Spiking neural models: from point processes to partial differential equations.

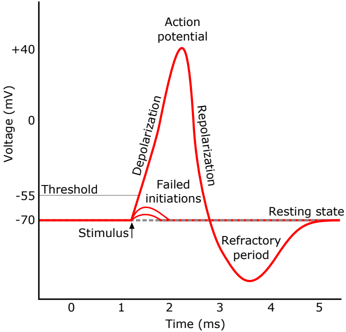
J. Chevallier

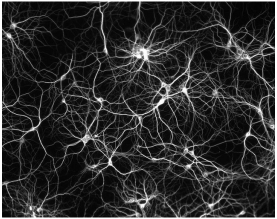
LJAD University of Nice

MathStatNeuro Workshop
Nice

2015/09/09

Biological context





- Physiological constraint: refractory period.
- Model interacting spiking neurons.

Microscopic modelling

Microscopic modelling of spike trains

Time point processes = random countable sets of times (points of $\mathbb{R}$ or $\mathbb{R}_+$).

- Point process: $N = \{T_i, i \in \mathbb{Z}\}$ s.t. $\cdots < T_0 \leq 0 < T_1 < \cdots$.
- Point measure: $N(dt) = \sum_{i\in\mathbb{Z}} \delta_{T_i}(dt)$. Hence, $\int f(t)N(dt) = \sum_{i\in\mathbb{Z}} f(T_i)$.

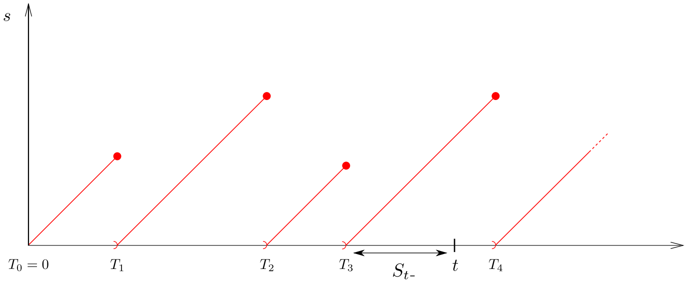
Microscopic modelling

Microscopic modelling of spike trains

Time point processes = random countable sets of times (points of $\mathbb{R}$ or $\mathbb{R}_+$).

- Point process: $N = \{T_i, i \in \mathbb{Z}\}$ s.t. $\cdots < T_0 \leq 0 < T_1 < \cdots$.
- Point measure: $N(dt) = \sum_{i\in\mathbb{Z}} \delta_{T_i}(dt)$. Hence, $\int f(t)N(dt) = \sum_{i\in\mathbb{Z}} f(T_i)$.
- Age process: $(S_{t-})_{t\geq 0}$.

Age = delay since last spike.



Microscopic modelling

Microscopic modelling of spike trains

Time point processes = random countable sets of times (points of $\mathbb{R}$ or $\mathbb{R}_+$).

- Point process: $N = \{T_i, i \in \mathbb{Z}\}$ s.t. $\cdots < T_0 \leq 0 < T_1 < \cdots$.
- Point measure: $N(dt) = \sum_{i\in\mathbb{Z}} \delta_{T_i}(dt)$. Hence, $\int f(t)N(dt) = \sum_{i\in\mathbb{Z}} f(T_i)$.
- Age process: $(S_{t-})_{t\geq 0}$.

Stochastic intensity

- Heuristically,

$$\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}\left(N([t, t+\Delta t]) = 1 \,|\, \mathscr{F}^N_{t-} \right),$$

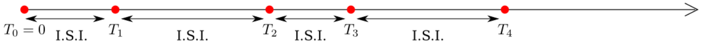
  where $\mathscr{F}^N_{t-}$ denotes the history of N before time t.
- Local behaviour: probability to find a new spike.
- May depend on the past (e.g. refractory period, excitation, inhibition).

