]: only TPFA

Price to pay to extend the convergence analysis of the pressure pressure model to the Gradient scheme framework:

⋆ the approximation of the mobility is centered

⋆ for $(\mathbf{x}, s) \in \Omega \times [0, 1]$, $\quad M^{\alpha}(\mathbf{x}, s) \in [M^{\min}, M^{\max}]$ $\quad$ with $M^{\max} \geq M^{\min} > 0$

Properties of a sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$

- **Coercivity** $$C_{\mathcal{D}} = \max_{v\in X_{\mathcal{D},\mathbf{0}}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^2(\Omega)}}{\|v\|_{\mathcal{D}}}$$ $\Rightarrow$ discrete Poincaré inequality

  $C_{\mathcal{D}_m}$ **remains bounded**

- **Consistency :** $S_{\mathcal{D}_m} \to 0$

$$\forall \varphi \in H_0^1(\Omega)\,, \quad S_{\mathcal{D}}(\varphi) = \min_{v\in X_{\mathcal{D},\mathbf{0}}} \left(\|\Pi_{\mathcal{D}} v - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^2(\Omega)^d}\right)$$

- **Limit-conformity :** $W_{\mathcal{D}_m} \to 0$

$$\forall \boldsymbol{\varphi} \in H_{\rm div}(\Omega)\,,\ W_{\mathcal{D}}(\boldsymbol{\varphi}) = \max_{u\in X_{\mathcal{D},\mathbf{0}}\setminus\{0\}} \frac{1}{\|u\|_{\mathcal{D}}} \left| \int_\Omega (\nabla_{\mathcal{D}} u(\mathbf{x})\cdot\boldsymbol{\varphi}(\mathbf{x}) + \Pi_{\mathcal{D}} u(\mathbf{x}) {\rm div}\boldsymbol{\varphi}(\mathbf{x}))\,{\rm d}\mathbf{x} \right|$$

- **Compactness** $$\forall \boldsymbol{\xi} \in \mathbb{R}^d\,,\ T_{\mathcal{D}}(\boldsymbol{\xi}) = \max_{v\in X_{\mathcal{D},\mathbf{0}}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}} v(\cdot + \boldsymbol{\xi}) - \Pi_{\mathcal{D}} v\|_{L^2(\mathbb{R}^d)}}{\|v\|_{\mathcal{D}}}$$

  $$\lim_{|\boldsymbol{\xi}|\to 0} \sup_{m\in\mathbb{N}} T_{\mathcal{D}_m}(\boldsymbol{\xi}) = 0$$

Convergence of the numerical scheme - sketch of proof

- Estimates on $\|\nabla_{\mathcal{D}} p^{\alpha}\|_{L^2(\Omega\times(0,T))^d}$, $\alpha = 1, 2$

- Estimate on a dual semi-norm of the discrete time derivative of $s_{\mathcal{D}}$

- Estimate on time and space translates of $s_{\mathcal{D}}$
  $\Rightarrow$ strong convergence of $s_{\mathcal{D}}$ in L^2

- Minty trick : $\lim S(., \Pi_{\mathcal{D}} p_c) = S(., \lim \Pi_{\mathcal{D}} p_c)$ where $p_c = p^1 - p^2$

- Convergence to the weak solution by consistency and limit-conformity

VAG scheme (*Vertex Approximate Gradient scheme*)

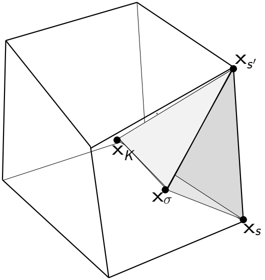
$X_{\mathcal{D}} = \{$ discrete value u_K at the cell centers $\mathbf{x}_K$ and u_s at the vertices $\mathbf{s}$ $\}$

- Tetrahedral submesh of each cell K

$$\mathbf{x}_\sigma = \sum_{s \in \mathcal{V}_\sigma} \frac{1}{\mathrm{Card}\mathcal{V}_\sigma} \mathbf{x}_s, \quad u_\sigma = \sum_{s \in \mathcal{V}_\sigma} \frac{1}{\mathrm{Card}\mathcal{V}_\sigma} u_s$$

