Topology and geometry of random projective submanifolds

Mémoire présenté par

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devant un jury composé de :

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Organization of the manuscript

This manuscript presents an overview of my contributions to the study of random submanifolds in real and complex algebraic geometry. The text is organized as follows.

- Chapter 1 serves as an introduction to the rest of the manuscript. We present some well-known results in random algebraic geometry, along with some of my own contributions. The results in this chapter are not presented in their most general form, nor do they aim to provide a comprehensive list; rather, they are intended primarily to give a flavor of the types of results that will be explored in the following sections.

- In Chapter 2, we present results obtained with Damien Gayet in a joint project started a couple of years ago [AG23, AG24b, AG24a, AG24c]. The goal is to study the Riemannian geometry of random complex subvarieties of a projective manifold. For such Riemannian manifolds we study the statistics of Riemannian quantities such as systole, diameter, curvature or spectral gap. The first interesting example is that of smooth complex plane curves in $\mathbf{P}^2$. Topologically, these are surfaces whose genus depends solely on the degree of the polynomial defining them. Such surfaces are endowed with the restriction of the ambient Fubini–Study metric: this metric on the surface depends on the choice of the polynomial, and selecting it randomly produces a model of random Riemannian surfaces. This yields a random surface model that is algebraic in nature, analogous to the Weil–Petersson model introduced by M. Mirzakhani, which, in contrast, is of hyperbolic nature.

- In Chapter 3, we present some results on the topology of random real algebraic varieties [Anc24, Anc23, Anc22b]. The origin of the questions studied in this chapter can be traced to the following question: How many real roots does a random polynomial with real coefficients have? When the polynomial is a homogeneous polynomial in several variables, its zero locus defines a real hypersurface in the real projective space $\mathbf{RP}^n$. Hilbert's 16th problem asks for a classification of such hypersurfaces for each degree. While this problem remains unsolved, random real algebraic geometry focuses on studying the statistics of the topology of these hypersurfaces. The aim is to discard rare events and to understand the expected topology, for example by looking at the Betti numbers. We present a rarefaction result for hypersurfaces with large Betti numbers, and an approximation result for real hypersurfaces in terms of lower-degree hypersurfaces. We also provide an existence result for hypersurfaces with prescribed topology obtained by probabilistic methods.

- In Chapter 4, we present results obtained with Thomas Letendre at first [AL21a, AL21b, AL23] and later with Louis Gass, Thomas Letendre et Michele Stecconi [AGLS25]. The goal is to study the volume of random real algebraic subvarieties within a real algebraic variety. The main result is the calculation of the higher-order moments of that random variable. Part of the chapter is devoted to explaining how this problem is related to a problem of interpolation on the one hand and compactification of configuration spaces on the other. We also show how the proof method we use is very general, and can allow us to calculate the higher order moments of other quantities. For example, we explain how this allows us to calculate the higher order moments of the number of critical points of a Gaussian function on a Riemannian manifold.

- In Chapter 5, we present a result with Yohann Le Floch [ALF22], which is slightly different in nature from the others discussed in the preceding sections. This result combines Berezin–Toeplitz quantization with Shiffman–Zelditch theory of random sections. More precisely, we study the distribution of the zero locus of a random section of an ample holomorphic line bundle, viewed in the sense of currents, after a Toeplitz operator has been applied to the section. By analyzing the distribution of the zeros of such random sections, we are able to recover the distribution of the zero locus of the main symbol f of the Toeplitz operator. This is in the spirit of geometric quantization: recovering information about the classical observable (in this case, the function f) from the information of the quantum observable (the Toeplitz operator associated with f). The results of this section are not presented in the introductory Chapter 1.

- Finally, in Chapter 6, we collect several questions related to the results presented in this text that we would like to tackle in the future.

List of publications

Here is the list of works presented in this manuscript:

[AGLS25] Zeros and Critical Points of Gaussian Fields: Cumulants Asymptotics and Limit Theorems (with L. Gass, T. Letendre and M. Stecconi). *Preprint.*

[AG24c] On the curvature of random complex submanifolds (with D. Gayet). *Submitted.*

[AG24a] Lower bound for the Cheeger constants of random complex curves (with D. Gayet). *Submitted.*

[AG24b] How curved is a random complex curve? (with D. Gayet). *Submitted.*

[AG23] Metric and spectral aspects of random complex divisors (with D. Gayet). *Submitted.*

[AL23] Multijet bundles and application to the finiteness of moments for zeros of Gaussian fields (with T. Letendre). To appear in *Analysis & PDE.*

[ALF22] Berezin-Toeplitz operators, Kodaira maps, and random sections (with Y. Le Floch). To appear in *American Journal of Mathematics.*

[Anc22b] Existence of real algebraic hypersurfaces with many prescribed components. To appear in *Mathematical Research Letters.*

[Anc23] On the topology of random real complete intersections. *The Journal of Geometric Analysis*, 33, 32 (2023).

[Anc24] Exponential rarefaction of maximal real algebraic hypersurfaces. *Journal of the European Mathematical Society* (26) 2024 no.4.

[AL21a] Roots of Kostlan polynomials: moments, strong law of large numbers and central limit theorem (with T. Letendre). *Annales Henri Lesbegue*, 4 (2021), 1659-1703.

[AL21b] Zeros of smooth stationary Gaussian processes (with T. Letendre). *Electronic Journal of Probabability*, 26 (2021), 1-81.

The following works are not presented here:

[AL24] Symplectic instability of Bézout's Theorem (with A. Lerario). *Israel Journal of Mathematics* Volume 261, pages 841-850, (2024).

[Anc22a] Random real branched covering of the projective line. *Journal of the Institut of Mathematics of Jussieu*, Volume 21, Issue 5 (2022), pp. 1783-1799.

[Anc21] Random sections of line bundles over real Riemann surfaces. *International Mathematics Research Notices*, Volume 2021, Issue 9, May 2021, Pages 7004-7059.

[Anc20a] Critical points of random branched coverings of the Riemann sphere. *Mathematische Zeitschrift*, 296, 1735-1750 (2020).

[Anc20b] Expected number of critical points of real Lefschetz pencils. *Annales de l'Institut Fourier*, 70 (2020) no. 3 p. 1085-1113.

Contents

Chapter 1

Introduction

1.1 Complex algebraic geometry

1.1.1 Topology of complex projective hypersurfaces

In this section, we recall some standard facts about the topology of high degree submanifolds of a complex projective manifold. We start by considering the case of degree d homogeneous polynomials $\mathbf{C}_d^{\text{hom}}[X_0, \ldots, X_n]$. The zero locus Z_P of such a polynomial defines a complex hypersurface in $\mathbf{P}^n$. Generically, a polynomial P vanishes transversally (i.e. the gradient of P is non-zero at any point of Z_P), so that Z_P is a smooth complex hypersurface of $\mathbf{P}^n$. However, there are polynomials that do not vanish transversally. Such polynomials define the so-called discriminant $\Delta_d \subset \mathbf{C}_d^{\text{hom}}[X_0, \ldots, X_n]$, which is given by

$$\Delta_d := \{P \in \mathbf{C}_d^{\text{hom}}[X_0, \ldots, X_n], Z_P \text{ is singular}\}.$$

By a theorem of Bertini, the discriminant Δ_d is a complex hypersurface of $\mathbf{C}_d^{\text{hom}}[X_0, \ldots, X_n]$ and therefore it has real codimension 2. In particular, $\mathbf{C}_d^{\text{hom}}[X_0, \ldots, X_n] \setminus \Delta_d$ is connected. Thus, any two polynomials $P_1, P_2 \in \mathbf{C}_d^{\text{hom}}[X_0, \ldots, X_n] \setminus \Delta_d$ can be joined by a path avoiding the discriminant and then, by the Ehresmann Theorem, Z_{P_1} and Z_{P_2} are isotopic. In particular, the topology of Z_P for $P \in \mathbf{C}_d^{\text{hom}}[X_0, \ldots, X_n] \setminus \Delta_d$ does not depend on P. The topology of smooth projective hypersurfaces is well understood. By the Lefschetz hyperplane Theorem [AF59], the i-th Betti number $b_i(Z_P)$ of Z_P equals the i-th Betti number of $\mathbf{P}^n$ for any $i \neq n-1$. The $(n-1)$-th Betti number $b_{n-1}(Z_P)$ can be explicitly computed using the adjunction formula, and one finds that $b_{n-1}(Z_P)$ is a degree n polynomial in d. In particular one has the asymptotics $b_{n-1}(Z_P) = d^n + O(d^{n-1})$ as $d \to \infty$.

The classical example to keep in mind is that of degree d smooth complex curves in $\mathbf{P}^2$: they are all smooth, connected, orientable surfaces of genus $\frac{1}{2}(d-1)(d-2)$.

This phenomenon holds in the much more general setting that we now introduce. Let X be a complex projective manifold of dimension n. Let L and E be respectively an ample line bundle and a rank r holomorphic vector bundle on X. We denote by $H^0(X, L^d \otimes E)$ the space of holomorphic global sections of $L^{\otimes d} \otimes E$. For sufficiently large d, the discriminant Δ_d of $H^0(X, L^d \otimes E)$ (i.e., the subspace of sections that do not vanish transversally) is a complex subvariety of $H^0(X, L^d \otimes E)$, and thus the zero set Z_s of a generic section s is a smooth complex submanifold of codimension r, whose topology does not depend on the choice of s. For example, if E is the trivial rank 1 vector bundle, then by Lefschetz hyperplane theorem we have $b_i(Z_s) = b_i(X)$ for $i \neq n-1$ and $b_{n-1}(Z_s) = (\int_X c_1(L)^n)d^n + O(d^{n-1})$.

1.1.2 Geometry of complex projective hypersurfaces

Recall that the complex projective space $\mathbf{P}^n$ is equipped with a natural Kähler metric: the Fubini–Study metric g_{FS}. Thus, any degree d complex hypersurface Z_P is naturally equipped with a Kähler metric, the restriction $g_{\mathrm{FS}|Z_P}$ of the Fubini–Study metric. The Riemannian geometry of $(Z_P, g_{\mathrm{FS}|Z_P})$ is quite mysterious. The only exception concerns the volume of $(Z_P, g_{\mathrm{FS}|Z_P})$: it equals $\frac{d}{(n-1)!}$, for any $d \in \mathbf{N}$. Let us give two proofs of the equality $\mathrm{Vol}(Z_P) = \frac{d}{(n-1)!}$. The first uses Wirtinger's Inequality [Wir36], which turns out to be an equality for complex objects: the Riemannian volume $\mathrm{Vol}(Z_P)$ of Z_P equals the symplectic volume, that is,

$$\mathrm{Vol}(Z_P) = \int_{Z_P} \frac{\omega_{\mathrm{FS}|Z_P}^{n-1}}{(n-1)!}.$$

The latter is a pure cohomological quantity (it is the integral of a closed differential form on a closed submanifold of $\mathbf{P}^n$) and, in particular, it does not depend on Z_P itself but only on its homology class $[Z_P] \in H_{2n-2}(\mathbf{P}^n, \mathbf{Z})$. One can then compute such volume when Z_P is the union of d complex hyperplanes and find the result. The other proof is of probabilistic nature. It uses Crofton Formula, which states that $(n-1)! \cdot \mathrm{Vol}(Z_P)$ equals the average number of intersection points between Z_P and a random projective line $\mathbf{P}^1 \subset \mathbf{P}^n$. Now, by Bézout Theorem, a generic projective line intersects Z_P in exactly d points, hence the result.

Except for the volume, all the other natural Riemannian quantities (diameter, systole, spectrum of the Laplacian,...) depend on the choice on P. Note also that the previous result about the volume is more generally true for complex submanifolds inside a projective manifold X equipped with some Kähler form ω: their volume depends only on their cohomology classes.

1.2 Real projective geometry

1.2.1 Topology of real projective hypersurfaces

We now turn to the problem of describing the topology of the real locus of real algebraic submanifolds within a real projective manifold. This problem turns out to be completely different with respect to the complex analogue. Such difference can be already seen in the case of degree d homogeneous real polynomials $\mathbf{R}_d^{\mathrm{hom}}[X_0, \ldots, X_n]$. We denote by $\mathbf{R}Z_P$ the real zeros of a polynomial P, which define a real hypersurface in $\mathbf{RP}^n$. Generically, a polynomial P vanishes transversally on $\mathbf{RP}^n$, so that $\mathbf{R}Z_P$ is a smooth hypersurface of $\mathbf{RP}^n$. Like in the complex case, there are polynomials that do not vanish transversally. Such polynomials define the so-called real discriminant $\mathbf{R}\Delta_d \subset \mathbf{R}_d^{\mathrm{hom}}[X_0, \ldots, X_n]$, that is

$$\mathbf{R}\Delta_d := \{P \in \mathbf{C}_d^{\mathrm{hom}}[X_0, \ldots, X_n], \mathbf{R}Z_P \text{ is singular}\}.$$

By a real version of a theorem of Bertini, the real discriminant $\mathbf{R}\Delta_d$ is a semi-algebraic subset of $\mathbf{R}_d^{\mathrm{hom}}[X_0, \ldots, X_n]$ of codimension 1. Now, $\mathbf{R}_d^{\mathrm{hom}}[X_0, \ldots, X_n] \setminus \mathbf{R}\Delta_d$ is *not* connected. The connected components of $\mathbf{R}_d^{\mathrm{hom}}[X_0, \ldots, X_n] \setminus \mathbf{R}\Delta_d$ are called chambers. Inside a chamber, any two polynomials define diffeomorphic hypersurfaces, while different chambers generally represent different topologies. The example to keep in mind is the following. Let $P_t = X_0^2 + X_1^2 - tX_2^2$ where $t \in (-1, 1)$. For $t < 0$, $\mathbf{R}Z_{P_t}$ is empty, while, for $t > 0$, $\mathbf{R}Z_{P_t}$ is diffeomorphic to the circle $\mathbf{S}^1$.

This phenomenon is more generally true for real algebraic manifolds. Recall that a real algebraic manifold (X, c_X) consists of a smooth projective complex algebraic variety X together with an antiholomorphic involution $c_X : X \to X$, called the real structure. For example, the projective space $\mathbf{P}^n$ equipped with the standard conjugaison *conj* is a real algebraic manifold. More generally, the solutions to a generic system of homogeneous real polynomial equations in $n+1$ variables define a real algebraic manifold X inside $\mathbf{P}^n$, whose real structure is the restriction of *conj* to X. The real locus $\mathbf{R}X$ of a real algebraic manifold is the set of fixed points of the real structure, that is $\mathbf{R}X = \mathrm{Fix}(c_X)$. The real locus $\mathbf{R}X$ is either empty, or a finite union of n dimensional $\mathscr{C}^\infty$-manifolds, where n is the (complex) dimension of X. Let $\pi : (L, c_L) \to (X, c_X)$ be an ample real holomorphic line bundle over X, that is, an ample holomorphic line bundle L over X equipped with a real structure c_L such that $\pi \circ c_L = c_X \circ \pi$. The space of real sections $\mathbf{R}H^0(X, L)$ of L is by definition the set of holomorphic sections s of L such that $s \circ c_X = c_L \circ s$. For example, if $(X, L) = (\mathbf{P}^n, \mathcal{O}(d))$, then $\mathbf{R}H^0(X, L) = \mathbf{R}_d^{\text{hom}}[X_0, \dots, X_n]$.

As for the case of polynomials, the real discriminant $\mathbf{R}\Delta_d \subset \mathbf{R}H^0(X, L^d)$ separates $\mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$ into many connected components. Understanding the possible topologies of $\mathbf{R}Z_s$, $s \in \mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$, is one of the main problems in real algebraic geometry, originating directly from Hilbert's 16th problem [Wil78]. It is worth mentioning that, even the problem of describing all the possible topologies for plane curves in $\mathbf{RP}^2$ of degree $d \geq 8$ and for real surfaces in $\mathbf{RP}^3$ of degree $d \geq 5$ is completely open!

1.2.2 Geometry of real projective hypersurfaces

The Fubini–Study metric of $\mathbf{P}^n$ induces a Riemannian metric on $\mathbf{RP}^n$. More generally, given a real algebraic variety (X, c_X), any compatible Kähler metric ω (i.e. a Kähler metric such that $c_X^*\omega = -\omega$) induces a Riemannian metric on $\mathbf{R}X$. Thus, the real locus of a real algebraic submanifold of (X, c_X) becomes, by restriction of the ambient Riemannian metric, a Riemannian manifold. The real locus $\mathbf{R}Z_s$ of Z_s, with $s \in \mathbf{R}H^0(X, E \otimes L^d)$, is in general highly non-connected, hence many Riemannian quantities are either almost–meaningless (the diameter of a not–connected Riemannian manifold equals $+\infty$) or should be studied for any connected component, which seems at the same time unreachable and not very natural. A quantity which is insensitive to the number of connected components is the volume. Recall that, in the complex case, the volume only depends on the degree of the submanifold. For the real locus of a real algebraic submanifold this is no longer true: the volume of $\mathbf{R}Z_s$ depends on the choice of s. For example, for the family of conics given in the previous Section 1.2.1 one has, for $t < 0$, $\text{Vol}(\mathbf{R}Z_{P_t}) = 0$, while, for $t > 0$, one has that $\text{Vol}(\mathbf{R}Z_{P_t}) > 0$ and is increasing as t grows, at least for small values of t.

1.3 Probabilistic setting

The leitmotiv of this manuscript is to understand the topological and geometric quantities described in the previous sections when a real or complex projective submanifold is chosen randomly. To this end, we now describe the random model that will be used throughout most of the manuscript. The model we consider is the one introduced by B. Shiffman and S. Zelditch in their seminal paper [SZ99].

Let L be an ample line bundle over a complex projective manifold X of dimension n. Let h_L be a Hermitian metric on L. Recall that the curvature ω of h_L is defined as the following closed real $(1,1)$-form:

$$\omega = \frac{1}{2i\pi}\partial\bar{\partial}\log\|s\|^2_{h_L},$$

where s is any local non vanishing holomorphic section of L. We assume that the curvature of h_L is positive, that is, the $(1,1)$-form ω is a Kähler form.

Similarly, let (E, h_E) be a rank r Hermitian holomorphic vector bundle over X. The metrics on L and E induce a Hermitian metric $h^d := h_L^{\otimes d} \otimes h_E$ on $L^d \otimes E$ and also an $\mathscr{L}^2$-product on $H^0(X, L^d \otimes E)$. This is defined by

$$\langle s_1, s_2 \rangle_{\mathscr{L}^2} = \int_{x \in X} h_x^d(s_1(x), s_2(x))\frac{\omega^n}{n!} \tag{1.1}$$

for any pair of global sections $s_1, s_2 \in H^0(X, L^d \otimes E)$. Remark that a Hermitian product on a vector space induces a Lebesgue measure $\mathrm{d}s$ and a Gaussian measure $\mathrm{d}\mu_d$ defined by

$$\mu_d(A) = \frac{1}{(2\pi)^{N_d}} \int_{s\in A} e^{-\frac{\|s\|^2}{2}} \mathrm{d}s \tag{1.2}$$

for any Borel subset $A \subset H^0(X, L^d \otimes E)$. Here, N_d is the complex dimension of $H^0(X, L^d \otimes E)$. Equivalently, a random section s of $H^0(X, L^d)$ is of the form

$$s = \sum_{i=1}^{N_d} a_i s_i. \tag{1.3}$$

where $s_1, \dots, s_{N_d}$ is a unitary basis of $H^0(X, L^d \otimes E)$ and $a_i \sim \mathcal{N}_{\mathbf{C}}(0,1)$ are independent complex Gaussian random variables. A random section like the one above is an example of a Gaussian field. A natural object attached to a Gaussian field is its covariance kernel, which is the section of the bundle $(L^d \otimes E) \boxtimes (L^d \otimes E)^* \to X \times X$ defined by $\mathbf{E}(s(x) \otimes s(y)^*)$. A direct calculation shows that the covariance kernel of s is the Bergman kernel $B_d(x, y)$ associated with the Hermitian bundle $(L^d \otimes E, h^d)$ and ω. This is why the Bergman kernel will play an important role in this manuscript.

Example 1.3.1 (Projective space and polynomials)**.** For any point $x \in \mathbf{P}^n$, consider the following one dimensional vector space

$$\mathcal{O}(d)_x := \{\text{degree } d \text{ homogeneous forms over the line in } \mathbf{C}^{n+1} \text{ represented by } x\}.$$

These vector spaces define a holomorphic line bundle on $\mathbf{P}^n$, denoted by $\mathcal{O}(d)$. This line bundle is equipped with a Hermitian metric h_d, called the Fubini–Study metric, which defined by the following formula: for any $x \in \mathbf{P}^n$ and any $s \in \mathcal{O}(d)_x$, we have

$$h_d(s,s)^{1/2} = \frac{|s(v)|}{\|v\|^d},$$

where v is any (non-zero) vector in the line represented by x. The curvature of the metric h_1 of $\mathcal{O}(1)$ is the Fubini-Study form ω_{FS}.

The space of global sections $H^0(\mathbf{P}^n, \mathcal{O}(d))$ of $\mathcal{O}(d)$ can be identified with the vector space $\mathbf{C}_d^{\mathrm{hom}}[X_0, \dots, X_n]$ of degree d homogeneous polynomials in $n+1$ variables. With respect to the metric h_d and the volume form $\frac{\omega_{FS}^n}{n!}$, the $\mathscr{L}^2$-product on $H^0(\mathbf{P}^n, \mathcal{O}(d)) \simeq \mathbf{C}_d^{\mathrm{hom}}[X_0, \dots, X_n]$ is the Hermitian product that makes the family

$$\left\{ \sqrt{\frac{(d+n)!}{n!\alpha_0! \cdots \alpha_n!}} X_0^{\alpha_0} \cdots X_n^{\alpha_n} \right\}_{\alpha_0 + \dots + \alpha_n = d}$$

a unitary basis.

The previous definitions and constructions have a real algebraic analogue.

- Let (L, c_L) and (E, c_E) be respectively a real ample line bundle and a rank r real holomorphic vector bundle over a real algebraic manifold (X, c_X).
- We denote by $\mathbf{R}H^0(X, L^d \otimes E)$ the space of real holomorphic global sections of $L^d \otimes E$.
- Let h_L and h_E be compatible Hermitian metrics over L and E (meaning that $c_L^* h_L = \bar{h}_L$ and similarly for h_E). In this case, the $\mathscr{L}^2$-product defined by (1.1) restricts to a scalar product on $\mathbf{R}H^0(X, L^d \otimes E)$. A scalar product over a vector space defines a Lebesgue measure $\mathrm{d}s$ and a Gaussian measure $\mathrm{d}\mu_d$ defined by

$$\mu_d(A) = \frac{1}{\sqrt{2\pi}^{N_d}} \int_{s\in A} e^{-\frac{\|s\|^2}{2}} \mathrm{d}s \tag{1.4}$$

for any Borel subset $A \subset \mathbf{R}H^0(X, L^d \otimes E)$. Equivalently, a random real global section s of $\mathbf{R}H^0(X, L^d \otimes E)$ equals

$$s = \sum_{i=1}^{N_d} a_i s_i. \tag{1.5}$$

where $s_1, \dots, s_{N_d}$ is an orthonormal basis of $\mathbf{R}H^0(X, L^d \otimes E)$ and $a_i \sim \mathcal{N}(0,1)$ are independent Gaussian variables.

Example 1.3.2 (Kostlan polynomials)**.** When (X, c_X) is the n-dimensional projective space $(\mathbf{P}^n, conj)$ and L is the degree 1 real holomorphic line bundle $\mathcal{O}(1)$ equipped with the standard Fubini-Study metric, then the vector space $\mathbf{R}H^0(\mathbf{P}^n, \mathcal{O}(d))$ is isomorphic to the space $\mathbf{R}_d^{hom}[X_0, \dots, X_n]$ of degree d homogeneous real polynomials in $n+1$ variables and the $\mathscr{L}^2$-scalar product is the one which makes the family of monomials

$$\left\{ \sqrt{\binom{(n+d)!}{n!\alpha_0! \cdots \alpha_n!}} X_0^{\alpha_0} \cdots X_n^{\alpha_n} \right\}_{\alpha_0 + \cdots + \alpha_n = d}$$

an orthonormal basis.

A random polynomial with respect to the Gaussian probability measure induced by this scalar product is called a Kostlan polynomial [Kos93, SS93].

Remark 1.3.3. In all the text, we will be interested in the zero set Z_s (resp. $\mathbf{R}Z_s$) of a random holomorphic section $s \in H^0(X, L^d \otimes E)$ (resp. $s \in \mathbf{R}H^0(X, L^d \otimes E)$). For this reason, we could have chosen the projective space $\mathbf{P}H^0(X, L^d \otimes E) \cong \mathbf{P}^{N_d - 1}$ (resp. $\mathbf{P}\left(\mathbf{R}H^0(X, L^d \otimes E)\right) \cong \mathbf{RP}^{N_d-1}$) equipped with the probability measure induced by the Fubini–Study metric on $\mathbf{P}^{N_k - 1}$ (resp. $\mathbf{P}^{N_k-1}$) as a probability space. Indeed, any Hermitian product on $H^0(X, L^d \otimes E)$ (resp. scalar product on $\mathbf{R}H^0(X, L^d \otimes E)$) induces a Fubini–Study metric on the associated projective space. All the results and proofs would remain the same. We choose to work with the Gaussian measure for the sake of clarity, and because, when dealing with explicit computations, it is more convenient to work with vector spaces rather than projective spaces.

1.4 Some results in random complex projective geometry

1.4.1 Topology of random complex projective hypersurfaces

As explained in Section 1.1.1, the topology of a smooth projective complex hypersurface depends only on its degree. Indeed, for a smooth projective hypersurface Z_s for $s \in H^0(X, L^d)$ one has $b_i(Z_s) = b_i(X)$ for any $i \neq n-1$, by the Lefschetz hyperplane theorem, where n denotes the complex dimension of X. For the $(n-1)$-st Betti number one has the asymptotics

$$b_{n-1}(Z_s) = \left(\int_X c_1(L)^n \right) d^n + O(d^{n-1}),$$

as the degree d of the hypersurface tends to infinity. An analogue result also holds for complete intersections. Thus, from a random point of view, there is nothing to say about global topology of smooth projective complex hypersurfaces. Recently, D. Gayet has studied the local topology of random projective hypersurfaces [Gay24]. Indeed, given an open set $\Omega \subset X$, the topology of $Z_s \cap \Omega$ depends in general on s. For example, for any $k \in \{0, \dots, d\}$, it is easy to construct examples of degree d plane curves whose trace in a fixed ball in $\mathbf{P}^2$ has k connected components.

Theorem 1.4.1 (D. Gayet). *Let $\Omega \subset X$ be an open set. Then, for any $i \in \{0, \ldots, 2n-2\} \setminus \{n-1\}$, one has*

$$\lim_{d\to\infty} \frac{1}{d^n}\mathbf{E}[b_i(Z_s \cap \Omega)] = 0$$

while

$$\lim_{d\to\infty} \frac{1}{d^n}\mathbf{E}[b_{n-1}(Z_s \cap \Omega)] = \int_\Omega \omega^n.$$

1.4.2 Geometry of random complex projective hypersurfaces

The study of Riemannian invariants of projective manifolds has long been of significant interest. Among many notable results, we can mention the work of J.-P. Bourguignon, P. Li, and S.T. Yau [BLY94], in which they provide an upper bound for the first eigenvalue of the Laplacian on projective submanifolds, and the work of S.-T. Feng and G. Schumacher [FS99], in which they study the diameter of complex plane curves and establish an upper bound.

In a series of works with Damien Gayet [AG23, AG24b, AG24a, AG24c], we provide several bounds for various Riemannian quantities associated with random projective subvarieties. These results can be seen as part of the program initiated in [SZ99] on the understanding of random complex hypersurfaces of a complex projective manifold. As an example, I present here two results in the case of plane curves. These will be generalized in Section 2.2 to complex submanifolds of arbitrary codimension within a complex projective manifold. Recall that the systole of a Riemannian surface Z is the shortest length of a noncontractible loop in Z. We establish the following lower bound for the systole of a random plane curve:

Theorem 1.4.2 ([AG23]). *One has*

$$\mu_d\left[P \in \mathbf{C}_d^{\text{hom}}[X_0, X_1, X_2], \operatorname{syst}(Z_P, g_{\text{FS}|Z_P}) \geq d^{-4}\right] \underset{d\to\infty}{\to} 1$$

where $\operatorname{syst}(Z_P, g_{\text{FS}|Z_P})$ denotes the systole of $(Z_P, g_{\text{FS}|Z_P})$.

