

Cell Centered Finite Volume Schemes for Multiphase Flow Applications

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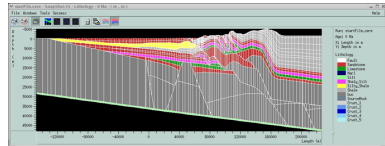
³ Université Paris Est

june 22-24th 2009

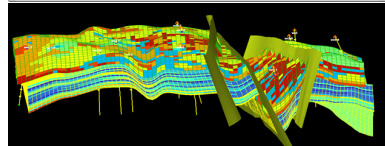
- 1 Applications and meshes
- 2 Flux formulation
 - The L and G schemes
- 3 Discrete variational framework
 - The GradCell scheme
 - The O scheme
- 4 Numerical Experiments

Applications

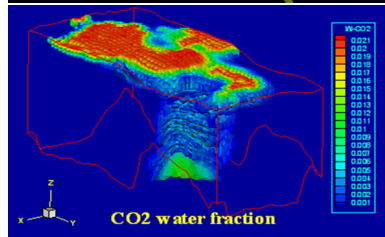
Basin Modeling



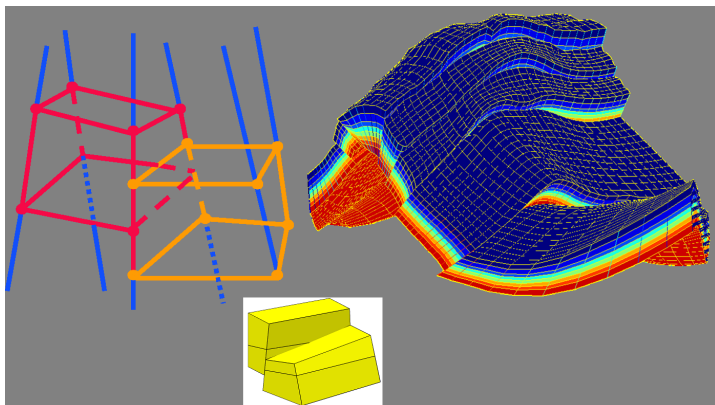
Reservoir simulation



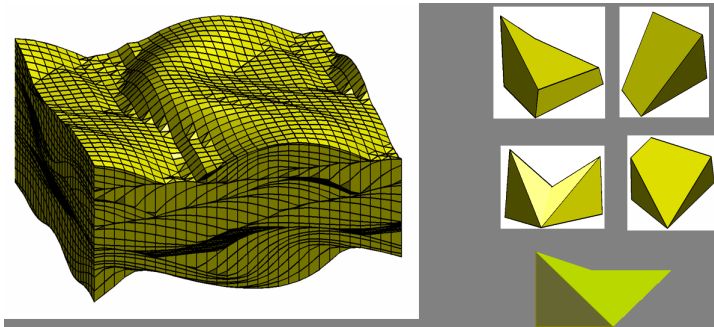
CO2 geological storage



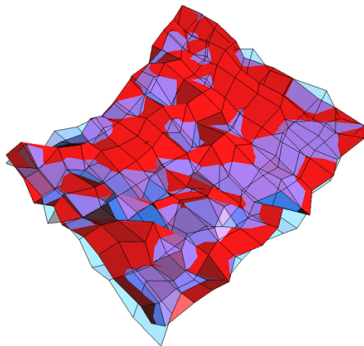
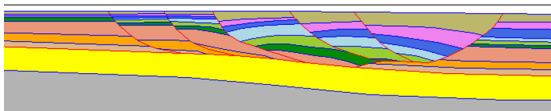
Meshes: corner point geometries with faults



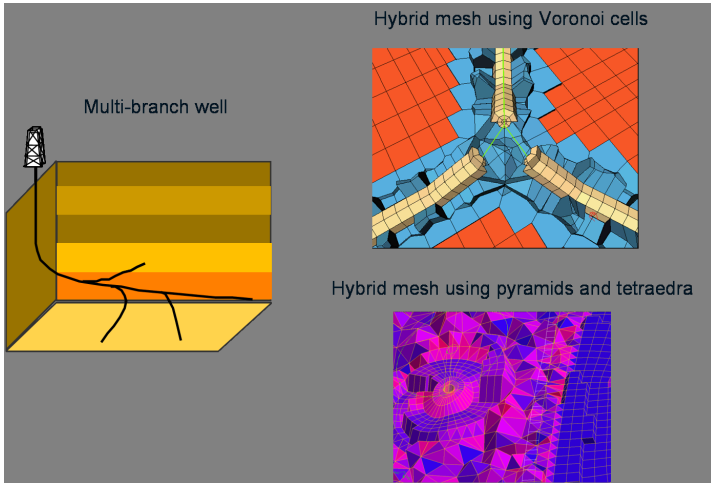
Meshes: corner point geometries with erosions



Meshes: basin geometries



Meshes: nearwell meshes



Difficulties: Geometry

- Degenerated cells due to erosion
- Dynamic mesh (basin models): the scheme must be recomputed at each time step
- Faults in basin models: geometry not always available (overlaps and holes)
- Conductive Faults in basin models
- General polyhedral cells
- Submeshes (dead cells)
- Local Grid Refinement
- Adaptive Mesh Refinement
- Boundary conditions

