A networked integrate-and-fire model of mean-field type

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Motivation

- Global behavior of a simple neural network
- Local description
  - each neuron $\rightsquigarrow$ noisy LIF (Leaky Integrate and Fire) model
  - kind of homogeneity in the population
- Interacting rules
  - symmetry $\rightsquigarrow$ mean-field interaction
  - instantaneous excitatory interaction
- Main question
  - emergence of massive simultaneous spikes depending on the intensity of the excitation
- Modeling taken from earlier works [Lewis and Rinzel (03); Ostojic, Brunel and Hakim (09)...]

General LIF model

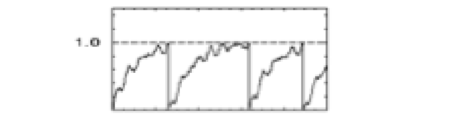
- Describe membrane potential of a neuron through threshold dynamics
- Subthreshold dynamics

$$\frac{d}{dt}V_t = -\lambda V_t + I_t^{\text{int}} + I_t^{\text{ext}}$$

  - λ connected with properties of the membrane
  - $I^{\text{int}} \rightsquigarrow$ current due to interactions with other cells
  - $I^{\text{ext}} \rightsquigarrow$ collective effect due to external phenomena
- Threshold $\rightsquigarrow$ spike whenever V reaches firing value V_F

$$\tau = \inf\{t \geq 0 : V_t \geq V_F\}$$

  - after τ (no refractory period) $\rightsquigarrow$ reset potential at $V_\tau = V_R$



Taken from W. Gerstner and W. Kistler, Spiking neuron models, 2002

Description of the currents

- Label the neurons $i = 1, \ldots, N$

$$\frac{d}{dt} V_t^i = -\lambda V_t^i + I_t^{\text{int},i} + I_t^{\text{ext},i}$$

  - $N \rightsquigarrow$ number of neurons

- Interaction current

$$I_t^{\text{int},i} = I^{\text{int}}(V_t^j, j \neq i)$$

  - depends on the states of the other neurons

- External current

$$I_t^{\text{ext},i} = \text{mean-trend}_t^i + \text{noise}_t^i$$

  - focus on the noise $\rightsquigarrow$ $\text{noise}_t^i = (\dot{W}_t^i)_{t \geq 0}$ white noise
  - very strong assumption $\rightsquigarrow$ start with independent noises
  - may think of correlated cases as well $\rightsquigarrow$ more complicated

Mean-field interaction

- Force symmetric interactions $\leftrightarrow$ interaction with the whole population (no privileged interactions between labels)
  - $I_t^{\text{int}}(V_t^j, j \neq i)$ depending on the empirical distribution

$$I_t^{\text{int}}(V^j, j \neq i) = I^{\text{int}}\Big(N^{-1} \sum_{j \neq i} \delta_{V^j}\Big)$$

- Subthreshold dynamics

$$V_t^i = V_0^i - \lambda \int_0^t V_s^i ds + I^{\text{int}}\Big(N^{-1} \sum_{j \neq i} \delta_{V^j}\Big) + W_t^i$$

- Asymptotic model when $N \to +\infty$? expect decorrelation between neurons as $N \to \infty$ + symmetry $\Rightarrow$ expect averaging

$$I^{\text{int}}\Big(\frac{1}{N} \sum_{j \neq i} \delta_{V_t^j}\Big) \sim I^{\text{int}}(\mathcal{L}(V_t))$$

- Typical neuron interacts with its own law $\rightsquigarrow$ McKean-Vlasov eq.

$$dV_t = -\lambda V_t dt + I^{\text{int}}(\mathcal{L}(V_t))dt + dW_t$$

Choice of the interaction functional

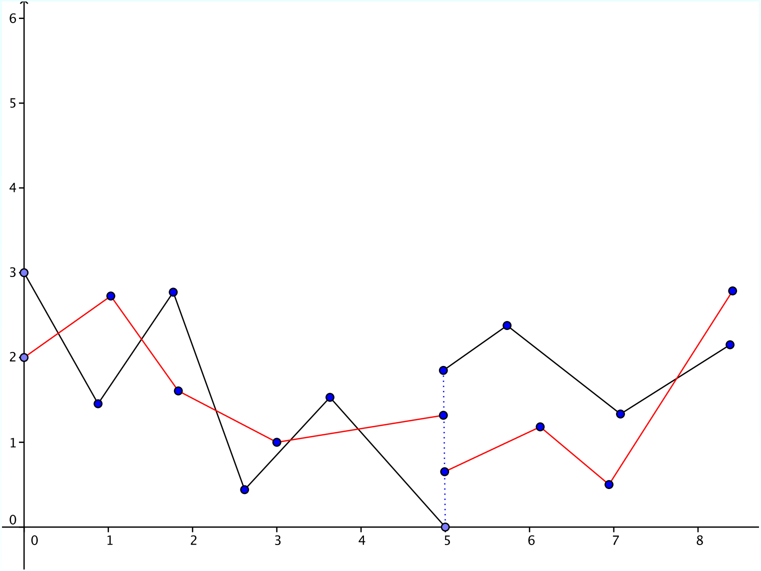
- Conclusion of previous slide $\rightsquigarrow$ instead of focusing on N neurons with mean-field interaction
  - focus on one typical neuron, characteristic of the population
  - licit for functionals I^{int} which are not too singular (make possible the passage to the limit)
- Model suggest in literature $\rightsquigarrow$ interactions
  - replace interaction currents by interaction pulses