Note that there are no non-trivial deterministic lower bounds for the systole of a complex plane curve; that is, for any $d \geq 2$ one has

$$\inf_{P\in\mathbf{C}_d^{hom}[X_0,X_1,X_2]} \operatorname{syst}(Z_P, g_{\text{FS}|Z_P}) = 0.$$

Indeed, let Q be the product of d linear forms vanishing at the same point $x \in \mathbf{P}^2$. Then, any non-contractible curve in the small smoothing of Z_Q can be isotopied near x to a curve with small length. More generally, if one considers a family of polynomials $\{P_t\}_{t\in[0,1)}$ with smooth zero locus that converges as $t \to 1$ to a general point of the discriminant locus $\Delta_d \subset \mathbf{C}_d^{hom}[X_0, X_1, X_2]$ (that is, the locus of polynomials defining singular complex curves), then one can observe a non–contractible loop γ_t of $Z(P_t)$ that shrinks more and more as $t \to 1$, until it becomes a double singular point at the limit $t = 1$. We can then see that the length of the loop γ_t goes to zero as $t \to 1$.

Remark that a theorem by M. Gromov [Gro96, 2.C] about the systole of compact real surfaces implies the following deterministic upper bound for the systole of a plane curve: there exists $C > 0$ such that for any d and any $P \in \mathbf{C}_d^{hom}[X_0, X_1, X_2]$,

$$\operatorname{syst}(Z_P, g_{\text{FS}|Z_P}) \leq Cd^{-\frac{1}{2}} \log d.$$

For the first eigenvalue of the Laplacian, we have the following lower bound.

Theorem 1.4.3 ([AG24a]). *One has*

$$\mu_d\left[P \in \mathbf{C}_d^{\text{hom}}[X_0, X_1, X_2], \lambda_1(Z_P) \geq d^{-10}\right] \underset{d\to\infty}{\to} 1$$

where $\lambda_1(Z_P)$ denotes the first non-zero eigenvalue of the Laplacian of $(Z_P, \omega_{\text{FS}|Z_P})$.

Further results will be given in Chapter 2, in which we will also explore the curvature and injectivity radius of random projective submanifolds. Additionally, we will provide a deterministic bound on the diameter of complex submanifolds of complex projective manifolds. In the case of plane curves, this bound improves upon the one given by S.-T. Feng and G. Schumacher in [FS99].

Theorem 1.4.4 ([AG23]). *There exists* $C > 0$*, such that for any* $P \in \mathbf{C}_d^{\text{hom}}[X_0, X_1, X_2]$ *one has*

$$\operatorname{diam}(Z_P, g_{\mathrm{FS}|Z_P}) \leq Cd^3.$$

The previous known bound by Feng–Schumacher was Cd^4. We were not able to produce better probabilistic estimates on the diameter of complex plane curves. We believe that it may be true that

$$\mu_d\left[P \in \mathbf{C}_d^{\text{hom}}[X_0, X_1, X_2], \operatorname{diam}(Z_P, g_{\mathrm{FS}|Z_P}) \leq d^2\right] \underset{d\to\infty}{\to} 1.$$

Also, up to now, the only known example of a sequence $(C_d)_{d\in\mathbf{N}}$ of degree d plane curves such that $\operatorname{diam}(C_d) \to \infty$ as $d \to \infty$ is a construction by F.A. Bogomolov [Bog94]. He constructed a sequence of curves such that $\operatorname{diam}(C_d) \geq \log d$. It would be interesting to construct other examples, where the diameter grows (at least) linearly.

1.5 Some results in random real projective geometry

1.5.1 Topology of random real projective hypersurfaces

Let (X, c_X) be a n-dimensional real algebraic manifold with non-empty real locus $\mathbf{R}X$. As explained in Section 1.2.1, in contrast to the complex setting, $\mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$ is *not* connected and this produces the following phenomenon: the topology of the real locus $\mathbf{R}Z_s$ of a real section $s \in \mathbf{R}H^0(X, L^d)$ depends on the choice of the section. Indeed, the real discriminant creates walls inside $\mathbf{R}H^0(X, L^d)$, and the Betti numbers of the real locus of a projective hypersurface change when we cross a wall.

This raises a natural question: what is the topology of $\mathbf{R}Z_s$, if we pick s at random? This question is moreover motivated by the fact that the number of connected components of $\mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$ can grow very fast as $d \to \infty$ and then a deterministic study of all the topologies seems out of reach. For example, when X is a toric variety (for example $X = \mathbf{P}^n$) the number of connected component of $\mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$ grows super-exponentially in d, see [OK00, Proposition 10].

Recall that a fundamental restriction on the topology of the real locus of a real algebraic variety is given by the Smith–Thom inequality [Tho65] which asserts that the total Betti number of the real locus $\mathbf{R}X$ of a real algebraic variety is bounded from above by the total Betti number of the complex locus:

$$\sum_{i=0}^{n} \dim H_i(\mathbf{R}X, \mathbf{Z}/2) \leq \sum_{i=0}^{2n} \dim H_i(X, \mathbf{Z}/2). \tag{1.6}$$

We will more compactly write $b_*(\mathbf{R}X) \leq b_*(X)$, where b_* denotes the total Betti number (i.e. the sum of the $\mathbf{Z}/2$-Betti numbers). A real algebraic manifold (X, c_X) is called *maximal* if $b_*(\mathbf{R}X) = b_*(X)$.

Among all the possible topologies of $\mathbf{R}Z_s$, $s \in \mathbf{R}H^0(X, L^d)$, maximal hypersurfaces are those that have always attracted attention in the study of the topology of real algebraic varieties. What is the probability to find a divisor defining a real algebraic maximal hypersurface? The next result shows that real algebraic almost maximal hypersurfaces are exponentially rare inside their linear series.

Theorem 1.5.1 ([Anc24]). *For any $a > 0$ there exists $c > 0$ such that*

$$\mu_d\big\{s \in \mathbf{R}H^0(X, L^d), b_*(\mathbf{R}Z_s) \geq b_*(Z_s) - ad^{n-1}\big\} = O(e^{-c\sqrt{d}\log d})$$

as $d \to \infty$. Here, the measure μ_d is the Gaussian measure defined in Equation (1.4).

Theorem 1.5.1 provides a moral explanation for why it is often difficult to construct examples of maximal hypersurfaces, and, in general, with large Betti numbers. I actually proved such result for random complete intersections of arbitrarily codimension, see [Anc23]. In Chapter 3 we will also see another rarefaction result for a larger class of hypersurfaces. Indeed we will see that there exists $\epsilon > 0$ such that

$$\mu_d\big\{s \in \mathbf{R}H^0(X, L^d), b_*(\mathbf{R}Z_s) \geq (1-\epsilon)b_*(Z_s)\big\} = O(d^{-\infty})$$

as $d \to \infty$, see Theorem 3.1.1. Here, the notation $O(d^{-\infty})$ stands for $O(d^{-k})$ for any $k \in \mathbf{N}$. Theorem 1.5.1 extends to any real algebraic varieties previous results of Gayet–Welschinger [GW11] (who proved the result in dimension 2) and Diatta-Lerario [DL22] (who proved the result for Kostlan polynomials, see Example 1.3.2). Theorem 1.5.1 is a consequence of a more precise approximation result (Theorem 3.1.2) that will be discussed and proved in Sections 3.1 and 3.1.1.

The previous results are concerned with rare events and extremal topologies. Regarding the expected topology of a random real hypersurface, we cite two fundamental results of D. Gayet and J.–Y. Welschinger [GW14b, GW16] and F. Nazarov and M. Sodin [NS16]. As before, we consider a random real holomorphic section $s \in \mathbf{R}H^0(X, L^d)$ of a real ample line bundle (L, c_L) over a n-dimensional real algebraic manifold (X, c_X).

Theorem 1.5.2 (Gayet–Welschinger). *There exist universal constants $0 < c_-(n,i) \leq c_+(n,i)$ such that*

$$c_-(n,i) \leq \liminf_{d\to\infty} \frac{1}{\sqrt{d}^n \mathrm{Vol}(\mathbf{R}X)} \mathbf{E}(b_i(\mathbf{R}Z_s)) \leq \limsup_{d\to\infty} \frac{1}{\sqrt{d}^n \mathrm{Vol}(\mathbf{R}X)} \mathbf{E}(b_i(\mathbf{R}Z_s)) \leq c_+(n,i).$$

Theorem 1.5.3 (Nazarov–Sodin). *There exists an universal constant $c(n) > 0$ such that*

$$\lim_{d\to\infty} \frac{1}{\sqrt{d}^n \mathrm{Vol}(\mathbf{R}X)} \mathbf{E}(b_0(\mathbf{R}Z_s)) = c(n).$$

We invite the reader to consult N. Anantharaman's Bourbaki seminar [Ana16] for a deeper insight into these results. Theorems 1.5.2 and 1.5.3 raise a natural question:

Question 1.5.4. Does the limit

$$\lim_{d\to\infty} \frac{1}{\sqrt{d}^n} \mathbf{E}(b_i(\mathbf{R}Z_s))$$

exist ?

The first open case to study is the convergence of $\frac{1}{d^2}\mathbf{E}(b_1(\mathbf{R}Z_s))$ for random real algebraic threefolds inside a real algebraic fourfold (X, c_X). Indeed, for a real algebraic surface (X, c_X) the convergence of the average of Betti numbers of real random curves is a consequence of Theorem 1.5.3 , while for a real algebraic threefold (X, c_X) the convergence of the average of Betti numbers of real random surface are consequence of Theorem 1.5.3 and of the convergence of the (renormalized) average of the Euler characteristic $\frac{1}{\sqrt{d}^3}\mathbf{E}(\chi(\mathbf{R}Z_s))$. The convergence of the Euler characteristic in any dimension is due to S. S. Podkoryov for random hypersurfaces in the projective space [Pod98], P. Bürgisser [Bür07] for random complete intersections in the projective space and to T. Letendre in the general case [Let16].

We finish this section by stating a deterministic result that I obtained using probabilistic methods. For any smooth manifolds S and Σ, we denote by $\mathcal{N}_\Sigma(S)$ the number of connected components of S that are diffeomorphic to Σ.

Theorem 1.5.5 ([Anc22b]). *Let X be a real projective manifold of dimension n equipped with an ample real holomorphic line bundle. For any r-codimensional closed submanifold Σ of $\mathbf{R}^n$ with trivial normal bundle, there exists $c > 0$ and $d_0 \in \mathbf{N}$ such that for any $d \geq d_0$ there exists $s_1, \ldots, s_r \in \mathbf{R}H^0(X, L^d)$ with*

$$\mathcal{N}_\Sigma (\mathbf{R}Z_{s_1} \cap \cdots \cap \mathbf{R}Z_{s_r}) \geq cd^n.$$

The assumption of a trivial normal bundle comes from the fact that each of these connected components will be realized inside a small ball B in $\mathbf{R}X$. By trivializing $\mathbf{R}L^d$ over a ball B and considering the sections $(s_1, \ldots, s_r)$ restricted to B and with respect to this trivialization, the connected component diffeomorphic to Σ will thus be the zero locus of a function $F : B \to \mathbf{R}^r$. Now, the zero locus of a function always has a trivial normal bundle.

It is worth noticing that the order d^n appearing in Theorem 1.5.5 is optimal. Indeed, by Smith–Thom inequality the total Betti number $b_*(\mathbf{R}Z_{s_1} \cap \cdots \cap \mathbf{R}Z_{s_r})$ of $\mathbf{R}Z_{s_1} \cap \cdots \cap \mathbf{R}Z_{s_r}$ is smaller than the total Betti number of $Z_{s_1} \cap \cdots \cap Z_{s_r}$. The latter grows as vd^n, where v is a constant depending only on the Chern classes of L and X.

In Section 3.2, we will present an application of Theorem 1.5.5 in symplectic geometry, along with a proof outline that uses Hörmander's spectral function associated with the eigenfunctions of the Laplacian on the unit sphere.

1.5.2 Geometry of random real projective hypersurfaces

As already mentioned in Section 1.2.2, unlike the complex case, the volume of the real locus of a real projective submanifold depends on the choice of the polynomial/section of the bundle, and not just on its degree. For example, for the family of conics $P_t(X_0, X_1, X_2) = X_0^2 + X_1^2 + tX_2^2$, we have that the volume of $\mathbf{R}Z_{P_t}$ goes to zero as $t \to 0$. What is the expected volume of a random real projective submanifold? E. Kostlan [Kos93] proved the expected volume of a random degree d real projective hypersurface in $\mathbf{P}^n$ is $\sqrt{d}$ and more generally that the expected volume of a random complete intersection defined by r independent random polynomials $P_1, \ldots, P_r \in \mathbf{R}_d^{\text{hom}}[X_0, \ldots, X_n]$ is $\sqrt{d}^r \text{Vol}(\mathbf{RP}^{n-r})$. This was later generalized by P. Bürgisser [Bür07] for random complete intersections in $\mathbf{P}^n$ of multi-degree $d_1, \ldots, d_r$. In the general setting of random real submanifolds in (X, c_X) defined by $s \in \mathbf{R}H^0(X, L^d \otimes E)$, where (E, c_E) and (L, c_L) are respectively a rank r real holomorphic vector bundle and a real ample line bundle over (X, c_X), T. Letendre proved that the expected volume of $\mathbf{R}Z_s$ is equivalent to $\sqrt{d}^r \text{Vol}(\mathbf{R}X) \frac{\text{Vol}(\mathbf{S}^{n-r})}{\text{Vol}(\mathbf{S}^n)}$ as $d \to \infty$. Later, D. Armentano, J.-M. Azaïs, F. Dalmao and J. R. León [AADL18] (for polynomial systems of maximal rank) and T. Letendre and M. Puchol [Let19, LP19] (for the general case) computed the variance of the volume of random real submanifolds. In this context, the main results we proved with L. Gass, T. Letendre, and M. Stecconi [AGLS25] imply the following result about the computation of the moments $m_p(\text{Vol}(\mathbf{R}Z_s)) := \mathbf{E}(|\text{Vol}(\mathbf{R}Z_s) - \mathbf{E}(\text{Vol}(\mathbf{R}Z_s))|^p)$.

Theorem 1.5.6 ([AGLS25]). *Let $s \in \mathbf{R}H^0(X, L^d \otimes E)$ be a random real holomorphic section, where (E, c_E) and (L, c_L) are respectively a rank r real holomorphic vector bundle and a real ample line bundle over a n-dimensional real projective manifold (X, c_X). Then, for any $p \geq 2$, the following holds as $d \to +\infty$:*

$$m_p(\text{Vol}(\mathbf{R}Z_s)) = w_p d^{\frac{p}{2}(r-\frac{n}{2})} \sigma_{n,r}^p \text{Vol}(\mathbf{R}X)^{\frac{p}{2}} + o(d^{\frac{p}{2}(r-\frac{n}{2})})$$

where $\sigma_{n,r} > 0$ is a universal constant depending only on n and r, and w_p is the p-th moment of the standard Gaussian variable $\mathcal{N}(0,1)$.

Corollary 1.5.7 ([AGLS25]). *Under the hypotheses of Theorem 1.5.6, for any sequence $(\epsilon_d)_d$ of positive real numbers and for any $p \in \mathbf{N}$, we have the following concentration around the* *expected value*

$$\mu_d\left(d^{-\frac{r}{2}}|\mathrm{Vol}(\mathbf{R}Z_s)-\mathbf{E}(\mathrm{Vol}(\mathbf{R}Z_s))|>\epsilon_d\right)=O\left(\left(d^{\frac{n}{4}}\epsilon_d\right)^{-2p}\right).$$

By the method of moments (see [Bil95, Section 30]), Theorem 1.5.6 also provides a Central Limit Theorem for the volume of the real zero locus of random real holomorphic sections. This generalizes the CLT for Kostlan polynomials, proved by D. Armentano, J.-M. Azaïs, F. Dalmao, and J. R. León [AADL22], to any real algebraic manifold, using different methods.

Remark that the special case of Theorem 1.5.6 given by $(X, c_X) = (\mathbf{P}^1, conj)$ has a historical interest. Indeed, in this case we are counting real roots of random (Kostlan) polynomials and a natural question is to compute the higher moments. This was done by T. Letendre and myself [AL21a]. Let us denote by N_P the number of real roots of a degree d real polynomial P. E. Kostlan [Kos93] proved that the average number of real roots of a degree d random polynomial equals $\sqrt{d}$, while F. Dalmao [Dal15] proved that $\mathrm{Var}(N_P) = \sigma^2\sqrt{d} + o(\sqrt{d})$, as $d \to \infty$, where $\sigma > 0$. Together with T. Letendre, we proved the following result.

Theorem 1.5.8 ([AL21a])**.** *There exists $\sigma > 0$ such that for any $p \geq 2$*

$$m_p(N_P) = d^{\frac{p}{4}}\sigma^p w_p + o(d^{\frac{p}{4}}),$$

where w_p is the p-th moment of the standard Gaussian variable $\mathcal{N}(0,1)$.

The previous asymptotics of the higher moments of N_P allowed us in particular to recover the Central Limit Theorem for real zero of Kostlan polynomials proved by F. Dalmao [Dal15], that is, we obtained

$$\left(d^{\frac{1}{4}}\sigma\right)^{-1}\left(N_P-\sqrt{d}\right)\xrightarrow[d\to+\infty]{}\mathcal{N}(0,1)$$

in distribution. Actually, we proved Theorem 1.5.8 and CLT for the number of real zeros of random sections of line bundles over real projective curves.

Theorem 1.5.6 will be discussed and generalized in Section 4, where we will also make the connection between the problem of moments and an interpolation problem. Such interpolation problem will be itself solved by constructing a suitable compactification of the configuration space of p points on a given manifold, together with a vector bundle that we called the multijets bundle.

We will also show how these techniques can be used to study Gaussian functions over smooth manifolds, solving the question about the finiteness (and the computation) of the moments of the number of critical points of such random functions.

Chapter 2

Geometry of random complex submanifolds

In this chapter, we describe joint works with Damien Gayet [AG23, AG24b, AG24a, AG24c] in which we study the Riemannian geometry of random complex projective submanifolds.

2.1 Case of complex curves

Smooth complex projective hypersurfaces in $\mathbf{P}^n$ have the remarkable property that their diffeomorphism type depends only on their degree. For example, smooth complex curves of degree d in $\mathbf{P}^2$ are compact Riemann surfaces of genus $\frac{1}{2}(d-1)(d-2)$. When equipped with the restriction of the Fubini–Study metric, these complex hypersurfaces become Riemannian manifolds, whose geometry strongly depends on the chosen hypersurfaces. In this chapter, we study some of their metric properties when the hypersurface is taken at random.

A source of inspiration and motivation for this question in the case $n = 2$ is provided by M. Mirzhakani's model of random hyperbolic surfaces [Mir10]. Let us briefly recall this model. Let $\mathcal{M}_g$ be the moduli space of compact hyperbolic surfaces of genus g. This space is an orbifold of real dimension $6g-6$ and can be equipped with a natural symplectic form, the Weil–Petersson form ω_{WP}. It turns out that the volume of $\mathcal{M}_g$ with respect to this symplectic form is finite, and therefore, the renormalized volume form associated with ω_{WP} provides a natural probability measure $\mu_{\mathrm{WP},g}$ on $\mathcal{M}_g$. Since the foundational work of M. Mirzhakani [Mir10], there has been a great deal of effort in the study of the geometry of a random hyperbolic surface $X \in \mathcal{M}_g$, as $g \to \infty$. The most studied observables in this model were the diameter, the systole, the injectivity radius, and the spectrum of the Laplace operator, especially the first eigenvalue. As an example, we can present two results concerning the geometry of random hyperbolic surfaces. M. Mirzakhani proved in this context [Mir10, Theorem 4.2]: there exists $\epsilon_0 > 0$ and $0 < c < C$ such that for any $\epsilon \leq \epsilon_0$ and every $g \geq 2$,

$$c\epsilon^2 \leq \mu_{WP,g}\left[\mathrm{syst}(X,h) \leq \epsilon\right] \leq C\epsilon^2,$$

while N. Anantharaman and L. Monk [AM24] proved that, for any $\epsilon > 0$,

$$\mu_{WP,g}\left[\lambda_1(X,h) \geq \frac{1}{4} - \epsilon\right] \underset{g\to\infty}{\to} 1.$$

The goal we set with D. Gayet was to study such observables for a more algebraic class of metrics on a surface: the Fubini–Study metrics, which are defined as the restriction of the ambient metric of $\mathbf{P}^2$ to complex plane curves. As will be shown below, although the inspiration for the questions to be studied comes from the random hyperbolic surface model, the techniques of study are entirely different. This is primarily due to the fact that the Riemannian surfaces we obtain are *never* of constant curvature.

Let us consider an example that illustrates very well the types of pathologies one might encounter in the study of the geometry of curves in the complex plane. Consider the union of d generic projective lines in $\mathbf{P}^2$. This is a reducible curve with $\frac{d(d-1)}{2}$ singular points, corresponding to the intersection points of any two distinct lines among the d lines. The other points on the curve are smooth and have constant positive curvature equal to 2. Consider now an arbitrarily small generic perturbation of this complex curve. The resulting complex curve is smooth and has a large region of points with positive curvature. Moreover, by the Gauss–Bonnet formula, we have

$$\int_{Z_P} K(x)\mathrm{dvol}(x) = 2 - (d-1)(d-2), \tag{2.1}$$

where K denotes the Gaussian curvature, which implies that the curvature at points close to the previous singularities is very negative. Such an example shows that, for any $d \geq 2$

$$\sup_{P \in \mathbf{C}_d^{\mathrm{hom}}[X_0,X_1,X_2]} \frac{\mathrm{Vol}(K>0)}{\mathrm{Vol}(Z_P)} = 1,$$

and

$$\inf_{P \in \mathbf{C}_d^{\mathrm{hom}}[X_0,X_1,X_2]} \inf_{x \in Z_P} K(x) = -\infty.$$

Actually, any degree d curve which is close to the discriminant Δ_d (that is, close to a singular curve) has points of very negative curvature. Additionally, any generic degree d plane curve always has $3d(d-2)$ points of curvature equal to 2, that are the inflection points. This fact follows from the observation that the inflection points are exactly the points where both the polynomial P and its Hessian (the determinant of the Hessian matrix) vanish. Now, if $\deg P = d$, then $\deg \mathrm{Hess}(P) = 3(d-2)$, so that the result follows from Bézout's theorem. Remark also that 2 is actually the maximal value for the curvature of a complex curve [Nes77].

These extreme phenomena are the source of the difficulties in studying the Riemannian geometry of complex plane curves and also the cause of the poor (or expected poor) estimates for the systole and first eigenvalue of the Laplacian of random complex curves with respect to random hyperbolic surfaces (compare Theorems 1.4.2 and 1.4.3 stated in Section 1.4.2 with the corresponding results in the Weil–Petersson case stated above). The first step is to study the statistics of the curvature of complex plane curves and attempt to isolate the most extreme phenomena from a probabilistic perspective.

The first result in this direction is a probabilistic bound on the sup-norm of the curvature of a random complex plane curve.

Theorem 2.1.1 ([AG23])**.**

$$\mu_d\left[\sup_{x \in Z_P} |K(x)| \leq d^5\right] \underset{d\to\infty}{\to} 1$$

where $K := K_{Z_P}$ denotes the Gaussian curvature of Z_P.

Theorems 1.4.2, 1.4.3 and 2.1.1 will be generalized in higher dimensions in Section 2.2 and they will be proved in Sections 2.3 and 2.4.

In order to state our next theorem, we need the following notation. Let Z be a $\mathscr{C}^\infty$ surface equipped with a metric g of finite area. For any Borel subset $A \subset \mathbf{R}$, let us define

$$\overline{\mathrm{vol}}(K_Z \in A) = \frac{\mathrm{Vol}_g\{x \in Z, K_Z(x) \in A\}}{\mathrm{Vol}_g Z}. \tag{2.2}$$

Our next result asserts that, statistically, a uniform portion of the area of a random curve of large degree has controlled negatively pinched curvature.

Theorem 2.1.2 ([AG24b]). *There exists $c > 0$ such that*

$$\forall d \gg 1,\ \mu_d\left[P \in \mathbf{C}_d^{hom}[X_0, X_1, X_2],\ \overline{\mathrm{vol}}\left(K_{Z_P} \in \left[-4d, -\frac{d}{8}\right]\right) > c\right] \geq c.$$

Idea of proof of Theorem 2.1.2. It is known that locally around a point $x_0 \in \mathbf{P}^2$, after a rescaling by $\sqrt{d}$, a random plane curve of very high degree d "looks like" a random complex curve in the unit ball of $\mathbf{C}^2$ defined as the zero locus of the Bargmann–Fock field f (which is a particular example of random analytic function). This is the probabilistic analogue of the well-known fact that on a small ball of radius $\frac{1}{\sqrt{d}}$ around a point $x_0 \in \mathbf{P}^2$, after adequate rescaling, the Bergman kernel associated with $\mathcal{O}(d)$ converges to a universal limit [MM07, DLM06].

Remark that, while doing the local $\sqrt{d}$-rescaling the curvature is multiplied by $1/d$ while the area is multiplied by d. We prove that with positive probability the zero locus of the Bargmann–Fock field f is $\mathscr{C}^\infty$-close to the zero locus of $f_0(z, w) = zw - \frac{1}{4}$. We then prove that the area of Z_{f_0} where the curvature is between $[-4, -\frac{1}{8}]$ is strictly positive, say equal to $c > 0$. This means that, given $x_0 \in \mathbf{P}^2$ the probability that

$$\mathrm{Vol}\left\{x \in Z_P \cap B\left(x_0, \frac{1}{\sqrt{d}}\right), K_{Z_P}(x) \in [-4d, -\frac{d}{8}]\right\} \geq c/d$$

is bounded from below by a constant independent of d. Finally, we show that we can choose ϵd^2 points $\{x_i\}_{i=1,\dots,\epsilon d^2}$ in $\mathbf{P}^2$ such that the balls of radius $\frac{1}{\sqrt{d}}$ around these points are pairwise disjoint and such that the probability of the event

$$\left\{\forall i \in \{1, \dots \epsilon d^2\}, \mathrm{Vol}\{x \in Z_P \cap B\left(x_i, \frac{1}{\sqrt{d}}\right), K_{Z_P}(x) \in [-4d, -\frac{d}{8}]\} \geq c/d\right\}$$

is bounded from below by a constant independent of d. This estimates, together with the fact that if $P \in \mathbf{C}_d^{\mathrm{hom}}[X_0, X_1, X_2]$ then $\mathrm{Vol}(Z_P) = d$, ends the proof. □

The last theorem presented in this section computes exactly the average of the proportion of the area of the random complex curve with prescribed curvature.

Theorem 2.1.3 ([AG24b]). *Let $0 < r < R$. There exists a $\varphi_{r,R} \in]0, 1[$ such that*

$$\forall d \geq 2,\ \mathbf{E}_{\mu_d}\left[\overline{\mathrm{vol}}\Big(K_{Z_P} \in [2 - Rd, 2 - rd]\Big)\right] = \varphi_{r,R},$$

where $\overline{\mathrm{vol}}$ is defined above by (2.6).

The proof of Theorem 2.1.3 is similar to the proof of Theorem 2.5.1 about the holomorphic bisectional curvature of random projective submanifolds given in Section 2.5.1. One consequence of Theorem 2.1.3 is the following.

Corollary 2.1.4 ([AG24b]). *Let $\ell \in]-\infty, 2\pi[$. Then,*

$$\forall \eta \in]0, 1[, \quad \mu_d\left[\overline{\mathrm{vol}}\Big(K_{Z_P} \in [\ell, 2\pi]\Big) > \eta\right] \underset{d\to\infty}{\to} 0.$$

The last assertion implies that the event that a uniform proportion of the area of the curve has a non negative curvature becomes rarer and rarer for large degrees. We stress again that for any complex plane curve, there always exist points with positive curvature. Corollary 2.1.4 will be improved and generalized in higher dimensions in Section 2.5.

2.2 General setting

One advantage of the model we look at is that it allows the problem to be posed in any dimension and codimension without any major effort. Indeed, the questions we look at will be posed for complex subvarieties of any codimension within any complex projective manifold. For example, taking k random polynomials of degree $d_1, \ldots, d_k$, we obtain a codimension k random complete intersection in $\mathbf{P}^n$ whose topology only depends on $d_1, \ldots, d_k$ but whose geometry highly depends on the chosen polynomials. In general, we will take a random section s of $L^{\otimes d} \otimes E$, where L and E are respectively a rank 1 and a rank k holomorphic vector bundle over a projective manifold X and we will study the geometry of the zero set of s as $d \to \infty$ and when L is ample. In the following, we generalize the results that we presented in Sections 1.4.2 and 2.1 to this more general setting. We start by recalling some notation we have already encountered in the introduction.

