Heteroclinic for a 6-dimensional reversible system occurring in orthogonal domain walls in convection

Gérard Iooss

Laboratoire J.A.Dieudonné, I.U.F., Université Côte d'Azur, CNRS,

Parc Valrose, 06108 Nice cedex 2, France

iooss.gerard@orange.fr

Abstract

A six-dimensional reversible normal form system occurs in Bénard-Rayleigh convection between parallel planes, when we look for domain walls intersecting orthogonally (see Buffoni et al [1]). This leads to study analytically the system

$$\begin{aligned} \frac{d^4A}{dx^4} &= A(1-A^2-gB^2) \\ \frac{d^2B}{dx^2} &= \varepsilon^2 B(-1+gA^2+B^2), \end{aligned}$$

for $x \in \mathbb{R}$, and looking for a heteroclinic connection between the two equilibria $M_- : (A,B) = (1,0)$ and $M_+ : (A,B) = (0,1)$, each corresponding to a system of convective rolls. In [1] such a heteroclinic is shown to exist, on which $0 \leq B \leq 1$, with no uniqueness result and no possibility to use it for a persistence result under a reversible perturbation. The lack of normal hyperbolicity in $(A,B) = (0, 1/\sqrt{g})$ of equilibria obtained at the limit $\varepsilon = 0$, is the main problem. The 3-dimensional unstable manifold of M_- is built for $0 \leq B \leq \frac{1-c\varepsilon^{4/5}}{\sqrt{g}}$, while, in solving a certain 4th-order differential equation independent of ε (occuring in [15], [2]), we overcome the lack of hyperbolicity in building the stable manifold of M_+ for $\frac{1-c\varepsilon^{4/5}}{\sqrt{g}} \leq B \leq 1$. We use [1] for proving that the two manifold intersect. Then the two 3-dimensional manifolds intersect transversally, leading to the existence, uniqueness and analyticity in (ε, g) of the heteroclinic, for which we give estimates of $A(x), B(x)$ and their derivatives. We finally study the properties of the linearized operator along the heteroclinic, allowing to prove (in [9]) the persistence of the heteroclinic under perturbation, corresponding to the existence of orthogonal domain walls in the Bénard-Rayleigh convection problem.

Key words: Reversible dynamical systems, Invariant manifolds, Bifurcations, Heteroclinic connection, Domain walls in convection

1 Introduction and Results

In this work we study the following 6th order reversible system

$$\begin{aligned} A^{(4)} &= A(1-A^2-gB^2) \\ B'' &= \varepsilon^2 B(-1+gA^2+B^2), \end{aligned} \tag{1}$$

where A and B are real functions of $x \in \mathbb{R}$. This system occurs in the search for domain walls intersecting orthogonally, in a fluid dynamic problem such as the Bénard-Rayleigh convection between parallel horizontal plates (see subsection 1.1 and all details in [1]). The heteroclinic we are looking for, corresponds to the connection between rolls on one side and rolls oriented orthogonally on the other side. The system (1) has been also introduced by Manneville and Pomeau in [15], obtained after formal physical considerations using symmetries.

We would like to find analytically a heteroclinic connection ($g > 1$, ε small) such that

$$\begin{aligned} A_*(x), B_*(x) &> 0, \\ (A_*(x), B_*(x)) &\to \left\{ \begin{array}{l} (1,0) \text{ as } x \to -\infty \\ (0,1) \text{ as } x \to +\infty \end{array} \right. . \end{aligned}$$

By a variational argument Boris Buffoni et al [1] prove the existence of such an heteroclinic orbit, for any $g > 1$, and ε small enough. This type of elegant proof does not unfortunately allow to prove the persistence of such heteroclinic curve under reversible perturbations of the vector field. This is our motivation for producing analytic arguments, proving such an existence, uniqueness and smoothness in parameters (ε, g) of this orbit (in particular analyticity in g), however for limited values $10/9 < g \leq 2$, fortunately including physical interesting ones. Then we study the linearized operator along the heteroclinic curve, allowing to attack the problem of existence of orthogonal domain walls in convection (see [9] and Remark 37).

