Spike-train description: process or Gibbs? (Overview of issues in stochastic modeling)

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Two modeling issues in neuroscience

Spike train

Sequence of spikes of one neuron

$$\cdots\ \omega_{-n-1}\ \omega_{-n} \cdots \omega_{-1}\ \omega_0\ \omega_1\ \cdots\ \omega_n\ \omega_{n+1} \cdots$$

$\omega_i = 0$ or 1 (simplicity).

Brain mapping

Sequence of spike configurations of regions of N neurons (rasters)

$$\cdots \begin{matrix} \omega_N(t_1) \\ \vdots \\ \omega_1(t_1) \end{matrix} \cdots \begin{matrix} \omega_N(t_n) \\ \vdots \\ \omega_1(t_n) \end{matrix} \cdots = \cdots \underline{\omega}(t_1) \cdots \underline{\omega}(t_N) \cdots$$

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Basic tenets

Stochastic description due to signal variability

Full description = probability measure

Spike trains: Measure on $\{0,1\}^{\mathbb{Z}}$ = stochastic process

Brain mappings:

- *Evolving approach:* Stochastic process on $\Omega_N^{\mathbb{Z}}$ with $\Omega_N = \{0,1\}^N$ (e.g. Galves-Löcherbach).
- *"Equilibrium" approach:* Register has stabilized, law of $\underline{\omega}(t)$ independent of t = measure on Ω_N.

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Describing probability measures

μ probability measure on a product space $X^{\mathbb{L}}$, $\mathbb{L}$ countable
= Countable family of random variables $(X_i)_{i \in \mathbb{L}}$

E.g:

- Spike trains: $X = \{0, 1\}$, $\mathbb{L} = \mathbb{Z}$ (or part thereof)
- Brain mappings - evolving: $X = \Omega_N$, $\mathbb{L} = \mathbb{Z}$
- Brain mappings - equilibrium: $X = \{0, 1\}$, $\mathbb{L} = \{1, \ldots, N\}$

Key issue: efficient characterization of μ.
Complementary points of view:

- *Modeling:* Definition of μ on the basis of available info
- *Decoding:* Extracting info from realizations of μ

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First aspect: Averages

Consider functions of *finitely many* random variables

$$F_j \;=\; F_j\big(X_{i_1^j}, \ldots, X_{i_{n_j}^j}\big) \qquad j = 1, \cdots, K$$

$[\{i_1^j, \ldots, i_{n_j}^j\}$ = (finite) window for $F_j]$

Consider the corresponding expected values

$$\alpha_j \;=\; \mathbb{E}(F_j) \qquad j = 1, \cdots, K \tag{1}$$

Issues:

- *Modeling:* $\alpha_1, \ldots, \alpha_K$ known; choose μ satisfying (1)
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Truncated averages

Related aspect: correlation between far away regions:

$$\mathbb{E}(F\,\tau_n G) - \mathbb{E}(F)\,\mathbb{E}(G) \;\sim\; \Phi(n)$$

τ_n = translation a distance n

- Expected: $\Phi(n) \xrightarrow[N\to\infty]{} 0$ (asymptotic independence)
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Finitely many averages do *not* uniquely determine a process
Which of the possible measures should we choose?

Standard answer: The one with *maximal entropy*

- Proposed by Boltzmann, re-proposed by Jaynes
- Succesful in statistical mechanics (ensembles!)
- Leads to variational approaches
- Problem:
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  - However: relevant information as size $\to \infty$ (thermodynamic limit)

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Review on entropy

If ν is a probability measure on a finite space Γ

$$S_\Gamma(\mu) \;=\; -\sum_{x\in\Gamma} \mu(x)\,\ln \mu(x)$$

$[0 \ln 0 = 0]$. In particular, if μ is the uniform measure:

$$S(\text{uniform}) \;=\; \ln|\Gamma|$$

Want to find μ maximizing S among the family

$$\mathcal{P}(\alpha_1,\ldots,\alpha_K) \;=\; \{\mu : \mathbb{E}_\mu(F_1) = \alpha_1\,,\ldots,\,\mathbb{E}_\mu(F_K) = \alpha_K\}$$

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Canonical ensembles

Lagrange multipliers $\lambda_1, \ldots, \lambda_K$ lead to extremization of

$$\mathcal{L}\big[\{\mu(x)\}_{x\in\Gamma}, (\lambda_i)_{i=1,\ldots,K}\big] = -\sum_x \mu(x)\, \ln \mu(x) + \sum_{j=1}^{K} \lambda_j \Big[\alpha_j - \sum_x F_j(x)\, \mu(x)\Big]$$

The solution is

$$\mu(x) \;=\; \frac{1}{Z} \exp\Big[-\sum_{j=1}^{K} \lambda_j\, F_j(x)\Big] \tag{2}$$

with λ_j tuned so that

$$\alpha_j \;=\; \sum_{x\in\Gamma} F_j(x)\, \mu(x) \qquad j = 1, \cdots, K \tag{3}$$

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Canonical ensembles (cont.)