Let X be a complex projective manifold of dimension $n \geq 1$ equipped with a Hermitian ample holomorphic line bundle $(L, h_L) \to X$ with positive curvature ω. Let $g_\omega = \omega(\cdot, i\cdot)$ be the associated Kähler metric. Let $(E, h_E) \to X$ be a holomorphic vector bundle of rank r equipped with a Hermitian metric h_E. The space $H^0(X, E \otimes L^d)$ of holomorphic sections of $E \otimes L^{\otimes d}$ is non trivial for d large enough, and can be equipped with the $\mathscr{L}^2$ Hermitian product

$$(s,t) \in (H^0(X, E \otimes L^d))^2 \mapsto \langle s, t \rangle = \int_X \langle s(x), t(x) \rangle_{h_d} \frac{\omega^n}{n!}, \tag{2.3}$$

where $h_d := h_E \otimes h_L^d$. This product induces a Gaussian measure μ_d over $H^0(X, E \otimes L^d)$, that is for any Borelian $A \subset H^0(X, E \otimes L^d)$,

$$\mu_d(A) = \int_{s \in A} e^{-\frac{1}{2}\|s\|^2} \frac{\mathrm{d}s}{(2\pi)^{N_d}}, \tag{2.4}$$

where N_d denotes the complex dimension of $H^0(X, E \otimes L^d)$ and $\mathrm{d}s$ denotes the Lebesgue measure associated to the Hermitian product (2.3).

Let $\Delta_d \subset H^0(X, E \otimes L^d)$ be the discriminant subset, that is the set of sections s such that there exists x in Z_s where $\nabla s(x)$ is not onto. Recall that Δ_d is a complex subvariety of positive codimension, and that for any $s \in H^0(X, E \otimes L^d) \setminus \Delta_d$, the zero set $Z_s \subset X$ is a compact smooth complex submanifold of X of codimension r. Moreover, still outside Δ_d, the diffeomorphism class of Z_s depends only on d. For any s we equip Z_s with the restriction $g_{\omega|Z_s}$ of the Kähler metric g_ω. Then, $(Z_s, g_{\omega|Z_s})$ can be seen as a fixed smooth manifold with a random metric.

Recall that for an n-dimensional Riemannian manifold (M, g) the Berger k-systole of (X, g) is defined as

$$\mathrm{syst}(X,g) = \begin{cases} \inf\{\mathrm{length}(c),\ c \text{ non contractible} \\ \text{smooth closed curve in } X\} & \text{if } \pi_1(X) \neq \{1\} \\ +\infty & \text{if } \pi_1(X) = \{0\} \end{cases}$$

and for any $k \in \{1, \cdots, n\}$,

$$\mathrm{syst}_k(X,g) = \begin{cases} \frac{1}{2}\inf\{\mathrm{diam}(H) | H_k(X, \mathbf{R}) \ni [H] \neq 0\} & \text{if } H_k(X,\mathbf{R}) \neq \{0\} \\ +\infty & \text{if } H_k(X,\mathbf{R}) = \{0\} \end{cases}.$$

Note that $\mathrm{syst} \leq \mathrm{syst}_1$.

If $X = \mathbf{P}^n$ and $E = X \times \mathbf{C}$, then by the Lefschetz hyperplane theorem, for any odd $k \neq n-1$, any degree $d \geq 1$ and any generic section $s \in H^0(X, L^d)$, one has $H_k(Z_s, \mathbf{R}) = 0$, so that $\mathrm{syst}_k(Z_s) = +\infty$. This is also true for complete intersections. The same theorem implies also that for $n \geq 3$, Z_s is simply connected, so that $\mathrm{syst} = +\infty$ in this case.

A first result of [AG23] provides probabilistic lower bounds for these systoles:

Theorem 2.2.1 ([AG23]). *Let X be a compact smooth complex manifold of dimension $n \geq 2$ equipped with an ample holomorphic line bundle $(L,h) \to X$ endowed with a Hermitian metric h with positive curvature and Kähler metric g_ω, and with (E,h_E) a rank r holomorphic vector bundle. Then, for any sequence $(a_d)_d$ converging to 0, there exists a constant $C>0$ and a positive integer d_0, such that for any $k \in \{1,\cdots,2n-2r\}$,*

$$\forall d \geq d_0,\ \mu_d \left[\operatorname{syst}_k(Z_s, g_{\omega|Z_s}) \geq \frac{a_d}{C\sqrt{\log d}} d^{-\frac{3n+1}{2}}\right] \geq 1 - C(a_d + \frac{1}{d}).$$

The same holds for the systole. Here, μ_d denotes the Gaussian measure defined by (2.4).

Theorem 2.2.1 in the special case of complex plane curves imples Theorem 1.4.2 stated in Section 1.4.2, since one always has $\operatorname{syst} \leq \operatorname{syst}_1$.

Remark that in [Gay22, Theorem 1.16 and Corollary 1.21] it was proven that there exists $c>0$ such that for any d large enough,

- if $n-r=1$, $\mu_d\left[\operatorname{syst}(Z_s, g_{\omega|Z_s}) \leq \frac{1}{\sqrt{d}}\right] \geq c$;
- if n is odd, $\mu_d\left[\operatorname{syst}_{n-1}(Z_s, g_{\omega|Z_s}) \leq \frac{1}{\sqrt{d}}\right] \geq c$.

We also give an estimate of the sectional curvature K of a random complex submanifold:

Theorem 2.2.2 ([AG23]). *Under the hypotheses of Theorem 2.2.1, for any sequence $(a_d)_d$ converging to zero, there exists $C>0$ such that*

$$\forall d \gg 1,\ \mu_d \left[\sup_{\substack{x\in Z_s \\ P \in \operatorname{Grass}(2, T_x Z_s)}} |K(P)| \leq \frac{C}{a_d^2} d^{\frac{3}{2}(n+1)} \log^2 d\right] \geq 1 - C(a_d + \frac{1}{d}).$$

The latter and the systole estimate allow us to give a statistical control of the injectivity radius of the complex submanifolds:

Theorem 2.2.3 ([AG23]). *Under the hypotheses of Theorem 2.2.1, for any sequence $(a_d)_d$ converging to zero, there exists $C>0$ such that*

$$\forall d \gg 1,\ \mu_d \left[\operatorname{inj}(Z_s, g_{\omega|Z_s}) \geq \frac{a_d}{C d^{\frac{3n+1}{2}} \log d}\right] \geq 1 - C(a_d + \frac{1}{d}).$$

Finally, we prove a control for the spectral gap. Recall that if (M,g) is a compact smooth Riemannian manifold, if Δ is its associated Laplace–Beltrami operator, then its spectrum is positive, infinite, discrete and unbounded. The first non-zero eigenvalue $\lambda_1 = \lambda_1(M,g)$ is called the *spectral gap* and is of great importance in various problems. For instance, the heat kernel decreases with the speed $\sqrt{\lambda_1}$.

Theorem 2.2.4 ([AG23]). *Under the hypotheses of Theorem 2.2.1, for any sequence $(a_d)_d$ converging to zero, there exists $C>0$ such that*

$$\mu_d \left[\lambda_1(Z_s, g_{\omega|Z_s}) \geq \exp\left(-\frac{C}{a_d} d^{\frac{3n+3+12r}{4}} \log d\right)\right] \geq 1 - C(a_d + \frac{1}{d}).$$

We observe that for complex curves, the lower bound obtained for the spectral gap is much better (polynomial instead of exponential), see Theorem 1.4.3 and Corollary 2.4.3. This is due to the fact that for complex curves we are able to control the Cheeger constant (which we will define in Section 2.4) and that the Cheeger constant itself allows us to control the spectral gap. In all dimensions, we are not able to control the Cheeger constant, so we will use another technique, which is to control the curvature and diameter and to use a theorem of M. Gromov. These two proofs will be explained in Section 2.4.

2.3 Proof of the systole, sectional curvature, and injectivity radius bounds

Systole

The estimate for the systole relies on the following simple idea, which comes from the implicit function theorem: if, locally around any point $x \in Z_s$, the submanifold Z_s is a graph over a ball of radius R centered at x, then any non-contractible loop in X has length greater than R. Indeed, let γ be any loop passing through x. Then there are two possible cases: either γ is included in the (the graph over) the ball of radius R, either it is not included. In the first case, the loop γ is contractible, since it is completely included in a ball, in the second case, the length of γ is bigger then the diameter of the ball, in particular it is strictly bigger than R.

In order to prove Theorem 2.2.1, we then need to control how R depends on s. Such control can be given by a quantitative implicit function theorem which provides a lower bound for R in terms of:

- an upper bound of the $\mathscr{C}^2$ norm of s,
- a lower bound of its gradient over Z_s.

The lower bound of the gradient of s over Z_s is a form of quantitative transversality: our goal is not only for s to vanish transversely, but also for it to vanish "sufficiently" transversely. In other words, we do not merely want s to avoid lying on the discriminant, but we want to ensure that s lies sufficiently far from it. Thus, our lower bound is controlled by the distance of s to the discriminant locus Δ_d. We are able to control the distance to the discriminant using peak sections [Tia90]. Moreover, one can estimate the probability that s is far enough to Δ_d. The fact that Δ_d is an algebraic set is important here, since the volume of ϵ-neighborhood of algebraic sets can be estimated using [BC13, Theorem 21.1]. As for the upper bound on the $\mathscr{C}^2$ norm, direct deterministic estimates can be obtained using estimates on the Bergman kernel. We obtain better probabilistic estimates using Lévy's concentration arguments.

Sectional curvature

The curvature estimate of Theorem 2.2.2 is given also by similar estimates. Indeed, one can show that the sectional curvature of Z_s at a point x can be controlled by the sectional curve of X and by the norm of ∇s and $\nabla^2 s$ at x. More precisely we obtained the following result.

Proposition 2.3.1 ([AG23]). *Let (X, g) be a compact Kähler manifold, $(E, h) \to X$ be a rank r holomorphic Hermitian vector bundle. Then, there exists $C > 0$ (depending only on $n = \dim X$ and r) such that for any transverse holomorphic section $s \in H^0(X, E)$ the following holds. If $Z = \{s = 0\} \subset X$, then, for any $x \in Z_s$, the sectional curvature K of $(Z, g_{|Z})$ at x satisfies*

$$\max_{P \in \mathrm{Grass}_{\mathbf{R}}(2, T_x X)} |K(x, P) - K^X(x, P)| \leq C \|(\nabla s \nabla s^*)^{-1}\|^2 (\|\nabla^2 s\| + \|\nabla s\|)^2 \|\nabla s\|^2,$$

where $T_x M$ and E_x are identified with their dual through their metric and the right-hand-side is evaluated at x.

In order to prove Theorem 2.2.2, one can then control the right-hand side of the inequality in Proposition 2.3.1. This is done by controlling the gradient and the $\mathscr{C}^2$ norm of s, as in the case of the systole.

Injectivity radius

For the proof of the injectivity radius bound, we use the following result by P. E. Ehrlich [Ehr74].

Theorem 2.3.2. *Let (M, g) be a compact smooth Riemannian manifold. Assume that there exists $k > 0$, such that the sectional curvature is bounded above by k. Then*

$$\mathrm{inj}(M, g) \geq \min\{\frac{\pi}{\sqrt{k}}, \frac{1}{2}\mathrm{syst}(M, g)\}.$$

Thus, Theorem 2.2.3 is a direct consequence of Theorems 2.2.1, 2.2.2 and 2.3.2.

2.4 Proof of the spectral gap lower bound

To control the spectral gap, we use two different methods: one that works very well for complex curves (providing better estimates), and the other that works in any dimension.

2.4.1 Spectral gap for complex curves

Recall the definition of the Cheeger constant $h(S, g)$ of a Riemannian surface (S, g):

$$h(S, g) = \inf \frac{\ell(\gamma)}{\min\{\mathrm{vol}(A), \mathrm{vol}(B)\}}$$

where the infimum is taken among all smooth curves γ which separates S into two disjoint and non-empty parts A and B and where $\ell(\gamma)$ is the length of γ. Let us point out that γ does not necessarily have to be connected.

Remark that there is no non-trivial deterministic lower bound for $h(Z_P)$, that is, for any $d \geq 2$, one has

$$\inf_{P \in \mathbf{C}_d^{\mathrm{hom}}[X_0, X_1, X_2]} h(Z_P) = 0.$$

Indeed, let P_1 and P_2 be two polynomials of degree d_1 and d_2, respectively, with $d_1 + d_2 = d$. Suppose that Z_{P_1} and Z_{P_2} are smooth and intersect transversally (both are generic conditions). Then the degree d polynomial $P = P_1 P_2$ is such that Z_P is a reducible curve with $d_1 d_2$ double points, denoted by $x_1, \ldots, x_{d_1 d_2}$. Note that, by construction, $Z_P \setminus \{x_1, \ldots, x_{d_1 d_2}\}$ is not connected. Now, for any small $\epsilon > 0$, we can take a small generic perturbation Q_ϵ of P such that Z_{Q_ϵ} contains $d_1 d_2$ small loops (these are the vanishing cycles collapsing to $x_1, \ldots, x_{d_1 d_2}$ when Q_ϵ tends to P) whose length is smaller than ϵ, and which separate Z_{Q_ϵ} into two parts whose volumes are close to the volumes of Z_{P_1} and Z_{P_2}.

Our main result about the Cheeger constant is the following probabilistic lower bound.

Theorem 2.4.1 ([AG24a])**.** *One has*

$$\mu_d \left[h(Z_P) \geq \frac{1}{d^5} \right] \underset{d \to \infty}{\to} 1$$

where μ_d denotes the Gaussian measure on the space $\mathbf{C}_d^{\mathrm{hom}}[X_0, X_1, X_2]$.

The Cheeger constant $h(S, g)$ of a Riemannian surface (S, g) is related with the spectral gap $\lambda_1(S, g)$ by the well-known Cheeger inequality $\lambda_1 \geq \frac{h^2}{4}$ [Che71]. In particular Theorem 2.4.1 implies Theorem 1.4.3.

The previous results actually hold in the much more general setting of random complex curves within complex projective manifolds:

Theorem 2.4.2 ([AG24a]). *Let X be a n-dimensional projective manifold, (L, h) be a positive Hermitian line bundle and (E, h_E) be a rank $n-1$ Hermitian line bundle. Let s be a random section of $L^d \otimes E$ and $(a_d)_d$ be a sequence of positive real numbers converging to 0. Then, there exists $C > 0$ such that*

$$\mu_d \left[h(Z_s) \geq \frac{a_d}{C d^{\frac{5n-1}{2}} \sqrt{\log d}} \right] \geq 1 - C(\sqrt{a_d} + \frac{1}{d}).$$

Using Cheeger inequality, we obtain:

Corollary 2.4.3 ([AG24a]). *Let $(a_d)_d$ be a sequence of positive real numbers converging to 0. Under the hypotheses of Theorem 2.4.2, there exists $C > 0$ such that*

$$\mu_d \left[\lambda_1(Z_s) \geq \frac{a_d^2}{C d^{5n-1} \log d} \right] \geq 1 - C(\sqrt{a_d} + \frac{1}{d}).$$

Let us give an idea of proof of Theorem 2.4.2. Let s be a holomorphic section of $E \otimes L^d$ and consider the Riemann surface Z_s. Given a not necessarily connected curve γ in Z_s that separates Z_s into two disjoint and non-empty parts A and B, the goal is to provide a lower bound for

$$h(\gamma) := \frac{\ell(\gamma)}{\min\{\mathrm{vol}(A), \mathrm{vol}(B)\}}.$$

There are two possibilities for the curve γ:

- If every connected component of γ is a *small* contractible loop, we estimate $h(\gamma)$ using an isoperimetric inequality that relates the length of each loop τ of γ to the area of the disk it bounds (see, for example, [Fia41] or [Top99, Corollary 1 (iii)]). Here, *small* means that each such loop is contained within a ball of radius $\left(\sup_{x \in Z_s} |K(x)|\right)^{-\frac{1}{2}}$, where K is the Gaussian curvature of Z_s. The reason we only consider *small* loops is that the isoperimetric inequality applied to a loop τ is given by

  $$\ell(\tau)^2 \geq \mathrm{vol}(D_\tau)(4\pi - K_+ \mathrm{vol}(D_\tau)),$$

  where K_+ is an upper bound of the curvature, and D_τ is the disk bounded by τ. If $\mathrm{vol}(D_\tau)$ is too large, the isoperimetric inequality reduces to the trivial inequality $\ell(\tau)^2 \geq 0$. In order to control $\mathrm{vol}(D_\tau)$, we use the Bishop–Gromov Inequality [BC01, Corollary 4, page 256], which, together with Theorem 2.2.2 and with the isoperimetric inequality stated above, ensures that with very high probability, any small loop τ of γ satisfies $\ell(\tau)^2 \geq 2\pi \mathrm{vol}(D_\tau)$. The choice of the radius $\left(\sup_{x \in Z_s} |K(x)|\right)^{-\frac{1}{2}}$ is precisely to obtain this inequality. From this inequality, a straightforward computation shows that for any such γ, we have $h(\gamma) \geq \frac{2\pi}{\sqrt{d}}$.

- If there is at least one connected component of γ that is not a *small* contractible loop, then this connected component is either non-contractible or a *big* contractible loop. In the first case, the length of this component must be greater than the systole of Z_s. In the second case, the length is at least $\left(\sup_{x \in Z_s} |K(x)|\right)^{-\frac{1}{2}}$. On the other hand, $\min\{\mathrm{vol}(A), \mathrm{vol}(B)\}$ is always smaller than $\mathrm{vol}(Z_s)$, which is a deterministic and computable quantity of order $O(d^n)$.

Theorem 2.4.2 then follows from the previous two cases, together with the bounds of the curvature and of the systole given respectively by Theorems 2.2.2 and 2.2.1. $\square$

2.4.2 Spectral gap in higher dimensions

In any dimension, there is an analogous definition of the Cheeger constant, and Cheeger's inequality still holds. However, we are unable to control the Cheeger constant in higher dimensions. To circumvent this problem, we use the following general estimate due to M. Gromov [HKP16, Theorem 1.2.1]:

Theorem 2.4.4 (Gromov)**.** *Let (M,g) be a n-dimensional compact Riemannian manifold, whose Ricci curvature is bounded from below by $-\kappa$, with $\kappa > 0$. Then there exists C, depending only on the dimension n of M, such that*

$$\lambda_1(M) \geq C^{1+\mathrm{diam}(M)\sqrt{\kappa}}\mathrm{diam}(M)^{-2}.$$

Theorem 2.2.4 regarding the spectral gap then follows from the probabilistic bound on the sectional curvature of Z_s (Theorem 2.2.2), along with the following deterministic upper bound on the diameter of a projective submanifold:

Theorem 2.4.5 ([AG23])**.** *There exists $C > 0$ such that*

$$\forall d \gg 1, \forall s \in H^0(X, E \otimes L^d),\ \mathrm{diam}(Z_s) \leq Cd^{3r}.$$

The proof of Theorem 2.4.5 is similar to the one given by Feng and Schumacher [FS99] for complex plane curves. We provide a better bound for complex plane curves, namely d^3 instead of d^4, and adapt their proof to our more general setting, proving the bound in any codimension and for any complex projective manifold.

Sketch of proof. Given $s \in H^0(X, E \otimes L^d)$, we want to bound from above the diameter of (Z_s, g_{Z_s}). Let us fix once for all an embedding $X \subset \mathbf{P}^N$. As $g \leq c_1 g_{\mathbf{P}^N}|_X$ for some $c_1 > 0$, it is enough to bound from above the diameter of $(Z_s, g_{\mathbf{P}^N}|_{Z_s})$. Equivalently, we need to bound the distance between any pair of points $p, q \in Z_s$ from above. To this end, we consider a generic $\mathbf{P}^{N-n+r+1}$ passing through p and q, which defines an $(r+1)$–dimensional complex submanifold $S = X \cap \mathbf{P}^{N-n+r+1}$ along with a curve $C_s = Z_s \cap \mathbf{P}^{N-n+r+1}$. We have the inequality $\mathrm{dist}_{Z_s}(p,q) \leq \mathrm{dist}_{C_s}(p,q)$, so that the result will follow from the inequality

$$\mathrm{dist}_{C_s}(p,q) \leq cd^{3r}. \tag{2.5}$$

In order to prove (2.5), we will explicitly construct a path between p and q in C_s whose length is bounded from above by cd^3. Here is the construction of such a path. Let $\pi : \mathbf{P}^N \dashrightarrow \mathbf{P}^1$ be a generic linear projection and

$$u : C_s \to \mathbf{P}^1$$

be the restriction of $\pi : \mathbf{P}^N \dashrightarrow \mathbf{P}^1$ to C_s. One can show that the degree of u is bounded from above by $D = c_2 d^r$, where c_2 only depends on the embedding $X \subset \mathbf{P}^N$ and on L and E. We can assume that u is a simple branched covering, meaning that all the branch points are simple (i.e. of multiplicity 2), and that p and q are not branch points. Let $p_j \in C_s$, for $j \in \{1, \ldots, \ell\}$, denote the branch points. We denote by $q_j = u(p_j) \in \mathbf{P}^1$ the critical values. After an arbitrarily small perturbation of π, we can suppose that the q_j's are distinct and that no three of them are contained in a closed geodesic. We choose an auxiliary point

$$x \in \mathbf{P}^1 \setminus \{q_1, \cdots, q_\ell\}$$

and for each $j \in \{1, \cdots, \ell\}$ we consider an arc of geodesic S_j from x to q_j. Then, $\mathbf{P}^1 \setminus \cup_j S_j$ is contractible, and then the cover

$$u : u^{-1}(\mathbf{P}^1 \setminus \cup_j S_j) \to \mathbf{P}^1 \setminus \cup_j S_j$$

is the degree D trivial cover, consisting of D copies $F_1, \dots, F_D$ with $F_i \simeq \mathbf{P}^1 \setminus \cup_j S_j$, called the sheets of the covering. We denote by $S_j^{(i)}$ and by x_i the copies of S_j and of x appearing in the i–th sheet, respectively. As the branched covering $u : C_s \to \mathbf{P}^1$ has only simple ramifications, any branch point p_j is contained in the closure of precisely two sheets. Any two sheets whose closures share a branch point are called *adjacent*.

Lemma 2.4.6. *If a and b are two points lying in two adjacent sheets, then there is a path in C_s of length bounded by $c_3 d^{2r}$ that goes from a to b.*

Proof of Lemma 2.4.6. Let $F_{i(a)}$ and $F_{i(b)}$ be the sheets containing a and b and let $p_{j(a,b)}$ be the branch point connecting the closures $\overline{F}_{i(a)}$ and $\overline{F}_{i(b)}$. We can then go from a to $x_{i(a)}$ following a path which is contained in $u^{-1}(\gamma_1)$, where γ_1 is a geodesic going from $u(a)$ to x. Then we can go from $x_{i(a)}$ to p_{ab} following $S_{j(a,b)}^{(i(a))}$, which is, by construction, contained in $u^{-1}(S_{j(a,b)})$. As the point $p_{j(a,b)}$ lies in $\bar{F}_{i(b)}$, we can follow backwards $S_{j(a,b)}^{(i(b))}$ and go from $p_{j(a,b)}$ to $x_{i(b)}$. Such path is again contained in $u^{-1}(S_{j(a,b)})$. Finally, we can go from x_b to b following a path which is contained in $u^{-1}(\gamma_2)$, where γ_2 is a geodesic going from $u(b)$ to x. Using classical tools from semialgebraic geometry, one shows that each of these four paths has length bounded from above by $c_4 d^{2r}$, so the distance between a and b is bounded from above by $4c_4 d^{2r}$. We have then proved that any pair of points lying in two adjacent sheets can be joint by a path of length smaller than $c_3 d^{2r}$. □

Lemma 2.4.6 implies that any pair of points lying on two k–adjacent sheets (two sheets are called k–adjacent if one can move from one to the other by passing through at most k branch points) can be joined by a curve of length smaller than $c_3 d^{2r} k$. Since any rwo sheets are $(D-1)$–adjacent (recall that D is the degree of the covering) and $D = c_2 d^r$, this implies (2.5) and hence the result. □

2.5 Curvature of random projective submanifolds

Let (X, ω) be a projective manifold equipped with a Kähler form ω. We denote by $g = g_\omega$ the induced metric. Recall that for any $x \in X$ and any two complex lines $\sigma, \sigma' \subset T_x X$ (that is, for any $\sigma, \sigma' \in \mathbf{P}(T_x X)$), the holomorphic bisectional curvature at x along σ and σ' equals

$$\mathrm{HBC}_X(\sigma, \sigma') = R(v, Jv, v', Jv'),$$

where R denotes the Riemannian curvature tensor associated with g, the vector v (resp. v') is a unit vector of σ (resp. σ') and J denotes the complex structure of the tangent space $T_x X$. Recall also that the holomorphic sectional curvature is defined by $\mathrm{HSC}_X(\sigma) := \mathrm{HBC}_X(\sigma, \sigma)$. Similarly, the Ricci curvature is defined by $\mathrm{Ric}_X(\sigma) = \sum_{i=1}^n \mathrm{HBC}_X(\sigma, \sigma_i)$ and the scalar curvature by $\mathrm{Scal}_X = \sum_{i=1}^n \mathrm{Ric}_X(\sigma_i)$, where $\sigma_1, \dots, \sigma_n$ are pairwise orthogonal complex lines of $T_x X$. For complex curves, all these curvatures coincide. In general, the holomorphic bisectional curvature determines the holomorphic sectional curvature, the Ricci curvature, and the scalar curvature.

We study these curvatures for complex submanifolds Z of X with respect to the induced metric $g_{|Z}$. For any $a \in \mathbf{R}$, and any such $Z \subset X$ we denote by

$$\mathrm{Vol}(\mathrm{HBC}_Z < a) := \mathrm{Vol}_g \left\{ z \in Z, \sup_{\sigma, \sigma' \in \mathbf{P}(T_z Z)} \mathrm{HBC}_Z(\sigma, \sigma') < a \right\}$$

the volume occupied by points where the holomorphic bisectional curvature is strictly smaller than a and by

$$\overline{\mathrm{vol}}(\mathrm{HBC}_Z < a) := \frac{\mathrm{Vol}\,(\mathrm{HBC}_Z < a)}{\mathrm{Vol} Z}. \tag{2.6}$$

the proportion of points of X where the holomorphic bisectional curvature is smaller than a. Similarly, we define $\overline{\mathrm{vol}}\,(\mathrm{HSC}_Z < a)$, $\overline{\mathrm{vol}}\,(\mathrm{Ric}_Z < a)$ and $\overline{\mathrm{vol}}\,(\mathrm{Scal}_Z < a)$.

Before stating the main result, let us briefly recall the setting and the notations. Let $(E, h_E) \to X$ be a rank r Hermitian holomorphic vector bundle over X and $(L, h_L) \to X$ be a Hermitian holomorphic line bundle with positive curvature. For any $s \in H^0(X, L^d \otimes E)$, we denote by Z_s the zero locus of s, which we equip with the restriction of the ambient Kähler metric g of X. The space of global sections $H^0(X, L^d \otimes E)$ is equipped with the Gaussian measure μ_d defined by Equation (1.2). The main theorem is the following.

Theorem 2.5.1 ([AG24c])**.** *There exists $C > 0$ such that, for any sequence $(a_d)_{d\in\mathbf{N}}$ of positive reals verifying $a_d = o(d)$, the following holds:*

1. *(holomorphic bisectional curvature) If $3r \geq 2n - 1$,*
$$\forall d \gg 1,\ \mathbf{E}[\overline{\mathrm{vol}}(\mathrm{HBC}_{Z_s} < -a_d)] \geq 1 - C\left(\frac{a_d + C}{d}\right)^{3r-2n+2}.$$
2. *(holomorphic sectional curvature) If $2r \geq n$,*
$$\forall d \gg 1,\ \mathbf{E}[\overline{\mathrm{vol}}(\mathrm{HSC}_{Z_s} < -a_d)] \geq 1 - C\left(\frac{a_d + C}{d}\right)^{2r-n+1}.$$
3. *(Ricci curvature)* $\forall d \gg 1,\ \mathbf{E}[\overline{\mathrm{vol}}(\mathrm{Ric}_{Z_s} < -a_d)] \geq 1 - C\left(\frac{a_d + C}{d}\right)^{r(n-r)-(n-r-1)}.$
4. *(Scalar curvature)* $\forall d \gg 1,\ \mathbf{E}[\overline{\mathrm{vol}}(\mathrm{Scal}_{Z_s} < -a_d)] \geq 1 - C\left(\frac{a_d + C}{d}\right)^{\frac{1}{2}r(n-r)(n-r+1)}.$

The condition on the codimension that appears in the case of holomorphic sectional and bisectional curvature in the theorem is optimal. In fact, if these relations are not satisfied, then, for every point of the submanifold, there will always be directions in which the holomorphic sectional and bisectional curvature are equal to that of the ambient manifold, which can be positive (e.g. for $X = \mathbf{P}^n$).