- Constant gradient on each tetrahedra $\mathcal{T}$

$$\nabla_{\mathcal{T}} u = \sum_{s \in \mathcal{V}_\sigma} (u_s - u_K) g^s_{\mathcal{T}}$$



Piecewise constant gradient in $L^2(\Omega)^d$ $\quad \nabla_{\mathcal{D}} u = \nabla_{\mathcal{T}} u$ on each tetrahedra $\mathcal{T}$

Reconstruction operator

$$\Pi_{\mathcal{D}} u(\mathbf{x}) = u_K \text{ on } \Omega_K, u_s \text{ on } \Omega_{Ks}, \text{ with } K = \Omega_K \cup \bigcup_{s \in \mathcal{V}_K \setminus \partial\Omega} \Omega_{K,s}$$

VAG is vertex-centered : unknowns u_K can be eliminated from the linear system

Variational formulation and fluxes

$$a_{\mathcal{D}}(u,w) = \int_\Omega \Lambda(\mathbf{x})\nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} w(\mathbf{x})\ d\mathbf{x} = \int_\Omega f(\mathbf{x})\ \Pi_{\mathcal{D}} w(\mathbf{x})\ d\mathbf{x} \text{ for all } w \in X_{\mathcal{D},\mathbf{0}}.$$

Finite Element nodal basis: η_K, $K \in \mathcal{M}$, η_s, $s \in \mathcal{V}$.

$\mathcal{V}_K$: set of nodes of the cell K.

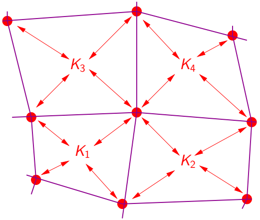
$$\begin{aligned} a_{\mathcal{D}}(u,w) &= \sum_{K\in\mathcal{M}} \sum_{s\in\mathcal{V}_K} \left(\int_K -\Lambda(\mathbf{x})\nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla\eta_s(\mathbf{x}) d\mathbf{x}\right)\left(w_K - w_s\right), \\ &= \sum_{K\in\mathcal{M}} \sum_{s\in\mathcal{V}_K} F_{K,s}(u)\ \left(w_K - w_s\right) \end{aligned}$$

with the fluxes

$$F_{K,s}(u) = \int_K -\Lambda(\mathbf{x})\nabla_{\mathcal{D}} u \cdot \nabla\eta_s(x) d\mathbf{x} = \sum_{s\in\mathcal{V}_K} T_K^{s,s'}(u_K - u_{s'}).$$

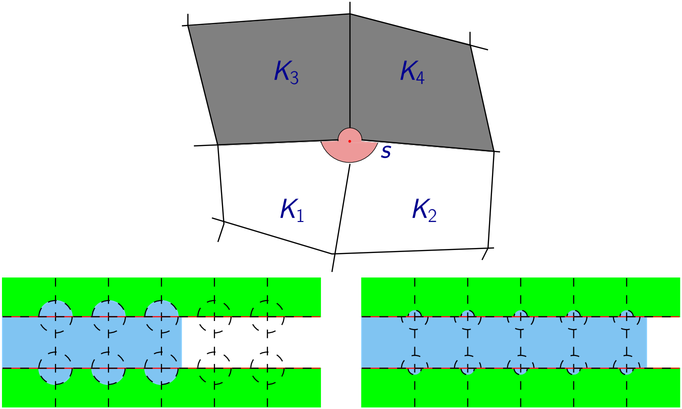
Equivalent discrete conservation laws

$$\begin{cases} \displaystyle\sum_{s\in\mathcal{V}_K} F_{K,s}(u) = \int_{\Omega_K} f(\mathbf{x})\, d\mathbf{x} \text{ for all } K \in \mathcal{M}, \\ \displaystyle\sum_{K\in\mathcal{M}_s} -F_{K,s}(u) = \sum_{K\in\mathcal{M}_s} \int_{\Omega_{K,s}} f(\mathbf{x})\, d\mathbf{x} \text{ for all } \mathbf{s} \in \mathcal{V} \setminus \partial\Omega \end{cases}$$