1.1 Origin of system (1)

The Bénard-Rayleigh convection problem is a classical problem in fluid mechanics. It concerns the flow of a three-dimensional viscous fluid layer situated between two horizontal parallel plates and heated from below. Upon increasing the difference of temperature between the two plates, the simple conduction state looses stability at a critical value of the temperature difference corresponding to a critical value $\mathcal{R}_c$ of the Rayleigh number. Beyond the instability threshold, a convective regime develops in which patterns are formed, such as convective rolls, hexagons, or squares. Observed patterns are often accompanied by defects.

We start with the Navier-Stokes-Boussinesq (N-S-B) *steady* system of PDE's, applying spatial dynamics with x as "time" (as introduced by K.Kirchgässner in [13], adapted for N-S equations in [10], and more generally in [7]) and considering solutions $2\pi/k$ periodic in y (coordinate parallel to the wall). We show in [1] that near criticality a 12-dimensional center manifold reduction to a reversible system applies for $(\mathcal{R}, k)$ close to $(\mathcal{R}_c, k_c)$, $\mathcal{R}$ being the Rayleigh number, and k_c the critical wave number. This high dimension of the center manifold may be explained as follows. Due to the equivariance of the system under horizontal shifts, the eigenvectors of the linearized problem are of the form $\exp i(\pm k_1 x \pm k_2 y)$, the factor being only function of $k^2 = k_1^2 + k_2^2$ (invariance under rotations). It results that, for eigenvectors independent of x corresponding to a 0 eigenvalue in the spatial dynamics formulation, the eigenvalue is double in general (make $\pm k_1 \to 0$). Now, at criticality, $k = k_c$ corresponds to two different values of k_2 merging towards k_c, which doubles the dimension, making a quadruple 0 eigenvalue with complex and complex-conjugate eigenvectors. Hence we already have a dimension 8 invariant subspace for the 0 eigenvalue, with two 4×4 Jordan blocks. This corresponds to convective rolls of amplitude A and $\overline{A}$ at $x = -\infty$. Now for eigenvectors independent of y corresponding to eigenvalues $\pm ik$ in the spatial dynamics formulation it is shown in [6] that they are simple in general, and give double eigenvalues $\pm ik_c$ for $k = k_c$ with amplitudes B and $\overline{B}$ respectively. Hence this adds 4 dimensions to the central space, so finally obtaining a 12-dimensional central space. Now we restrict the study to solutions invariant under reflection $y \to -y$ (the change y into $-y$ changing A in $\overline{A}$ and not changing B), which constitutes an *invariant subspace for the full system.* This restricts the study to *real* amplitudes A and the full system reduces to a 8-dimensional sub-center manifold, such that $A \in \mathbb{R}$ and $B \in \mathbb{C}$ are the amplitudes of the rolls respectively at $x = -\infty$, and $x = +\infty$. Moreover, for the full system, we keep

i) the reversibility symmetry: $(x, A, B) \to (-x, A, \overline{B})$,

ii) the equivariance under shifts by half of a period in y direction, leading to the symmetry: $(A, B) \to (-A, B)$.

Now, in [1] we use a normal form reduction up to cubic order, and rewrite the system as one real 4th order differential equation for A, and a second order complex differential equation for B. In addition to the above symmetries, the normal form commutes in particular with the symmetry: $(A, B) \to (A, Be^{i\phi})$, for any $\phi \in \mathbb{R}$.

Handling the full N-S-B equations, in [1] the authors show that the study leads to a small perturbation of the reduced system of amplitude equations (1). More precisely, after a suitable scaling (see [1] and more details in [9]), and denoting by $(\varepsilon^2 A_0, \varepsilon^2 B_0)$ rescaled amplitudes $(A, Be^{-ik_c x})$, and after a rescaling of the coordinate x, we obtain the system

$$\begin{aligned} A_0^{(4)} &= k_- A_0'' + A_0\left(1 - \frac{k_-^2}{4} - A_0^2 - g|B_0|^2\right) + \widehat{f}, \\ B_0'' &= \varepsilon^2 B_0(-1 + gA_0^2 + |B_0|^2) + \widehat{g}, \end{aligned} \tag{2}$$