A consequence of Theorem 2.5.1 is the existence of complex submanifolds whose almost entire volume has negative curvature.

Corollary 2.5.2 ([AG24c])**.** *There exists $C > 0$ such that, for any sequence $(a_d)_{d\in\mathbf{N}}$ of positive reals verifying $a_d = o(d)$, the following holds:*

1. *(holomorphic bisectional curvature) If $3r \geq 2n-1$, there exists a sequence $(s_d)_{d\geq d_0}$ with $s_d \in H^0(X, E \otimes L^d)$ such that $Z_{s_d} \subset X$ is a smooth $(n-r)$-submanifold and*
$$\overline{\mathrm{vol}}(\mathrm{HBC}_{Z_{s_d}} < -a_d) \geq 1 - C\left(\frac{a_d + C}{d}\right)^{3r-2n+2}.$$
2. *(holomorphic sectional curvature) If $2r \geq n$, there exists a sequence of sections $(s_d)_{d\geq d_0}$ with $s_d \in H^0(X, E \otimes L^d)$ such that $Z_{s_d} \subset X$ is a smooth $(n-r)$-submanifold and*
$$\overline{\mathrm{vol}}(\mathrm{HSC}_{Z_{s_d}} < -a_d) \geq 1 - C\left(\frac{a_d + C}{d}\right)^{2r-n+1}.$$
3. *(Ricci curvature) There exists a sequence $(s_d)_{d\geq d_0}$ with $s_d \in H^0(X, E \otimes L^d)$ such that $Z_{s_d} \subset X$ is a smooth $(n-r)$-submanifold and*
$$\overline{\mathrm{vol}}(\mathrm{Ric}_{Z_s} < -a_d) \geq 1 - C\left(\frac{a_d + C}{d}\right)^{r(n-r)-r+n+1}.$$

4. *(Scalar curvature) There there exists a sequence $(s_d)_{d\geq d_0}$ with $s_d \in H^0(X, E\otimes L^d)$ such that $Z_{s_d} \subset X$ is a smooth $(n-r)$-submanifold and*

$$\overline{\mathrm{vol}}(\mathrm{Scal}_{Z_{s_d}} < -a_d) \geq 1 - C\left(\frac{a_d + C}{d}\right)^{\frac{1}{2}(n-r)(n-r+1)}.$$

Remark that for any $d \in \mathbf{N}^*$ and any r, there exists a degree d codimension r projective submanifold in $\mathbf{P}^n$ for which the bisectional curvature is not negative everywhere. Note also that any complex curve in $\mathbf{P}^2$ has points with positive curvature. These are the inflexion points, and generically there are $3d(d-2)$ of them. Recently J.-P. Mohsen [Moh22] proved that

1. if $4r \geq 3n-1$ then, for every sufficiently large d, there exists a complete intersections Z_d of degree d and codimension r such that $g_{|Z_d}$ has negative holomorphic bisectional curvature in X.
2. if $3r \geq 2n$, the same holds for the holomorphic sectional curvature.
3. if $r \geq 2$, the same holds for the Ricci curvature.
4. For $n \geq 3$, the same holds for the scalar curvature.

For the proof of his theorem, J.-P. Mohsen used Donaldson's method [Don96], which is a subtle construction of holomorphic sections with a prescribed lower positive bound of the norm of their derivatives. Our proof does not use Donaldson's construction at all. In fact, from a probabilistic point of view, Donaldson's sections are exponentially rare. Moreover, we need weaker dimension and codimension conditions. On the other side, we cannot recover his result.

2.5.1 Idea of the proof of Theorem 2.5.1

The first ingredient in the proof of Theorem 2.5.1 is a Kac–Rice type formula [AW09] which expresses $\mathbf{E}[\mathrm{Vol}(\mathrm{HBC}_{Z_s} < -a_d)]$ as an integral over X of an explicit density function. The formula we found is as follows:

$$\mathbf{E}[\mathrm{Vol}(\mathrm{HBC}_{Z_s} < -a_d)] = \int_{x\in X} \frac{1}{\det\left(B_d(x,x)\right)}\left(\int_{s\in\ker \mathrm{ev}_x} \mathbf{1}_{\mathrm{HBC}_{Z_s}<-a_d}(x)\|\nabla s(x)\|^2 \mathrm{d}\mu_{\ker \mathrm{ev}_x}(s)\right)\mathrm{d}x$$

where $B_d(x,x)$ is the Bergman kernel of $L^d\otimes E$ along the diagonal, and $\ker \mathrm{ev}_x$ denotes the space of sections $s \in H^0(X, E\otimes L^d)$ vanishing at x. Let us now take a closer look at the function $s \in \ker \mathrm{ev}_x \mapsto \mathbf{1}_{\mathrm{HBC}_{Z_s}<-a_d}(x)$. The average value of this function equals the probability that a submanifold Z_s passing through x satisfies the condition

$$\sup_{\sigma,\sigma'\in\mathbf{P}(T_xZ_s)} \mathrm{HBC}_{Z_s}(\sigma,\sigma') < -a_d.$$

A first step to estimate this probability is to find a formula that expresses the holomorphic bisectional curvature of Z_s at x in terms of s and its derivatives:

Proposition 2.5.3 ([AG24c]). *Let (X, g, J) be a Kähler manifold, $F \to X$ be a complex vector bundle of rank r, and $s \in \mathscr{C}^\infty(X, F)$ a section of F vanishing transversely along a J-complex submanifold Z_s. Then, for any $x \in Z_s$ and any $\sigma, \sigma' \in \mathbf{P}(T_xZ_s)$,*

$$\mathrm{HBC}_{Z_s}(\sigma,\sigma') = \mathrm{HBC}_X(\sigma,\sigma') - 2(\nabla^F sG(\nabla^F s)^*)^{-1}((\nabla^F)^2_{v,v'}s)((\nabla^F)^2_{v,v'}s),$$

where everything is computed at x. Here, v and v' are unitary vector of σ and σ', and $G : T^*X \to TX$ is the isomorphism given by $g(G(\alpha),\cdot) = \alpha$, $\forall\alpha \in T^*X$.

Remark 2.5.4. It may be worth explaining the term $(\nabla^F sG(\nabla^F s)^*)^{-1}((\nabla^F)^2_{v,v'}s)((\nabla^F)^2_{v,v'}s)$ in more detail. The composition $\nabla^F sG(\nabla^F s)^*$ is a map from F^* to F, so that $(\nabla^F sG(\nabla^F s)^*)^{-1}$ is a map from F to F^*. Thus, $(\nabla^F sG(\nabla^F s)^*)^{-1}((\nabla^F)^2_{v,v'}s)((\nabla^F)^2_{v,v'}s)$ should be read as follows: first, apply $(\nabla^F)^2_{v,v'}s \in F$ to $(\nabla^F sG(\nabla^F s)^*)^{-1}$, which gives an element of F^*; then contract this element with $(\nabla^F)^2_{v,v'}s$ to obtain a scalar.

This proposition generalizes results of L. Ness [Nes77] (who proved the formula for the case of plane curves) and A. Vitter [Vit74] (who proved the formula for hypersurfaces in $\mathbf{P}^n$). It is very close in spirit to the classical formulas expressing the curvature of a submanifold in terms of its second fundamental form, and should be compared to Griffiths' formula for the curvature of a holomorphic subbundle, see [GH94, Section 5].

Let us now take a closer look to Proposition 2.5.3. Using Bergman kernel estimates, one can show that the average over $s \in H^0(X, E \otimes L^d)$ of $\left\|\nabla^2 s(x)\right\|^2$ is of order $O(d^{n+2})$, while the average of $\|\nabla s(x)\|^2$ is of order $O(d^{n+1})$. In particular, one expects

$$(\nabla sG(\nabla s)^*)^{-1}((\nabla)^2_{v,v'}s)((\nabla)^2_{v,v'}s) \tag{2.7}$$

to be of order d. Moreover, it can be verified that the quantity (2.7) is always non-negative (as is the case for the second fundamental form in Griffiths' formula, which ensures that the curvature becomes more negative for holomorphic subbundles). This suggests that for a point x, the holomorphic bisectional curvature HBC_{Z_s} is typically negative and of order d. In particular, the probability that a submanifold Z_s passing through x satisfies the condition

$$\sup_{\sigma,\sigma' \in \mathbf{P}(T_x Z_s)} \mathrm{HBC}_{Z_s}(\sigma, \sigma') < -a_d$$

is close to 1, since $a_d = o(d)$.

Let us now explain why the exponent that appears in Theorem 2.5.1 for the holomorphic bisectional curvature is $3r - 2n + 2$. A consequence of Proposition 2.5.3 is that if

$$\min_{\substack{v,v' \in T_x Z_s \\ \|v\|=\|v'\|=1}} \left\|\nabla^2_{v,v'} s\right\|^2 > (a_d + \|\mathrm{HBC}_X\|)\|\nabla s\|, \tag{2.8}$$

then the holomorphic bisectional curvature HBC_{Z_s} at x is smaller than a_d. We relate the left-hand side of (2.8) to the distance $\mathrm{dist}(\nabla^2 s, \Omega)$ from $\nabla^2 s$ to a discriminant locus Ω within the space $\mathrm{Bil}(T_x Z_s, E \otimes L^d)$ of complex bilinear forms from $T_x Z_s$ to $E \otimes L^d$. The advantage of working with a distance is that the probability that $\mathrm{dist}(\nabla^2 s, \Omega) < \epsilon$ corresponds to the Gaussian volume of the ϵ-neighborhood Ω^ϵ of Ω. Now, the discriminant Ω is a complex subvariety of $\mathrm{Bil}(T_x Z_s, E \otimes L^d)$, and thus the volume of its ϵ-neighborhood can be estimated in terms of its codimension [BL23]. More precisely, we have $\mathrm{Vol}(\Omega^\epsilon) \simeq \epsilon^{2\mathrm{codim}\Omega}$, as $\epsilon \to 0$. We prove that the codimension of Ω is $3r - 2n + 2$, which therefore brings out the exponent we were looking for.

For the holomorphic sectional, Ricci, and scalar curvature, the proof follows a similar approach to that of the holomorphic bisectional curvature. The exponents in the statement for these curvatures also correspond to the codimension of a discriminant locus within a suitably chosen space of maps.

Chapter 3

Topology of real algebraic submanifolds

In this chapter, I describe my works [Anc24, Anc23] about the topology of random real algebraic submanifolds as well as an existence result [Anc22b] obtained using probabilistic methods.

Recall that a real algebraic manifold (X, c_X) is the data of a smooth complex projective variety X and of an antiholomorphic involution $c_X : X \to X$, called the real structure. The real locus $\mathbf{R}X$ of a real algebraic variety is the set of fixed points of the real structure, that is $\mathbf{R}X = \mathrm{Fix}(c_X)$. The real locus $\mathbf{R}X$ is either empty, or a finite union of n dimensional $\mathscr{C}^\infty$-manifolds, where n is the (complex) dimension of X. The study of the topology of real algebraic varieties is a central topic in real algebraic geometry since the works of Harnack and Klein on the topology of real algebraic curves [Har76, Kle76], the famous Hilbert's 16th problem [Wil78], and the celebrated Nash–Tognoli theorem [Nas52, Tog73], which asserts that given a closed $\mathscr{C}^\infty$-manifold M, there exists a real algebraic manifold (X, c_X) such that $\mathbf{R}X$ is diffeomorphic to M. A vague modern formulation of Hilbert's 16th problem is the following: describe the connected components of $\mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$, where $(L, c_L) \to (X, c_X)$ is an real ample holomorphic line bundle over X, and $\mathbf{R}\Delta_d$ denotes the real discriminant. This can be made more precise in at least two ways.

Problem 3.0.1. Provide a list of the connected components of $\mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$ and, for each connected component $\mathcal{C}$ of $\mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$, describe the topology of $\mathbf{R}Z_s$ for $s \in \mathcal{C}$.

Problem 3.0.2. Provide a list of the connected components of $\mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$ and, for each connected component $\mathcal{C}$ of $\mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$, describe the topology of the pair $(\mathbf{R}X, \mathbf{R}Z_s)$ for $s \in \mathcal{C}$.

The difference between Problem 3.0.1 and Problem 3.0.2 can already be seen for quartics in $\mathbf{P}^2$. By multiplying two real polynomials of degree 2 whose real locus is a circle (and by perturbe a little bit the resulting polynomial, so that it defines a smooth real plane quartic), one can obtain either two concentric circles or two circles placed side by side. The real locus of these two quartics is abstractly the same (a union of two circles), but it is different as a submanifold of $\mathbf{RP}^2$. The original Hilbert's 16th is Problem 3.0.2 for $X = \mathbf{P}^2$ and $L^d = \mathcal{O}(d)$. In general, both Problems 3.0.1 and 3.0.2 seem to be beyond reach. A complete classification is known only for curves of degree $d \leq 7$ in $\mathbf{P}^2$ [Vir84], for surfaces of degree $d \leq 4$ in $\mathbf{P}^3$ [Kha76], for threefolds of degree $d \leq 3$ in $\mathbf{P}^4$ [FK10], and for curves of bidegree (a, b) in $\mathbf{P}^1 \times \mathbf{P}^1$ for a few values of a and b. One of the challenges in solving these problems in full generality is of a numerical nature: the number of connected components of $\mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$ grows super-exponentially as $d \to \infty$, see [OK00]. Typically, to study Problems 3.0.1 and 3.0.2, the first step is to look for a priori topological restrictions on real algebraic varieties, and the second step is, once the topological restrictions are known, to try to construct concrete examples. A fundamental constraint on the topology of the real locus in terms of the complex one is given by the Smith–Thom inequality [Tho65], which asserts that the total Betti number of the real locus $\mathbf{R}X$ of a real algebraic variety is bounded from above by the total Betti number of the complex locus:

$$\sum_{i=0}^{n} \dim H_i(\mathbf{R}X, \mathbf{Z}/2) \leq \sum_{i=0}^{2n} \dim H_i(X, \mathbf{Z}/2). \tag{3.1}$$

We will write more compactly $b_*(\mathbf{R}X) \leq b_*(X)$, where b_* denotes the total Betti number (i.e. the sum of the $\mathbf{Z}/2$-Betti numbers). For a real algebraic curve C, the inequality (3.1) is known as Harnack's inequality [Har76, Kle76], and it states that $b_0(\mathbf{R}C) \leq g+1$, where g is the genus of C.

A real algebraic variety which realizes the equality in (3.1) is called *maximal*.

Question 3.0.3. Given a real linear series of divisors in X, what is the probability to find a divisor defining a real algebraic maximal hypersurface?

The answer I gave to this question in [Anc24] is the following: real algebraic maximal hypersurfaces are exponentially rare inside their linear system. This is the content of Theorem 1.5.1 already stated in Section 1.5.1 that we will generalize in the next section.

3.1 Rarefaction and approximation results

We equip the real ample line bundle (L, c_L) with a smooth real Hermitian metric h of positive curvature ω (we recall that *real* means $c_L^* h = \bar{h}$). Recall that such a Hermitian metric induces an $\mathscr{L}^2$-scalar product on the space of real global sections $\mathbf{R}H^0(X, L^d)$ as well as a Gaussian probability measure μ_d on it, see Section 1.3.

The zero locus of a real global section s_d of L^d is denoted by Z_{s_d} and its real locus by $\mathbf{R}Z_{s_d}$. The total Betti number $b_*(Z_{s_d})$ of Z_{s_d} verifies the asymptotic $b_*(Z_{s_d}) = v(L)d^n + O(d^{n-1})$, as $d \to \infty$, where $v(L) := \int_X c_1(L)^n$ is the so-called volume of the line bundle L (see, for example, [GW14a, Lemma 3]). If the total Betti number $b_*(\mathbf{R}Z_{s_d})$ of the real loci of a sequence of real algebraic hypersurfaces verifies the same asymptotics, then the hypersurfaces are called *asymptotically maximal*. The existence of asymptotically maximal hypersurfaces is known in many cases, for example when the ambient space is a real algebraic surface [GW11, Theorem 5], a projective space [IV07], or a toric variety [Ber06, Theorem 1.3]. The first result of this section implies that asymptotically maximal hypersurfaces are very rare in their linear system. :

Theorem 3.1.1 ([Anc24]). *For some (hence any) $\epsilon > 0$ small enough, we have*

$$\mu_d\big\{s \in \mathbf{R}H^0(X, L^d), b_*(\mathbf{R}Z_s) \geq (1-\epsilon)b_*(Z_s)\big\} = O(d^{-\infty})$$

as $d \to \infty$.

The notation $O(d^{-\infty})$ stands for $O(d^{-k})$ for any $k \in \mathbf{N}$ and the measure μ_d is the Gaussian measure defined in Equation (1.4). Note that actually we have more than "asymptotically maximal hypersurfaces are very rare". Indeed, asymptotically maximal hypersurfaces corresponds to the asymptotics $b_*(\mathbf{R}Z_s) = (\int_X c_1(L)^n)d^n + O(d^{n-1})$, while in Theorem 3.1.1 we consider bigger subsets of $\mathbf{R}H^0(X, L^d)$ of the form $b_*(\mathbf{R}Z_s) \geq (1-\epsilon)d^n$, for some ϵ small enough.

Theorem 3.1.1 (as well as Theorem 1.5.1 presented in Section 1.5.1) is a consequence of a general "approximation theorem" which states that, for some $\alpha < 1$, with very high probability, the zero locus of a real section of L^d is diffeomorphic to the zero locus of a real section of $L^{\lfloor \alpha d \rfloor}$, where $\lfloor \alpha d \rfloor$ is the greatest integer less than or equal to αd. More precisely we have:

Theorem 3.1.2 ([Anc24]). *Let (X, c_X) be a real algebraic variety and (L, c_L) be a real Hermitian line bundle of positive curvature.*

1. *There exists a positive $\alpha_0 < 1$ such that for any $\alpha_0 < \alpha < 1$ the following happens: the probability that, for a real section s of L^d, there exists a real section s' of $L^{\lfloor \alpha d \rfloor}$ such that the pairs $(\mathbf{R}X, \mathbf{R}Z_s)$ and $(\mathbf{R}X, \mathbf{R}Z_{s'})$ are isotopic, is at least $1 - O(d^{-\infty})$, as $d \to \infty$.*

2. *For any $k \in \mathbf{N}$ there exists $c > 0$ such that the following happens: the probability that, for a real section s of L^d, there exists a real section s' of L^{d-k} such that the pairs $(\mathbf{R}X, \mathbf{R}Z_s)$ and $(\mathbf{R}X, \mathbf{R}Z_{s'})$ are isotopic, is at least $1 - O(e^{-c\sqrt{d}\log d})$, as $d \to \infty$. If moreover the real Hermitian metric on L is analytic, this probability is at least $1 - O(e^{-cd})$, as $d \to \infty$*

Let us see how Theorem 3.1.2 directly implies Theorem 3.1.1 (and also Theorem 1.5.1 stated in Section 1.5.1).

Proof of Theorems 1.5.1 and 3.1.1. We prove Theorem 3.1.1 using the first assertion of Theorem theorem approximation; Theorem 1.5.1 is proved using the second assertion of Theorem 3.1.2 in a similar way.

Recall that we want to prove that for small enough $\epsilon > 0$, we have

$$\mu_d\{s \in \mathbf{R}H^0(X, \oplus_{i=1}^m L^d), b_*(\mathbf{R}Z_s) < (1-\epsilon) b_*(Z_s)\} = 1 - O(d^{-\infty})$$

as $d \to \infty$. Let α_0 be given by point *(1)* of Theorem 3.1.2 and denote $\delta_0 = 1 - \alpha_0$. By Theorem 3.1.2, for any $0 < \delta < \delta_0$, the real zero locus of a global section s of L^d is diffeomorphic to the real zero locus of a global section s' of $L^{\lfloor (1-\delta)d \rfloor}$ with probability $1 - O(d^{-\infty})$. Now, by Smith–Thom inequality (see Equation (3.1)), the total Betti number $b_*(\mathbf{R}Z_{s'})$ of the real zero locus a generic section s' of $L^{\lfloor (1-\delta)d \rfloor}$ is smaller or equal than $b_*(Z_{s'})$, which, in turns, has the asymptotic $b_*(Z_{s'}) = \left(\int_X c_1(L)^n\right)\left(\lfloor (1-\delta)d \rfloor\right)^n + O(d^{n-1})$. In particular, with probability $1 - O(d^{-\infty})$, the total Betti number $b_*(\mathbf{R}Z_s)$ of the real zero locus of a section s of L^d is smaller than $\left(\int_X c_1(L)^n\right)\left(\lfloor (1-\delta/2)d \rfloor\right)^n$, as $d \to \infty$. Choosing ϵ so that $(1-\epsilon) > (1-\delta/2)^n$ we have the result. ☐

Remark 3.1.3. Theorem 3.1.2 implies not only that maximal hypersurfaces are rare, but that “maximal configurations” are. For instance, the probability that a real algebraic curve in a surface has a deep nest of ovals is exponentially small (roughly speaking a nest of ovals means severals ovals one inside the other). More recently, I extended Theorem 3.1.2 for random real complete intersections inside a real algebraic variety [Anc23].

When $(X, c_X) = (\mathbf{P}^n, conj)$, $L = \mathcal{O}(1)$ and the Hermitian metric on L equals the Fubini-Study metric (that is for the case of the so-called Kostlan polynomials, see Example 1.3.2), a low degree approximation property was proven by Diatta and Lerario [DL22], so that Theorem 3.1.2 is a natural generalization of their result. We stress that in [DL22], it is essential that the considered real algebraic variety is the projective space and that the metric on $\mathcal{O}(1)$ is the Fubini-Study one (and not, for instance, a small perturbation of it). Indeed, in this situation:

1. the induced $\mathscr{L}^2$-scalar product on $\mathbf{R}_d^{\text{hom}}[X_0, \dots, X_n]$ is invariant under the action of the orthogonal group $O(n+1)$, which acts on the variables $X_0, \dots, X_n$ (equivalently, the group $O(n+1)$ acts by real holomorphic isometries on $(\mathbf{P}^n, conj)$);

2. there exists a canonical $O(n+1)$-invariant decomposition of the space of polynomials $\mathbf{R}_d^{\text{hom}}[X_0, \dots, X_n] = \bigoplus_{d-\ell \in 2\mathbf{N}} V_{d,\ell}$, where $V_{d,\ell}$ is the space of homogeneous harmonic polynomials of degree ℓ, thanks to which it is possible to define projections of degree d polynomials into lower degree ones.

These two properties, together with the classification [Kos93] of the $O(n+1)$-invariant scalar products on $\mathbf{R}_d^{\text{hom}}[X_0,\dots,X_n]$, are fundamental for the proof of the results in [DL22]. This is a very special feature of Kostlan polynomials that is not true in general. Indeed,

1. on a general real algebraic variety equipped with a Kähler metric ω the group of holomorphic isometries is trivial;

2. given a real Hermitian holomorphic line bundle $L \to X$ there is no canonical decomposition of $\mathbf{R}H^0(X,L^d)$.

Hence, in order to obtain an approximation property for sections of line bundles on a general real algebraic variety, we have to use a different strategy, that we will explain in more details in Section 3.1.1. In particular, in our proof, in constrast to the case of Kostlan polynomials [DL22], the complex locus of the variety X (and not only the real one) plays a fundamental role. Indeed, we will consider real subvarieties of X with empty real locus and we will study the real sections of L^d that vanish along these subvarieties. These real sections are the fundamental tool for our low degree approximation property.

During the proof of Theorem 3.1.2, we need to understand how much we can perturb a real section $s \in \mathbf{R}H^0(X,L^d)$ without changing the topology of its real locus. This leads us to study two quantities related to the discriminant $\mathbf{R}\Delta_d \subset \mathbf{R}H^0(X,L^d)$, that is the subset of sections which do not vanish transversally along $\mathbf{R}X$. More precisely, we will consider the volume (with respect to the Gaussian measure μ_d) of ϵ-neighborhoods of $\mathbf{R}\Delta_d$ and the function "distance to the discriminant", for which we obtained the following asymptotic formula.

Proposition 3.1.4 ([Anc24]). *Let* $\text{dist}_{\mathbf{R}\Delta_d}(s)$ *be the distance (induced by the* $\mathscr{L}^2$*-scalar product* (1.1)*) from a section* s *to the discriminant* $\mathbf{R}\Delta_d$*. Then we have the uniform estimate:*

$$\text{dist}_{\mathbf{R}\Delta_d}(s) = \min_{x\in\mathbf{R}X}\left(\frac{\|s(x)\|^2_{h^d}}{d^n}+\frac{\|\nabla s(x)\|^2_{h^d}}{d^{n+1}}\right)^{1/2}\left(\pi^{n/2}+O(1/d)\right)$$

as $d\to\infty$*.*

This formula is a generalization of a result of Raffalli [Raf14] in the context of polynomials (in that case, the formula is exact and not just asymptotic) and has already been used in [DL22]; however in our general framework the lack of symmetries makes the proof of Proposition 3.1.4 more delicate and requires the use of the Bergman kernel estimates along the diagonal [Tia90, Zel98, MM07].

3.1.1 Idea of the proof of Theorem 3.1.2

In this section, we sketch the proof of the second assertion of Theorem 3.1.2, the proof of the first assertion is similar.

Recall that we aim to prove that, with high probability, the real vanishing locus of a real section s of L^d is ambient isotopic to the real vanishing locus of a real section s' of L^{d-k}. To do so, we will define a linear map $A : \mathbf{R}H^0(X,L^d) \to \mathbf{R}H^0(X,L^{d-k})$ (the "approximation map") and prove that with high probability the real vanishing locus of s is ambient isotopic to the real vanishing locus of $A(s)$.

Step 1: construction of the approximation map. The first fact we use is the existence of a real section σ of L^k, for some suitable even $k \in 2\mathbf{N}$ large enough, with the properties that σ vanishes transversally and $\mathbf{R}Z_\sigma = \emptyset$. In order to obtain such a section σ, we consider an integer m such that L^m is very ample and a basis $\{s_1,\dots,s_N\}$ is a basis of $\mathbf{R}H^0(X,L^m)$. Then, any general small perturbation σ of the section $\sum_{i=1}^N s_i^{\otimes 2}$ of L^{2m} vanishes transversally and has empty real vanishing locus.

Let us define $\mathbf{R}H_{d,\sigma}$ to be the vector space of real sections $s \in \mathbf{R}H^0(X, L^d)$ such that s vanishes along the vanishing locus Z_σ of σ. We also denote by $\mathbf{R}H^{\perp}_{d,\sigma}$ the $\mathscr{L}^2$-orthogonal complement of $\mathbf{R}H_{d,\sigma}$. Then, for any section $s \in \mathbf{R}H^0(X, L^d)$, there exists an unique decomposition $s = s^{\perp}_{\sigma} + s^0_\sigma$ with $s^0_\sigma \in \mathbf{R}H_{d,\sigma}$ and $s^{\perp}_\sigma \in \mathbf{R}H^{\perp}_{d,\sigma}$. Moreover, one can easily show that the section $s^0_\sigma \in \mathbf{R}H_{d,\sigma}$ can be uniquely written as $\sigma \otimes A(s)$, for some real section $A(s)$ of L^{d-k}. This rule defines the approximation map $A : \mathbf{R}H^0(X, L^d) \to \mathbf{R}H^0(X, L^{d-k})$.

Step 2: study of the $\mathscr{C}^1$–norm of s^0_σ and $s^{\perp}_\sigma$. We now want to prove that $\mathbf{R}Z_s$ is isotopic to $\mathbf{R}Z_{A(s)}$ with high probability. This is done using two arguments, one being deterministic and the other being probabilistic.