Difficulties: complex physics

- Heterogeneous anisotropic media
- Dispersion (full tensor,time and space dependent)
- Multiphase Darcy Flows
- Complex closure laws: thermodynamical equilibrium, geochemistry, Kinetics
- Thermics
- Geomechanics

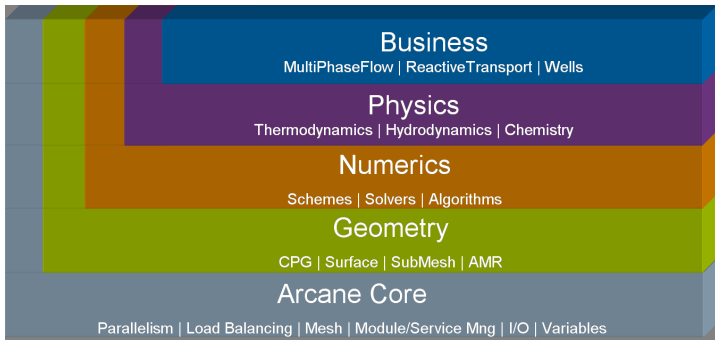
Motivations of cell centered schemes for compositional multiphase Darcy flow applications

- N_c primary unknowns per cell for multiphase compositional flows
- Explicit linear fluxes
- Easier to combine TPFA and MPFA
- Existing Efficient Preconditioners like CPR-AMG
 - Adapted to fully or semi implicit discretizations of multiphase compositional Darcy flows

- But “compact” MPFA VF schemes are non symmetric on general meshes
 - Possible lack of robustness due to mesh and diffusion coefficients dependent coercivity (linear solver, pressure convergence) and monotonicity (non linear solver)

ArcGeoSim platform

- Based on Arcane Platform co-developped by CEA-DAM and IFP



Cell centered schemes currently implemented in ArcGeoSim

- L and G schemes
- O scheme
- GradCell scheme

Model problem

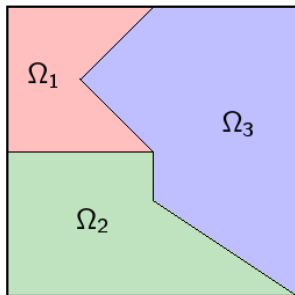
- Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal domain
- For $f \in L^2(\Omega)$, consider the following problem:

$$\begin{cases} -\operatorname{div}(\nu \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- Let $a(u, v) = \int_{\Omega} \nu \nabla u \cdot \nabla v$. The weak formulation reads

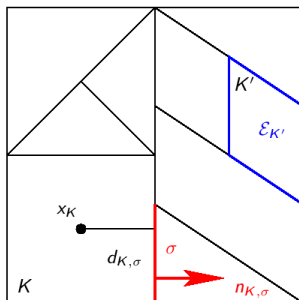
$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u, v) = \int_{\Omega} f v \text{ for all } v \in H_0^1(\Omega) \quad (\Pi)$$

Model problem



- Let $\{\Omega_i\}_{1 \leq i \leq N_\Omega}$ be a partition of Ω into bounded polygonal sub-domains
- $\nu|_{\Omega_i}$ smooth and $\nu(x)$ is s.p.d. for a.e. $x \in \Omega$

Polyhedral admissible meshes



\mathcal{T}_h : set of cells K

$\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$: set of inner and boundary faces σ

m_σ : surface of the face σ

m_K : volume of the cell K

Discrete function space V_h

- V_h : space of piecewise constant functions on \mathcal{T}_h

$$v_h(x) = v_K \text{ for all } x \in K$$

- Equip V_h with the following **discrete H_0^1 norm**:

$$\forall v_h \in V_h, \quad \|v_h\|_{V_h} = \left(\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}} |\gamma_\sigma(v_h) - v_K|^2 \right)^{1/2}$$

- using the following trace reconstruction at the faces σ

$$\begin{cases} \gamma_\sigma(v_h) = \frac{v_K d_{L,\sigma} + v_L d_{K,\sigma}}{d_{L,\sigma} + d_{K,\sigma}} & \text{if } \sigma = \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h^i, \\ \gamma_\sigma(v_h) = 0 & \text{if } \sigma \in \mathcal{E}_h^b. \end{cases}$$

Finite Volume Scheme

- Let $F_{K,\sigma}(u_h)$ denote a conservative linear approximation of

$$\int_{\sigma} \nu \nabla u \cdot n_{K,\sigma}$$

conservativity: $F_{K,\sigma}(u_h) + F_{L,\sigma}(u_h) = 0, \quad \sigma = \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h^i.$

- The finite volume scheme reads

find $u_h \in V_h$ such that

$$-\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_h) = \int_K f \quad \text{for all } K \in \mathcal{T}_h.$$

Discrete variational formulation

- For all $u_h, v_h \in V_h$, let

$$\begin{aligned}
 a_h(u_h, v_h) &= \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_h) (\gamma_\sigma(v_h) - v_K) \\
 &= \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h} F_{K,\sigma}(u_h) (v_L - v_K) - \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^b} F_{K,\sigma}(u_h) v_K
 \end{aligned}$$

- The finite volume scheme is equivalent to:

find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \int_{\Omega} f v_h \text{ for all } v_h \in V_h.$$