$$I_t^{\text{int}}(V^j, j \neq i) = \frac{\alpha}{N} \sum_{j \neq i} \sharp\{V_{t-}^j = V_F\}$$
$$= \frac{\alpha}{N} \sharp\{\text{spiking neurons } \neq i \text{ at } t\}$$

  - $\alpha > 0 \Leftrightarrow$ instantaneous self-excitatory interaction
- Questions: Asymptotic model and influence of parameter α?

Example for V_F−potential

- Consider V_F minus the potential $\rightsquigarrow$ threshold is zero

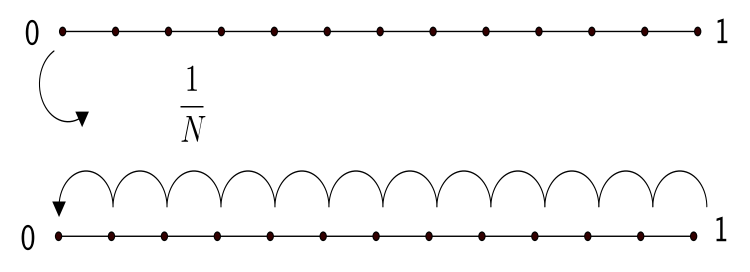


Cascade phenomenon

- Cascade phenomenon in the modelling!

- Example: runaway behavior if reset ($V_R = 0$, $V_F = 0$) $\rightsquigarrow$ plot V_F-potential

    - choose $N+1$ neurons, $\alpha = (N+1)/N$ and $V_0^i = i/N$, $i = 0, \ldots, N$,



    - particles keep jumping!

    - $\alpha < (N+1)/N \Rightarrow$ no way for defaulting twice at same time

Guess for the limit model

- Recall the subthreshold dynamics

$$V_t^i = V_0^i - \lambda \int_0^t V_s^i ds + \frac{\alpha}{N} \sum_{j \neq i} \sharp\{\text{neuron}(j) \text{ spiked before } t\} + W_t^i$$

- Heuristics $\rightsquigarrow$ same formal reasoning as a for a regular interaction current

$$I_t^{\text{int}}(V^j, j \neq i) \underset{N \to +\infty}{\sim} \alpha \mathbb{E}(M_t)$$

  - M_t = number of spikes for typical neuron up to t

- Subthreshold dynamics for typical neuron as $N \to \infty$

$$V_t = V_0 - \lambda \int_0^t V_s ds + \alpha \mathbb{E}(M_t) + W_t$$

  - $M_t = \sharp\{t \geq 0 : V_{t-} = V_F\}$ depends on V!
  - Mathematical questions: well-posedness, qualitative behavior?

Interpretation of the mean-field interaction

- Subthreshold dynamics

$$V_t = V_0 - \lambda \int_0^t V_s ds + \alpha \mathbb{E}(M_t) + W_t$$

  - firing value $V_F = 1$, reset (after spiking) $V_R = 0$

- Crucial mathematical question: what class of admissible solutions?

  - class of solutions dictates regularity for $\mathbb{E}(M_t) \rightsquigarrow$ physical interpretation?

$$\begin{aligned}
&\mathbb{E}(M_{t+h} - M_t) \\
&\sim_{N=\infty} \text{probability of default in } [t, t+h] \\
&\sim_{N<\infty} \text{proportion of companies default in } [t, t+h]
\end{aligned}$$

  - $\mathbb{E}(M_t)$ is allowed to jump $\leftrightarrow$ large proportion of neurons may spike at the same time

  - may stand for massive simultaneous spikes in the system

Instantaneous firing rate

- Meaning for requiring $e : t \mapsto \mathbb{E}(M_t)$ to be differentiable?