Some classical point processes in neuroscience

- Poisson process: $\lambda_t = \lambda(t)$ (no refractory period).

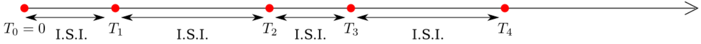
Some classical point processes in neuroscience

- Poisson process: $\lambda_t = \lambda(t)$ (no refractory period).
- Renewal process: $\lambda_t = f(S_{t-}) \Leftrightarrow$ i.i.d. ISIs. (refractory period)



Some classical point processes in neuroscience

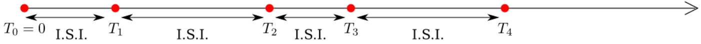
- Poisson process: $\lambda_t = \lambda(t)$ (no refractory period).
- Renewal process: $\lambda_t = f(S_{t-}) \Leftrightarrow$ i.i.d. ISIs. (refractory period)



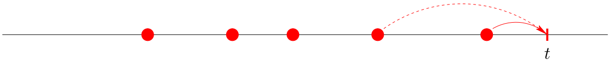
- Hawkes process: $\lambda_t = \Phi\left(\int_0^{t-} h(t-x)N(dx)\right)$.

Some classical point processes in neuroscience

- Poisson process: $\lambda_t = \lambda(t)$ (no refractory period).
- Renewal process: $\lambda_t = f(S_{t-}) \Leftrightarrow$ i.i.d. ISIs. (refractory period)

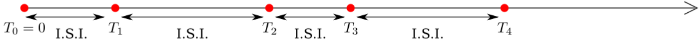


- Hawkes process: $\lambda_t = \Phi\left(\underbrace{\int_0^{t-} h(t-x)N(dx)}_{\sum_{\substack{T\in N\\ T<t}} h(t-T)}\right).$



Some classical point processes in neuroscience

- Poisson process: $\lambda_t = \lambda(t)$ (no refractory period).
- Renewal process: $\lambda_t = f(S_{t-}) \Leftrightarrow$ i.i.d. ISIs. (refractory period)

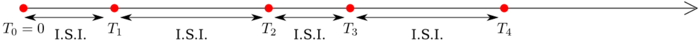


- Multivariate HP: $\lambda_t^i = \Phi\left(\int_0^{t-} h_{i\to i}(t-x)N^i(dx) + \sum_{j\neq i} \int_0^{t-} h_{j\to i}(t-x)N^j(dx) \right).$

  $(i = 1, \ldots, n)$

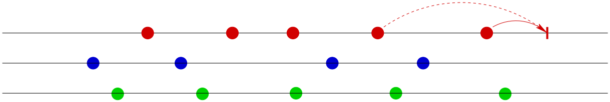
Some classical point processes in neuroscience

- Poisson process: $\lambda_t = \lambda(t)$ (no refractory period).
- Renewal process: $\lambda_t = f(S_{t-}) \Leftrightarrow$ i.i.d. ISIs. (refractory period)



- Multivariate HP: $\lambda_t^i = \Phi\left(\int_0^{t-} h_{i\to i}(t-x)N^i(dx) + \sum_{j\neq i} \int_0^{t-} h_{j\to i}(t-x)N^j(dx) \right).$

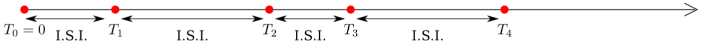
  $(i = 1, \dots, n)$



$i \to i$

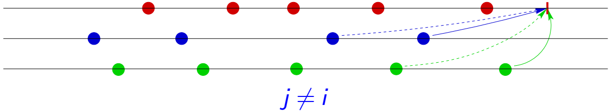
Some classical point processes in neuroscience

- Poisson process: $\lambda_t = \lambda(t)$ (no refractory period).
- Renewal process: $\lambda_t = f(S_{t-}) \Leftrightarrow$ i.i.d. ISIs. (refractory period)