- The deterministic part consists in proving that the "orthogonal component" $s^{\perp}_\sigma$ of s is very small in $\mathscr{C}^\infty$-topology along the real locus $\mathbf{R}X$. This is proved using the logarithmic and partial Bergman kernel theory [Sun21, CM17, Fin24]. It allows to prove that the $\mathscr{C}^1$–norm (actually, the $\mathscr{C}^\infty$) along $\mathbf{R}X$ of a section $s^{\perp}_\sigma \in \mathbf{R}H^{\perp}_{d,\sigma}$ of unit $\mathscr{L}^2$-norm is exponentially small in the degree d. The geometric reason why this is true is that the space $\mathbf{R}H^{\perp}_{d,\sigma}$ is generated by the peak sections [Tia90] which have a peak on Z_σ. Such peak sections are exponentially small outside a ball of radius $\sim \frac{\log d}{\sqrt{d}}$ centered at their peak [MM15]. Now, as $Z_\sigma \cap \mathbf{R}X = \emptyset$, this implies that the pointwise $\mathscr{C}^\infty$-norm of these peak sections is very small along $\mathbf{R}X$.

- The probabilistic part consists in proving that, with high probability, the $\mathscr{C}^1$-norm of s is big enough along $\mathbf{R}X$, so that s vanishes transversally along $\mathbf{R}X$ in a precise quantitative way. This is proved by showing that, with high probability, the $\mathscr{L}^2$-distance from s to the real discriminant is big enough and then by relating this distance with the $\mathscr{C}^1(\mathbf{R}X)$-norm of s. In the first part we use Bergman kernel estimates and an argument borrowed from [DL22, Section 4] which consists in computing the volume of a tube around the discriminant.

Step 3: end of the proof. From the previous two points, we conclude that with high probability, s is a $\mathscr{C}^1$-small pertubation of s^0_σ along $\mathbf{R}X$. As such, Thom's Isotopy Lemma [Mat12] implies that the pairs $(\mathbf{R}X, \mathbf{R}Z_s)$ and $(\mathbf{R}X, \mathbf{R}Z_{s^0_\sigma})$ are isotopic. Finally, recalling that the section s^0_σ can be written as $\sigma \otimes A(s)$, for some real section $A(s)$ of L^{d-k}, and that σ does not have any real zero, we have the equality $\mathbf{R}Z_{s^0_\sigma} = \mathbf{R}Z_{A(s)}$, and hence the result. $\square$

3.2 An existence result using probabilistic tools

The goal of this section is to prove Theorem 1.5.5 stated in Section 1.5.1. Recall that Theorem 1.5.5 ensures that, given any real algebraic manifold (X, c_X) of dimension n with non-empty real locus, an ample line bundle (L, c_L) on (X, c_X), and an r-codimensional closed submanifold Σ of $\mathbf{R}^n$ with trivial normal bundle, there exists $c > 0$ such that, for any d large enough, there exist $s_1, \dots, s_r \in \mathbf{R}H^0(X, L^d)$ with the property that $\mathbf{R}Z_{s_1} \cap \cdots \cap \mathbf{R}Z_{s_r}$ contains at least cd^n connected components diffeomorphic to Σ.

Before proving Theorem 1.5.5, we give an application of it in symplectic geometry. Recall that the complex projective space $\mathbf{P}^n$ is equipped with a natural symplectic form ω_{FS}, called the Fubini-Study symplectic form. Any smooth complex hypersurface of $\mathbf{P}^n$ inherits, by restriction, a symplectic form. Recall also that a Lagrangian submanifold of a symplectic manifold M is a smooth, closed, half-dimensional submanifold of M, for which the restriction of the symplectic form is zero everywhere. For example, $\mathbf{RP}^n$ is a Lagrangian of $\mathbf{P}^n$, the real locus of a real algebraic hypersurface of $\mathbf{P}^n$ is a Lagrangian of its complex locus, and, more generally, the real locus of any real algebraic manifold equipped with a real Kähler form. As an application of Theorem 1.5.5, we obtain an easy proof of the following result by D. Gayet [Gay22].

Theorem 3.2.1. *For any smooth closed hypersurface Σ of $\mathbf{R}^n$, there exists $c>0$ and $d_0 \in \mathbf{N}$ such that, for any $d \geq d_0$, any degree d smooth complex hypersurface of $\mathbf{P}^n$ contains at least cd^n pairwise disjoint Lagrangians diffeomorphic to Σ.*

Proof. Theorem 1.5.5 provides smooth complex hypersurfaces of degree d with at least cd^n pairwise disjoint Lagrangians diffeomorphic to Σ (as the real locus of a real algebraic hypersurface is Lagrangian). As, from a symplectic point of view, all degree m smooth complex hypersurfaces of $\mathbf{P}^n$ are isomorphic, this implies that *any* degree d smooth complex hypersurface contains at least cd^n pairwise disjoint Lagrangians diffeomorphic to Σ. □

It is worth noting that the order d^n appearing in Theorem 3.2.1 is optimal when the Euler characteristic of Σ is nonzero. Indeed, if Z_d denotes a smooth degree d complex hypersurface of $\mathbf{P}^n$, then the rank of $H_{n-1}(Z_d, \mathbf{Z})$ grows exactly as d^n when $d \to \infty$, and if $\chi(\Sigma) \neq 0$, then the disjoint Lagrangians diffeomorphic to Σ are linearly independent in $H_{n-1}(Z_d, \mathbf{Z})$. To prove the last assertion, since all these Lagrangians are disjoint, it is enough to show that they are non-zero in $H_{n-1}(Z_d, \mathbf{Z})$. To this end, note that the complex structure of Z_d provides an isomorphism between the tangent bundle and the normal bundle of any Lagrangian submanifold. Thus, the self-intersection of a Lagrangian equals its Euler characteristic, which is non-zero by hypothesis.

3.2.1 Idea of proof of Theorem 1.5.5

To prove Theorem 1.5.5, we construct a particular probability measure ν_d on $\mathbf{R}H^0(X, L^d)$ for which the expected value $\mathbf{E}_{\nu_d}\left(\mathcal{N}_\Sigma(\mathbf{R}Z_{s_1} \cap \cdots \cap \mathbf{R}Z_{s_r})\right)$ of $\mathcal{N}_\Sigma(\mathbf{R}Z_{s_1} \cap \cdots \cap \mathbf{R}Z_{s_r})$ is at least cd^n, where c is a positive constant independent of d and $\mathcal{N}_\Sigma$ denoted the number of connected components diffeomorphic to Σ. Indeed, this estimate on the expected value directly implies Theorem 1.5.5.

Remark 3.2.2. It is important to note that the probability measure ν_d used in this result is distinct from the natural Gaussian probability measure μ_d defined in Section 1.3 and considered throughout the manuscript. As mentioned before, for the probability measure μ_d the order of the average of the Betti numbers is $\sqrt{d}^n$, here instead it is of order d^n (which is the maximal order granted by the Smith–Thom inequality). We describe the construction of the probability measure in Section 3.2.2.

The main technical result in the proof of Theorem 1.5.5 is the following:

Proposition 3.2.3 ([Anc22b])**.** *There exist $R, c_\Sigma > 0$ and d_0 such that for any point $x \in \mathbf{R}X$ and any $d \geq d_0$, we have*

$$\nu_d\left\{\mathcal{N}_\Sigma\left(\mathbf{R}Z_{s_1} \cap \cdots \cap \mathbf{R}Z_{s_r} \cap B_{\mathbf{R}X}\big(x, \frac{R}{d}\big)\right) \geq 1\right\} \geq c_\Sigma.$$

Here, $B_{\mathbf{R}X}\big(x, \frac{R}{d}\big)$ denotes the ball in $\mathbf{R}X$ around x with radius $\frac{R}{d}$ with respect to a Riemannian metric on $\mathbf{R}X$ that we will fix later. By covering $\mathbf{R}X$ with approximately d^n disjoint balls of radius R/d and using Proposition 3.2.3, we obtain the desired lower bound for the expected value of $\mathcal{N}_\Sigma(\mathbf{R}Z_{s_1} \cap \cdots \cap \mathbf{R}Z_{s_r})$.

Corollary 3.2.4. *There exists c (depending only on X and Σ) such that*

$$\mathbf{E}_{\nu_d}\left(\mathcal{N}_\Sigma(\mathbf{R}Z_{s_1} \cap \cdots \cap \mathbf{R}Z_{s_r})\right) \geq cd^n.$$

The proof of Proposition 3.2.3 is established using the so-called *barrier method*, a technique introduced by Nazarov–Sodin in [NS09] and later refined by Gayet–Welschinger in [GW14b]. Roughly, the barrier method applied to our setting consists in two steps: first, it is shown that for each $x \in \mathbf{R}X$ there exist real holomorphic sections $s_1, \dots, s_r$ such that $\mathcal{N}_\Sigma\left(\mathbf{R}Z_{s_1} \cap \dots \cap \mathbf{R}Z_{s_r} \cap B\big(x, \frac{R}{d}\big)\right) \geq 1$. Then it is shown that these sections can be perturbed quantitatively so that the event

$$\left\{\mathcal{N}_\Sigma\left(\mathbf{R}Z_{s_1} \cap \dots \cap \mathbf{R}Z_{s_r} \cap B_{\mathbf{R}X}\big(x, \frac{R}{d}\big)\right) \geq 1\right\}$$

happens with uniform positive probability.

Note that the scale $1/d$ appearing in Proposition 3.2.3 is crucial for us, as it allows us to have approximately d^n disjoint balls, and thus to obtain Theorem 1.5.5. This is the point where the "ad hoc" measure ν_d is used. It is worth mentioning that very recently, the technique I used to prove Theorem 1.5.5 (i.e., using the barrier method for a well-chosen probability measure) was employed by Lerario and Stecconi [LS24] to prove the existence of real algebraic hypersurfaces with "many" singularities.

The rest of the chapter is devoted to the explanation of the construction of the measure ν_d, to the reason why the scale $1/d$ occurs, and to the proof of Proposition 3.2.3. As we will see, eigenfunctions of the Laplacian of the round sphere will play a key role.

3.2.2 Construction of the measure ν_d

Suppose for simplicity that L is very ample and fix once for all a real algebraic embedding

$$\iota : (X, c_X) \hookrightarrow (\mathbf{P}^N, conj)$$

induced by L. By definition, we have $\iota^*\mathcal{O}(1) = L$ so that $\iota^*\mathcal{O}(d) = L^d$. Let $\mathbf{R}_d^{\text{hom}}[X_0, \dots, X_N]$ be the space of real homogeneous polynomials of degree d in $N+1$ variables. This space coincides with the space of global algebraic sections of the line bundle $\mathcal{O}(d)$ on $\mathbf{P}^N$ which are defined over $\mathbf{R}$. By construction, for any polynomial $P \in \mathbf{R}_d^{\text{hom}}[X_0, \dots, X_N]$, its restriction $P_{|X}$ is a real algebraic section of L^d. The probability measure ν_d on $\mathbf{R}H^0(X, L^d)$ we aim to construct will be the image, under the restriction map, of a probability measure on $\mathbf{R}_d^{\text{hom}}[X_0, \dots, X_N]$, which we will continue to denote by ν_d, with a slight abuse of notation. The construction is as follows. Consider $\mathbf{S}^N$ as the double covering of $\mathbf{RP}^N$. We fix on $\mathbf{S}^N$ the standard round metric $g_{\mathbf{S}^N}$. We define an $\mathscr{L}^2$-scalar product on $\mathbf{R}_d^{\text{hom}}[X_0, \dots, X_N]$ by

$$\langle P_1, P_2 \rangle_{\mathscr{L}^2} = \int_{\mathbf{S}^N} P_1 P_2 \mathrm{dvol}_{\mathbf{S}^N}. \tag{3.2}$$

This scalar product induces a Gaussian measure ν_d on $\mathbf{R}_d^{\text{hom}}[X_0, \dots, X_N]$ defined by

$$\nu_d(U) = \frac{1}{\sqrt{\pi}^{N_d}} \int_U e^{-\|P\|^2_{\mathscr{L}^2}} \mathrm{d}P \tag{3.3}$$

for any open set $U \subset \mathbf{R}_d^{\text{hom}}[X_0, \dots, X_N]$, where $N_d = \dim \mathbf{R}_d^{hom}[X_0, \dots, X_N]$.

Now, let $M \subset \mathbf{S}^N$ be the induced double covering of $\mathbf{R}X$. For any r-tuple of real polynomials $\underline{P} = (P_1, \dots, P_r) \in \mathbf{R}_d^{hom}[X_0, \dots, X_N]^r$ we denote by $\mathbf{R}V_{\underline{P}}$ the zero locus in $\mathbf{S}^N$ defined by $\{P_1 = 0\} \cap \dots \cap \{P_r = 0\}$ and by $\mathbf{R}Z_{\underline{P}}$ the complete intersection in M defined by $\mathbf{R}V_{\underline{P}} \cap M$. Fix any point $x \in M$. To prove Proposition 3.2.3, it is sufficient to show that

$$\nu_d\left\{\underline{P} \in \mathbf{R}_d^{\text{hom}}[X_0, \dots, X_N]^r, \mathcal{N}_\Sigma\left(\mathbf{R}Z_{\underline{P}} \cap B_M\big(x, \frac{R}{d}\big)\right) \geq 1\right\} \geq c_\Sigma. \tag{3.4}$$

where the probability measure ν_d is defined in (3.3). This will be shown in Section 3.2.5.

Remark 3.2.5. The scalar product (3.2) could also have been defined as follows. Let h_d be the Fubini-Study metric on the line bundle $\mathcal{O}(d)$ and let g_{FS} be the Riemannian Fubini-Study metric on $\mathbf{RP}^N$. Then, for any pair of real polynomials $P_1, P_2 \in \mathbf{R}_d^{hom}[X_0, \dots, X_N]$ we consider the product

$$\langle P_1, P_2 \rangle_{\mathscr{L}^2} = \int_{\mathbf{RP}^N} h_d(P_1, P_2) \mathrm{dvol}_{\mathrm{FS}}. \tag{3.5}$$

Up to a constant the scalar products (3.5) and (3.2) are the same.

The difference between the scalar product (3.5) and the one defined in Section 1.3 and considered in the rest of the manuscript is that the scalar product (3.5) is defined by integration on the real locus $\mathbf{RP}^n$ instead of the complex locus.

3.2.3 Eigenfunctions of the Laplacian and the spectral function

Let (M, g) be a n-dimensional closed Riemannian manifold. Recall that the spectrum of the Laplace–Beltrami operator $\Delta_{(M,g)} : \mathscr{L}^2(M) \to \mathscr{L}^2(M)$ is a discrete and unbounded subset of the set $\mathbf{R}_{\geq 0}$ of positive real numbers. Let us denote by E_λ the eigenspace associated with the eigenvalue λ. Weyl's law [Wey11] computes the asymptotic number of eigenvalues (counted with multiplicities) that are less than or equal to a given positive real number λ, as $\lambda \to \infty$. This asymptotics reads as follows: let us denote by $N(\lambda) = \sum_{\mu \leq \lambda} \dim E_\mu$, then

$$N(\lambda) = c_n \mathrm{Vol}(M) \lambda^{n/2}(1 + o(1))$$

as $\lambda \to \infty$. Here $c_n = \frac{\mathrm{Vol}(B_{\mathbf{R}^n}(0,1))}{(2\pi)^n}$, in particular it is a universal constant depending only on the dimension.

Let $f_{\mu,1}, \dots, f_{\mu, \dim E_\mu}$ be an $\mathscr{L}^2$-orthonormal basis of E_μ. Denote by $f_\mu : M \times M \to \mathbf{R}$ the function defined by $f_\mu(x, y) = \sum_{i=1}^{\dim E_\mu} f_{\mu,i}(x) f_{\mu,i}(y)$. The Hörmander spectral function is the function $K_\lambda : M \times M \to \mathbf{R}$ defined by $K_\lambda = \sum_{\mu \leq \lambda} f_\mu$, that is, the kernel of the projector operator onto $\bigoplus_{\mu \leq \lambda} E_\mu$.

Remark 3.2.6. By construction, one has $\int_{x \in M} K_\lambda(x, x) \mathrm{dvol} = N(\lambda)$. In particular, Hörmander spectral function asymptotics will be a refinement of Weyl's law.

Before stating the asymptotic estimate of K_λ, let us fix the following useful and standard notation: for any $v \in \mathbf{R}^n$ (with small norm) and any $x \in M$, denote by $x + v \in M$ the points $\exp_x(A(v))$, where A is an isometry from $\mathbf{R}^n$ to T_xM and $\exp_x : T_xM \to M$ denotes the exponential map.

Theorem 3.2.7 (Hörmander [Hör68])**.** *For any fixed $R > 0$, any point $x \in M$ and any $u, v \in B_{\mathbf{R}^n}(0, R)$, the function*

$$K_{\lambda,x}(u, v) := \lambda^{-n/2} K_\lambda\left(x + \frac{u}{\sqrt{\lambda}}, x + \frac{v}{\sqrt{\lambda}}\right)$$

converges in the $\mathscr{C}^\infty$-topology as $\lambda \to \infty$ to a function $K : B_{\mathbf{R}^n}(0, R) \times B_{\mathbf{R}^n}(0, R) \to \mathbf{R}$ which is independent of x. The limit function K is explicit and given by

$$K(u, v) = \int_{B_{\mathbf{R}^n}(0,1)} e^{i\langle u - v, \xi \rangle} \mathrm{d}\xi.$$

In particular, the Hörmander spectral function is universal at the scale $1/\sqrt{\lambda}$.

Remark 3.2.8 (Spectral function vs. Bergman kernel)**.** The roles played by the parameter λ, Weyl's law, and Hörmander spectral function in Riemannian geometry should be compared respectively with those played by the parameter d of a tensor power L^d of a positive line bundle L, the asymptotic Riemann–Roch theorem, and the Bergman kernel associated with L in complex geometry. Recall that asymptotic Riemann–Roch theorem for a positive line bundle L over a n dimensional complex projective manifold X says that

$$\dim H^0(X, L^d) \sim_{d\to\infty} \frac{(c_1(L))^n}{n!} d^n = \mathrm{Vol}(X) d^n,$$

while Bergman kernel asymptotics says that at the scale $\frac{1}{\sqrt{d}}$ around each point $x \in X$, the Bergman kernel B_d associated with L^d has a universal limit function that only depends on the dimension, see [DLM06, MM07]. More precisely, if one consider normal coordinates around x and a well-chosen local trivialization of L, then $\frac{1}{\sqrt{d}^n} B_d(x + \frac{u}{\sqrt{d}}, x + \frac{v}{\sqrt{d}})$ converges in the $\mathscr{C}^\infty$–topology as $d \to \infty$ to the function $\exp\Big(-\frac{1}{2}|u-v|^2\Big)$.

3.2.4 Why is the scaling $1/d$?

Here we explain why the size of the ball considered in Proposition 3.2.3 is $1/d$. Let us specialize the discussion of the previous section to eigenfunctions and Hörmander's spectral function for the case of the round sphere $\mathbf{S}^N$. In this case, it is known that the spectrum of the Laplacian on the round sphere is the set $\{\ell(\ell+N-2)\}_{\ell\in\mathbf{N}}$. It turns out that the space $E_{\ell(\ell+N-2)}$ coincides with the space of functions on the sphere that can be written as $P_{|\mathbf{S}^N}$, where $P : \mathbf{R}^{N+1} \to \mathbf{R}$ is a harmonic homogeneous polynomial of degree ℓ, that is, $P \in \mathbf{R}^{\mathrm{hom}}_\ell[X_0, \dots, X_N]$ and $\Delta_{\mathbf{R}^N} P = 0$. These functions are the so-called spherical harmonics. Moreover any degree d homogenous polynomial $P \in \mathbf{R}^{\mathrm{hom}}_d[X_0, \dots, X_N]$ can be uniquely written as $P = P_d + \|X\|^2 P_{d-2} + \|X\|^4 P_{d-4} + \dots$, where $P_\ell \in E_{\ell(\ell+N-2)}$ is a degree ℓ homogeneous harmonic polynomial and $\|X\|^2 = X_0^2 + \cdots + X_N^2$. Remark that the function $\|X\|^2 = X_0^2 + \cdots + X_N^2$ equals 1 on the sphere, so that, for any $P \in \mathbf{R}^{\mathrm{hom}}_d[X_0, \dots, X_N]$, one has

$$P_{|\mathbf{S}^N} = \sum_{\substack{\ell\in\mathbf{N} \\ d-\ell\in 2\mathbf{N}}} P_\ell.$$

Thus, the restriction on the sphere of $\mathbf{R}^{\mathrm{hom}}_d[X_0, \dots, X_N]$ is precisely the direct sum of the space of eigenfunctions associated with all the eigenvalues less than or equal to $d(d+N-2)$ (in fact, one considers only the eigenvalues $\ell(\ell+N-2)$, where ℓ as the same parity of d, but this detail is not crucial). In particular, Theorem 3.2.7 states that the spectral function associated with the round sphere exhibits universality at the scale $\frac{1}{\sqrt{d(d+N-2)}} \sim \frac{1}{d}$, as $d \to \infty$.

3.2.5 Idea of the proof of Proposition 3.2.3

As already mentioned, the proof of Proposition 3.2.3 uses the so-called barrier method [NS09]. We give the idea of the proof in the case $r = 1$; the general case is done in the same way. Let us fix a point x in $M \subset \mathbf{S}^N$ (which we recall is the double cover of $\mathbf{R}X \subset \mathbf{RP}^n$) and consider the ball $B_M\big(x, \frac{R}{d}\big)$. First, let us denote by $F_d : B_{\mathbf{R}^n}(0, R) \to \mathbf{R}$ the random function obtained by the composition of a rescaling map from a fixed ball $B_{\mathbf{R}^n}(0, R)$ to $B_M\big(x, \frac{R}{d}\big)$ with a degree d random polynomial $P \in (\mathbf{R}^{\mathrm{hom}}_d[X_0, \dots, X_N], \nu_d)$. We prove that F_d converges in probability to a limit Gaussian function $F : B_{\mathbf{R}^n}(0, R) \to \mathbf{R}$, as $d \to \infty$, whose covariance kernel K is given by

$$K(u, v) = \int_{B_{\mathbf{R}^N}(0,1)} e^{i\langle u-v, \xi\rangle} \mathrm{d}\xi$$

for any $u, v \in B_{\mathbf{R}^n}(0, R)$. This follows from Theorem 3.2.7. Indeed, as seen in Sections 3.2.3 and 3.2.4, the covariance kernel of the random polynomial $P \in (\mathbf{R}_d^{\text{hom}}[X_0, \dots, X_N], \nu_d)$ coincides with the Hörmander spectral function associated with the round sphere, and the latter converges to K when restricted to $B_M\left(x, \frac{R}{d}\right)$, and after rescaling by a factor d, according to Theorem 3.2.7. Now, the convergence of the covariance kernels implies the convergence in probability of the Gaussian functions, see for example [LS19, Theorem 5].

Since we know that F_d converges in probability to F, to prove (3.4) (and hence Proposition 3.2.3), it is sufficient to show that the probability that the zero set of F is diffeomorphic to Σ is strictly positive. This is equivalent to showing that the *support* of F contains a function whose zero locus is diffeomorphic to Σ. By definition, the support of F is the set of functions $f \in \mathscr{C}^\infty(B_{\mathbf{R}^n}(0, R), \mathbf{R})$ such that for any neighborhood U of f in $\mathscr{C}^\infty(B_{\mathbf{R}^n}(0, R), \mathbf{R}^k)$, the probability that F lies in U is strictly positive. We actually show that the support of F contains all the smooth functions defined on the closure $\bar{B}_{\mathbf{R}^n}(0, R)$ of the ball. This uses the explicit universal limit function K of the Hörmander spectral function. □

Chapter 4

Volume of random submanifolds

The first part of this chapter (Sections 4.1.2-4.1.3) describes joint work with T. Letendre [AL23] and addresses two different but related problems. The first one is to define a natural notion of multijet for a $\mathscr{C}^k$ function on $\mathbf{R}^n$, generalizing the usual concept of a k-jet. By multijet we mean that we want to consider a collection of jets at different points in $\mathbf{R}^n$ and patch them together in a relevant way. The second one is to find natural conditions on a Gaussian field $f : M \to \mathbf{R}^r$ defined on an n-dimensional smooth Riemannian manifold ensuring that the $(n-r)$-dimensional volume of $f^{-1}(0)$ admits finite higher moments. One way to tackle this second problem is by considering the multijet of the random field f.

In the second part of the chapter (Section 4.2), we will present a recent work with L. Gass, T. Letendre, and M. Stecconi [AGLS25], in which we calculate the precise asymptotics of the moments of the volume of the zero set of a Gaussian field. In particular, in Section 4.2, we prove Theorem 1.5.6, which was introduced in the manuscript's introduction.

4.1 Multijets bundle and moments of the zeros of Gaussian fields

4.1.1 Gaussian fields

In this section, we briefly recall some notations and conventions concerning Gaussian vectors. We will mostly consider centered random vectors in finite-dimensional vector spaces, so we restrict ourselves to this setting. In the following, V is a finite-dimensional real vector space.

Definition 4.1.1 (Gaussian vector)**.** We say that a random vector X with values in V is a *centered Gaussian vector* if, for all $\eta \in V^*$, the real random variable $\eta(X)$ is a Gaussian in $\mathbf{R}$ with $\mathbf{E}(\eta(X)) = 0$.

In particular, a centered Gaussian vector in V has finite moments up to any order. Let us assume that V is endowed with a scalar product $\langle \cdot\,, \cdot \rangle$. Then for all $v \in V$, we set $v^* = \langle v\,, \cdot \rangle \in V^*$.

Definition 4.1.2 (Variance operator)**.** Let X be a centered Gaussian vector in $(V, \langle \cdot\,, \cdot \rangle)$, then its *variance operator* is the non-negative self-adjoint endomorphism $\mathrm{Var}X = \mathbf{E}(X \otimes X^*)$ of V. We say that X is *non-degenerate* if $\mathrm{Var}X$ is invertible.

Recall that a centered Gaussian vector in $(V, \langle \cdot\,, \cdot \rangle)$ is completely determined by its variance.

Definition 4.1.3 (Gaussian field and covariance kernel)**.** Let $E \to M$ be a vector bundle over some manifold M, we say that a random section s of $E \to M$ is a *centered Gaussian* *field* if for all $m \geq 1$ and all $x_1, \dots, x_m$ the random vector $(s(x_1), \dots, s(x_m))$ is a centered Gaussian. We say that this field is *non-degenerate* if $s(x)$ is non-degenerate for all $x \in M$.

The covariance kernel K of a Gaussian field s is the section of the bundle $E \boxtimes E^* \to M \times M$ defined by $K(x, y) = \mathbf{E}(s(x) \otimes s(y)^*)$.

The covariance kernel completely determines a centered Gaussian field.

4.1.2 Geometric issue for the computation of moments

In this section, we explain the geometric nature that lies behind the problem of the computation of the higher moments associated with a Gaussian field. In Sections 4.1.3 and 4.1.4 we will see how to solve this problem.

We place ourselves in a simple setting, but one that already includes all the difficulties. Let H be a finite-dimensional vector space of smooth functions from a closed Riemannian manifold M of dimension n to $\mathbf{R}^n$. For instance, one might consider M as the unit sphere, and H as the space of polynomial functions from M to $\mathbf{R}^n$ of degree at most d, or as the space of functions whose components are sums of eigenfunctions of the Laplacian, corresponding to a bounded set of eigenvalues. Let us endow H with a scalar product, which induces a Gaussian probability measure μ on H. A random function $f \in (H, \mu)$ is an example of Gaussian field. Under reasonable assumptions on H, the Gaussian function f almost-surely vanishes transversally, so that its zero set consists of a finite number of points. One would like to count this random set of points. The first quantity one would look at is then the expected number of such points. Such expected number can be computed by the so-called Kac–Rice formula, see [AW09]. We give a geometric interpretation of the Kac–Rice formula.

For $f \in H$ we denote by Z_f the zero set and by N_f the number of zeros of f. Then, by definition, $\mathbf{E}(N_f) = \int_H N_f \mathrm{d}\mu$, which can be rewritten as $\mathbf{E}(N_f) = \int_H \left(\sum_{x \in Z_f} 1\right) \mathrm{d}\mu$. The idea now is to "exchange" the sum with the integral. In order to do so, one defines the incidence manifold

$$\Sigma = \{(f, x) \in H \times M, f(x) = 0\}.$$

Assuming that for any $x \in M$ the evaluation map $\mathrm{ev}_x : f \in H \mapsto f(x) \in \mathbf{R}$ is non-zero, one can check that Σ is a smooth manifold. The incidence manifold is equipped with two natural projections, one being onto H, denoted by π_H, and the other one onto M, denoted by π_M. The fibers of π_H are generically finite, and one has $\pi_H^{-1}(f) = Z_f$. The fibers of π_M are vector spaces, and one has $\pi_M^{-1}(x) = \ker \mathrm{ev}_x$. The map π_H being generically finite, one has a well-defined measure $\pi_H^* \mu$ on Σ and, with respect to this measure, one has the equality

$$\mathbf{E}(N_f) = \int_H \left(\sum_{x \in Z_f} 1 \right) \mathrm{d}\mu = \int_\Sigma \pi_H^* \mathrm{d}\mu.$$

The Coarea Formula [Fed69, Theorem 3.2.3] allows to rewrite the right-hand side of the previous equality as an integral over the base M and over the fibers of π_M. More precisely one obtains

$$\int_\Sigma \pi_H^* \mathrm{d}\mu = \int_M \left(\int_{f \in \ker \mathrm{ev}_x} \frac{\left|\det^\perp \mathrm{d}_x f\right|}{\left|\det^\perp \mathrm{ev}_x\right|} \mathrm{d}\mu_{|\ker \mathrm{ev}_x} \right) \mathrm{d}x$$

where $\det^\perp$ denotes the normal Jacobian.