$$\text{probability of default in } [t, t+h] \sim e'(t)h$$

  - $e' \leftrightarrow$ instantaneous firing rate

- Subthreshold dynamics if differentiability

$$dV_t = -\lambda V_t dt - \alpha e'(t)dt + dW_t$$

  - stock. diff. eq. $\rightsquigarrow$ stochastic calculus and regularizing effect
  - $\mathbb{P}(V_t \in dy) = p(t,y)dy, \quad t > 0, \quad y < 1$

- Fokker Planck equation

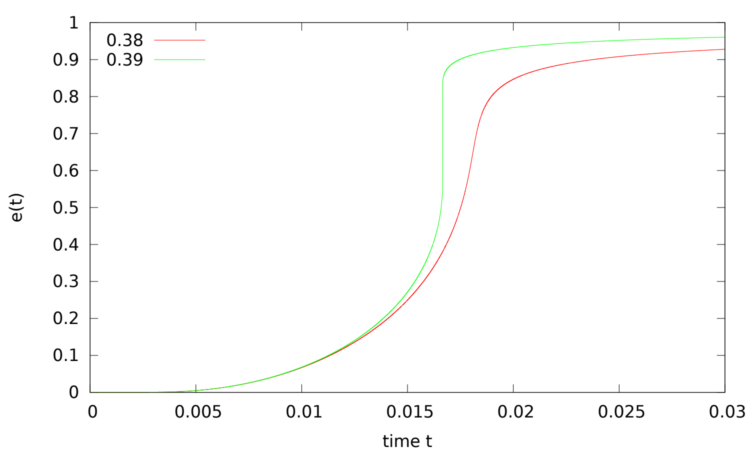
$$\partial_t p(t,y) + \partial_y[(-\lambda y + \alpha e'(t))p(t,y)] - \frac{1}{2}\partial^2_{yy} p(t,y) = e'(t)\delta_0$$

  - $p(t,1) = 0$ and $\partial_y p(t,1) = -\frac{1}{2}e'(t)$
  - control of e' $\Leftrightarrow$ control of the mass near 1

Solvability of the regular model

- Existence of regular solutions in arbitrary time?
  - avoid blow-up of e' in finite time?
  - $\Leftrightarrow$ avoid massive defaults?
- Caceres, Carrillo, Perthame (2011)
  - for any $\alpha > 0$, $\exists V_0 > 0$ such that blow-up in finite time!
- D., Inglis, Rubenthaler and Tanré (2014)
  - for $V_0 > 0$, $\exists!$ solution without blow-up for α small enough
  - explicit (but non-optimal) bounds on critical values α
- Brownian example: $\lambda = 0$ and $V_0 = 0.8$ ($V_F = 1$, $V_R = 0$)
  - existence of regular solutions if $\alpha \leq 0.10$
  - no regular solutions if $\alpha \geq 0.54$
  - numerically, critical value $\sim 0.38\ldots$
- Exemple O-U $\lambda \to \infty \Rightarrow$ critical $\alpha \to 1$ ($\Leftrightarrow \lambda$ fixed and $\sigma \to 0$)

Illustration



Convergence and solutions with blow-up

- Limit of particle system $\Rightarrow$ generates a solution but with a possible blow-up
  - $\alpha \ll 1$ uniqueness guarantees that the limit has no bow-up
  - use of the M1 topology on space of càd-làg paths (weaker than J1)
- Description of the jumps of $e(t) = \mathbb{E}(M_t)$ when blow-up?

$$\Delta e(t) = e(t) - e(t-) \geq \delta_0$$
$$\Leftrightarrow \forall \delta \leq \delta_0, \text{ kick due to particles in } (1-\delta, 1] \geq \delta$$

$$\Delta e(t) = \sup\Big\{\delta_0 : \forall \delta \leq \delta_0, \underbrace{\alpha \int_{1-\delta}^{1} p(t-, y) dy}_{\text{kick due to particles in } (1-\delta,1]} \geq \delta\Big\}$$

  - restart with density $p(t, y) = p\big(t-, y + \Delta e(t)\big)$ for y near 0
- Uniqueness? regularization of e just after default?

Model with a common noise

- Common source of noise in dynamics of the neurons companies

$$V_t^i = V_0^i - \lambda \int_0^t V_s^i ds + I_t^i + W_t^i + W_t^0$$

- Mean-field modeling

$$V_t = V_R - \lambda \int_0^t V_s ds + \alpha \mathbb{E}(M_t | W^0) + W_t + W_t^0$$

  - ∘ same $\alpha \rightsquigarrow$ two different kinds of plots competition with common noise