Assumptions

- **Flux consistency in \mathcal{Q}** : for all $\varphi \in \mathcal{Q}$ with $\varphi_h = \{\varphi(x_K)\}_{K \in \mathcal{T}_h}$,

$$\lim_{h \rightarrow 0} \left(\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \frac{d_{K,\sigma}}{m_\sigma} |F_{K,\sigma}(\varphi_h) - m_\sigma \langle \nu \nabla \varphi \rangle_K \cdot n_{K,\sigma}|^2 \right) = 0$$

- **Coercivity of the bilinear form a_h** :

$$\forall v \in V_h, \quad a_h(v, v) \gtrsim \|v\|_{V_h}^2$$

Well-posedness

- Using the coercivity of a_h and the discrete Poincaré inequality:

$$\begin{aligned}\|u_h\|_{V_h}^2 &\lesssim a_h(u_h, u_h) = \int_{\Omega} f u_h \\ &\lesssim \|f\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)} \\ &\lesssim \|f\|_{L^2(\Omega)} \|u_h\|_{V_h}\end{aligned}$$

Stability of the scheme in V_h norm: $\|u_h\|_{V_h} \lesssim \|f\|_{L^2(\Omega)}$

Convergence

- Consider a sequence of admissible FV discretizations with $h \rightarrow 0$.
- Discrete Rellich theorem: there exist a subsequence and $\tilde{u} \in H_0^1(\Omega)$ s.t. $u_h \rightarrow \tilde{u}$ in $L^2(\Omega)$ and $\tilde{\nabla}_h u_h \rightharpoonup \nabla \tilde{u}$ weakly in $(L^2(\Omega))^d$ with

$$(\tilde{\nabla}_h v_h)_K = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (\gamma_\sigma(v_h) - v_K) n_{K,\sigma}$$

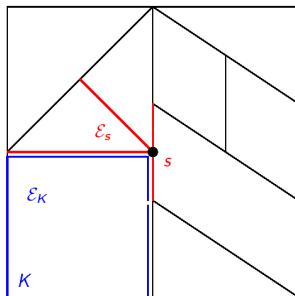
- Using the coercivity and the flux consistency assumptions and the density of \mathcal{Q} in $H_0^1(\Omega)$ we can deduce that

$$\tilde{u} = u.$$

The L and G schemes

- Consistent vertex-based gradient reconstruction to obtain fluxes
- L scheme (see [Aavatsmark et al., 2007], [Aavatsmark et al., 2008])
 - monotonicity enhancing tuning
- G scheme: (see [Agélas et al., 2009])
 - coercivity-enhancing tuning
- Convergence analysis: see [Agélas et al., 2009]

Groups

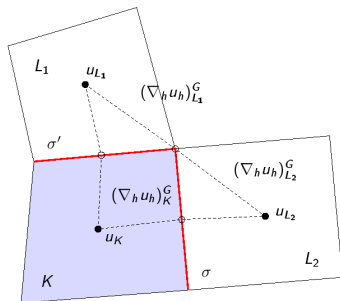


$$\mathcal{G} = \{G \subset \mathcal{E}_K \cap \mathcal{E}_s \text{ s.t. } \text{card}G = d\}$$

L-interpolation

$$G = \{\sigma, \sigma'\}$$

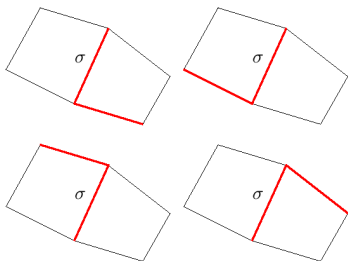
$$\mathcal{T}_G = \{K, L_1, L_2\}$$



- Piecewise linear interpolation of u_h on K, L_1, L_2
- Full continuity of potential and normal fluxes at the faces σ and σ'
- Amounts to solve a linear system $A^G X = b$ of size d with X the vector of the d components of $(\nabla_h u_h)_K^G$.

Fluxes

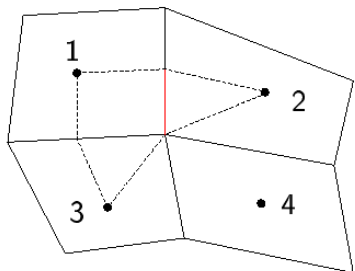
$$\mathcal{G}_\sigma = \{G \in \mathcal{G} \mid \sigma \in G\}$$



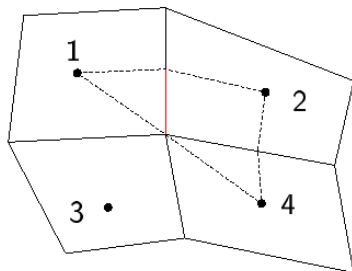
- $F_{K,\sigma}^G = m_\sigma \nu_K (\nabla_h u_h)_K^G \cdot n_{K,\sigma}$
- $F_{K,\sigma} = \sum_{G \in \mathcal{G}_\sigma} \theta_\sigma^G F_{K,\sigma}^G$ with $\sum_{G \in \mathcal{G}_\sigma} \theta_\sigma^G = 1$, $\theta_\sigma^G \geq 0$
- **L scheme:** θ_σ^G are fixed, choice of the groups G to enhance the monotonicity of the scheme ([Aavatsmark et al., 2007])
- **G scheme:** keep all the groups and choose the θ_σ^G to enhance the coercivity of the scheme