where ε^4 is proportional to $\mathcal{R} - \mathcal{R}_c$, the coefficient $g > 1$ is function of the Prandtl number and is the same as introduced and computed in [6], k_- comes from the freedom left to the wave number of the rolls at $-\infty$, defined as

$$k = k_c(1 + \varepsilon^2 k_-),$$

and $\widehat{f}$ and $\widehat{g}$ are perturbation terms, smooth functions of their arguments, coming

i) from the rest of the cubic normal form, at least of order ε^2 for $\widehat{f}$, and at least of order ε^3 for $\widehat{g}$;

ii) from higher order terms not in normal form, and not autonomous (because of the introduction of $Be^{-ik_c x}$ rescaled as $\varepsilon^2 B_0$ in (2)), and of order ε^4 for $\widehat{f}$, and of order ε^6 for $\widehat{g}$. Without k_-, $\widehat{f}$, and $\widehat{g}$, this is the system (1), with $B_0 \in \mathbb{C}$ replacing B, and $|B_0|^2$ replacing B^2. The truncation leading to (1) allows to take B real, since the phase of B_0 does not play any role in the dynamics for (1). The two different wave numbers of the rolls, close to the critical value k_c are left free for the full problem, however they do not appear in the present proof of the heteroclinic, even though they are important for the final proof of existence of the orthogonal domain walls (see Remark 37 in section 6). It should be noticed that the system (2), without $\widehat{f}$ and $\widehat{g}$, was obtained a long time ago by Pomeau-Manneville in [15], however they did not deal with the full N-S-B system, and only considered cases with identical wave numbers at infinities, while it is shown in [9] that some cubic terms, not existent in [15], as $\varepsilon^2(A_0^2A_0'' - A_0A_0'^2)$ in $\widehat{f}$ and $i\varepsilon^3 B_0A_0A_0'$ in $\widehat{g}$ are crucial for the determination of the solutions of the full problem, with different wave numbers at infinities (see Remark 37).

1.2 Sketch of the method and results

From now on let us consider the system (1). The equilibrium $(A, B) = (0, 1)$ of the system (1) gives an approximation of convection rolls parallel to the wall (periodic in the x direction, with fixed phase) bifurcating for Rayleigh numbers $\mathcal{R} > \mathcal{R}_c$ close to $\mathcal{R}_c$, whereas the equilibrium $(A, B) = (1, 0)$ of the system (1) gives the same convection rolls (periodic in the y direction) rotated by an angle $\pi/2$ with the phase fixed by the imposed reflection symmetry. A heteroclinic orbit connecting these two equilibria provides then an approximation of orthogonal domain walls (see Figure 2).

We set $\delta = (g-1)^{1/2}$. The idea here might be to use the arc of equilibria $A^2 + B^2 = 1$, which exists for $\delta = 0$, connecting end points $M_- = (1, 0)$ and $M_+ = (0, 1)$, and to prove that for suitable values of δ (> 0 but close to 0), the 3-dimensional unstable manifold of M_- intersects transversally the 3-dimensional stable manifold of M_+, both staying on a 5 dimensional invariant manifold $\mathcal{W}_{\varepsilon,\delta}$. However, for $\delta = 0$ the situation in M_+ is very degenerated, with a quadruple 0 eigenvalue for the linearized operator, while it is a double eigenvalue for M_-. Then for δ close to 0, a 5-dimensional center-stable invariant manifold starting from M_+ needs to intersect a four-dimensional center-unstable manifold starting from M_-. We are not able to prove this. Moreover, for $\delta \neq 0$ but close to 0, we cannot prove that the 3-dimensional unstable manifold of M_- exists from $B = 0$ until B reaches a value close enough to 1. In fact, we may fortunately notice that the physically interesting values of δ are not close to 0 (see Remark 7). So that we prefer to play with ε.