One can compute the quantity

$$\int_{f \in \ker \mathrm{ev}_x} \frac{\left|\det^\perp \mathrm{d}_x f\right|}{\left|\det^\perp \mathrm{ev}_x\right|} \mathrm{d}\mu_{|\ker \mathrm{ev}_x}$$

in terms of the covariance kernel of f. For example, in the case of polynomials, one can rewrite such a quantity in terms of the Bergman kernel along the diagonal, and then performs (asymptotic) computations using the known asymptotic of the Bergman kernel.

Let us now try to do the same trick for the computation of higher moments of N_f. By definition, $\mathbf{E}(N_f^p) = \int_H N_f^p \mathrm{d}\mu$, which can be rewritten as $\mathbf{E}(N_f) = \int_H \left(\sum_{(x_1,\dots,x_p)\in Z_f^p} 1\right) \mathrm{d}\mu$. One then defines the incidence manifold

$$\Sigma_p = \{(f, x_1, \cdots, x_p) \in H \times (M^p \setminus \Delta_p), f(x_1) = \cdots = f(x_p) = 0\},$$

where

$$\Delta_p = \{(x_1, \dots, x_p) \in M^p, \exists i \neq j \text{ such that } x_i = x_j\}$$

denotes the diagonal of M^p. Here is the first difference between expected value and higher moments (and here is the actual source of almost all the difficulties in computing higher moments): in order for Σ_p to be a manifold, one really had to remove the diagonal Δ_p from the definition of the incidence manifold Σ_p. Indeed, for a point on the diagonal $\underline{x} \in \Delta_p$ the evaluation map $\mathrm{ev}_{\underline{x}} : f \in H \mapsto (f(x_1), \dots, f(x_p)) \in \mathbf{R}^p$ cannot be surjective so that the equation $f(x_1) = \cdots = f(x_p) = 0$ does not cut transversally the incidence manifold at this point.

Still, the manifold Σ_p is equipped with the two natural projections as before and performing the same kind of computations as before, one obtains

$$\mathbf{E}(N_f^p) = \int_{\underline{x}\in M^p} \rho_p(\underline{x})\mathrm{d}\underline{x} \tag{4.1}$$

with

$$\rho_p(\underline{x}) := \int_{f\in\ker \mathrm{ev}_{\underline{x}}} \frac{\prod_{i=1}^p \left|\det^\perp \mathrm{d}_{x_i} f\right|}{\left|\det^\perp \mathrm{ev}_{\underline{x}}\right|} \mathrm{d}\mu_{|\ker \mathrm{ev}_{\underline{x}}}.$$

Now, the function $\underline{x} \in M^p \mapsto \det^\perp \mathrm{ev}_{\underline{x}}$ vanishes along the diagonal. This is equivalent to the fact that, along the diagonal, $\mathrm{ev}_{\underline{x}}$ is never surjective. Therefore, it is not clear that the function ρ_p is (locally) integrable on M^p. The problem we face is twofold: first, we need to prove that such a function is always integrable, which would guarantee the finiteness of higher-order moments; and second, we must compute the integral.

To address the first issue, our approach draws on techniques from algebraic geometry, specifically the resolution of singularities: if M is an algebraic variety and the function ρ_p is a rational function on M^p whose indeterminacy locus is the diagonal Δ_p, then one could replace M^p with a "larger" variety $C_p[M]$ obtained by a sequence of blow-ups of the diagonal. On this larger variety $C_p[M]$ the function ρ_p extends to a well-defined regular function, which is, in particular, continuous, hence locally integrable.

It is instructive to study the problem in the case $p = 2$. In this case, we can see that the compactification $C_2[M]$ is nothing but the blow-up of $M \times M$ along the diagonal. The idea is that when two points x_1 and x_2 are approaching each other, then in local coordinates, the equations $f(x_1) = 0$ and $f(x_2) = 0$ are equivalent to $f(x_1) = 0$ and $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$. When x_2 converges to x_1 along a direction $v \in T_{x_1}M$, then $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ converges to $\mathrm{d}_v f(x_1)$. One can define the 2-*multijets* $\mathrm{mj}_2(f, z)$ of f at a point z of $\mathrm{Bl}_{\Delta_2}(M \times M)$. Intuitively $\mathrm{mj}_p(f, z)$ is $\left(f(x_1), \frac{f(x_2)-f(x_1)}{x_2-x_1}\right)$ if $z = (x_1, x_2) \in M^2 \setminus \Delta$ and it is equal to $(f(x_1), \mathrm{d}_v f(x_1))$ if we are at a point $z = (x_1, v)$ on the exceptional divisor of $\mathrm{Bl}_{\Delta_2}(M \times M)$. The 2-multijets form a vector bundle $\mathcal{MJ}_2(M)$ over $C_2[M]$, which we call the multijets bundle.

We consider the following partial compactification of the incidence manifold:

$$\overline{\Sigma}_2 = \{(f, \underline{x}) \in H \times C_2[M], \mathrm{mj}_2(f, \underline{x}) = 0\}.$$

One can show that $\overline{\Sigma}_2$ is a smooth manifold that fibers over $C_2[M]$ and such that over $M^2 \setminus \Delta \subset C_2[M]$ equals Σ_p. Applying the coarea formula trick to the fibration $\overline{\Sigma}_2 \to C_2[M]$, one can directly show the finiteness of the second moment of N_f.

The main objective of this chapter is to explain how these ideas can be applied to establish the finiteness of the higher moments. In practice, the problem is local in nature, specifically concerning the local integrability of certain functions and the definition of multijets. More precisely, we need to consider the case where $M = \mathbf{R}^n$. Once this local problem is resolved, it is quite easy to extend the approach to address the more global problem of functions (or sections of vector bundles) defined on manifolds.

4.1.3 Moments: solution in dimension 1

In this section, we show how to prove finiteness of the moments in dimension 1. This result was established in [AL21b]. In this case, the solution is simpler compared to higher dimensions and lacks many of the associated complexities. However, challenges still remain in one dimension, and the solution to this problem provides valuable insights that pave the way for addressing the higher-dimensional case.

Let $f : \mathbf{R} \to \mathbf{R}$ be smooth centered Gaussian field. Recall that centered means that $\mathbf{E}(f(x)) = 0$ for any $x \in \mathbf{R}$. Let $p \in \mathbf{N}$. Suppose that for any $m \in \mathbf{N}$ and any $k_1, \ldots, k_m \in \mathbf{N}$ with $\sum_{i=1}^m (k_i + 1) = p$ the vector

$$(f(x_1), f'(x_1), \ldots, f^{(k_1)}, \ldots, f(x_m), f'(x_m), \ldots, f^{(k_m)})$$

is a non-degenerate Gaussian. Remark that this non-degeneracy condition can be translated as follows. Let $(f_i)_{i\in\mathbf{N}}$ be a family of smooth functions, so that $f = \sum_{i\in\mathbf{N}} a_i f_i$, where a_i are i.i.d. Gaussian variables $\mathcal{N}(0,1)$. Let H be $\mathscr{C}^\infty$-closure of the space of functions generated by $(f_i)_{i\in\mathbf{N}}$. Then, the previous non-degeneracy conditions is equivalent to the surjectivity of the evaluation maps

$$g \in H \mapsto (g(x_1), g'(x_1), \ldots, g^{(k_1)}, \ldots, g(x_m), g'(x_m), \ldots, g^{(k_m)})$$

for any $m \in \mathbf{N}$ and any $k_1, \ldots, k_m \in \mathbf{N}$ with $\sum_{i=1}^m (k_i + 1) = p$ and any pairwise distinct $x_1, \ldots, x_m \in \mathbf{R}$.

Denote by $\eta = \sum_{x\in f^{-1}(0)} \delta_x$ the random Radon measure associated with the zero set of f. For any test function ϕ (i.e. smooth function with compact support) we have

$$\langle \eta, \phi \rangle = \sum_{x \in f^{-1}(0)} \phi(x).$$

For any $\underline{x} = (x_1, \ldots, x_p) \in \mathbf{R}^p$ let $\phi^{\otimes p}(\underline{x}) = \prod_{i=1}^p \phi(x_i)$. Similarly, let $\eta^{\otimes p} = \sum_{\underline{x}\in f^{-1}(0)^p} \delta_{\underline{x}}$ be the random Radon measure on $\mathbf{R}^p$ so that

$$\mathbf{E}(\langle \eta^{\otimes p}, \phi^{\otimes p} \rangle) = \mathbf{E}(\langle \eta, \phi \rangle^p).$$

We also denote by $\eta^{[p]} = \sum_{\underline{x}\in f^{-1}(0)^p \setminus \Delta_p} \delta_{\underline{x}}$. Moments of order p of the linear statistics $\langle \eta, \phi \rangle$ can be easily expressed in terms of $\left\{\mathbf{E}(\eta^{[k]})\right\}_{1\le k\le p}$. The latter are called factorial moments. The advantage to consider the factorial moments is that these have an analytical expression given by Kac–Rice formula. More precisely, for any p, we have the equality $\mathbf{E}(\eta^{[p]}) = \rho_p(\underline{x})\mathrm{d}\underline{x}$ as measures on $\mathbf{R}^p$, where

$$\rho_p(\underline{x}) = \frac{\mathbf{E}(\prod_{i=1}^p |f'(x_i)| \mid \forall i f(x_i) = 0)}{\det \operatorname{Var}(f(x_1, \ldots, f(x_p)))^{1/2}}, \tag{4.2}$$

where $\mathbf{E}(\cdot \mid \cdot)$ stands for the conditional expectation. In the literature, the function ρ_p is called the *p-point function* or *Kac–Rice density*. The general idea to compute moments in [AL21a, AL21b] is the following:

1. Local integrability of the maps ρ_k for any $k \in \{1, \dots, p\}$. For this, one needs to understand the singularity of ρ_k along the diagonal.

2. Clustering property for the maps ρ_k. This informally means that if $(\underline{x}, \underline{y}) \in \mathbf{R}^{a+b}$ are such that $\min_{i,j} \operatorname{dist}(x_i, y_j)$ is large, then $\rho_{a+b}(\underline{x}, \underline{y}) = \rho_a(\underline{x})\rho_b(\underline{y}) + \text{error}$, where the error term becomes smaller and smaller as the distance between the points of $\underline{x}$ and those of $\underline{y}$ increases.

3. Combinatorics: write the p-th central moment as the integral of a polynomial in $(\rho_k)_{1\le k\le p}$ and compute.

In the following discussion, we focus on the first point (the two other points will be discussed in Section 4.2). At first glance, the function ρ_p is not defined on the diagonal. We provide a conceptual proof that, in fact, the function ρ_p is continuous everywhere. Let us consider $F(\underline{x}) = (f(x_1), \dots, f(x_p)) = \operatorname{ev}_{\underline{x}}(f)$, which, as a reminder, is non-degenerate on $\mathbf{R}^p \setminus \Delta_p$ but which is degenerate on the diagonal. With this notation, we have

$$\rho_p(\underline{x}) = \frac{\mathbf{E}(\det \mathrm{d}F(\underline{x}) \mid F(\underline{x}) = 0)}{\det \operatorname{Var}(F(\underline{x}))^{1/2}}.$$

Formally, we can replace F by any smooth Gaussian field G such that the equality

$$F^{-1}(0) \setminus \Delta_p = G^{-1}(0) \setminus \Delta_p$$

holds in distribution, and we will obtain

$$\rho_p(\underline{x}) = \frac{\mathbf{E}(\det \mathrm{d}G(\underline{x}) \mid G(\underline{x}) = 0)}{\det \operatorname{Var}(G(\underline{x}))^{1/2}}.$$

The idea is to find G which is non-degenerate on $\mathbf{R}^p$: this automatically implies that ρ_p is continuous on M^p. In order to do this, we use Hermite interpolation: there exists a unique degree $p-1$ polynomial $P \in \mathbf{R}_{p-1}[X]$ such that

$$(P^{(k)}(x_i))_{\substack{1\le i\le m\\ 0\le k\le k_i}} = (f^{(k)}(x_i))_{\substack{1\le i\le m\\ 0\le k\le k_i}}.$$

We denote this polynomial by $j(f, \underline{x})$. Note that $\underline{x} \mapsto j(f, \underline{x})$ is a Gaussian field taking values in $\mathbf{R}_{p-1}[X]$, since the map $f \mapsto j(f, \underline{x})$ is linear. One can also check that it is smooth. Moreover, by construction, for any $\underline{x} \in \mathbf{R}^p \setminus \Delta_p$ we have

$$F(\underline{x}) = 0 \iff j(F, \underline{x}) = 0.$$

Finally, $j(f, \cdot)$ is non-degenerate on the whole $\mathbf{R}^p$. Thus, $j(f, \cdot)$ is the Gaussian field G we were looking for. We have therefore shown that ρ_p is continuous everywhere, which proves the finiteness of the pth-moment.

4.1.4 Multijets bundle and a compactification of the configuration space

In order to deal with the moments in higher dimension, one would like to perform the same construction as in dimension 1, replacing $\mathbf{R}$ with $\mathbf{R}^n$. The first problem that appears is that polynomial interpolation is rather ill-behaved in $\mathbf{R}^n$ Also, one can show that $\rho_2(x, y)$ behaves like $\|x - y\|^{2-n}$ as $y \to x$. Thus, in general, we can then hope that ρ_p is $\mathscr{L}^1_{\mathrm{loc}}$ but not $\mathscr{C}^0$. The first problem will be addressed by considering Kergin interpolation and constructing multijet bundles. The second problem is solved by considering a partial compatification $C_p[\mathbf{R}^n]$ of $(\mathbf{R}^n)^p \setminus \Delta_p$, where Δ_p denotes the diagonal of $(\mathbf{R}^n)^p$. This is the core of the section.

Let us start by recalling Kergin interpolation [Ker80], which is a multivariate polynomial interpolation suited to our needs. For more background on polynomial interpolation in $\mathbf{R}^n$, we refer to the survey [Lor00].

Recall the notion of k-dimensional simplex:

$$\sigma_k = \{\underline{t} = (t_0, \dots, t_k) \in [0,1]^{k+1}, \sum_{i=0}^{k} t_i = 1\} \subset \mathbf{R}^{k+1}. \tag{4.3}$$

The simplex σ_k is a subset of $\{\underline{t} \in \mathbf{R}^{k+1} \mid \sum t_i = 1\}$, and we denote by ν_k the (k-dimensional) Lebesgue measure on this hyperplane, normalized so that $\nu_k(\sigma_k) = \frac{1}{k!}$.

For any $\underline{x} = (x_0, \dots, x_k) \in (\mathbf{R}^n)^{k+1}$, we denote by $\sigma(\underline{x})$ the convex hull of the x_i and we define $\upsilon_{\underline{x}} : \underline{t} \mapsto \sum_{i=0}^{k} t_i x_i$ from σ_k onto $\sigma(\underline{x})$.

Definition 4.1.4 (Divided differences)**.** Let $\underline{x} = (x_i)_{0 \le i \le k} \in (\mathbf{R}^n)^{k+1}$ and let f be a $\mathscr{C}^k$ function defined on some open neighborhood of $\sigma(\underline{x})$ in $\mathbf{R}^n$. We define the *divided difference* of f at $\underline{x}$ by:

$$f_{[x_0,\dots,x_k]} = \int_{\sigma_k} D^k_{\upsilon_{\underline{x}}(\underline{t})} f \,\mathrm{d}\nu_k(\underline{t}) \in \mathrm{Sym}^k(\mathbf{R}^n),$$

that is, as the average of $D^k f$ over $\sigma(\underline{x})$ with respect to the pushed-forward measure $(\upsilon_{\underline{x}})_*(\nu_k)$. Here, $\mathrm{Sym}^k(\mathbf{R}^n)$ denotes the space of symmetric k-linear forms on $\mathbf{R}^n$ and $D^k_x f \in \mathrm{Sym}^k(\mathbf{R}^n)$ stands the k-th differential of f at x.

Remark 4.1.5.

- If $\underline{x} = (x, \dots, x)$ for some $x \in \mathbf{R}^n$ then $f_{[x,\dots,x]} = \frac{1}{k!} D^k_x f$.
- When $n = 1$, Definition 4.1.4 coincides with the classical definition of divided differences, under the canonical isomorphism $\mathrm{Sym}^k(\mathbf{R}) \simeq \mathbf{R}$. This is known as the Hermite–Gennochi formula [MM80].

Proposition 4.1.6 (Kergin interpolation, see [MM80])**.** *Let $\underline{x} \in (\mathbf{R}^n)^p$ and let f be a function of class $\mathscr{C}^{p-1}$ defined on some neighborhood of $\sigma(\underline{x})$ in $\mathbf{R}^n$. There exists a unique polynomial $K(f, \underline{x}) \in \mathbf{R}_{p-1}[X_1, \dots, X_n]$ such that, for all non-empty subset $I \subset \{1, \dots, p\}$, we have*

$$f_{[\underline{x}_I]} = K(f, \underline{x})_{[\underline{x}_I]}.$$

Moreover,

$$K(f, \underline{x}) = \sum_{k=1}^{p} f_{[x_1,\dots,x_k]}(X - x_1, \dots, X - x_{k-1}), \tag{4.4}$$

where $X = (X_1, \dots, X_n)$.

Remark 4.1.7. In particular, Proposition 4.1.6 implies the following.

- If x appears with multiplicity at least $k+1$ in $\underline{x}$, then:

$$D^k_x f = k! f_{[\underbrace{x, \dots, x}_{k+1 \text{ times}}]} = k!(K(f, \underline{x}))_{[\underbrace{x, \dots, x}_{k+1 \text{ times}}]} = D^k_x(K(f, \underline{x})).$$

- The map $P \mapsto (P_{[x_1,\dots,x_j]})_{1 \le j \le p}$ is an isomorphism from the space $\mathbf{R}_{p-1}[X_1, \dots, X_n]$ to $\bigoplus_{j=0}^{p-1} \mathrm{Sym}^j(\mathbf{R}^n)$ whose inverse map is given by

$$(S_j)_{0 \le j \le p-1} \mapsto \sum_{j=0}^{p-1} S_j(X - x_1, \dots, X - x_j)$$

  where $X = (X_1, \dots, X_n)$.

Definition 4.1.8 (Kergin polynomial)**.** The polynomial $K(f, \underline{x})$ from Proposition 4.1.6 is called the *Kergin interpolating polynomial* of f at $\underline{x}$.

Example 4.1.9. If $n = 1$, then $K(f, \underline{x})$ is the Hermite interpolating polynomial of f at $\underline{x} \in \mathbf{R}^p$. If $\underline{x} = (x, \dots, x)$, then $K(f, \underline{x})$ is the Taylor polynomial of order $p-1$ of f at $x \in \mathbf{R}^n$.

We begin by defining the multijet of a function f for a point $\underline{x} \notin \Delta_p$. Intuitively, the multijet of f at $\underline{x} = (x_1, \dots, x_p) \notin \Delta_p$ is the list of value of f at $x_1, \dots, x_p$. As discussed in Section 4.1.3, in one dimension, the space where the multijets at $\underline{x}$ take their values was identified with the vector space $\mathbf{R}_{p-1}[X]$ through Hermite interpolation. In higher dimension, we are tempted to do the same using Kergin interpolation. However, the space $\mathbf{R}_{p-1}[X_1, \dots, X_n]$ is too large to be the space of multijets at $\underline{x}$. To circumvent this problem, remark that for $\underline{x} \notin \Delta_p$, the evaluation map $\mathrm{ev}_{\underline{x}} : \mathbf{R}_{p-1}[X_1, \dots, X_n] \to \mathbf{R}^p$ is surjective. Denote by $\mathcal{G}(\underline{x}) = \ker \mathrm{ev}_{\underline{x}} \in \mathrm{Gr}_p(\mathbf{R}_{p-1}[X_1, \dots, X_n])$ the kernel of the evaluation map at $\underline{x}$, which is a codimension p linear subspace of $\mathbf{R}_{p-1}[X_1, \dots, X_n]$. This defines a smooth map $\mathcal{G} : (\mathbf{R}^n)^p \setminus \Delta_p \to \mathrm{Gr}_p(\mathbf{R}_{p-1}[X_1, \dots, X_n])$ with the property that

$$(f(x_1), \dots, f(x_p)) = (g(x_1), \dots, g(x_p)) \iff K(f, \underline{x}) - K(g, \underline{x}) \in \mathcal{G}(\underline{x}).$$

The space of multijets at $\underline{x} \notin \Delta_p$ is defined as $\mathcal{MJ}_{\underline{x}} = \mathbf{R}_{p-1}[X_1, \dots, X_n]/\mathcal{G}(\underline{x})$ and the multijets of a function f at a point $\underline{x} \notin \Delta_p$ is the element of $\mathcal{MJ}_{\underline{x}}$ defined by $\mathrm{mj}_p(f, \underline{x}) = K(f, \underline{x}) \bmod \mathcal{G}(\underline{x})$.

The main difference compared to the one-dimensional case is that we cannot extend $\mathcal{G}$ as a continuous map on $(\mathbf{R}^n)^p$, as shown by the following example.

Example 4.1.10. For $n = 2 = p$, the Grassmannian $\mathrm{Gr}_p(\mathbf{R}_{p-1}[X_1, \dots, X_n])$ is the set of lines in $\mathbf{R}_1[X_1, X_2]$. Taking $x = R(\cos\theta, \sin\theta) \in \mathbf{R}^2 \setminus \{0\}$ the reader can check that $\mathcal{G}(0, x) = \mathrm{Span}(X_1 \sin\theta - X_2 \cos\theta)$ which does not converge as $R \to 0$. However, in this case, the map $x \in \mathbf{R}^2 \setminus \{0\} \mapsto \mathcal{G}(0, x)$ extends to the blow-up $\mathrm{Bl}_0\mathbf{R}^2$ of $\mathbf{R}^2$ at 0 and similarly $\mathcal{G}$ extends smoothly to $\mathrm{Bl}_{\Delta_2}(\mathbf{R}^2 \times \mathbf{R}^2)$.

The previous example suggests that, even though $\mathcal{G}$ does not extend smoothly to $(\mathbf{R}^n)^p$, it might extend to a larger space. We will construct a smooth manifold $C_p[\mathbf{R}^n]$ that contains $(\mathbf{R}^n)^p \setminus \Delta_p$ as a dense open subset, and to which $\mathcal{G}$ extends smoothly. In this way, the manifold $C_p[\mathbf{R}^n]$ will be equipped with a multijet bundle.

Such a larger manifold $C_p[\mathbf{R}^n]$ is constructed as follows. Consider the graph of the map $\mathcal{G}$:

$$(\mathbf{R}^n)^p \setminus \Delta_p \simeq \{(\underline{x}, \mathcal{G}(\underline{x})) \mid \underline{x} \in (\mathbf{R}^n)^p \setminus \Delta_p\} =: \Gamma.$$

The set Γ is a smooth algebraic subvariety of $((\mathbf{R}^n)^p \setminus \Delta_p) \times \mathrm{Gr}_p(\mathbf{R}_{p-1}[X_1, \dots, X_n]))$ which can be embedded in $(\mathbf{R}^n)^p \times \mathrm{Gr}_p(\mathbf{R}_{p-1}[X_1, \dots, X_n]))$. One can now remark that the map $\mathcal{G}$ extends to the closure $\overline{\Gamma} \subset (\mathbf{R}^n)^p \times \mathrm{Gr}_p(\mathbf{R}_{p-1}[X_1, \dots, X_n]))$ of Γ by projecting onto $\mathrm{Gr}_p(\mathbf{R}_{p-1}[X_1, \dots, X_n]))$. One can observe that $\overline{\Gamma}$ is not smooth as soon as $n \geq 2$ and $p \geq 3$. We define $C_p[\mathbf{R}^n]$ as a resolution of singularities of $\overline{\Sigma}$, see [Hir64]. Thus, $C_p[\mathbf{R}^n]$ is not uniquely defined; however, this is not an issue for applications involving the computation of moments. Any choice of $C_p[\mathbf{R}^n]$ constructed as described, will suffice to solve the problem. Over any such compactification $C_p[\mathbf{R}^n]$, one can define the multijets bundle as follows.

Definition 4.1.11 (Vector bundle of multijets)**.** Let $n \geq 1$ and $p \geq 1$, the *vector bundle of multijets of order p on $\mathbf{R}^n$* is the smooth vector bundle of rank p over $C_p[\mathbf{R}^n]$ defined by:

$$\mathcal{MJ}_p(\mathbf{R}^n) = (\mathbf{R}_{p-1}[X_1, \dots, X_n] \times C_p[\mathbf{R}^n])/\mathcal{G}.$$

Recalling the definition of Kergin polynomials given by 4.1.8, we can now define the p-multijet of a $\mathscr{C}^{p-1}$ function on $\mathbf{R}^n$.

Definition 4.1.12 (Multijet of a function)**.** Let $f \in \mathscr{C}^{p-1}(\mathbf{R}^n)$ and $z \in C_p[\mathbf{R}^n]$, the *multijet of f at z* is the element of $\mathcal{MJ}_p(\mathbf{R}^n)_z$ defined as:

$$\mathrm{mj}_p(f, z) = K(f, \pi(z)) \bmod \mathcal{G}(z),$$

where $\pi : C_p[\mathbf{R}^n] \to (\mathbf{R}^n)^p$ is the composition of the resolution map $C_p[\mathbf{R}^n] \to \overline{\Gamma}$ with the projection onto the first factor, given by the inclusion $\overline{\Gamma} \subset (\mathbf{R}^n)^p \times \mathrm{Gr}_p(\mathbf{R}_{p-1}[X_1, \ldots, X_n])$.

Thus, the p-multijet of any $\mathscr{C}^{p-1}$ function on $\mathbf{R}^n$ induces a continuous section of the multijets bundle $\mathcal{MJ}_p(\mathbf{R}^n) \to C_p[\mathbf{R}^n]$. We can summarize the construction we have done in this section by the following theorem.

Theorem 4.1.13 ([AL23])**.** *Let $n \geq 1$ and $p \geq 1$. There exist a smooth manifold $C_p[\mathbf{R}^n]$ of dimension np without boundary and a smooth vector bundle $\mathcal{MJ}_p(\mathbf{R}^n) \to C_p[\mathbf{R}^n]$ of rank p with the following properties.*

1. *There exists a smooth proper surjection $\pi : C_p[\mathbf{R}^n] \to (\mathbf{R}^n)^p$ such that $\pi^{-1}((\mathbf{R}^n)^p \setminus \Delta_p)$ is a dense open subset of $C_p[\mathbf{R}^n]$, and π restricted to $\pi^{-1}((\mathbf{R}^n)^p \setminus \Delta_p)$ is a diffeomorphism onto $(\mathbf{R}^n)^p \setminus \Delta_p$.*
2. *There exists a smooth map $\mathrm{mj}_p : \mathscr{C}^{p-1}(\mathbf{R}^n) \times C_p[\mathbf{R}^n] \to \mathcal{MJ}_p(\mathbf{R}^n)$ such that:*
   - *for all $z \in C_p[\mathbf{R}^n]$, the map $\mathrm{mj}_p(\cdot, z) : \mathscr{C}^{p-1}(\mathbf{R}^n) \to \mathcal{MJ}_p(\mathbf{R}^n)_z$ is linear and surjective;*
   - *for all $f \in \mathscr{C}^{l+p-1}(\mathbf{R}^n)$, the section $\mathrm{mj}_p(f, \cdot)$ of $\mathcal{MJ}_p(\mathbf{R}^n) \to C_p[\mathbf{R}^n]$ is $\mathscr{C}^l$.*
3. *Let $z \in C_p[\mathbf{R}^n]$ be such that $\pi(z) = (x_1, \ldots, x_p) \notin \Delta_p$ then for all $f \in \mathscr{C}^{p-1}(\mathbf{R}^n)$ we have:*
$$\mathrm{mj}_p(f, z) = 0 \iff \forall i \in \{1, \ldots, p\}, f(x_i) = 0.$$
4. *Let $z \in C_p[\mathbf{R}^n]$ be such that $\pi(z)$ is obtained as a permutation of $(y_1, \ldots, y_1, \ldots, y_m, \ldots, y_m)$ where y_j appears exactly $(k_j + 1)$ times and $y_1, \ldots, y_m$ are pairwise distinct vectors in $\mathbf{R}^n$. Then, there exists a linear surjection $\Theta_z : \prod_{i=1}^m \mathcal{J}_{k_j}(\mathbf{R}^n)_{y_j} \to \mathcal{MJ}_p(\mathbf{R}^n)_z$ such that*
$$\forall f \in \mathscr{C}^{p-1}(\mathbf{R}^n), \qquad \mathrm{mj}_p(f, z) = \Theta_z \left(\mathrm{jet}_{k_1}(f, y_1), \ldots, \mathrm{jet}_{k_m}(f, y_m)\right).$$
   Here, $\mathcal{J}_k(\mathbf{R}^n)_y$ denotes the space of k-jets at y.