L-Groups (exemple for an half edge in 2D)

- ▶ Finite Volume MPFA L method :



group g_1



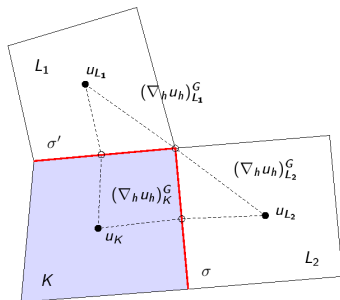
group g_2

if $|t_1^{g_1}| < |t_2^{g_2}|$, *group g_1* is chosen, else *group g_2* is chosen

G-Weights I

$$G = \{\sigma, \sigma'\}$$

$$\mathcal{T}_G = \{K, L_1, L_2\}$$



- Let $\mathcal{H}_{h,G} = \{u_K, u_{L_1}, u_{L_2}\}$
- Set $a_{h,G}(u, v) = F_{K,\sigma}^G(u)(v_{L_1} - v_K) + F_{K,\sigma'}^G(u)(v_{L_2} - v_K)$
- Define $\gamma_G = \inf_{\{u \in \mathcal{H}_{h,G}, \|u\|_{V_h} = 1\}} a_{h,G}(u, u)$

G-Weights II

- For a given $\epsilon > 0$ define

$$\begin{cases} f_\epsilon(x) = \frac{\epsilon^2}{\epsilon - x} & \text{if } x < 0, \\ f_\epsilon(x) = x + \epsilon & \text{otherwise,} \end{cases}$$

- The weights are defined as

$$\theta_\sigma^G = \frac{f_\epsilon(\gamma_G)}{\sum_{G' \in \mathcal{G}_\sigma} f_\epsilon(\gamma_{G'})} \quad \forall G \in \mathcal{G}, \forall \sigma \in \mathcal{G}.$$

- The larger γ_G , the more the group G contributes!

Convergence analysis of L and G schemes

- Fluxes consistency in Q :
 - require to assume the stability of the L-interpolation i.e. $\|(A^G)^{-1}\| \geq \beta$

- Coercivity assumption: mesh and ν dependent
 - sufficient computable condition using the submatrix around each vertex.

GradCell scheme: using a discrete variational framework [Agélas et al., 2008]

- Discrete variational formulation

$$a_h(u_h, v_h) = \int_{\Omega} f v_h \text{ for all } v_h \in V_h.$$

- with a_h based on two cellwise constant gradients and stabilized by residuals

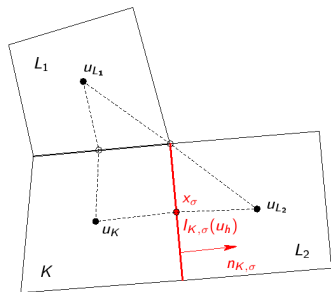
$$\begin{aligned} a(u_h, v_h) &= \sum_{K \in \mathcal{T}_h} m_K \nu_K (\nabla_h u_h)_K \cdot (\tilde{\nabla}_h v_h)_K \\ &+ \sum_{K \in \mathcal{T}_h} \eta_K \sum_{\sigma \in \mathcal{E}_K} \frac{m_{\sigma}}{d_{K,\sigma}} R_{K,\sigma}(u_h) R_{K,\sigma}(v_h) \end{aligned}$$

Discrete gradients reconstructions

- The cellwise constant gradients are obtained via the Green formula and trace reconstructions

$$(\nabla_h \mathbf{v}_h)_K = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (I_{K,\sigma}(\mathbf{v}_h) - v_K) \mathbf{n}_{K,\sigma}$$

$$(\tilde{\nabla}_h \mathbf{v}_h)_K = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (\gamma_\sigma(\mathbf{v}_h) - v_K) \mathbf{n}_{K,\sigma}$$



- Residuals:

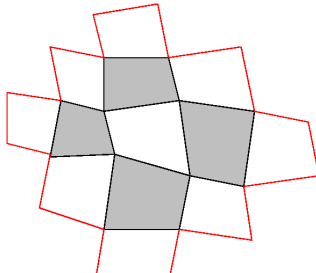
$$R_{K,\sigma}(\mathbf{v}_h) = I_{K,\sigma}(\mathbf{v}_h) - v_K - (\nabla_h \mathbf{v}_h)_K \cdot (\mathbf{x}_\sigma - \mathbf{x}_K)$$

Fluxes

- Fluxes can be derived from the bilinear form using:

$$a_h(u_h, v_h) = \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h} F_{K,\sigma}(u_h)(v_L - v_K) - \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^b} F_{K,\sigma}(u_h) v_K$$

- Stencil of the scheme: neighbours of the neighbours
- Example for topologically cartesian grids
 - 13 cells in 2D
 - 21 cells in 3D



Convergence analysis

- Consistency of $\nabla_h u_h$ for piecewise smooth functions with normal flux continuity: stability of the L-interpolation $\|(A^G)^{-1}\| \geq \beta$
- Coercivity assumption: $a(v_h, v_h) \gtrsim \|v_h\|_{V_h}^2$

Remarks

- Remark 1: a linear interpolation $I_{K,\sigma}$ can be used for smooth diffusion tensors ν
- Remark 2: choosing $\tilde{\nabla}_h u_h = \nabla_h u_h$ yields a symmetric coercive scheme but at the price of a larger stencil (21 in 2D and 81 in 3D) see [Eymard and Herbin, 2007].