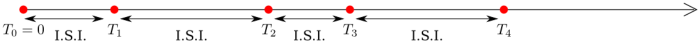
- Multivariate HP: $\lambda_t^i = \Phi\left(\int_0^{t-} h_{i\to i}(t-x)N^i(dx) + \sum_{j\neq i} \int_0^{t-} h_{j\to i}(t-x)N^j(dx) \right).$

  $(i = 1, \ldots, n)$



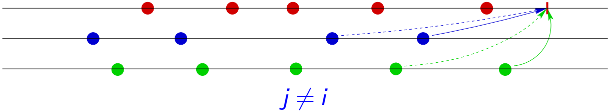
Some classical point processes in neuroscience

- Poisson process: $\lambda_t = \lambda(t)$ (no refractory period).
- Renewal process: $\lambda_t = f(S_{t-}) \Leftrightarrow$ i.i.d. ISIs. (refractory period)



- Multivariate HP: $\lambda_t^i = \Phi\left(\int_0^{t-} h_{i\to i}(t-x)N^i(dx) \right.$

$(i = 1, \ldots, n)$ $\left. + \sum_{j\neq i} \int_0^{t-} h_{j\to i}(t-x)N^j(dx) \right).$



Interaction function $h_{j\to i} \leftrightarrow$ synaptic weight of neuron j over neuron i.

Age structured equations (K. Pakdaman, B. Perthame, D. Salort, 2010)

- Age = delay since last spike.
- $u(t,s) = \begin{cases} \text{probability density of finding a neuron with age } s \text{ at time } t. \\ \text{ratio of the neural population with age } s \text{ at time } t. \end{cases}$

$$\begin{cases} \frac{\partial u(t,s)}{\partial t} + \frac{\partial u(t,s)}{\partial s} + \Psi(s, X(t))\, u(t,s) = 0 \\ u(t,0) = \int_0^{+\infty} \Psi(s, X(t))\, u(t,s)\, ds. \end{cases} \qquad \text{(PPS)}$$

Key Parameter

$$X(t) = \int_0^t h(t-x) u(x,0) dx \quad \text{(global neural activity)}$$

Age structured equations (K. Pakdaman, B. Perthame, D. Salort, 2010)

- Age = delay since last spike.
- $u(t,s) = \begin{cases} \text{probability density of finding a neuron with age } s \text{ at time } t. \\ \text{ratio of the neural population with age } s \text{ at time } t. \end{cases}$

$$\begin{cases} \frac{\partial u(t,s)}{\partial t} + \frac{\partial u(t,s)}{\partial s} + \Psi(s, X(t))\, u(t,s) = 0 \\ u(t,0) = \int_0^{+\infty} \Psi(s, X(t))\, u(t,s)\, ds. \end{cases} \qquad \text{(PPS)}$$

Key Parameter

$$X(t) = \int_0^t h(t-x) u(x,0) dx \quad \text{(global neural activity)}$$

$$X(t) \quad \longleftrightarrow \quad \int_0^{t-} h(t-x) N(dx).$$

Age structured equations (K. Pakdaman, B. Perthame, D. Salort, 2010)

- Age = delay since last spike.
- $u(t,s) = \begin{cases} \text{probability density of finding a neuron with age } s \text{ at time } t. \\ \text{ratio of the neural population with age } s \text{ at time } t. \end{cases}$

$$\begin{cases} \frac{\partial u(t,s)}{\partial t} + \frac{\partial u(t,s)}{\partial s} + \Psi(s, X(t))\, u(t,s) = 0 \\ u(t,0) = \int_0^{+\infty} \Psi(s, X(t))\, u(t,s)\, ds. \end{cases} \qquad \text{(PPS)}$$

Key Parameter

$$X(t) = \int_0^t h(t-x)u(x,0)dx \quad \text{(global neural activity)}$$

$$X(t) \quad \longleftrightarrow \quad \int_0^{t-} h(t-x)N(dx).$$

- Cacérès et al. (2015): Study of the analogy with univariate Hawkes process. Not conclusive.