We may observe that, after changing the coordinate x in $\overline{x} = \varepsilon x$, we obtain the new system

$$\begin{aligned} \varepsilon^4 \frac{d^4 A}{d\overline{x}^4} &= A(1 - A^2 - gB^2) \\ \frac{d^2 B}{d\overline{x}^2} &= B(B^2 + gA^2 - 1), \end{aligned} \tag{3}$$

where the limit $\varepsilon \to 0$ is singular, and gives indeed a non smooth heteroclinic solution such that

(i) for x running from $-\infty$ to 0, then (A, B) varies from $(1, 0)$ to $(0, \frac{1}{\sqrt{g}})$ on the ellipse $A^2 + gB^2 = 1$, while

(ii) for x running from 0 to $+\infty$, then (A, B) varies from $(0, \frac{1}{\sqrt{g}})$ to $(0, 1)$, satisfying, in the original coordinate x, the differential equation

$$\frac{dB}{dx} = \frac{\varepsilon}{\sqrt{2}}(1 - B^2).$$

The two manifolds $A = \widetilde{A_*} = (1 - gB^2)^{1/2}$, and $A = 0$ are named "slow

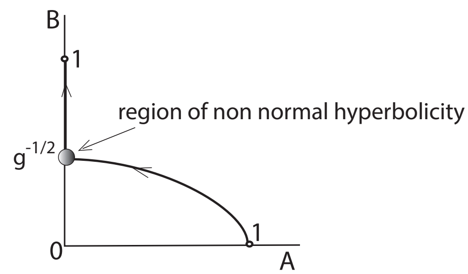

Figure 1: Critical manifold

manifolds" in litterature (see [4],[14]). We might then think to use Fenichel's theorems [4] on the system (1) for ε close to 0. For the part (i) of the curve, where $x \in (-\infty, 0]$, the set of equilibria, here $A^2 + gB^2 = 1$, is not normally hyperbolic at the end point $(A, B) = (0, 1/\sqrt{g})$ (see in section 3 eigenvalues of the linear operator $\boldsymbol{L}_\delta$ corresponding to $\widetilde{A_*} = 0$). For the second part (ii) of the curve, where $x \in [0, +\infty)$, the set of equilibria $(A, B) = (0, B)$ is also not normally hyperbolic for $B = 1/\sqrt{g}$ (the 4 remaining eigenvalues are such that $\lambda^4 = 1 - gB^2$ which cancel for $B = 1/\sqrt{g}$). The normal hyperbolicity is essential in Fenichel's theorems, so we cannot use them directly. However we may use normal hyperbolicity up to a small neighborhood of $(A, B) = (0, 1/\sqrt{g})$, as this is done in sections 3 for finding the unstable manifold of $(A, B) = (1, 0)$ in a neighborhoof of the slow manifold $A = \widetilde{A_*}$, and in section 4.4 for finding the stable manifold of $(A, B) = (0, 1)$ in a neighborhood of the slow manifold $A = 0$.

The neighborhood of $(A, B) = (0, 1/\sqrt{g})$ not reached by the method above has a size of order $\mathcal{O}(\varepsilon^{4/5})$. We could think to use a geometric analysis, as Krupa et al did in [14], where a blow-up method is used for getting a system independent of ε. Indeed the scaling

$$\begin{aligned} A &= K^2\varepsilon^{2/5}\overline{A}, \\ B &= \frac{1}{\sqrt{g}}(1 + \frac{K^4}{2}\varepsilon^{4/5}\overline{B}), \\ x &= \frac{z}{K\varepsilon^{1/5}}, \end{aligned}$$

with $\overline{B} = z$, since at main order

$$\overline{B}'' = 0, \ \overline{B}(0) = 0,$$

leads, at main order, to

$$\frac{d^4\overline{A}}{dz^4} = -\overline{A}(\overline{A}^2 + z), \ z \in [-a_-, +a_+], \tag{4}$$

which is independent of ε. However, the work of [14] is made in 2 dimensions, while we have here the 6-dimensional system (1). It results that the nice pictures of [14] would be very hard to transpose here. In addition, we need to satisfy boundary values (also independent of ε) coming on the left side from the connection with the unstable manifold, and from the right side from the connection with the stable manifold.