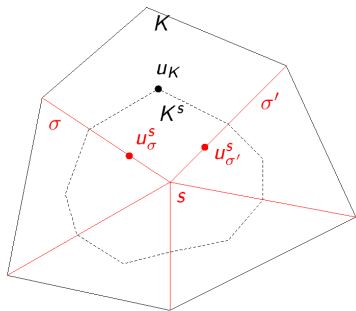
Discrete gradient

Piecewise constant gradient on each K^s :

$$(\nabla_h u)_K^s = \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} (u_\sigma^s - u_K) g_{K,\sigma}^s$$

with $g_{K,\sigma}^s \in \mathbb{R}^d$ such that

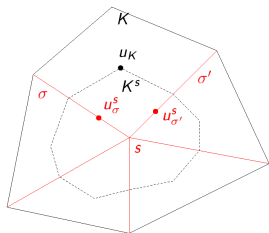
$$\sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} v \cdot (x_\sigma^s - x_K) g_{K,\sigma}^s = v \quad \text{for all } v \in \mathbb{R}^d.$$



Usual O scheme I

- Assume that $\{x_\sigma^s - x_K\}_{\sigma \in \mathcal{E}_s \cap \mathcal{E}_K}$ defines a basis of \mathbb{R}^d
- then $(\nabla_h u)_K^s$ is uniquely defined by the gradient of the linear interpolation of (u_K, x_K) , $(u_\sigma^s, x_\sigma^s)_{\sigma \in \mathcal{E}_s \cap \mathcal{E}_K}$.

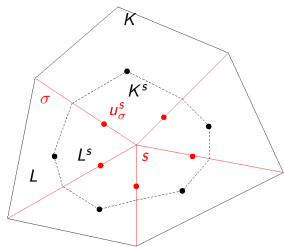
Subfluxes: $F_{K,\sigma}^s(u_h) = m_\sigma^s \nu_K (\nabla_h u)_K^s \cdot n_{K,\sigma}$



Usual O scheme II: Hybrid Finite volume scheme

$$\left\{ \begin{array}{l} \sum_{\sigma \in \mathcal{E}_K} \left(\sum_{s \in \mathcal{V}_\sigma} F_{K,\sigma}^s(u_h) \right) = \int_K f(x) dx \quad \text{for all } K \in \mathcal{T}_h, \\ F_{K,\sigma}^s(u_h) = -F_{L,\sigma}^s(u_h) \quad \text{for all } s \in \mathcal{V}_\sigma, \sigma = \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h^i. \end{array} \right.$$

$(u_\sigma^s)_{\sigma \in \mathcal{E}_s}$ are eliminated around each vertex s in terms of the cell centered unknowns around s .



Usual O scheme: discrete hybrid variational formulation

Bilinear form on \mathcal{H}_h :

$$\mathcal{H}_h = \left\{ (u_K)_{K \in \mathcal{T}_h}, (u_\sigma^s)_{\sigma \in \mathcal{E}_s, s \in \mathcal{V}_h}, \text{ s. t. } u_\sigma^s = 0 \text{ for all } \sigma \in \mathcal{E}_h^b \right\}.$$

$$a_h(u, v) = \sum_{K \in \mathcal{T}_h} \sum_{s \in \mathcal{V}_K} m_K^s (\nabla_h u)_K^s \cdot \nu_K (\tilde{\nabla}_h v)_K^s$$

with

$$(\tilde{\nabla}_h u)_K^s = \frac{1}{m_K^s} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} m_\sigma^s (u_\sigma^s - u_K) n_{K,\sigma}.$$

Finite volume scheme: find $u_h \in \mathcal{H}_h$ such that

$$a_h(u_h, v) = \sum_{K \in \mathcal{T}_h} v_K \int_K f(x) dx \quad \text{for all } v \in \mathcal{H}_h.$$

Generalization: discrete hybrid variational formulation

Bilinear form on \mathcal{H}_h :

$$a_h(u, v) = \sum_{K \in \mathcal{T}_h} \sum_{s \in \mathcal{V}_K} \left(m_K^s (\nabla_h u)_K^s \cdot \nu_K (\tilde{\nabla}_h v)_K^s \right. \\ \left. + \alpha_K^s \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \frac{m_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u) R_{K,\sigma}^s(v) \right)$$

with

$$R_{K,\sigma}^s(u) = u_\sigma^s - u_K - (\nabla_h u)_K^s \cdot (x_\sigma^s - x_K).$$

Finite volume scheme: find $u_h \in \mathcal{H}_h$ such that

$$a_h(u_h, v) = \sum_{K \in \mathcal{T}_h} v_K \int_K f(x) dx \quad \text{for all } v \in \mathcal{H}_h.$$

Hybrid Finite Volume scheme

$$a_h(u, v) = \sum_{K \in \mathcal{T}_h} \sum_{s \in \mathcal{V}_K} \sum_{\sigma \in \mathcal{E}_s \cap \mathcal{E}_K} \sum_{\sigma' \in \mathcal{E}_s \cap \mathcal{E}_K} (T_K^s)_{\sigma, \sigma'} (u_{\sigma'}^s - u_K) (v_{\sigma'}^s - v_K),$$

