

TP MAM5 Dicitéisation Volume Fini des EDPs :
finite volume discretization of the 1D Laplacian with Dirichlet and Neumann boundary conditions.

We consider the following problem

$$(P) \begin{cases} -u''(x) = f(x) & \text{on } (0, L), \\ u(0) = u_D, \\ -u'(L) = g, \end{cases}$$

which has a unique solution in $H^1(0, 1)$ for all $u_D \in \mathbb{R}$, $g \in \mathbb{R}$, $f \in L^2(0, L)$.

We consider the following subdivision of the interval $(0, L)$ with $N + 1$ points :

$$x_{1/2} = 0 < x_{3/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N-1/2} < x_{N+1/2} = L.$$

Keeping the notations of the course, the finite volume discretization of the interval $(0, L)$ consists of the set of N cells $\kappa_i = (x_{i-1/2}, x_{i+1/2})$ for $i = 1, \dots, N$, and of the cell centers $x_i = \frac{x_{i-1/2} + x_{i+1/2}}{2}$ for $i = 1, \dots, N$. We also set $x_0 = 0$ and $x_{N+1} = L$, $h_{i+1/2} = |x_{i+1} - x_i|$ for $i = 0, \dots, N$, and $h_i = |x_{i+1/2} - x_{i-1/2}|$ for $i = 1, \dots, N$. Finally, we set $h = \max_{i=1, \dots, N} h_i$.

- (1) Let us consider the N discrete unknowns u_i approximating $u(x_i)$ for $i = 1, \dots, N$. Write the discrete fluxes $f_{i+1/2}$ approximating $-u'(x_{i+1/2})$, $i = 0, \dots, N$, and taking into account the boundary conditions for $i = 0$ and $i = N$.

Write the finite volume discretization of (P) consisting of N discrete conservation equations on the cells κ_i , $i = 1, \dots, N$ using the previous fluxes.

- (2) Write the square matrix A_h of size N and the right hand side $S_h \in \mathbb{R}^N$ such that the finite volume scheme is equivalent to $A_h U_h = S_h$ where $U_h \in \mathbb{R}^N$ is such that $U_{h,i} = u_i$, $i = 1, \dots, N$.
- (3) We consider a uniform mesh with $h = h_i = \frac{L}{N}$ for all $i = 1, \dots, N$. Let us consider the fonction

$$u(x) = e^{\sin(\pi x)},$$

which is the exact solution of the previous problem with $u_D = u(0) = 1$, $g = -u'(L)$, and

$$f(x) = -u''(x) = \pi^2 \left(\sin(\pi x) - \cos^2(\pi x) \right) e^{\sin(\pi x)},$$

Program the functions u , u' and f , then compute $u_D = u(0)$ and $g = -u'(L)$.

- (4) Implement the function VF computing the vector $X \in \mathbb{R}^N$ containing the N cell centers such that $X(i) = x_i$, $i = 1, \dots, N$, the square matrix A_h , the right hand side S_h , and the solution U_h of the discrete FV problem taking into account the uniform mesh with $h = h_i = \frac{L}{N}$ for all $i = 1, \dots, N$. The function VF will return the vector X and the solution U_h . For $L = 1$, plot the exact solution and the FV solution on the same figure for $L = 1$ for $N = 5, 10, 20, 40$.

- (5) Let us set $L = 1$ and let us define $e_i = u(x_i) - u_i$ for all $i = 0, \dots, N$, setting $u_0 = u_D$.

Plot in log log scale the discrete L^2 and H_0^1 errors

$$\|e_h\|_{L^2(0,L)} = \sqrt{\sum_{i=1}^N h(e_i)^2},$$

and

$$\|e_h\|_{1,h} = \sqrt{\frac{(0 - e_1)^2}{h_{1/2}} + \sum_{i=1}^{N-1} \frac{(e_i - e_{i+1})^2}{h_{i+1/2}}},$$

as a function of h for increasing values of $N = 10, 20, 40, 80, 160, 320, 640, 1280$.

What do you conclude about the convergence rates of the FV volume scheme in both norms?