Moreover we need to provide *precise estimates* (in function of ε) of the interval of values for B, between the value reached by the unstable manifold, via the standard method, and the value reached (backwards) by B for the stable manifold. The critical value $1/\sqrt{g}$ is included in this finite interval, and this finiteness is essential for extending the existence of the stable manifold on the full interval, until it meets the unstable manifold.

In section 2.2 we see that there are 3 unstable eigendirections starting from $M_- = (1, 0)$, and 3 stable eigendirections in $M_+ = (0, 1)$. The difficulty in the proof of Theorem 1 is to obtain a precise estimate for the existence of the 3-dimensional unstable manifold of M_-, where the coordinate B varies from 0 to a neighborhood of $1/\sqrt{g}$, and to obtain a precise estimate for the existence of the 3-dimensional stable manifold of M_+ until B varies from 1 (backwards) to a neighborhood of $1/\sqrt{g} = 1/\sqrt{1+\delta^2}$, while A stays close to 0. For approaching the closest possible to $B = 1/\sqrt{g}$, we use the first integral of (1), which implies that both invariant manifolds are included in a 5-dimensional invariant manifold. We are able to obtain the unstable manifold of M_- for $0 \leq B \leq \frac{1-c\varepsilon^{4/5}}{\sqrt{g}}$, while we first obtain the stable manifold of M_+ for $\frac{1+c'\varepsilon^{4/5}}{\sqrt{g}} \leq B \leq 1$. For extending the existence of the stable manifold in the gap of size of order $\varepsilon^{4/5}$, we need to solve the 4th order differential equation (4), independent of ε, also found in [15] and [2], after rescaling, where the boundary conditions, also independent of

ε, come from the 2 times 2 parameters introduced by each invariant manifolds arriving in $\pm a_{\pm}$.

We use a precise estimate on a_+ for being able to extend the domain of existence of the stable manifold, for B in the interval $\frac{1-c\varepsilon^{4/5}}{\sqrt{g}} \leq B \leq 1$. Using results of [1] the two manifolds intersect. We prove the following

Theorem 1 *Let us choose $1/3 \leq \delta \leq 1$, then for ε small enough, the 3-dim unstable manifold of M_- intersects transversally the 3-dim stable manifold of M_+, except maybe for a finite set of values of δ. The connecting curve which is obtained is unique (see Remark 5). Moreover its dependency in parameters (ε, δ) is analytic. In addition we have $B(x) > 0$ and $B'(x) > 0$ on $(-\infty, +\infty)$. For $x \to -\infty$ we have $(A-1, A', A'', A''', B, B') \to 0$ at least as $e^{\varepsilon\delta x}$, while for $x \to +\infty$, $(A, A', A'', A''') \to 0$ at least as $e^{-\sqrt{\frac{\delta}{2}}x}$, and $(B-1, B') \to 0$ at least as $e^{-\sqrt{2}\varepsilon x}$.*

Moreover we also have important estimates as follows, extensively used in [9].

Corollary 2 *For $x \in (-\infty, 0]$ and choosing $\delta^* < \delta$, there exists $c > 0$ independent of ε small enough, such that the heteroclinic curve satisfies*

$$\begin{aligned} |A(x) - \sqrt{1-(1+\delta^2)B(x)}| &\leq c\varepsilon^{2/5}B(x)e^{\varepsilon\delta^* x} \\ |A^{(m)}(x)| &\leq c\varepsilon^{3/5}B(x)e^{\varepsilon\delta^* x},\ m = 1, 2, 3. \end{aligned}$$

Corollary 3 *For $x \in [0, +\infty)$ and $\delta_* = \frac{1}{10}\delta^{2/5}$, there exists $c > 0$ independent of ε small enough, such that the heteroclinic curve satisfies*

$$|A^{(m)}(x)| \leq c\varepsilon^{2/5}e^{-\delta_*\varepsilon^{1/5}x},\ m = 0, 1, 2, 3.$$