Let us define the following subfluxes:

$$F_{K, \sigma}^s(u) = \sum_{\sigma' \in \mathcal{E}_s \cap \mathcal{E}_K} (T_K^s)_{\sigma, \sigma'} (u_{\sigma'}^s - u_K),$$

Hybrid Finite volume scheme:

$$\left\{ \begin{array}{l} \sum_{\sigma \in \mathcal{E}_K} \left(\sum_{s \in \mathcal{V}_\sigma} F_{K, \sigma}^s(u_h) \right) = \int_K f(x) dx \quad \text{for all } K \in \mathcal{T}_h, \\ F_{K, \sigma}^s(u_h) = -F_{L, \sigma}^s(u_h) \quad \text{for all } s \in \mathcal{V}_\sigma, \sigma = \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h^i \end{array} \right.$$

Convergence analysis

- Consistency of $\nabla_h \varphi_h$ for $\varphi \in C_c^\infty(\Omega)$ with

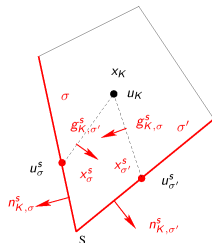
$$\varphi_h = \left(\varphi(x_K), \varphi(x_\sigma^s) \right)_{K \in \mathcal{T}_h, \sigma \in \mathcal{E}_s, s \in \mathcal{V}_h}$$

- Coercivity assumption: $a(v_h, v_h) \gtrsim \|v_h\|_{\mathcal{H}_h}^2$

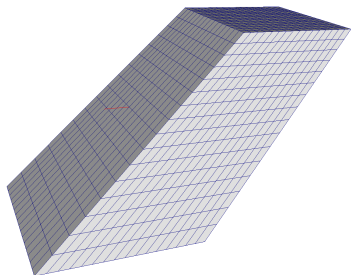
$$\|v\|_{\mathcal{H}_h} = \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \sum_{s \in \mathcal{V}_\sigma} \frac{m_K^s}{(d_{K,\sigma})^2} (v_\sigma^s - v_K)^2 \right)^{1/2}.$$

Symmetric unconditionally coercive cases

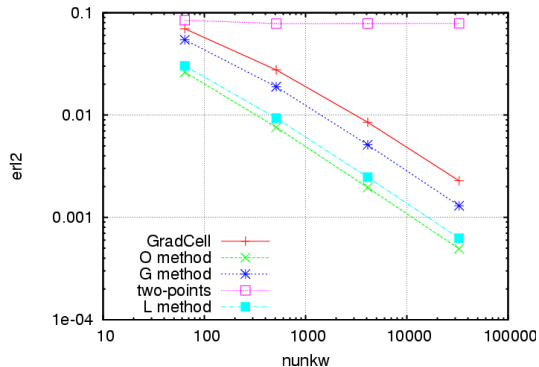
- Simplicial cells
- Parallelepipedic cells
 - for an ad hoc choice of x_σ^s one has $(\nabla_h u)_K^s = (\tilde{\nabla}_h u)_K^s$ i.e. $g_{K,\sigma}^s \parallel n_{K,\sigma}$
- Symmetrization (C. Lepotier)
 - Choose $(\nabla_h u)_K^s := (\tilde{\nabla}_h u)_K^s$
 - Fluxes are not consistent except for the above cases



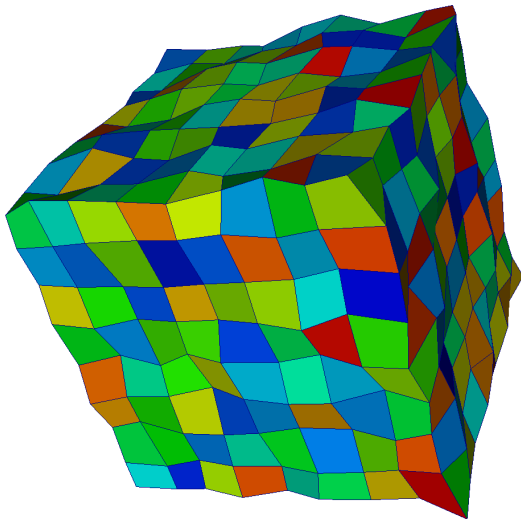
Hexahedral meshes, Identity permeability tensor,
 $u(x, y, z) = \sin(\pi x) \sin(\pi y) \sin(\pi z)$