Using exactly the same construction made in this section, multijets can be built on any smooth manifold, not just on $\mathbf{R}^n$. Also, the construction can be applied to sections of vector bundles of any rank, and not just for functions. A variation on Theorem 4.1.13 is to define holomorphic multijets for holomorphic maps, see [AL23, Theorem 8.2].

Remark 4.1.14. In Theorem 4.1.13, Condition 1 means that we can consider $(\mathbf{R}^n)^p \setminus \Delta_p$ as a dense open subset in $C_p[\mathbf{R}^n]$. Condition 2 are properties that we expect any reasonable notion of multijet to satisfy. Condition 3 means that, as in the previous discussions, if $\pi(z) \notin \Delta_p$ then $\mathcal{MJ}_p(\mathbf{R}^n)_z = \mathscr{C}^{p-1}(\mathbf{R}^n)/\sim$, where $f \sim g$ if and only if $f(x_i) = g(x_i)$ for all $i \in \{1, \ldots, p\}$. Condition 4 means that, more generally, $\mathrm{mj}_p(f, z)$ only depends on the collection of jets $\left(\mathrm{jet}_{k_j}(f, y_j)\right)_{1 \leq j \leq m}$. In particular, $\mathrm{mj}_p(f, z)$ still makes sense if f is only $\mathscr{C}^{k_j}$ on some neighborhood of y_j. This last condition also means that we can think of $\mathrm{mj}_p(f, z)$ intuitively as a family of p independent linear combinations of partial derivatives of f, up to order k_j at y_j.

Remark 4.1.15. The manifold $C_p[\mathbf{R}^n]$ is what is called in the literature a "partial compactification" of the configuration space $(\mathbf{R}^n)^p \setminus \Delta_p$. Partial compactification of configuration spaces are built to understand how a configuration (ordered or not) of p distinct points can degenerate as these points converge toward one another. They are usually obtained by blowing up various pieces of the diagonal. Points in the exceptional locus then correspond to singular configurations, with some extra data encoding along which paths regular configurations are allowed to degenerate in order to reach this singular configuration. The hope is that the extra data attached to singular configurations is enough to lift the singularities of the problem under consideration. The simplest example of this kind is the blow-up $\mathrm{Bl}_{\Delta_2}(\mathbf{R}^n)^2$ discussed above. More evolved examples are the space defined by Le Barz in [LB88], the compactification of Fulton–MacPherson [FM94] (see also [AS94]), Olver's multispace [Olv01], the polydiagonal compactification of Ulyanov [Uly02], the construction of Evain [Eva05] using Hilbert schemes, and many others.

In dimension $n = 1$, most of the partial compactifications of configuration spaces that we found in the literature coincide and can be used to define multijets, see for example [Anc21] where Olver's multispace is used. In higher dimension they are different and none of them exactly suited our needs for the problem of the computation of the moments. Thus to the best of our knowledge, the manifold $C_p[\mathbf{R}^n]$ in Theorem 4.1.13 is a new addition to the previous list.

4.1.5 Application of the multijets: finiteness of the moments

The goal of this section is to provide simple conditions on the field f that ensure the finiteness of the p-th moments of its linear statistics in any dimension and codimension. These conditions fall into two categories: we require the field to be sufficiently regular and to be non-degenerate in the following sense.

Definition 4.1.16 (p-non-degeneracy)**.** Let $p \geq 1$ and let $f : \Omega \to \mathbf{R}^r$ be a $\mathscr{C}^p$ centered Gaussian field. We say that the field f is *p-non-degenerate* if for all $x \in \Omega$ the centered Gaussian vector

$$(f(x), D_x f, \dots, D_x^p f) \in \bigoplus_{k=0}^{p} \mathrm{Sym}^k(\mathbf{R}^n) \otimes \mathbf{R}^r$$

is non-degenerate, where $\mathrm{Sym}^k(\mathbf{R}^n)$ denotes the space of symmetric k-linear forms on $\mathbf{R}^n$ and $D_x^k f \in \mathrm{Sym}^k(\mathbf{R}^n) \otimes \mathbf{R}^r$ stands the k-th differential of f at x.

Recall that, given $f : \Omega \to \mathbf{R}^r$ a non-degenerate Gaussian field, its zero set Z is almost-surely $(n-r)$-rectifiable. As such, it admits a well-defined $(n-r)$-dimensional volume measure dvol_Z induced by the Euclidean metric on $\mathbf{R}^n$. We denote by η the random Radon measure on Ω defined by:

$$\langle \eta, \phi \rangle = \int_Z \phi \, \mathrm{dvol}_Z \tag{4.5}$$

for any continuous function ϕ on Ω with compact support.

Theorem 4.1.17 ([AL23])**.** *Let $\Omega \subset \mathbf{R}^n$ be open, let $f : \Omega \to \mathbf{R}^r$ be a centered Gaussian field and let η be defined as in Equation (4.5). If f is $\mathscr{C}^p$ and $(p-1)$-non-degenerate then*

$$\mathbf{E}(|\langle \eta, \phi \rangle|^p) < +\infty$$

for all $\phi \in \mathscr{L}^\infty(\Omega)$ with compact support.

Sketch of the proof. The idea is as in dimension 1, and it was also sketched in Section 4.1.2. The Kac–Rice formula for the p-moment associated with the zero set of f is singular along the diagonal of $(\mathbf{R}^n)^p$ since the Gaussian field

$$F : (x_1, \dots, x_p) \mapsto (f(x_1), \dots, f(x_p))$$

is degenerate on the diagonal. We replace the Gaussian field F with $\mathrm{mj}_p(f,\cdot)$. The latter has the property that its zeros set in $(\mathbf{R}^n)^p \setminus \Delta_p$ coincides with the zero set of F on $(\mathbf{R}^n)^p \setminus \Delta_p$, but it has the advantage that it extends to a non-degenerate Gaussian field over $C_p[\mathbf{R}^n]$. By applying the Kac–Rice formula to $\mathrm{mj}_p(f,\cdot)$ we obtain the result. □

Example 4.1.18. Let us give some examples of fields satisfying the assumptions of Theorem 4.1.17.

- The Bargmann–Fock field, i.e. the smooth Gaussian field on $\mathbf{R}^n$ whose covariance kernel is $(x,y) \mapsto e^{-\frac{\|x-y\|^2}{2}}$, satisfies the hypotheses of Theorem 4.1.17.
- Let $f : \mathbf{R}^n \to \mathbf{R}$ be a stationary $\mathscr{C}^p$ centered Gaussian fields. If the support of its spectral measure has non-empty interior then f is $(p-1)$-non-degenerate.
- If $(f_i)_{1\leq i\leq r}$ are r independent $(p-1)$-non-degenerate $\mathscr{C}^p$ Gaussian fields then so is $f = (f_1, \dots, f_r)$.
- The Berry field, i.e. the smooth stationary Gaussian field f on $\mathbf{R}^n$ whose spectral measure is the uniform measure on $\mathbf{S}^{n-1}$, is 1-non-degenerate but not 2-non-degenerate. Indeed it almost surely satisfies $\Delta f + f = 0$, so that $(f(x), D_x f, D_x^2 f)$ is degenerate for all $x \in \mathbf{R}^n$.

We can consider the same question in a more geometric setting. Let (M,g) be a Riemannian manifold of dimension $n \geq 1$ without boundary and let $E \to M$ be a smooth vector bundle of rank $r \in \{1,\dots,n\}$ over M. Let s be a centered Gaussian field on M with values in E, in the sense that s is a random section of $E \to M$ such that for all $m \geq 1$ and all $x_1,\dots,x_m \in M$ the random vector $(s(x_1),\dots,s(x_m))$ is a centered Gaussian. We assume that s is almost surely $\mathscr{C}^1$ and that $\det \mathrm{Var}(s(x)) > 0$ for all $x \in M$. As in the Euclidean setting, $Z = s^{-1}(0)$ is almost surely $(n-r)$-rectifiable. As before, we denote by η the random Radon measure on M defined by integrating over Z with respect to the $(n-r)$-dimensional volume measure dvol_Z induced by g. For all $\phi \in \mathscr{L}^\infty(M)$ with compact support, we define the linear statistic $\langle \eta, \phi\rangle$ as in Equation (4.5). Remark that, if the centered Gaussian field s is $\mathscr{C}^p$, then its p-jet $\mathrm{jet}_p(s,x)$ is a centered Gaussian for all $x \in M$. We say that s is *p-non-degenerate* if, for all $x \in M$, the centered Gaussian vector $\mathrm{jet}_p(s,x) \in \mathscr{J}_p(M,E)_x$ is non-degenerate.

By using multijets spaces associated with M, the same proof as in Theorem 4.1.17 yields the following theorem.

Theorem 4.1.19 ([AL23])**.** *Let $p \geq 1$, let s be a centered Gaussian field on M with values in E and let η be defined as in Equation (4.5). If s is $\mathscr{C}^p$ and $(p-1)$-non-degenerate then $\mathbf{E}(|\langle \eta, \phi\rangle|^p) < +\infty$ for all $\phi \in \mathscr{L}^\infty(M)$ with compact support.*

Very similar arguments can show the finiteness of moments for other quantities related to a Gaussian field. The following theorem, for example, shows the finiteness in the case of critical points.

Theorem 4.1.20 (Finiteness of moments for critical points)**.** *Let M be a smooth manifold without boundary. Let $f : M \to \mathbf{R}$ be a centered Gaussian field and let ν_D denote the counting measure of its critical locus. Let $p \geq 1$, we assume that f is $\mathscr{C}^{2p}$ and $(2p-1)$-non-degenerate. Then, for all $\phi \in \mathscr{L}_c^\infty(M)$, we have $\mathbf{E}|\langle \nu_D, \phi\rangle|^p) < +\infty$.*

The finiteness results presented in this section were proved independently and with different proofs by L. Gass and M. Stecconi [GS24]. Their idea is to compare the Kac–Rice densities ρ_p of the field f with those of a well-chosen Gaussian polynomial P. Then they deduce the finiteness results of the moments associated with f from the ones associated with

P, the latter being a consequence of Bézout's Theorem. Our proof follows a different path, as it relies on the multijet bundle that we defined in Theorem 4.1.13. As sketched in the proof of Theorem 4.1.17, our idea is to observe the zero set of $F : (x_1, \dots, x_p) \mapsto (f(x_1), \dots, f(x_p))$ in the configuration space $\Omega^p \setminus \Delta_p$ is exactly the vanishing locus of the multijet $\mathrm{mj}_p(f, \cdot)$ restricted to $\Omega^p \setminus \Delta_p \subset C_p[\Omega]$. Instead of working with F which degenerates along Δ_p, we work with the field $\mathrm{mj}_p(f, \cdot)$ that we built to be non-degenerate everywhere. Then, we deduce Theorem 4.1.17 from the Kac–Rice Formula applied to the p-multijet of f and a compactness argument.

Theorem 4.1.17 for $r = 1$ was proved in [AAD+23] by different methods. In particular, their method does not seem to apply for $r > 1$ and for Theorem 4.1.20.

4.2 From finiteness to computation

Let $f : \mathbf{R}^n \to \mathbf{R}^k$ be a stationary centered Gaussian field. By stationary, we mean that, for all $t \in \mathbf{R}^n$, the process $x \to f(x + t)$ is distributed as f.

A centered Gaussian field is completely characterized by its covariance kernel $\mathbf{E}(f(x)f(y))$. The stationarity condition implies that the covariance kernel is characterized by the covariance function $\kappa(x) = \mathbf{E}(f(0)f(x))$. Let us denote by $\mathbf{B}_R$ the ball of radius R centered in 0 in $\mathbf{R}^n$.

Theorem 4.2.1 ([AGLS25]). *Assume that f is of class $\mathscr{C}^p$, is $(p-1)$-non-degenerate and the first $2p$ derivatives of its covariance function are in $\mathscr{L}^2(\mathbf{R}^n)$. Then, there exists a positive constant $\sigma > 0$ such that*

$$m_p(\mathrm{Vol}(Z_f \cap \mathbf{B}_R)) = \sigma^p w_p R^{\frac{pn}{2}}(1 + o(1))$$

as $R \to \infty$. Here, m_p denotes the p-th centered moment, and $w_p = m_p(\mathcal{N}(0, 1))$ is the p-th centered moment of the standard Gaussian variable.

Sketch of the proof. By Kac–Rice formula, for any $k \in \{1, \dots, p\}$, we can write

$$\mathbf{E}(\mathrm{Vol}(Z_f \cap \mathbf{B}_R)^k) = \int_{\mathbf{B}_R^k} \rho_k(\underline{x})\mathrm{dvol}$$

where $\rho_k(\underline{x})$ has the analogue expression as in Equation (4.2), replacing the derivatives with the differential. The function ρ_k is called the k-point function (or the Kac–Rice density) associated with f. The fundamental part in the proof of the theorem is a clustering property for the ρ_k. This informally means that if $(\underline{x}, \underline{y}) \in M^{a+b}$ are such that $\min_{i,j} \mathrm{dist}(x_i, y_j)$ is large, then $\rho_{a+b}(\underline{x}, \underline{y}) = \rho_a(\underline{x})\rho_b(\underline{y}) + \text{error}$, where the error term becomes smaller and smaller as the distance between the points of $\underline{x}$ and those of $\underline{y}$ increases. I will not prove the clustering formula here, since the proof involves many estimates. The vague idea is as follows: one has to express the function ρ_k in terms of the covariance function κ and to use the assumptions about κ to prove the clustering property (it is precisely at this point that the stationarity and decay assumptions of κ are used). Also, it is worth emphasizing that during the proof of such clustering property, we use the result of moment finiteness given by the Theorem 4.1.17.

Once one has the clustering property of the function ρ_k, the proof of the theorem follows the ideas of [AL21a, AL21b]. It is as follows. Write

$$m_p(\mathrm{Vol}(Z_f \cap \mathbf{B}_R)) = \sum_{k=0}^{p} \binom{p}{k} (-1)^k \mathbf{E}(\mathrm{Vol}(Z_f \cap \mathbf{B}_R)^k)\mathbf{E}(\mathrm{Vol}(Z_f \cap \mathbf{B}_R))^{p-k}. \tag{4.6}$$

All the terms $\mathbf{E}(\mathrm{Vol}(Z_f)^k)$ can be written as integrals over $(\mathbf{B}_R)^k$ of the k-point function ρ_k. Thus,

$$m_p(\mathrm{Vol}(Z_f \cap \mathbf{B}_R)) = \int_{\underline{x} \in (\mathbf{B}_R)^p} \lambda_p(\underline{x})\mathrm{dvol}$$

where the density λ_p is a polynomial in the k-point function $(\rho_k)_{1\leq k\leq p}$.

In order to understand the integral of λ_p, we split $(\mathbf{B}_R)^p$ as follows. For any point $\underline{x} = (x_1, \ldots, x_p) \in (\mathbf{B}_R)^p$, we define a graph whose vertices are the integers $\{1, \ldots, p\}$, with an edge between i and j if and only if $i \neq j$ and the distance from x_i to x_j is small (in a precise quantitative way, which depends on R and κ and related to the clustering property). The connected components of this graph yield a partition $\mathcal{I}(\underline{x}) \in \mathcal{P}_p$ (where $\mathcal{P}_p$ is the set of partitions of $\{1, \ldots, p\}$) encoding how the $(x_i)_{1\leq i\leq p}$ are clustered in $\mathbf{B}_R \subset \mathbf{R}^p$. Denoting by $A_R(\mathcal{I}) = \{\underline{x} \in (\mathbf{B}_R)^p \mid \mathcal{I}(\underline{x}) = \mathcal{I}\}$, we have:

$$m_p(\mathrm{Vol}(Z_f \cap \mathbf{B}_R)) = \sum_{\mathcal{I}\in\mathcal{P}_p} \int_{A_R(\mathcal{I})} \lambda_p \mathrm{dvol}.$$

Denoting by $|\mathcal{I}|$ the cardinality of $\mathcal{I}$ (i.e. the number of clusters), one shows that the integral of λ_p over $A_R(\mathcal{I})$ is bounded by $O(R^{n|\mathcal{I}|})$.

The key point now, which uses the clustering property, is to show that if $\mathcal{I}$ contains a singleton, then

$$\int_{A_R(\mathcal{I})} \lambda_p \mathrm{dvol}$$

is significantly smaller than $O(R^{n|\mathcal{I}|})$, and in fact, it becomes negligible in the asymptotic behavior of $m_p(\mathrm{Vol}(Z_f \cap \mathbf{B}_R))$.

Thus, the main contribution in $m_p(\mathrm{Vol}(Z \cap \mathbf{B}_R))$ comes from the integral of λ_p over the sets $A_R(\mathcal{I})$ indexed by partitions $\mathcal{I} \in \mathcal{P}_p$ that does not contain singletons and for which $|\mathcal{I}|$ is large. If p is even, these correspond precisely to the partitions of $\{1, \ldots, p\}$ into pairs. For any such partition, the contribution can be computed explicitly using the clustering property. This contribution is shown to be asymptotically equivalent to $\sigma^p R^{\frac{pn}{2}}$ as R goes to infinity. Finally, recall that the p-th moment w_p of the standard Gaussian distribution appears as the cardinality of the set of partitions of $\{1, \ldots, p\}$ into pairs. The case p odd is treated similarly. □

By applying the method of moments (see [Bil95, Section 30]), we obtain the following Central Limit Theorem.

Corollary 4.2.2. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a smooth stationary centered Gaussian field. Suppose that f is p-non-degenerate for any $p \in \mathbf{N}$ and that the covariance function of f and its derivatives are in $\mathscr{L}^2(\mathbf{R}^n)$. Then,*

$$\frac{\mathrm{Vol}(Z_f \cap \mathbf{B}_R) - \mathbf{E}(\mathrm{Vol}(Z_f \cap \mathbf{B}_R)))}{\sigma R^{\frac{n}{2}}}$$

converges in distribution to $\mathcal{N}(0, 1)$ as $R \to \infty$.

Remark 4.2.3. The clustering property that we established in [AGLS25] and used to calculate the moments, although it can be seen as a technical point, is a very useful and potentially valuable tool for describing other quantities attached to a Gaussian field. For example, the clustering property that I proved in one dimension in [Anc21] and with T. Letendre in [AL21b] was crucially used respectively by R. Feng and D. Yao and by R. Feng, F. Götze and D. Yao to prove that the point process "smallest distance between the zeros of a Gaussian function" converges in probability towards a Poisson process [FY24, FGY24].

The same type of proof as in Theorem 4.2.1 yields Theorem 1.5.6 given in Section 1.5.2 (that is the computation of the volume of random real algebraic submanifolds) as well as the following theorem about the cardinality of the critical points $\mathrm{Crit}(f)$ of f.

Theorem 4.2.4 ([AGLS25])**.** *Assume that f is of class $\mathscr{C}^{p+1}$, is p-non-degenerate and the first $2p+2$ derivatives of its covariance function are in $\mathscr{L}^2(\mathbf{R}^n)$. Then, there exists a positive constant $\sigma > 0$ such that*

$$m_p(\mathrm{Card}(\mathrm{Crit}(f) \cap \mathbf{B}_R)) = \sigma^p w_p R^{\frac{pn}{2}}(1 + o(1))$$

as $R \to \infty$. Here, w_p is the p-th centered moment of the standard Gaussian variable $\mathcal{N}(0,1)$.

In particular, if f is smooth and the covariance function of f and its derivatives are in $\mathscr{L}^2(\mathbf{R}^n)$, then

$$\frac{\mathrm{Card}(\mathrm{Crit}(f) \cap \mathbf{B}_R) - \mathbf{E}(\mathrm{Card}(\mathrm{Crit}(f) \cap \mathbf{B}_R))}{\sigma R^{\frac{n}{2}}}$$

converges in distribution to $\mathcal{N}(0,1)$ as $R \to \infty$.

Chapter 5

Random sections under Toeplitz operators

In this section, we will describe a work in collaboration with Yohann Le Floch [ALF22].

The motivation for the work comes from the following inverse problem: given the action of a quantum observable on random quantum states, can one recover properties of the underlying classical observable? This question is part of the broader goal of studying the quantum footprints of classical observables, a topic that has been the subject of intense research in recent decades. Specifically, we are interested in the following inverse problem: If $f \in \mathscr{C}^\infty(X, \mathbf{R})$ is a classical observable on a phase space X and $T : \mathcal{H} \to \mathcal{H}$ is a quantum observable quantizing f, acting on a Hilbert space $\mathcal{H}$, which properties of f can be derived from the study of T? This type of inverse problems is often seen from a spectral point of view, as in the seminal article by M. Kac [Kac66] dealing with the spectrum of the Laplacian on a planar domain, and the numerous works that it inspired (see for instance the surveys [DH13]). We work in a semiclassical context, where the Hilbert spaces and quantum observables depend on a small parameter $\hbar$, and we are interested in the limit $\hbar \to 0$, known as the semiclassical limit. Inverse spectral problems in this setting have been extensively studied by various authors; see, for instance, the recent review [VN21] and the references therein. Here we propose another approach, based on the observation of the action of T on quantum states obtained as random combinations of pure states: from the observation of this action for a large number of realizations of the random state, can one infer some properties of f? In [ALF22], we answer this last question positively. More precisely, we are able to recover all the regular levels of f from the zeros of certain random holomorphic sections of (a large power of) a holomorphic line bundle over X.

Although our initial objective was based on the Berezin–Toeplitz quantization, the results we obtained fall naturally into the subject of the distribution of zeros of random sections. Trying to understand the distribution of the zeros of random polynomials of large degree is a classical topic, dating back to the 1930s [LO38, BP31] at least, and popularized by the seminal article by M. Kac [Kac43]. A higher dimensional and more geometric setting has been introduced and investigated by B. Shiffman and S. Zelditch [SZ99, BSZ00, SZ10]. They considered high powers of an ample line bundle over a projective manifolds, and studied the current defined by the zero locus of a random section of such line bundles, proving several equidistribution results. Since the seminal paper by B. Shiffman and S. Zelditch [SZ99], the study of the distribution of zeros of random sections has been intensively investigated. For example, we can mention the work of T.-C. Dinh and N. Sibony [DS06], where they introduced an alternative approach that provides an estimate of the speed of convergence of zeros to the equilibrium distribution, or the result by D. Coman and G. Marinescu [CM15], where they generalized the results of B. Shiffman and S. Zelditch to random sections of big line bundles. We recommend the recent surveys [BCHM18] and [SZ23] for further reading.

5.1 Framework

We work in the context of geometric quantization [Sou66, Kos73] and Berezin–Toeplitz operators [Ber75, BMS94, Cha03, MM12]. This means that the phase space X is a compact Kähler manifold and that the quantum observables are operators acting on spaces of holomorphic sections $H^0(X, L^d)$, where $L \to X$ is a positive line bundle; the semiclassical limit is $d \to +\infty$ (in this setting the small parameter $\hbar$ corresponds to d^{-1}). For each $d \in \mathbf{N}$, the space $H^0(X, L^d)$ is finite-dimensional and carries a natural $\mathscr{L}^2$ Hermitian product $\langle \cdot, \cdot \rangle_{\mathscr{L}^2}$ induced by the choice of a positively curved Hermitian metric h on L as described in Section 1.3. In turns, this induces a Gaussian measure μ_d defined by Equation (1.2).

5.1.1 Berezin–Toeplitz operators.

To any classical observable $f \in \mathscr{C}^\infty(X, \mathbf{R})$, one can naturally associate a sequence of operators $T_d(f) : H^0(X, L^d) \to H^0(X, L^d)$ as follows. Let $\mathscr{L}^2(X, L^d)$ be the Hilbert space obtained as the closure of $\mathscr{C}^\infty(X, L^d)$ with respect to $\langle \cdot, \cdot \rangle_{\mathscr{L}^2}$, and let

$$\Pi_d : \mathscr{L}^2(X, L^d) \to H^0(X, L^d)$$

be the orthogonal projector from this space to the space of holomorphic sections. Then

$$T_d(f) : s \in H^0(X, L^d) \mapsto \Pi_d(fs) \in H^0(X, L^d).$$

This is an instance of Berezin–Toeplitz operator with principal symbol f. More generally, Berezin–Toeplitz operators are operators of the form

$$T_d = \Pi_d f(\cdot, d) + R_d : H^0(X, L^d) \to H^0(X, L^d)$$

where $(f(\cdot, d))_{d \in \mathbf{N}}$ is a sequence of elements of $\mathscr{C}^\infty(X, \mathbf{R})$ with an asymptotic expansion of the form

$$f(\cdot, d) = f_0 + d^{-1} f_1 + d^{-2} f_2 + \dots$$

for the $\mathscr{C}^\infty$ topology, and the operator norm of R_d is a $O(d^{-N})$ for every $N \in \mathbf{N}$. The first term f_0 in the asymptotic expansion of $f(\cdot, d)$ is called the *principal symbol* of T_d.

5.1.2 Currents

In order to state our main results, we need to use the language of currents. These can be seen as the form-valued analogues of distributions. Let M be a smooth manifold of dimension n. Denote by $\mathscr{D}^q(M)$ the space of smooth q-forms with compact support. A current of degree p on M is an element of the topological dual $\mathscr{D}'_p(M) := \mathscr{D}^{n-p}(M)'$ of the space $\mathscr{D}^{n-p}(M)$. Similarly, if M is a complex manifold of complex dimension n, one can define a current of bidegree (p, q) as an element of the dual of the space of smooth $(n-p, n-q)$-forms. We say that a sequence $(\eta_d)_{d \in \mathbf{N}}$ of currents of degree p *converges weakly in the sense of currents* to a current η of degree p if and only if for every compactly supported $(n-p)$-form φ, the sequence $(\langle \eta_d, \varphi \rangle)_{d \in \mathbf{N}}$ converges to $\langle \eta, \varphi \rangle$.

Example 5.1.1. Here are the two major examples of currents that we will consider below.

1. A locally integrable differential form $\eta \in \Omega^p(M)$ defines a current by the formula

$$\langle \eta, \varphi \rangle = \int_M \varphi \wedge \eta$$

for any $\varphi \in \mathscr{D}^{n-p}(M)$.

2. The integration current η_Z associated with a smooth oriented submanifold Z of codimension p of M, defined by

$$\langle \eta_Z, \varphi \rangle = \int_Z \varphi$$

for any $\varphi \in \mathscr{D}^{n-p}(M)$. Note that if Z is complex submanifold of complex dimension p inside an n dimensional complex manifold, then the current of integration along Z is a (p,p)-current. Actually, in the complex case, even when Z is a closed analytic subset of M, the integration along the regular locus of Z still defines a current, see [Lel57].

The Kähler form ω as well as the integration current $\eta_s := \eta_{Z_s}$ along the zero divisor $Z_s = \{s = 0\}$ of any holomorphic section $s \in H^0(X, L^d)$ define currents of bidegree $(1,1)$ on X.

The expectation $\mathbf{E}(\eta_s)$ of the integration current associated with the zero locus Z_s of the random holomorphic section s is itself a $(1,1)$-current defined by

$$\langle \mathbf{E}(\eta_s), \varphi \rangle := \mathbf{E}[\langle \eta_s, \varphi \rangle] = \int_{s \in H^0(X,L^d)} \langle \eta_s, \varphi \rangle \mathrm{d}\mu_k(s).$$

Note that one can associate a cohomology class to the current η_s, and one has the equality between cohomology classes $[\omega] = \frac{1}{d}[\eta_s]$. This implies that $[\omega] = \frac{1}{d}[\mathbf{E}(\eta_s)]$. It is then natural to compare the currents ω and $\mathbf{E}(\eta_s)$. This was done by B. Shiffman and S. Zelditch in [SZ99].