Remark 4 *It should be noticed that we show at Lemma 33 that, in the middle of the heteroclinic, $A(0) = \mathcal{O}(\varepsilon^{2/5})$ and for $x \in (0, +\infty)$, $A(x)$ oscillates, staying of order $\mathcal{O}(\varepsilon^{2/5})$, while $B(0) = 1/\sqrt{g}$ and $B(x)$ grows monotonically until 1.*

Remark 5 *Using symmetries of the system: $A \mapsto \pm A$, $B \mapsto \pm B$ and reversibility symmetry: $(A(x), B(x)) \mapsto (A(-x), B(-x))$, we find 8 heteroclinics. Two are connecting M_- to M_+ with opposite dynamics, two others connect $-M_-$ to M_+, two connect M_- to $-M_+$, and two connect $-M_-$ to $-M_+$. The one which interests us is the only one connecting M_- to M_+ with the dynamics running from M_- to M_+.*

Remark 6 *It should be noticed that the study made in [15] on the heteroclinic solution for the system (1) uses asymptotic analysis, suggesting the existence of the heteroclinic, later proved mathematically in [1]. Contrary to these previous works, using asymptotic analysis on the full real line, the precise estimate which is obtained for a_+(see (4)) is essential here, for getting a rigorous result.*

Remark 7 *Values of* δ *such that* $0.476 \leq \delta$ *include values obtained for* δ *in the Bénard-Rayleigh convection problem where* $g = 1 + \delta^2$ *is function of the Prandtl number* $\mathcal{P}$ *(as computed in [6]). With rigid-rigid, rigid-free, or free-free boundaries the minimum values of* g *are respectively* ($g_{\min} = 1.227$, 1.332, 1.423) *corresponding to* $\delta_{\min} = 0.476$, 0.576, 0.650. *The restriction in Theorem 1 corresponds to* $1 < g \leq 2$. *Then, the eligible values for the Prandtl number are respectively* $\mathcal{P} > 0.5308, > 0.6222, > 0.8078$.

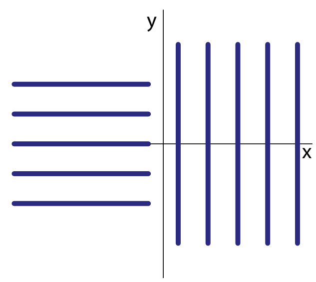

Figure 2: Orthogonal domain wall

The Schedule of the paper is as follows: in section 3 we prove at Lemma 13 the existence of the 3-dimensional unstable manifold of $M_- = (1, 0)$ for $B \in [0, (1 - \alpha_-^2\delta^2)/\sqrt{1 + \delta^2}]$ with $\varepsilon = \nu_-\alpha_-^{5/2}$, ν_- being independent of ε, for $x \in (-\infty, -x_*]$.

In section 4 we prove at Lemma 27 the existence of the 3-dimensional stable manifold of $M_+ = (0, 1)$ for $B \in [(1 + \alpha_+^2\delta^2)/\sqrt{1 + \delta^2}, 1]$, and $x \in [x_*^+, +\infty)$.

In section 5, we solve a certain 4th-order differential equation, on the finite interval $[-x_*, x_*^+]$ which, once rescaled, is independent of ε, so that we are able to extend the existence of the stable manifold for $B \in [(1-\alpha_-^2\delta^2)/\sqrt{1 + \delta^2}), (1+\alpha_+^2\delta^2)/\sqrt{1 + \delta^2})]$, A_0 still of order $\mathcal{O}(\alpha_+)$. Then we use results of [1] to control the existence of the intersection, and then prove the transverse intersection of the two manifolds. This ends the proof of Theorem 1.

In section 6 we give, in Lemma 35, properties of the linearized operator along the heteroclinic, which are necessary to prove a persistence result under a reversible perturbation for the heteroclinic in the 8-dimensional space (with $B \in \mathbb{C}$). This allows to prove the existence of orthogonal domain walls in convection as made in [9].