The grid

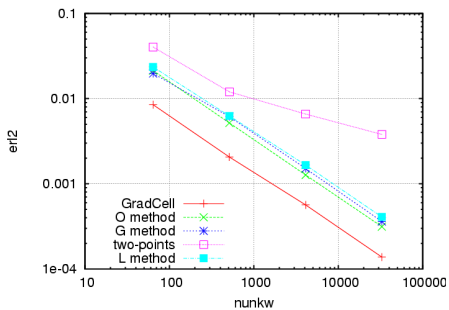


Randomly distorted hexahedral meshes I

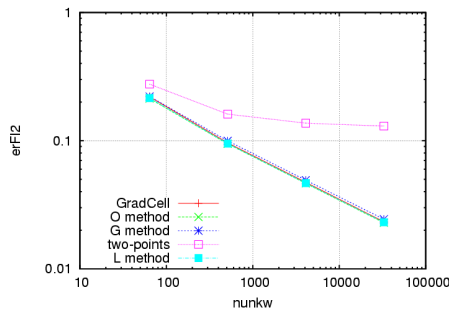


$$\nu = I$$

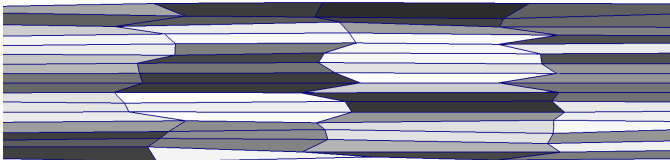
L^2 error on pressure



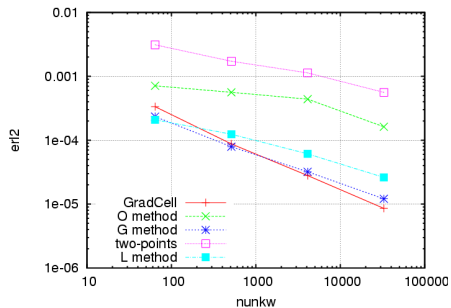
L^2 error on fluxes



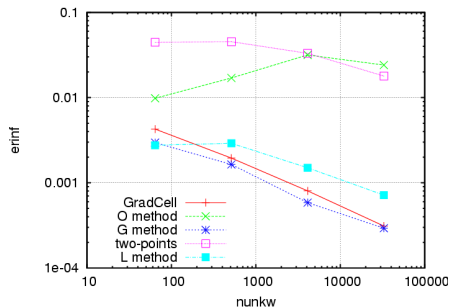
Randomly distorted hexahedral meshes II



$$\nu = I$$

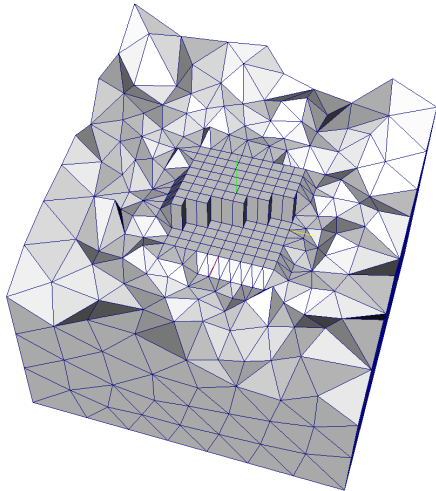


L_2 error for pressure

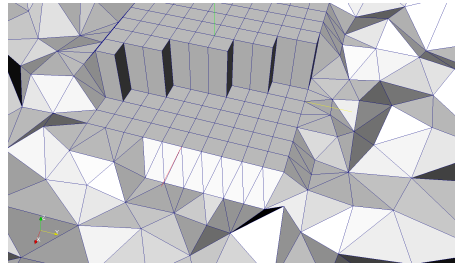


L_∞ error for pressure

Hybrid meshes

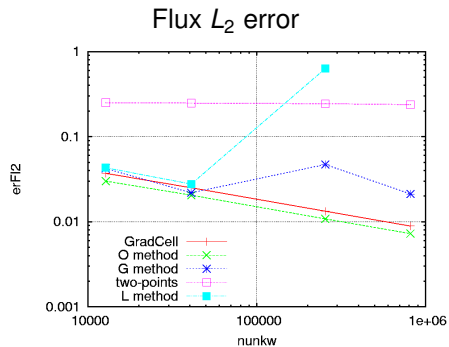
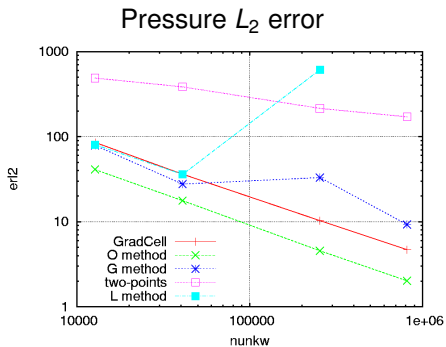


Slice of the mesh



Zoom in

Hybrid meshes, Isotropic tensor ν .



Hybrid meshes, Isotropic tensor ν , performance with AMG

Number of GMRES iterations vs grid size $\left(\frac{\|r\|}{\|r_0\|} < 10^{-9}\right)$.

| No of cells | O | L | G | GradCell | 2-point |
|-------------|----|---|----|----------|---------|
| 12723 | 7 | — | 16 | 8 | 7 |
| 40847 | 7 | — | 85 | 9 | 7 |
| 254645 | 11 | — | — | 15 | 8 |
| 813368 | 8 | — | — | 11 | 8 |

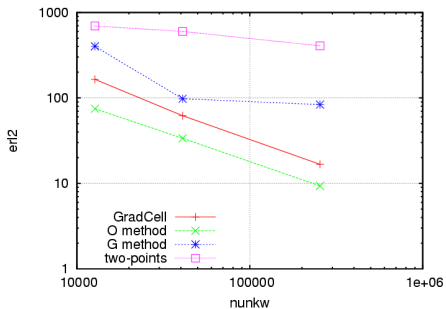
Hybrid meshes, Isotropic tensor ν , coercivity

$$\gamma_h = \frac{a_h(u_h - \pi_h(u), u_h - \pi_h(u))}{\|u_h - \pi_h(u)\|_{V_h}^2}$$