Université Nice Sophia Antipolis

Age structured equations (K. Pakdaman, B. Perthame, D. Salort, 2010)

- Age = delay since last spike.
- $u(t,s) = \begin{cases} \text{probability density of finding a neuron with age } s \text{ at time } t. \\ \text{ratio of the neural population with age } s \text{ at time } t. \end{cases}$

$$\begin{cases} \dfrac{\partial u(t,s)}{\partial t} + \dfrac{\partial u(t,s)}{\partial s} + \Psi(s, X(t))\, u(t,s) = 0 \\ u(t,0) = \displaystyle\int_0^{+\infty} \Psi(s, X(t))\, u(t,s)\, ds. \end{cases} \tag{PPS}$$

Key Parameter

$$X(t) = \int_0^t h(t-x) u(x,0) dx \quad \text{(global neural activity)}$$

$$X(t) \quad \longleftrightarrow \quad \int_0^{t-} h(t-x) N(dx).$$

- Cacérès et al. (2015): Study of the analogy with univariate Hawkes process. Not conclusive.
- This system has been designed to describe a population.

Propagation of chaos: a tool to link the two scales

Mean field n-particle system

- The particles are dependent.
- Homogeneous interactions scaled by $1/n$.
- The dynamics is described by a system of n equations.

Propagation of chaos: a tool to link the two scales

Mean field n-particle system

- The particles are dependent.
- Homogeneous interactions scaled by $1/n$.
- The dynamics is described by a system of n equations.

Asymptotic when $n \to +\infty$

- The particles are independent.
- Their distribution is described by one non-linear PDE.

Propagation of chaos: a tool to link the two scales

Mean field n-particle system

- The particles are dependent.
- Homogeneous interactions scaled by $1/n$.
- The dynamics is described by a system of n equations.

Asymptotic when $n \to +\infty$

- The particles are independent.
- Their distribution is described by one non-linear PDE.

Mean field in neuroscience

- Mean-field I&F equation (Treves, 1993; Brunel and Hakim, 1999; Delarue et al., 2015)
- Mean field equation for Hawkes processes (Delattre et al., 2015) and HP with variable length memory (Galves and Löcherbach, 2015).

Université Nice Sophia Antipolis

Generalized Hawkes processes

Renewal process

$$\lambda_t = f(S_{t-})$$

Multivariate HP

$$\lambda_t^i = \Phi\left(\sum_{j=1}^{n} \int_0^{t-} h_{j\to i}(t-x) N^j(dx)\right)$$

Generalized Hawkes processes

$$\underbrace{\begin{array}{cc} \text{Renewal process} & \text{Multivariate HP} \\ \lambda_t = f(S_{t-}) & \lambda_t^i = \Phi\left(\sum_{j=1}^{n} \int_0^{t-} h_{j\to i}(t-x) N^j(dx)\right) \end{array}}_{\text{mix}}$$

Generalized Hawkes processes

$$\underbrace{\begin{array}{cc} \text{Renewal process} & \text{Multivariate HP} \\ \lambda_t = f(S_{t-}) & \lambda_t^i = \Phi\left(\sum_{j=1}^{n} \int_0^{t-} h_{j\to i}(t-x) N^j(dx) \right) \end{array}}_{\text{mix}}$$

Age dependent Hawkes process (n-neurons system)

It is a multivariate point process $(N^i)_{i=1,..,n}$ with intensity given for all $i = 1, \ldots, n$ by

$$\lambda_t^i = \Psi\left(S_{t-}^i, \frac{1}{n} \sum_{j=1}^{n} \int_0^{t-} h(t-z) N^j(dz) \right). \quad \text{“}h_{j\to i} = \frac{1}{n} h\text{”}$$

- Example: $\Psi(s,x) = \Phi(x)\mathbb{1}_{s\geq\delta} \rightsquigarrow$ strict refractory period of length δ.