In summary, what is new in this paper?

i) Existence of the unstable manifold of M_-, analytic in δ, while coordinate B is an increasing function for $x \in (-\infty, -x_*]$, varying from 0 to a value $\mathcal{O}(\varepsilon^{4/5})-$ close to $1/\sqrt{g}$. Existence of the stable manifold of M_+, analytic in δ, while coordinate B is an increasing function for $x \in [x_*^+, +\infty)$, varying from a value $\mathcal{O}(\varepsilon^{4/5})-$ close to $1/\sqrt{g}$, to 1.

ii) Justification and resolution backwards of the intermediate 4th order differential equation (4) independent of ε, already introduced in [15] and [2], but now on a bounded interval with boundary conditions independent of ε, the solution being analytic in δ.

iii) On the heteroclinic, $B_0(x)$ satisfies $B_0'(x) > 0$. Estimates for coordinates in $\mathbb{R}^6$ are established, which are essential for a further study on the persistence under perturbations of the heteroclinic, as for the Bénard-Rayleigh convection problem.

iv) Study of the conditions for the invertibility of the linearized operator, along the heteroclinic, useful for any perturbation result.

Remark 8 *There are many lengthy awful calculations in this work. However, they are necessary for getting precise estimates.*

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2 General properties of the system

2.1 Global invariant manifold $\mathcal{W}_{\varepsilon,\delta}$

Let us define coordinates in $\mathbb{R}^6$ as

$$(A_0, A_1, A_2, A_3, B_0, B_1) = (A, A', A'', A''', B, B').$$

The first observation is that we have the first integral

$$W = \varepsilon^2 (A'^2)'' - 3\varepsilon^2 A''^2 - B'^2 + \frac{\varepsilon^2}{2}(A^2 + B^2 - 1)^2 + \varepsilon^2 \delta^2 A^2 B^2, \tag{5}$$

as noticed in [15], where W is used in an energy functional, used later in [1]. Then, for containing the end points $M_\pm$, our heteroclinic should satisfy

$$2\varepsilon^2 A_1 A_3 - \varepsilon^2 A_2^2 - B_1^2 + \frac{\varepsilon^2}{2}(A_0^2 + B_0^2 - 1)^2 + \varepsilon^2 \delta^2 A_0^2 B_0^2 = 0. \tag{6}$$

Since our purpose is to find B_0 growing from 0 to 1, we extract the positive square root (needs to be justified later):

$$B_1 = \{2\varepsilon^2 A_1 A_3 - \varepsilon^2 A_2^2 + \frac{\varepsilon^2}{2}(A_0^2 + B_0^2 - 1)^2 + \varepsilon^2 \delta^2 A_0^2 B_0^2\}^{1/2},$$

which defines a *5-dimensional invariant maniford* $\mathcal{W}_{\varepsilon,\delta}$ valid for any $\delta > 0$, which should contain the heteroclinic curve that we are looking for.

For $\delta > 0$, we find the singular points (where a tangent hyperplane is not defined)

$$\begin{aligned} (A_0, B_0) &= (\pm 1, 0),\ A_1 = A_2 = A_3 = B_1 = 0 \\ (A_0, B_0) &= (0, \pm 1),\ A_1 = A_2 = A_3 = B_1 = 0. \end{aligned} \tag{7}$$

For $\delta = 0$, singular points constitute the circle

$$A_0^2 + B_0^2 = 1,\ A_1 = A_2 = A_3 = B_1 = 0. \tag{8}$$

Remark 9 *We do not emphasize here on the hamiltonian structure of system (1) since this does not help our understanding. On the contrary, the reversibility property is inherited from the original physical problem and is still valid for the perturbed system (2). Moreover, if we consider perturbation terms as* $\varepsilon^2(A_0^2 A_0'' - A_0 A_0'^2)$ *in* $\widehat{f}$ *and* $i\varepsilon^3 B_0 A_0 A_0'$ *in* $\widehat{g}$*, we cannot find a new first integral analogue to (6), while the system is still reversible.*

2.2 Linear study of the dynamics

2.2.1 Neighborhood of $M_- = (1, 0)$

The eigenvalues of the linearized operator at M_- are such that $\lambda^4 = -2$ or $\lambda^2 = \varepsilon^2\delta^2$, hence they are $\pm 2^{-1/4}(1 \pm i)$ and $\pm\varepsilon\delta$. This gives a 3-dimensional unstable manifold, and a 3-dimensional stable manifold, originating from M_-.