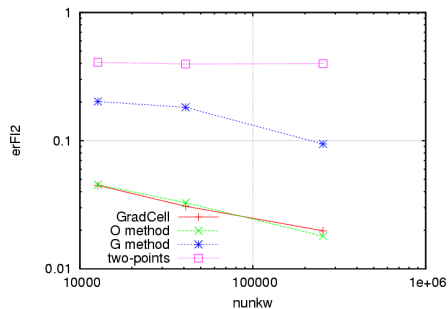
| No of cells | O | L | G | GradCell | 2-point |
|-------------|--------------|--------------|--------------|--------------|--------------|
| 12723 | $8.07e - 01$ | $1.35e - 01$ | $3.63e - 01$ | $1.45e + 00$ | $7.94e - 01$ |
| 40847 | $8.29e - 01$ | $8.52e - 02$ | $8.65e - 01$ | $1.51e + 00$ | $8.02e - 01$ |
| 254645 | $8.10e - 01$ | $7.20e - 06$ | $1.77e - 02$ | $1.47e + 00$ | $8.09e - 01$ |
| 813368 | $8.10e - 01$ | – | $3.40e - 02$ | $1.47e + 00$ | $8.21e - 01$ |

Anisotropic tensor $\nu = \text{diag}\{1, 1, 0.1\}$.

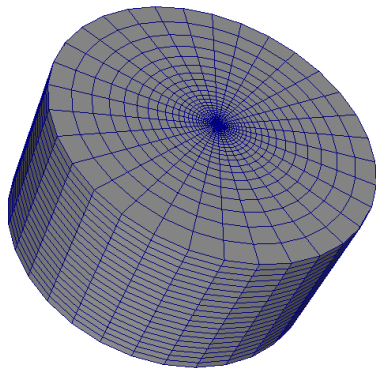
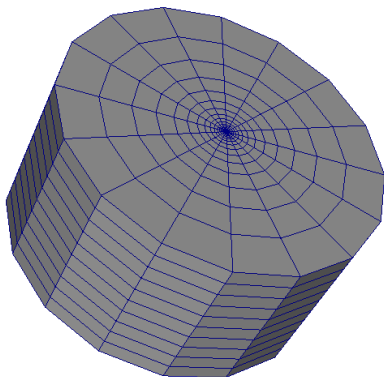
Pressure L_2 error



Flux L_2 error

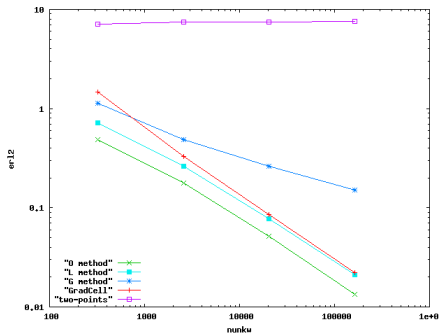


Nearwell radial grids

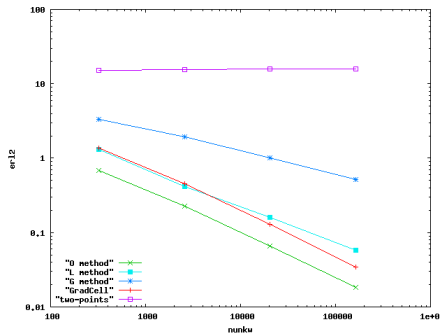


Anisotropic tensor $\nu = \text{diag}\{1, 1, 0.05\}$.

L^2 error on pressure



L^2 error on fluxes



Conclusion

- Case dependent results
- O scheme the most accurate but lacks of robustness for meshes with high aspect ratio (or anisotropy) combined with distorsion
- L and G schemes good on hexahedral meshes but may fail for Hybrid meshes
- GradCell exhibits the best robustness but requires two layers of communication in parallel

References



Aavatsmark, I., Eigestad, G., Mallison, B., and Nordbotten, J. (2008).
A compact multipoint flux approximation method with improved robustness.
Numer. Methods Partial Differential Equations, 24(5):1329–1360.



Aavatsmark, I., Eigestad, G., Mallison, B., Nordbotten, J., and ian, E. O. (2007).
A compact multipoint flux approximation method with improved robustness.
Numer. Methods Partial Differential Equations, 1(31).



Agélas, L., Di Pietro, D. A., and Droniou, J. (2009).
The G method for heterogeneous anisotropic diffusion on general meshes.
M2AN Math. Model. Numer. Anal.
Accepted for publication. Preprint available at <http://hal.archives-ouvertes.fr/hal-00342739/fr>.



Agélas, L., Di Pietro, D. A., Eymard, R., and Masson, R. (2008).
An abstract analysis framework for nonconforming approximations of anisotropic heterogeneous diffusion.
Preprint available at <http://hal.archives-ouvertes.fr/hal-00318390/fr>.
Submitted.



Agélas, L. and Masson, R. (2008).
Convergence of the finite volume MPFA O scheme for heterogenous anisotropic diffusion problems on general meshes.
In Eymard, R. and Hérard, J.-M., editors, *Finite Volumes for Complex Applications V*, pages 145–152. John Wiley & Sons.



Eymard, R. and Herbin, R. (2007).
A new collocated finite volume scheme for the incompressible Navier-Stokes equations on general non matching grids.
C. R. Math. Acad. Sci., 344(10):659–662.