$$\begin{cases} \displaystyle\sum_{s\in\mathcal{V}_K} F_{K,s}(u) = m_K f(\mathbf{x}_K) \text{ for all } K \in \mathcal{M}, \\ \displaystyle\sum_{K\in\mathcal{M}_s} -F_{K,s}(u) = \sum_{K\in\mathcal{M}_s} m_{K,s} f(\mathbf{x}_{K,s}) \text{ for all } \mathbf{s} \in \mathcal{V} \setminus \partial\Omega \end{cases}$$

The porous volume $m_{K,s}$ is taken from the surrounding cells proportionaly to the permeability of the cells



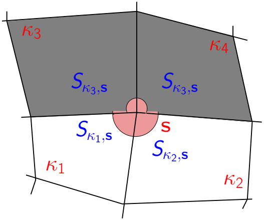
VAG discretization of the two phase flow model with upwinding

$$\mathcal{F}^{\alpha}_{K,s}(p^{\alpha}) = F_{K,s}(p^{\alpha}) + \rho^{\alpha} g F_{K,s}(z), \alpha = 1,2$$

S and M^{α} cellwise constant functions: S_K, M^{α}_K, $\alpha = 1,2$

$$S^n_K = S_K(p^n_{c,K}), \quad S^n_{K,s} = S_K(p^n_{c,s}).$$

$$\mathcal{G}^{\alpha}_{K,s}(p^{1,n}, p^{2,n}) = \begin{cases} M^{\alpha}_K(S^n_K)\mathcal{F}^{\alpha}_{K,s}(p^{\alpha,n}) & \text{if } \mathcal{F}^{\alpha}_{K,s}(p^{\alpha,n}) \geq 0, \\ M^{\alpha}_K(S^n_{K,s})\mathcal{F}^{\alpha}_{K,s}(p^{\alpha,n}) & \text{else} \end{cases}$$



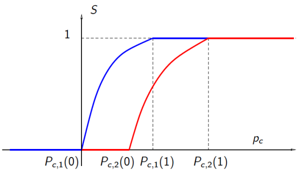
Equation for phase 1:

$$\begin{cases} m_K \phi_K \dfrac{S^n_K - S^{n-1}_K}{\Delta t} + \displaystyle\sum_{s \in \mathcal{V}_K} \mathcal{G}^{\alpha}_{K,s}(p^{1,n}, p^{2,n}) = 0, & K \in \mathcal{M}, \\ \displaystyle\sum_{K \in \mathcal{M}_s} m_{K,s}\phi_K \dfrac{S^n_{K,s} - S^{n-1}_{K,s}}{\Delta t} - \displaystyle\sum_{K \in \mathcal{M}_s} \mathcal{G}^{\alpha}_{K,s}(p^{1,n}, p^{2,n}) = 0, & s \in \mathcal{V} \setminus \partial\Omega. \end{cases}$$

Similar equation for phase 2.

Problem of non uniqueness of the solution p^1, p^2

Example: initial state with only phase 2:
p_c is not uniquely defined.



To avoid this singularity when solving the discrete nonlinear system:

Projections of $p_{c,K}$ on the interval:

$$\Big[P_{c,K}(0), P_{c,K}(1)\Big]$$

and of $p_{c,s}$ on

$$\Big[\min_{K\in\mathcal{M}_s} P_{c,K}(0), \max_{K\in\mathcal{M}_s} P_{c,K}(1)\Big].$$

