Course Finite Volume discretization of PDEs,
University Nice Sophia Antipolis

Exercize : finite volume discretization of a scalar hyperbolic equation with source terms.

Let Ω a bounded polygonal domain of $\mathbb{R}^2$ and u a function from $\Omega \times (0,T)$ to $\mathbb{R}$. We consider the following scalar hyperbolic problem :

$$\left\{\begin{array}{rll} \partial_t u(\mathbf{x},t) + \mathrm{div}\Big(f(u(\mathbf{x},t))\mathbf{V}(\mathbf{x})\Big) &= h(\mathbf{x})^+ f(c(\mathbf{x})) + h(\mathbf{x})^- f(u(\mathbf{x},t)) & \text{on } \Omega \times (0,T), \\ u(\mathbf{x},t) &= d(\mathbf{x}) & \text{on } \partial\Omega^- \times (0,T), \\ u(\mathbf{x},0) &= u^0(\mathbf{x}) & \text{on } \Omega, \end{array}\right.$$

with $u^0 \in L^\infty(\Omega)$, $c \in L^\infty(\Omega^+)$, $d \in L^\infty(\partial\Omega^-)$, $h \in L^\infty(\Omega)$, and $\mathbf{V} \in C^1(\overline{\Omega})$ such that

$$\mathrm{div}(\mathbf{V}) = h \text{ on } \Omega.$$

The function f is assumed non decreasing from $\mathbb{R}$ to $\mathbb{R}$ and to be Lipchitz on bounded sets. The domain Ω^+ is defined by

$$\Omega^+ = \{\mathbf{x} \in \Omega \text{ such that } h(\mathbf{x}) > 0\},$$

and the input boundary $\partial\Omega^-$ by

$$\partial\Omega^- = \{\mathbf{x} \in \partial\Omega \text{ such that } \mathbf{V}\cdot\mathbf{n} < 0\}$$

where $\mathbf{n}$ is the unit normal vector outward to the domain Ω. We have used the notations $h^+(\mathbf{x}) = \max(0, h(\mathbf{x}))$, $h^-(\mathbf{x}) = \min(0, h(\mathbf{x}))$. We recall that $h^+(\mathbf{x}) + h^-(\mathbf{x}) = h(\mathbf{x})$.

(1) We consider a conforming finite volume mesh of Ω such that $\partial\Omega^-$ is the union of a collection of boundary faces. Using the notations of the course, write the finite volume discretization of the hyperbolic equation using an Euler explicit time integration, and the upwind scheme for the fluxes. The initial condition will be approximated in all cells K by

$$u_K^0 = \frac{1}{|K|}\int_K u^0(\mathbf{x})d\mathbf{x}.$$

You will use the notations of the course and in particular

$$V_{K,\sigma} = \int_\sigma \mathbf{V}\cdot\mathbf{n}_{K,\sigma}d\sigma,$$

and

$$d_\sigma = \frac{1}{|\sigma|}\int_\sigma d(\mathbf{x})d\sigma, \quad c_K = \frac{1}{|K|}\int_K c(\mathbf{x})d\mathbf{x}, \quad h_K^+ = \int_K h^+(\mathbf{x})d\mathbf{x}, \quad h_K^- = \int_K h^-(\mathbf{x})d\mathbf{x}.$$

The discretization of the source term $h(\mathbf{x})^+ f(c(\mathbf{x})) + h(\mathbf{x})^- f(u(\mathbf{x},t))$ in the cell K will use only h_K^+, h_K^-, c_K, the unknown u_K taken explicit in time, and the function f.

(2) Using that $\mathrm{div}(\mathbf{V}) = h$ and that f is non decreasing and Lipchitz on bounded sets, derive the CFL condition on the time step in order to obtain the stability of the discretization.

(3) Assuming that the CFL condition is satisfied, write the maximum principle satisfied by the discrete solution u_K^n for all cells K and all time steps n function of c, d and u^0.