Limit process

Recall the intensities of the *n*-neurons system

$$\lambda_t^i = \Psi\left(S_{t-}^i, \frac{1}{n}\sum_{j=1}^n \int_0^{t-} h(t-z)N^j(dz)\right).$$

Limit process

Recall the intensities of the *n*-neurons system

$$\lambda_t^i = \Psi\left(S_{t-}^i, \frac{1}{n} \sum_{j=1}^{n} \int_0^{t-} h(t-z) N^j(dz) \right).$$

- Independence at the limit $\Rightarrow$ Law of Large Numbers.

Limit process

Recall the intensities of the n-neurons system

$$\lambda_t^i = \Psi\left(S_{t-}^i, \frac{1}{n}\sum_{j=1}^{n} \int_0^{t-} h(t-z) N^j(dz) \right).$$

- Independence at the limit $\Rightarrow$ Law of Large Numbers.

Limit process

It is a point process $\overline{N}$ with intensity given by

$$\overline{\lambda}_t = \Psi\left(\overline{S}_{t-}, \int_0^{t-} h(t-z)\mathbb{E}\left[\overline{N}(dz)\right] \right).$$

Limit process

Recall the intensities of the n-neurons system

$$\lambda_t^i = \Psi\left(S_{t-}^i, \frac{1}{n}\sum_{j=1}^{n} \int_0^{t-} h(t-z) N^j(dz) \right).$$

- Independence at the limit $\Rightarrow$ Law of Large Numbers.

Limit process

It is a point process $\overline{N}$ with intensity given by

$$\overline{\lambda}_t = \Psi\left(\overline{S}_{t-}, \int_0^{t-} h(t-z)\mathbb{E}\left[\overline{N}(dz)\right] \right).$$

- The intensity of $\overline{N}$ depends on the time and the age.

Link between the limit process and the (PPS) system

Recall the intensity of the limit process

$$\overline{\lambda}_t = \Psi\left(\overline{S}_{t-}, \int_0^{t-} h(t-z)\mathbb{E}\left[\overline{N}(dz)\right]\right).$$

Link between the limit process and the (PPS) system

Recall the intensity of the limit process

$$\overline{\lambda}_t = \Psi\left(\overline{S}_{t-}, \int_0^{t-} h(t-z)\mathbb{E}\left[\overline{N}(dz)\right]\right).$$

Proposition

If starting from a density, the distribution of the age $\overline{S}_{t-}$ admits a density denoted $u(t,\cdot)$ for all $t \geq 0$.
Moreover, u is the unique solution of the following (PPS) system

$$\begin{cases} \dfrac{\partial u(t,s)}{\partial t} + \dfrac{\partial u(t,s)}{\partial s} + \Psi(s, X(t))\, u(t,s) = 0, \\ u(t,0) = \displaystyle\int_{s\in\mathbb{R}} \Psi(s, X(t))\, u(t,s)\, ds, \end{cases}$$

where for all $t \geq 0$, $X(t) = \int_0^t h(t-z)u(z,0)dz$.

- This result is a consequence of a previous work (Cáceres et al., 2015).

Link between the limit process and the (PPS) system

Recall the intensity of the limit process

$$\overline{\lambda}_t = \Psi\left(\overline{S}_{t-}, \int_0^{t-} h(t-z)\mathbb{E}\left[\overline{N}(dz)\right]\right).$$

Proposition

If starting from a density, the distribution of the age $\overline{S}_{t-}$ admits a density denoted $u(t,\cdot)$ for all $t \geq 0$.
Moreover, u is the unique solution of the following (PPS) system

$$\begin{cases} \dfrac{\partial u(t,s)}{\partial t} + \dfrac{\partial u(t,s)}{\partial s} + \Psi(s, X(t))\, u(t,s) = 0, \\ u(t,0) = \displaystyle\int_{s\in\mathbb{R}} \Psi(s, X(t))\, u(t,s)\, ds, \end{cases}$$

where for all $t \geq 0$, $X(t) = \int_0^t h(t-z)u(z,0)dz$.