Comparison with Control Volume Finite Element Methods

Keep the cell center unknown

Decouple the computation of fluxes from the choice of the Control volumes

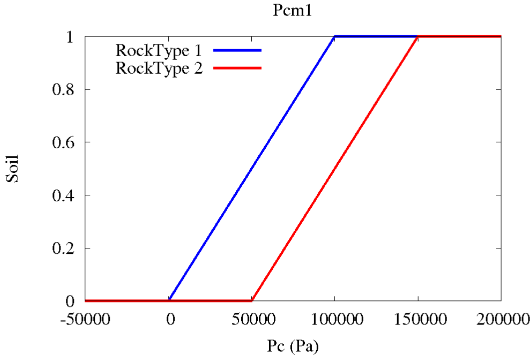
Fluxes always coercive

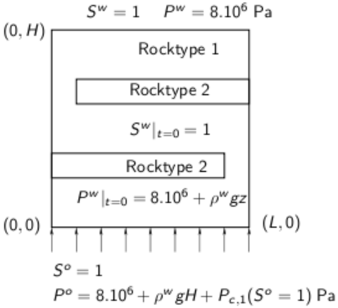
Choice of the porous volumes to match heterogeneities

Phase pressure unknowns to capture the jump of the saturations

Oil migration in a 2D basin with 2 barriers

Porous media with two rocktypes: $K_1 = K_2 = 1.10^{-12}\, m^2$, $\phi_1 = \phi_2 = 0.1$, $k^{\alpha}_{r,1} = k^{\alpha}_{r,2}$, $\alpha = w, o$, and the following $P^{-1}_{c,1}$, $P^{-1}_{c,2}$:



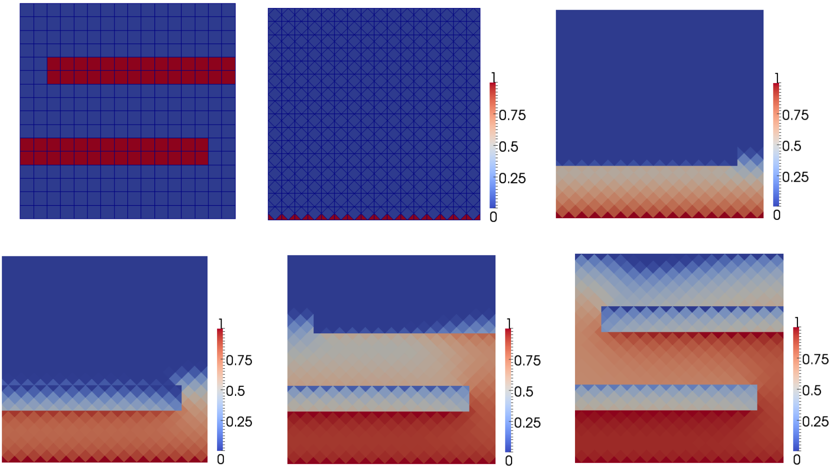


Density driven flow: $\rho^o = 800$, $\rho^w = 1000\ kg/m^3$,

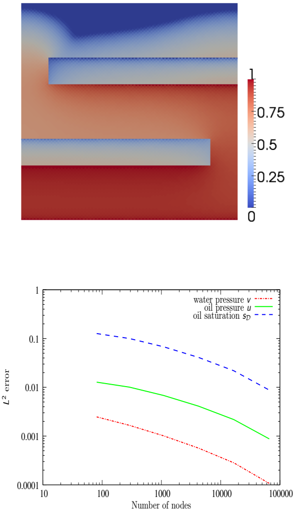
$$k^o_r(S^o) = (S^o)^2,\ \mu^o = 5.10^{-3},$$

$$k^w_r(S^w) = (S^w)^2,\ \mu^w = 1.10^{-3}.$$

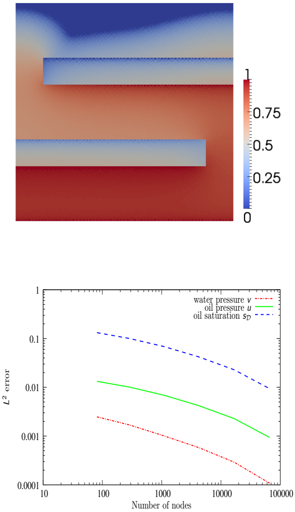
Oil migration in a 2D basin with 2 barriers