The system (2)–(3) has (usually) a unique solution
Comparison with statistical mechanics

- Measure μ = (macro)canonical ensemble
- "Temperature"=1
- λ_j = "field" conjugated to the "observable" F_j. Examples:
  - F = magnetization, λ = magnetic field
  - F = density, λ = chemical potencial
  - F = n-body term, λ = n-body coupling

Approach valid for finite configurations of neurons (or spikes)
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Entropy in the thermodynamic limit

Two problems appear if $|\Gamma| \to \infty$:

- Entropy blows up
- The limit of the measures (2)–(3) is not uniquely defined

Solutions:

- Pass to entropy density

$$s(\mu) = \lim \frac{S_\Gamma}{|\Gamma|}$$

  where S_Γ computed with the projection of μ on Γ
- Consider the measures *minimizing* the free energy

$$f(\mu) = \sum_j \mu(f_j) - s(\mu)$$

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Delicate points

- Entropy density = large deviation rate function
  = fluctuation control
  - Normal fluctuations (CLT): s strictly convex
  - s affine = abnormal fluctuations = critical points
    = 2nd-order phase transtions
- There may be several minimizing measures
  (1st-order phase transitions)
- These measures have pairwise zero *relative* entropy
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Decoding issues for averages

Ergodic hypothesis: Empirical averages → μ expectations

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$$\frac{1}{n}\sum_{\ell=1}^{n} F(\tau_\ell \omega) \xrightarrow[N\to\infty]{} \mu(F)$$

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Fine points

- If multiple measures, ergodicity only for extremal measures
- Ergodicity true for almost all realizations
- "Good" realizations can be studied with techniques of ergodic theory (symbolic dynamics)

Example (Takens and Verbitskiy, 2003): If

$$\mathcal{K}(\alpha_1, \ldots, \alpha_K) = \Big\{ (x_i) : \frac{1}{n} \sum_{i=1}^{n} F_j(x_i) \xrightarrow[n\to\infty]{} \alpha_j \,,\; j = 1, \ldots, K \Big\}$$

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topological entropy of $\mathcal{K}(\underline{\alpha})$ = max. entropy of measures in $\mathcal{P}(\underline{\alpha})$

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Second aspect: Transition probabilities

Machine-learning approach for spike or brain-mapping trains:

- Use first part of the train to develop “rules” to predict rest
- By recurrence: enough to predict *next* bit given “history”

That is, estimate the conditional probabilities w.r.t. past

$$P(X_n \mid X_{n-1}, X_{n-2}, \ldots)$$

through its law, defined by a function g such that

$$P\big(X_0 = \omega_0 \mid X_{-\infty}^{-1} = \omega_{-\infty}^{-1}\big) \;=\; g\big(\omega_0 \mid \omega_{-\infty}^{-1}\big)$$

Look for μ determined by (consistent with) this g:

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Regular g-measures

Relevant transitions must be insensible to the farther past:

g is a **regular g-function** if $\forall \epsilon > 0\, \exists n \geq 0$ such that

$$\left| g\big(\omega_0 \,\big|\, \sigma_{-1}^{-n}\, \omega_{-\infty}^{-n-1}\big) - g\big(\omega_0 \,\big|\, \sigma_{-\infty}^{-1}\big) \right| \;<\; \epsilon \qquad (4)$$

for every ω, σ

- (4) is equivalent to $g(\omega_0 \,|\, \cdot\,)$ continuous in product topology
- Additional, not relevant, non-nullness condition

A probability measure μ is a **regular g-measure** if it is consistent with some regular g-function

train μ thought as a process: past determines future (causality)

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Conditional probabilities

If the full train is available, why use only the past?

Learn to predict a bit using past *and future*!

X_n determined by finite-window probabilities

$$\mathbb{P}(X_n \mid X_{n-1}, X_{n-2}, \ldots ; X_{n+1}, X_{n+2}, \ldots)$$

through conditional laws determined by a function γ s.t.

$$P\big(X_0 = \omega_0 \,\big|\, X_{\{0\}^c} = \omega_{\{0\}^c}\big) \;=\; \gamma\big(\omega_0 \,\big|\, \omega_{\{0\}^c}\big)$$

Specification: γ satisfying certain compatibility condition

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Gibbs measures

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$$\left|\gamma\left(\omega_0 \;\middle|\; \omega_{-n}^{m} \sigma_{[n,m]^c}\right) - \gamma\left(\omega_0 \;\middle|\; \omega_{\{0\}^c}\right)\right| \;<\; \epsilon \tag{5}$$

for every σ, ω

- (5) is equivalent to $\gamma(\omega_0 \,|\, \cdot\,)$ continuous in product topology
- Additional, not relevant, non-nullness condition

A probability measure μ is a **Gibbs-measure** if it is consistent with some regular specification

Train μ thought as non-causal or with anticipation

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Link with stat mech

Theorem (Kozlov)

A specification is Gibbsian iff it has the Boltzmann form

$$\gamma\left(\omega_0 \mid \omega_{\{0\}^c}\right) = \exp\left\{-\sum_{A\ni 0} \phi_A(\omega_A)\right\}/Norm. ,$$

where $\{\phi_A\}$ (interaction) satisfy

$$\sum_{A\ni 0} \|\phi_A\|_\infty < \infty .$$

Questions, questions

Trains described/decoded as processes or as Gibbs?

Both setups give complementary information:

- Processes: ergodicity, coupling, renewal, perfect simulation
- Fields: Gibbs theory

Are these setups mathematically equivalent?

Is every regular g-measure Gibbs and viceversa?

Answers: It depends on the dependency on history (exterior):

- Yes, if it decays exponentially fast
- No if it decays too slowly
  - There are regular g-measures that are not Gibbs
  - Not known whether there are Gibbs that are not regular g (belief: yes!)

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For trains of spikes or evolving brain mapping

- What is more efficient: One-side or two-side conditioning?
- Neural interpretation of two-side conditioning?
- Possible use of multifractal analysis (spectrum)?

For equilibrium brain mapping

- Only Gibbs setup available (interaction). Is it appropriate?
- More generally, are asymptotic phenomena relevant?:
  - Gibbsianness
  - 1st-order phase transitions (multiple measures)
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