- This result is a consequence of a previous work (Cáceres et al., 2015).

From the n-neurons system to the PDE

Propagation of chaos

Fix k in $\mathbb{N}$. Then, the processes $N^1, \ldots, N^k$ of the n-neurons system behave at the limit when $n \to +\infty$ as i.i.d. copies of the limit process $\overline{N}$.

From the n-neurons system to the PDE

Propagation of chaos

Fix k in $\mathbb{N}$. Then, the processes $N^1, \dots, N^k$ of the n-neurons system behave at the limit when $n \to +\infty$ as i.i.d. copies of the limit process $\overline{N}$.

Theorem

If the ages at time 0 are i.i.d. with common density u^{in}, then for all $t \geq 0$,

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{(S^i_{t-})_{t \geq 0}} \xrightarrow[n \to \infty]{} u(t, \cdot),$$

where u is the unique solution of the (PPS) system with initial condition u^{in}.

From the n-neurons system to the PDE

Propagation of chaos

Fix k in $\mathbb{N}$. Then, the processes $N^1, \dots, N^k$ of the n-neurons system behave at the limit when $n \to +\infty$ as i.i.d. copies of the limit process $\overline{N}$.

Theorem

If the ages at time 0 are i.i.d. with common density u^{in}, then for all $t \geq 0$,

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{(S^i_{t-})_{t \geq 0}} \xrightarrow[n \to \infty]{} u(t, \cdot),$$

where u is the unique solution of the (PPS) system with initial condition u^{in}.

- Link between (PPS) and a well-designed microscopic model.

From the n-neurons system to the PDE

Propagation of chaos

Fix k in $\mathbb{N}$. Then, the processes $N^1, \dots, N^k$ of the n-neurons system behave at the limit when $n \to +\infty$ as i.i.d. copies of the limit process $\overline{N}$.

Theorem

If the ages at time 0 are i.i.d. with common density u^{in}, then for all $t \geq 0$,

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{(S^i_{t-})_{t\geq 0}} \xrightarrow[n\to\infty]{} u(t,\cdot),$$

where u is the unique solution of the (PPS) system with initial condition u^{in}.

- Link between (PPS) and a well-designed microscopic model.
- Goodness-of fit tests: Renewal and Hawkes processes.

What does it say about our neural network model ?

From a macroscopic point of view:

- The n-neurons system consists of homogeneously interacting age dependent Hawkes processes.
- At the limit, their intensity depends on the time and the age (refractory period).
- The distribution of their ages is given by the (PPS) system.

What does it say about our neural network model ?

From a macroscopic point of view:

- The n-neurons system consists of homogeneously interacting age dependent Hawkes processes.
- At the limit, their intensity depends on the time and the age (refractory period).
- The distribution of their ages is given by the (PPS) system.

From a microscopic point of view:

- Two given neurons interact with a "strength" scaled by $1/n$.
- At the limit, they behave as if they were independent.

What more ?

Moreover:

- The interaction functions $h_{j\to i}$ can be taken as i.i.d. random variables.
- A dependence with respect to what happened before time 0 can be added (e.g. the neurons have a common stimulus so that they spiked almost at the same time).

What more ?

Moreover:

- The interaction functions $h_{j \to i}$ can be taken as i.i.d. random variables.
- A dependence with respect to what happened before time 0 can be added (e.g. the neurons have a common stimulus so that they spiked almost at the same time).

Outlook:

- Fluctuations around the mean limit behaviour (Central Limit Theorem).
- Goodness of fit tests for both micro and macro models at the same time.
- Break independence with correlated synaptic weights (cf Faugeras and Maclaurin, 2014).