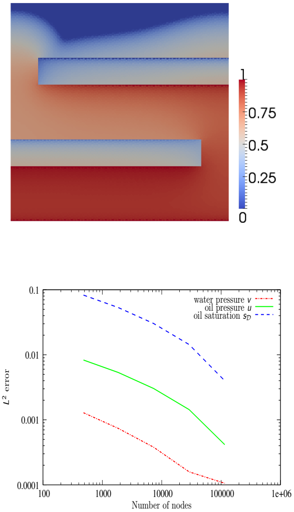
Oil migration in a 2D basin with 2 barriers



(a) Cartesian grid

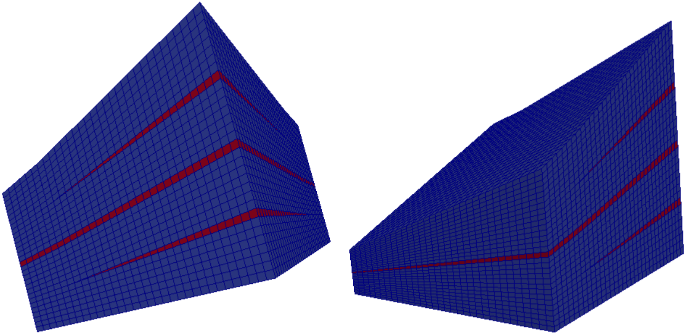


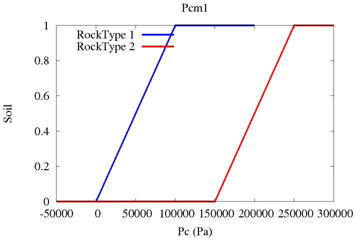
(b) quadrangular grid



(c) triangular grid

Oil migration in a 3D basin with barriers





Oil migration in a 3D basin with a barrier

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Oil migration in a 2D anisotropic heterogeneous basin

Density driven flow: $\rho^o = 850,\ \rho^w = 1000\ kg/m^3$.

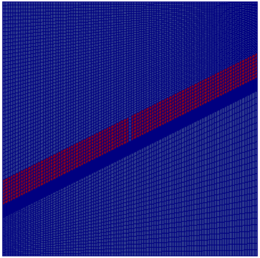
Permeability in the drain: $\Lambda(\mathbf{x}) = 10^{-12} \begin{pmatrix} 0.82 & -0.36 \\ -0.36 & 0.28 \end{pmatrix}$.

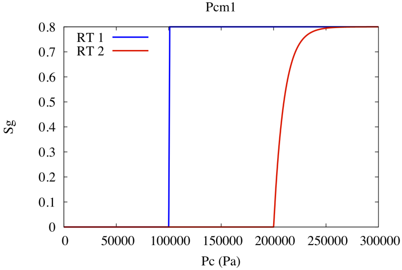
Permeability in the barrier: $\Lambda(\mathbf{x}) = 10^{-14} Id$.

Permeability in the fracture: $\Lambda(\mathbf{x}) = 10^{-11} Id$.

k_r^o and k_r^w: Corey laws with $S_{rw} = 0.2$, $S_{ro} = 0$ and exponents 2.

Capillary pressures for both rocktypes: Corey's laws





Oil migration in a 2D anisotropic heterogeneous basin

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Oil migration in a random media: $S_K(q) = \min(\max(\frac{q-\gamma_K 10^5}{10^5}, 0), 1)$, γ_K randomly chosen in $[-1, 1]$

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Conclusion

- Gradient schemes: general framework to analyse a large family of schemes, apply to some coupled nonlinear problems such as two phase Darcy flows

- the vertex-centered scheme VAG is well adapted to heterogeneous problems with different rocktypes thanks to

  - a pressure pressure formulation
  - interface unknowns capturing the saturation jump conditions at rocktype interfaces
  - an ad-hoc redistribution of the cell pore volumes