Theorem 5.1.2 (Shiffman–Zelditch). *The current $\frac{1}{d}\mathbf{E}(\eta_s)$ converges weakly in the sense of currents to the Kähler form ω.*

For $SU(2)$-polynomials, this result was obtained by E. Bogomolny, O. Bohigas and P. Leboeuf [BBL96].

Remark 5.1.3. In fact, in [SZ99] the authors proved more precise results than the one stated in Theorem 5.1.2. For example, they showed that for almost every sequence $\{s_d\}_d$ of sections of L^d, the sequence $\frac{1}{d}\nu_{s_d}$ of normalized integration currents converges to ω.

In order to give an idea of the proof of Shiffman–Zelditch's result, let us recall the Kodaira embedding theorem and Tian's asymptotic isometry theorem [Bou90, Tia90, Zel98].

Let $e_1, \ldots, e_{N_d}$ be any orthonormal basis of $H^0(X, L^d)$. The Kodaira embedding theorem says that, for d large enough, the base locus $\cap_{s \in H^0(X,L^d)} \{s = 0\}$ is empty and the Kodaira map

$$\Phi_d : x \in M \mapsto [e_1(x) : \cdots : e_{N_d}(x)] \in \mathbf{P}^{N_d - 1}$$

is an embedding. Let ω_{FS} be the Fubini–Study form of $\mathbf{P}^{N_d-1}$. G. Tian proved that Kodaira embeddings are asymptocally isometries as d goes to infinity.

Theorem 5.1.4 (Tian). *The differential form $\frac{1}{d}\Phi_d^* \omega_{\mathrm{FS}}$ converges in $\mathscr{C}^\infty$-topology to ω. More precisely, for any $d \in \mathbf{N}$, one has*

$$\left\| \frac{1}{d} \Phi_d^* \omega_{FS} - \omega \right\|_{\mathscr{C}^m} = O(d^{-1}).$$

Actually, G. Tian proved the previous theorem for the $\mathscr{C}^0$-topology, and then T. Bouche [Bou90], D. Catlin [Cat99], and S. Zelditch [Zel98] extended the result for the $\mathscr{C}^\infty$-topology. The proof of Theorem 5.1.2 is carried out in two steps: first, one proves by direct computation that $\mathbf{E}(\eta_s) = \Phi_d^* \omega_{\mathrm{FS}}$, and then one applies Tian's approximation theorem to obtain the desired convergence.

Remark 5.1.5. The equality $\mathbf{E}(\eta_s) = \Phi_d^* \omega_{\mathrm{FS}}$ is a particular case of the following more general phenomenon. Given a basepoint-free linear series V of $H^0(X, L^d)$ there is a well-defined map $\Phi_V : X \to \mathbf{P}^{\dim V - 1}$, which induces a Fubini–Study form $\Phi_V^* \omega_{\mathrm{FS}}$ on X. Similarly, by restricting the $\mathscr{L}^2$-Hermitian product to V, one obtains a random section $\eta_s^V \in V$. In this case, we always have the equality of currents $\mathbf{E}(\eta_s^V) = \Phi_V^* \omega_{\mathrm{FS}}$.

5.1.3 Random sections and Kodaira maps.

Given a Berezin–Toeplitz operator T_d, we study the zeros of $T_d s$ where s is a random holomorphic section of L^d of the form

$$s = \sum_{\ell=1}^{N_d} \alpha_\ell e_\ell, \quad \alpha_\ell \sim \mathcal{N}_{\mathbf{C}}(0,1) \text{ i.i.d.} \tag{5.1}$$

where $N_d = \dim H^0(X, L^d)$ and $(e_\ell)_{1\leq \ell \leq N_d}$ is any orthonormal basis of $H^0(X, L^d)$. Such random zeros are related to the properties of some Kodaira maps associated with T_d. Before defining those, we recall some facts about the standard Kodaira maps.

The main result we have obtained is analogous to Theorems 5.1.2 and 5.1.4, but in the case where the sections are twisted by a Berezin–Toeplitz operator. As we will see, there are actually differences between our results and Theorems 5.1.2 and 5.1.4 regarding when the principal symbol f of the Berezin–Toeplitz operator vanishes. Furthermore, random sections and Kodaira maps will allow us to recover the zero locus of f.

5.2 Main results

Let $e_1, \dots, e_{N_d}$ be any orthonormal basis of $H^0(X, L^d)$, and let T_d be a Berezin–Toeplitz operator with principal symbol $f \in \mathscr{C}^\infty(X, \mathbf{R})$. We consider the following "twisted" Kodaira map:

$$\Phi_{T_d} : X \dashrightarrow \mathbf{P}^{N_d - 1}, \quad x \mapsto [(T_d e_1)(x) : \cdots : (T_d e_{N_d})(x)], \tag{5.2}$$

which is well-defined outside the base locus $\bigcap_{s\in H^0(X,L^d)} \{T_d s = 0\}$. The result gives a natural sufficient condition on f for which the map Φ_{T_d} is everywhere well-defined.

Theorem 5.2.1 ([ALF22])**.** *Assume that the principal symbol f of T_d is a smooth, real-valued function which vanishes transversally. Then, for d large enough, the map Φ_{T_d} is well-defined on the whole X, that is $\bigcap_{s\in H^0(X,L^d)} \{T_d s = 0\} = \emptyset$.*

As a consequence of Theorem 5.2.1 we have that the pull-back of the Fubini–Study form ω_{FS} is a *smooth* form defined on the whole X (rather than just a current). The pull-backed forms $\Phi_{T_d}^* \omega_{FS}$ are usually also called Fubini–Study forms. The following theorem deals with the weak convergence of the normalized Fubini–Study forms.

Theorem 5.2.2 ([ALF22])**.** *Assume that the principal symbol $f \in \mathscr{C}^\infty(X, \mathbf{R})$ of T_d is a smooth function vanishing transversally. The sequence of smooth forms $\frac{1}{d}\Phi_{T_d}^* \omega_{FS}$ converges to ω weakly in the sense of currents.*

The next result estimates the error term $\frac{1}{d}\Phi_{T_d}^* \omega_{FS} - \omega$, which explicitly involves f.

Theorem 5.2.3 ([ALF22])**.** *Let $f \in \mathscr{C}^\infty(X, \mathbf{R})$ be a smooth function vanishing transversally, and let T_d be a Berezin–Toeplitz operator with principal symbol f. Then $\log f^2$ is locally integrable and*

$$\Phi_{T_d}^* \omega_{FS} - d\omega \underset{d\to+\infty}{\longrightarrow} i\partial\bar\partial \log f^2$$

in the sense of currents.

The following theorem highlights a striking difference compared to the Fubini-Study forms associated with the standard Kodaira maps: Theorem 5.1.4 asserts that in that case, there is convergence in the $\mathscr{C}^\infty$-topology, whereas for the twisted Kodaira maps, one does not even have pointwise convergence as soon as $f^{-1}(0) \neq \emptyset$.

Theorem 5.2.4 ([ALF22]). *Let $f \in \mathscr{C}^\infty(X, \mathbf{R})$ be a smooth function vanishing transversally, and let T_d be a Berezin–Toeplitz operator with principal symbol f. Then the sequence $\frac{1}{d}\Phi_{T_d}^*\omega_{FS}$ converges to ω locally uniformly on $X \setminus f^{-1}(0)$ in the $\mathscr{C}^\infty$ topology. However, $\frac{1}{d}\Phi_{T_d}^*\omega_{FS}$ does not converge to ω in the $\mathscr{C}^0$-topology (and even pointwise) on $f^{-1}(0)$. More precisely,*

$$\left(\frac{1}{d}\Phi_{T_d}^*\omega_{FS}\right)_x - \omega_x \underset{d\to+\infty}{\longrightarrow} \begin{cases} 0 \text{ if } f(x) \neq 0, \\ \frac{4i(\partial f \wedge \bar{\partial} f)_x}{|\mathrm{d}f(x)|_\omega^2} \text{ if } f(x) = 0 \end{cases}$$

where $|\cdot|_\omega$ is the metric on T^*X induced by the Kähler metric.

Theorem 5.2.4 shows that the zero locus of f plays a fundamental role in the non-convergence of the Fubini–Study forms $\frac{1}{d}\Phi_{T_d}^*\omega_{FS}$ to ω as differential forms. Indeed the difference $\left(\frac{1}{d}\Phi_{T_d}^*\omega_{FS}\right)_x - \omega_x$ exhibits two very different behaviors on and outside $f^{-1}(0)$. In order to further study these two regimes, a natural idea is to work on a smaller scale which allows us to localize around any given point. We show that the scale $d^{-\frac{1}{2}}$ is well-adapted to this problem. Remark that the scale $d^{-\frac{1}{2}}$ is natural in Kähler geometry (it is the scale at which the Bergman kernel displays its universality [BSZ00, DLM06, MM07]). At this scale, we are able to produce precise asymptotics for the difference $\frac{1}{d}\Phi_{T_d}^*\omega_{FS} - \omega$ in the sense of currents. This is the content of Theorem 5.2.5 (for the behavior on $f^{-1}(0)$) and Theorem 5.2.7 (for the behavior outside $f^{-1}(0)$). Recall that $n = \dim_{\mathbf{C}} X$.

Theorem 5.2.5 ([ALF22]). *Let $f \in \mathscr{C}^\infty(X, \mathbf{R})$ be a smooth function vanishing transversally, and T_d be a Berezin–Toeplitz operator with principal symbol f. Let φ be a smooth $(n-1, n-1)$-form on X. Then, for any $x \in f^{-1}(0)$ and any $R > 0$ we have*

$$\int_{B(x,\frac{R}{\sqrt{d}})} \left(\Phi_{T_d}^*\omega_{FS} - k\omega\right) \wedge \varphi = d^{-n+1}\frac{2F_\varphi(x)}{|\mathrm{d}f(x)|_\omega^2} C_n(R) + O(d^{-n+\frac{1}{2}}).$$

Here $B(x, \frac{R}{\sqrt{d}})$ is the geodesic ball of radius $\frac{R}{\sqrt{d}}$ around x, F_φ is the function defined as

$$i\partial f \wedge \bar{\partial} f \wedge \varphi = F_\varphi \frac{\omega^n}{n!}$$

and $C_n(R)$ is a positive (and explicit) universal constant, only depending on R and n.

The constant $C_n(R)$ in this statement is universal in the sense that it only depends on the dimension n and on R but it does not depend neither on f nor on φ nor on X. It equals

$$C_n(R) = \frac{2^n\pi^n(n-1)!}{(2n-2)!}\left(\sum_{\ell=0}^{n-1}\binom{n-\frac{3}{2}}{\ell}2^\ell R^{2\ell} - (1+2R^2)^{n-\frac{3}{2}}\right) \tag{5.3}$$

with $\binom{\alpha}{\ell} = \frac{\alpha(\alpha-1)\ldots(\alpha-\ell+1)}{\ell!}$ for $\alpha \in \mathbf{R}$, $\ell \in \mathbf{N}_{>0}$ and $\binom{\alpha}{0} = 1$. In particular,

$$C_1(R) = 2\pi\left(1 - \frac{1}{\sqrt{1+2R^2}}\right).$$

We showed that the constant $C_n(R)$ can also be expressed in terms of hypergeometric functions. Note that when $\varphi = \frac{\omega^{n-1}}{(n-1)!}$ the first order term in the expansion of Theorem 5.2.5 is universal (it depends neither on f nor on M, but only on n and R). More precisely:

Corollary 5.2.6 ([ALF22]). *Let $f \in \mathscr{C}^\infty(X, \mathbf{R})$ be a smooth function vanishing transversally, and let T_d be a Berezin–Toeplitz operator with principal symbol f. Then, for any $x \in f^{-1}(0)$ we have*

$$\int_{B(x,\frac{R}{\sqrt{d}})} \left(\Phi_{T_d}^*\omega_{FS} - d\omega\right) \wedge \frac{\omega^{n-1}}{(n-1)!} = d^{-n+1}C_n(R) + O(d^{-n+\frac{1}{2}})$$

where $C_n(R)$ is as in Equation (5.3).

The next theorem is the analogue of Theorem 5.2.5 in the case where the point x is not a zero of f.

Theorem 5.2.7 ([ALF22]). *Let $f \in \mathscr{C}^\infty(X, \mathbf{R})$ be a smooth function vanishing transversally, and T_d be a Berezin–Toeplitz operator with principal symbol f. Let φ be a smooth $(n-1, n-1)$-form on M. For any $x \notin f^{-1}(0)$ we have*

$$\int_{B(x,\frac{R}{\sqrt{d}})} \left(\Phi_{T_d}^* \omega_{FS} - k\omega\right) \wedge \varphi = d^{-n} R^{2n} L_\varphi(x) \mathrm{Vol}(B_{\mathbf{R}^{2n}}(0,1)) + O(d^{-n-\frac{1}{2}}).$$

Here $B(x, \frac{R}{\sqrt{d}})$ is the geodesic ball of radius $\frac{R}{\sqrt{d}}$ centered at x, and L_φ is the function defined as

$$i\partial\bar{\partial} \log f^2 \wedge \varphi = L_\varphi \frac{\omega^n}{n!}.$$

Remark that the quantity $d^{-n} R^{2n} \mathrm{Vol}(B_{\mathbf{R}^{2n}}(0,1))$ appearing in the previous result equals $\mathrm{Vol}(B_{\mathbf{C}^n}(0, R/\sqrt{d}))$. It is worth noting the two different behaviors of Theorems 5.2.5 and 5.2.7. Indeed, if $x \in f^{-1}(0)$, the order of magnitude of $\int_{B(x,\frac{R}{\sqrt{d}})} \left(\Phi_{T_d}^* \omega_{FS} - d\omega\right) \wedge \varphi$ is $O(d^{-n+1})$, whereas if $x \notin f^{-1}(0)$ this order is $O(d^{-n})$. This should be compared to Theorem 5.2.4, in which the differential forms $\frac{1}{d}\Phi_{T_d}^* \omega_{FS}$ did not converge exactly on the zero locus of f.

We now study how the action of a Berezin–Toeplitz operator affects the zeros of random sections $s \in H^0(X, L^d)$. Recall that, given a holomorphic section $s \in H^0(X, L^d)$, we denote by η_s the current of integration on the zero divisor $\{s = 0\}$. Given a Berezin–Toeplitz operator T_d, we are interested in the current-valued random variable $s \in H^0(X, L^d) \mapsto \eta_{T_d s}$. Recall that the expected value $\mathbf{E}[\eta_{T_d s}]$ of $\eta_{T_d s}$ is defined by the formula

$$\mathbf{E}[\langle \eta_{T_d s}, \varphi \rangle] = \int_{s \in H^0(X,L^d)} \left(\int_{\{T_d s = 0\}} \varphi \right) \mathrm{d}\mu_d(s)$$

for any smooth $(n-1, n-1)$-form φ. The following result is a generalization of Theorem 5.1.2, when the random section is perturbated by a Berezin–Toeplitz operator.

Theorem 5.2.8 ([ALF22]). *Let $f \in \mathscr{C}^\infty(X, \mathbf{R})$ be a smooth function vanishing transversally and let T_d be a Berezin–Toeplitz operator with principal symbol f. Then*

$$\frac{1}{d} \mathbf{E}[\eta_{T_d s}] \underset{d \to +\infty}{\longrightarrow} \omega$$

weakly in the sense of currents. Moreover, we have

$$\mathbf{E}[\eta_{T_d s}] - d\omega \underset{d \to +\infty}{\longrightarrow} \partial\bar{\partial} \log f^2$$

weakly in the sense of currents.

As shown in Theorems 5.2.5 and 5.2.7, by considering the scale $d^{-\frac{1}{2}}$, we can obtain much more precise asymptotics.

Theorem 5.2.9 ([ALF22]). *Let $f \in \mathscr{C}^\infty(X, \mathbf{R})$ be a smooth function vanishing transversally, and let T_d be a Berezin–Toeplitz operator with principal symbol f. Let $x \in M$. Let φ be a smooth $(n-1, n-1)$-form on M. For every $R > 0$,*

$$\int_{B(x,\frac{R}{\sqrt{d}})} \left(\mathbf{E}[\eta_{T_d s}] - d\omega\right) \wedge \varphi = \begin{cases} d^{-n+1} \frac{F_\varphi(x)}{\pi |\mathrm{d}f(x)|^2_\omega} C_n(R) + O(d^{-n+\frac{1}{2}}) & \text{if } x \in f^{-1}(0), \\ d^{-n} \frac{R^{2n} L_\varphi(x) \mathrm{Vol}(B_{\mathbf{R}^{2n}}(0,1))}{2\pi} + O(d^{-n-\frac{1}{2}}) & \text{if } x \notin f^{-1}(0). \end{cases}$$

Here $B(x, \frac{R}{\sqrt{d}})$, F_φ, L_φ and $C_n(R)$ are as in Theorems 5.2.5 and 5.2.7.

As for Theorems 5.2.5 and 5.2.7, it is worth noting the two different behaviors of

$$\int_{B(x,\frac{R}{\sqrt{d}})} (\mathbf{E}[\eta_{T_d s}] - d\omega) \wedge \varphi$$

for the cases $f(x) = 0$ and $f(x) \neq 0$. Indeed, in the first case, this is of order $O(d^{-n+1})$ whereas if $f(x) \neq 0$ this is of order $O(d^{-n})$. This suggests that the locus of zeros of $T_d s$ tends to concentrate a little more on $f^{-1}(0)$. This is confirmed by numerical simulations we made and that can be found in [ALF22, Section 5.2].

Remark 5.2.10. If T_d is a Berezin–Toeplitz operator with principal symbol f, then the operator $T_d - \lambda \mathrm{Id}$ is a Berezin–Toeplitz operator with principal symbol $f - \lambda$. So up to replacing f by $f - \lambda$, we can replace $f^{-1}(0)$ by any regular level set of f in the above statements and discussions.

To conclude, note that A. Drewitz, B. Liu and G. Marinescu [DLM23, DLM24] recently studied random holomorphic sections in a non-compact setting and for a wide class of principal symbols f. In this general context, they proved similar results as the above ones.

5.2.1 Idea of proof

The proofs of the main results follow from a careful analysis of the Schwartz kernel of the Berezin–Toeplitz operator $T_d^* T_d$. Let us denote by B_d the restriction to the diagonal of $X \times X$ of this kernel. We first prove that $B_d(x) = d^n \left(f^2 + d^{-1} b_1 + O(d^{-2})\right)$ where the remainder $O(d^{-2})$ is uniform on X and

$$b_1 = 2f \mathrm{Re}\,(g) + 2f\Delta f + \frac{r}{2} f^2 + \frac{1}{2} |\mathrm{d}f|^2_\omega.$$

where $g \in \mathscr{C}^\infty(X, \mathbf{R})$ denotes the subprincipal symbol of T_d and r is the scalar curvature of the Kähler form ω.

The key observation we made is the following: if f vanishes transversally, the kernel B_d on the diagonal is everywhere strictly positive, for d large enough. More precisely, there exists $c > 0$ such that, for d large enough and for any $x \in M$, we have $B_d(x) > cd^{n-1}$. Indeed, on $f^{-1}(0)$ one has $B_d(x) = d^{n-1} \left(b_1(x) + O(d^{-1})\right) = \frac{d^{n-1}}{2} |\mathrm{d}f(x)|^2_\omega > 0$. By continuity, b_1 is strictly positive in a neighborhood U of $f^{-1}(0)$. Let c be the minimum of b_1 on U and C be the minimum of $|f|^2$ on $M \setminus U$. We then have $B_d(x) \geq Cd^n + O(d^{n-1})$ outside U and $B_d(x) \geq cd^{n-1} + O(d^{n-2})$ on U. Hence the result.

Theorem 5.2.1 is a direct consequence of this positivity result. Indeed, Φ_{T_d} is well-defined at a point $x \in X$ if and only if $|\Phi_{T_d}(x)|^2 > 0$. A direct computation shows the equality $|\Phi_{T_d}(x)|^2 = B_d(x)$, hence the result.

For Theorem 5.2.2, one first proves the equality of forms $\Phi_{T_d}^* \omega_{\mathrm{FS}} = d\omega - i\partial\bar{\partial} \log B_d$. The positivity of B_d allows us to easily prove that $\frac{1}{d}\partial\bar{\partial} \log B_d$ converges to 0 weakly in the sense of currents as $d \to \infty$. A direct computation of $i\partial\bar{\partial} \log B_d$ yields Theorem 5.2.4.

In order to prove Theorems 5.2.5 and 5.2.7 one has to estimate

$$\int_{B(x,\frac{R}{\sqrt{d}})} \left(i\partial\bar{\partial} \log B_d\right) \wedge \varphi.$$

After a small manipulation on the error term, one reduces the problem into the estimate of

$$\int_{B(x,\frac{R}{\sqrt{d}})} \left(i\partial\bar{\partial} \log\left(f^2 + d^{-1} b_1\right)\right) \wedge \varphi. \tag{5.4}$$

If $f(x) \neq 0$, the leading term of (5.4) is $\int_{B(x,\frac{R}{\sqrt{d}})} \left(i\partial\bar{\partial}\log f^2\right) \wedge \varphi$. Then, Theorem 5.2.7 essentially follows from a Taylor expansion of the function L_φ around x.

If $f(x) = 0$, then both f^2 and $d^{-1}b_1$ are of order d^{-1} inside $B(x, \frac{R}{\sqrt{d}})$. One then expands $i\partial\bar{\partial}\log\left(f^2 + d^{-1}b_1\right)$ and shows that, up to an error term, the integral (5.4) equals

$$4i \int_{B(x,\frac{R}{\sqrt{d}})} \frac{d^{-1}|\mathrm{d}f|^2_\omega - 2f^2}{(2f^2 + d^{-1}|\mathrm{d}f|^2_\omega)^2} \partial f \wedge \bar{\partial} f \wedge \varphi. \tag{5.5}$$

Such an integral is more delicate to compute. We were able to estimate it using Hadamard's lemma and then making a well-chosen change of variables. In particular we showed that, up to error terms, the integral (5.5) is equal to

$$\frac{4d^{-n+1}F_\varphi(x)}{|\mathrm{d}f|^2_\omega - 2f^2} \int_{B_{\mathbf{R}^{2n}}(0,R)} \frac{1-2t_1^2}{(1+2t_1^2)^2} \,\mathrm{d}t_1 \cdots \,\mathrm{d}t_{2n}. \tag{5.6}$$

The constant $C_n(R)$ appearing in Theorem 5.2.5 is (half of) the value of the integral of (5.6). It is obtained using classical formulae involving hypergeometric functions.

Theorem 5.2.8 and Theorem 5.2.9 follow from Theorems 5.2.2, 5.2.3, 5.2.5 and 5.2.7 after a computation that shows that $\mathbf{E}[Z_{T_d s}]$ and $\Phi^*_{T_d}\omega_{FS}$ are equal as currents (see Remark 5.1.5).

Chapter 6

Further directions and open questions

Here, we collect a few questions related to the results presented in this manuscript that we would like to address in the future. Some of them already appeared explicitly in the text. We use the notations that we used all along the manuscript.

1. What are the laws of the various metric observables associated with a random complex plane curve, in particular $\mathrm{syst}(Z_P)$ and $\mathrm{diam}(Z_P)$?

2. It is known that random hyperbolic surfaces $X_g \in (\mathcal{M}_g, \mathbb{P}_{\mathrm{WP},g})$ converge to the hyperbolic plane $\mathbb{H}$ as $g \to \infty$ in the Benjamini–Schramm sense [Mon22]. This informally means that with high probability (i.e. outside of a set $\mathcal{N}_g \subset \mathcal{M}_g$ whose measure goes to 0 as $g \to \infty$) most of the points of a hyperbolic surface have the property that a big disk around them is isometric to a big disk in the hyperbolic plane. It would be interesting to study the Benjamini–Schramm convergence for random complex plane curves.

3. The first eigenvalue of the Laplacian of a complex plane curve is always bounded from above by 6 [BLY94]. It seems plausible to expect that with high probability the first eigenvalue of the Laplacian of a degree d random complex plane curve is bounded from above by a_d, where $(a_d)_{d\in\mathbf{N}}$ is a sequence of numbers tending to zero when d tends to infinity.

4. It is known that surfaces with negative curvature are Anosov. As explained in Section 2.1, complex plane curves have points of positive curvature. However, Corollary 2.1.4 shows that, for the typical complex plane curve, the area with positive curvature is very small. Are complex plane curves Anosov with high probability ?

5. Can one construct examples of sequences $(C_d)_{d\in\mathbf{N}}$ of degree d complex plane curves such that $\mathrm{diam}(C_d)$ grows at least linearly?

6. Is it true that the probability that a random complete intersection of sufficiently large codimension has negative holomorphic bisectional curvature tends to 1 as the degree of the complete intersection goes to infinity?

7. Given a real ample line bundle $L \to X$ over an n dimensional real algebraic manifold and $s \in \mathbf{R}H^0(X, L^d)$ a random section, does the limit

$$\lim_{d\to\infty} \frac{1}{\sqrt{d}^n} \mathbf{E}(b_i(\mathbf{R}Z_s))$$

exist ?

8. For a random real algebraic hypersurface Z_P in $\mathbf{P}^n$, the order of growth of $\mathbf{E}(b_i(\mathbf{R}Z_P))$ is $\sqrt{d}^n$. A result by Podkorytov [Pod98] shows that, for the Euler characteristic of a real algebraic surface in $\mathbf{RP}^3$, one has $\mathbf{E}[\chi(\mathbf{R}Z_P)] = -C\sqrt{d}^3$, with $C > 0$. This implies that $\mathbf{E}(b_0(\mathbf{R}Z_P)) < \mathbf{E}(b_1(\mathbf{R}Z_P))$. Is this true that a random real algebraic hypersurface in $\mathbf{P}^n$ verifies $\mathbf{E}(b_0(\mathbf{R}Z_P)) \leq \mathbf{E}(b_1(\mathbf{R}Z_P)) < \cdots \leq \mathbf{E}(b_{\lfloor\frac{n-1}{2}\rfloor}(\mathbf{R}Z_P))$? Recall that the Hard Lefschetz Theorem implies that for a n dimensional complex projective manifold X one has $b_0(X) \leq b_2(X) \leq \cdots \leq b_{2\lfloor\frac{n}{2}\rfloor}(X)$ and $b_1(X) \leq b_3(X) \leq \cdots \leq b_{2\lfloor\frac{n+1}{2}\rfloor-1}(X)$.

9. Let D be a divisor on a real algebraic variety X of dimension n with non-empty real locus $\mathbf{R}X$. For any $i \in \{0, \dots, n-1\}$ we define

$$v_i(D) = \limsup_{d\to\infty} \frac{1}{d^n} \sup_{Z\in|dD|} b_i(\mathbf{R}Z) \quad \text{and} \quad v(D) = \limsup_{d\to\infty} \frac{1}{d^n} \sup_{Z\in|dD|} b_*(\mathbf{R}Z)$$

where b_* denotes the total Betti number. Theorem 1.5.5 implies that $v_i(D) > 0$ if D is ample. In this case, the existence of asymptotically maximal hypersurfaces in $|dD|, d \gg 0$ is equivalent to the equality $v(D) = D^n$, where D^n denotes the top self-intersection number of D. Does there exist relations between the numbers $v_i(D)$, apart from the equality $v_i(D) = v_{n-i-1}(D)$ given by Poincaré duality?

10. Recall that, if D is a divisor of a complex manifold X then the volume of D is defined by the formula

$$\mathrm{Vol}(D) = \limsup_{d\to\infty} \frac{n!}{d^n} \dim H^0(X, \mathcal{O}(dD)).$$

A divisor D is big precisely when $\mathrm{Vol}(D) > 0$. If now we consider X and D to be defined over $\mathbf{R}$, can one characterize the bigness of D in terms of the quantities $v_i(D)$ defined in the previous point? If D is a big, are the numbers $v_i(D)$ positive for any $i \in \{0, \dots, n-1\}$?

11. The major technique for constructing real algebraic hypersurfaces inside $\mathbf{P}^n$ is Viro's patchworking [Vir84]. It would be interesting to understand in a quantitative sense how many possible topologies can be constructed using only patchworking. More precisely, what is the probability that the real zero locus of a real polynomial is isotopic to the real part of a hypersurface obtained by Viro's patchworking.

12. In Chapter 5, we described how to recognize the zero locus of a smooth function f over a projective manifold by considering random sections of the form $T_f s$. Can one recognize the critical points of f using the random sections $T_f s$?

13. Using the notation of Chapter 5, can one compute the asymptotics of the variance $\mathrm{Var}(\eta_{T_f s})$, as $d \to \infty$?

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