Course *Finite Volume discretization of PDEs*
University of Nice Sophia Antipolis

Exercize : finite volume discretization of the 1D convection diffusion equation.

Let us consider the following convection diffusion equation

$$\left\{\begin{array}{ll} cu'(x) - \nu u''(x) = 0 & \text{on } (0,1),\\ u(0) = u_D^0, & \\ u(1) = u_D^1, & \end{array}\right.$$

with $c \geq 0$ and $\nu > 0$. Using the notations of the course, we consider the following finite volume scheme $u_h \in V_h$ such that

$$f_{i+1/2} - f_{i-1/2} \quad = \quad 0,\ i = 1, \cdots, N, \tag{1}$$

setting $u_0 = u_D^0$, $u_{N+1} = u_D^1$ and with

$$f_{i+1/2} = \nu \frac{u_i - u_{i+1}}{h_{i+1/2}} + c\Big(\theta_{i+1/2} u_i + (1 - \theta_{i+1/2}) u_{i+1}\Big),\ i = 0, \cdots, N,$$

for given $\theta_{i+1/2} \in [1/2, 1]$.

(1) How can we call the discretization of the convection term for $\theta_{i+1/2} = 1$ and for $\theta_{i+1/2} = 1/2$ for all $i = 1, \cdots, N$.

Corrigé : For $\theta_{i+1/2} = 1$ the discretization of the convection flux is the upwind approximation (since $c \geq 0$), while for $\theta_{i+1/2} = 1/2$ it is the centered approximation.

(2) Give a condition on $\theta_{i+1/2}$, $i = 1, \cdots, N$, and c and h such that

$$u_i = \alpha_i u_{i+1} + (1 - \alpha_i) u_{i-1},$$

with $0 < \alpha_i < 1$ for all $i = 1, \cdots, N$. In the following we assume that this condition is satisfied.

Corrigé : The finite volume conservation equation in each cell i rewrites :

$$\Big(\frac{\nu}{h_{i+1/2}} + \frac{\nu}{h_{i-1/2}} + c(\theta_{i+1/2} + \theta_{i-1/2} - 1)\Big) u_i = \Big(\frac{\nu}{h_{i+1/2}} - c(1 - \theta_{i+1/2})\Big) u_{i+1} + \Big(\frac{\nu}{h_{i-1/2}} + c\theta_{i-1/2}\Big) u_{i-1}.$$

Since $\theta_{i+1/2} \in [1/2, 1]$ we note that the coefficient of u_i is always strictly positive. It results that the condition $\alpha_i \in (0, 1)$ is equivalent to

$$\frac{\nu}{h_{i+1/2}} - c(1 - \theta_{i+1/2}) > 0,$$

hence to

$$\theta_{i+1/2} > 1 - \frac{\nu}{c h_{i+1/2}} = 1 - \frac{1}{Pe_{i+1/2}},$$

where $Pe_{i+1/2} = \frac{c h_{i+1/2}}{\nu}$ is the local Peclet number (ratio of the convection term to the diffusion term at the scale of the cell). The best value of $\theta_{i+1/2}$ for the sake of accuracy is the closest to 1/2 (the centered approximation) which provides a second order accurate approximation. Hence we set

$$\theta_{i+1/2} = \max(1/2, 1 - \frac{1}{Pe_{i+1/2}}).$$

We remark that the centered scheme can be applied if $Pe_{i+1/2} \leq 2$ for all i meaning that the diffusion becomes basically dominant at the scale of the cell i. At fixed c and $\nu > 0$, it always happens if the size of the cell is small enough ie $h_{i+1/2} \leq 2\frac{\nu}{c}$.

(3) Show that the matrix of the scheme is an M-matrix and deduce that it admits a unique solution.

Corrigé : Let A denotes the matrix of the finite volume scheme. It is easy to check that

$$A_{i,i} > 0 \text{ for all } i = 1, \cdots, N,$$

that

$$A_{i,j} \leq 0 \text{ for all } i, j = 1, \cdots, N, \ i \neq j,$$

that

$$\sum_{j=1}^{N} A_{i,j} \geq 0, \text{ for all } i = 1, \cdots, N,$$

that there exists $i_1 = 1$ (or N) such that

$$\sum_{j=1}^{N} A_{i_1,j} > 0,$$

and that all i is connected to i_1 by non zero entries of the matrix A. It results by definition that A is an M-matrix.

Let us prove that it implies that A is a non-singular matrix. Let us assume that A is singular. Hence there exists $u_h \in V_h$ with $u_h \neq 0$ such that for all $i = 1, \cdots, N$

$$A_{i,i} u_i = \sum_{j=1, j\neq i}^{N} -A_{i,j} u_j.$$

Let us fix i_0 such that $\max_{j=1\cdots,N} u_j = u_{i_0}$ and assume that $u_{i_0} > 0$. Then we have

$$\Big(\sum_{j=1}^{N} A_{i_0,j}\Big) u_{i_0} = \sum_{j=1, j\neq i_0}^{N} A_{i_0,j}(u_{i_0} - u_j)$$

Since $u_{i_0} > 0$, it imposes that $\sum_{j=1}^{N} A_{i_0,j} = 0$. Hence there exists $j \neq i_0$ with $A_{i_0,j} < 0$ such that $u_j = u_{i_0}$ and $u_j = u_{i_0}$ for all j such that $A_{i_0,j} < 0$. By induction we will reach the line $i_1 = 1$ such that $\left(\sum_{j=1}^{N} A_{i_1,j}\right) > 0$ which implies that $u_{i_1} = u_{i_0} = \max_{j=1\cdots,N} u_j = 0$. Doing the same with $\min_{j=1\cdots,N} u_j < 0$ we conclude that $\min_{j=1\cdots,N} u_j = 0$ and hence that $u_h = 0$ and that the matrix A is non-singular.

(4) Prove that the solution of the scheme satisfies the following maximum principle

$$\min(u_D^0, u_D^1) \leq u_i \leq \max(u_D^0, u_D^1),$$

for all $i = 1, \cdots, N$.

Corrigé : To fix ideas, let us assume that there exists i_0 such that $u_{i_0} = \max_{i=1,\cdots,N} u_i > \max(u_D^0, u_D^1) = M$. Writing the finite volume equation in cell i_0 as

$$\alpha_{i_0}(u_{i_0+1} - u_{i_0}) + (1 - \alpha_{i_0})(u_{i_0-1} - u_{i_0}) = 0,$$

we deduce from $\alpha_{i_0} \in (0, 1)$ that $u_{i_0+1} = u_{i_0-1} = u_{i_0}$. By induction it results that $u_D^0 > M$ and $u_D^1 > M$ which is a contradiction. The same argument works for $\min(u_D^0, u_D^1)$.