2.2.2 Neighborhood of $M_+ = (0, 1)$

The eigenvalues of the linearized operator at M_+ are such that $\lambda^4 = -\delta^2$ or $\lambda^2 = 2\varepsilon^2$, hence defining $\delta' = \sqrt{\delta}$, the eigenvalues are $\pm 2^{-1/2}(1 \pm i)\delta'$, and $\pm\varepsilon\sqrt{2}$. This gives again a 3-dimensional unstable manifold and a 3-dimensional stable manifold originating from M_+.

All this implies that the 3-dimensional unstable manifold starting at M_- and the 3-dimensional stable manifold starting at M_+ which are both included into the 5−dimensional manifold $\mathcal{W}_{\varepsilon,\delta}$, give a good hope for these two manifolds to intersect along a heteroclinic curve...provided that they still exist as graphs with respect to B, "far" from the end points M_+ and M_-. The idea is to show that this occurs when δ is not too small and at most 1.

Remark 10 *The limit points* $M_- = (1, 0)$ *and* $M_+ = (0, 1)$ *have a degenerate situation for* $\delta = 0$*, because of the multiple* 0 *eigenvalue for the linearized operator. For* $\delta = 0$*, it is possible to build a family of 2-dim unstable invariant manifolds and a family of 2-dim stable manifolds along the arc of equilibria* $A^2 + B^2 = 1$*. For* $\delta > 0$ *and small, the perturbation gives two new 3-dim invariant manifolds, however their transversality is weaker and weaker as* $B \to 1$ *(so that Fenichel's theorem cannot apply). A more "serious" study would then be needed. However the physical interest is for values of* $\delta > 0$ *not too small, which cancels the physical interest of such a difficult question (see Remark 7).*

3 Unstable manifold of M_-

3.1 Choice of coordinates

Let us assume in this section $1/3 \leq \delta \leq 1$ and define η_0 and α_- such that

$$\begin{aligned} 0 &\leq B_0 \leq \sqrt{1-\eta_0^2\delta^2} = \left(\frac{1-\alpha^2\delta^2}{1+\delta^2}\right)^{1/2}, \quad \eta_0^2 = \frac{1+\alpha^2}{1+\delta^2}, \\ \frac{1}{\sqrt{g}} &= \frac{1}{\sqrt{1+\delta^2}} < \eta_0 < \frac{1}{\delta}, \ \alpha\delta < 1. \end{aligned}$$

α will be determined later, as a power of ε. Now, we define

$$\widetilde{A_*}^2 \stackrel{def}{=} 1-(1+\delta^2)B_0^2, \tag{9}$$

then

$$\alpha\delta \leq \widetilde{A_*} \leq 1. \tag{10}$$

Let us define the following coordinates in $\mathbb{R}^6$:

$$Z = (\widetilde{A_*}+\widetilde{A_0}, A_1, A_2, A_3, B_0, B_1)^t. \tag{11}$$

Remark 11 *The assumption $B_0 \geq 0$ comes from the result of [1] where $0 \leq B_0(x) \leq 1$ along the heteroclinic; $\widetilde{A_*}$ is just the first part of the "singular" heteroclinic found for the system singular for $\varepsilon = 0$ (3). The occurence of $\widetilde{A_*}$ is also linked with a formal computation of an expansion of the heteroclinic in powers of ε, which gives $\widetilde{A_*}$ as the principal part of A_0, valid for $B_0 < (1+\delta^2)^{-1/2} = 1/\sqrt{g}$. We expect to build the unstable manifold until this limit value.*

Remark 12 *We put a condition $1/3 \leq \delta$ in the purpose to have not too small values for δ, and to include known computed values of the coefficient $g = 1+\delta^2$, in the convection problems, with different boundary conditions (see Remark 7 and [6]). The restriction $\delta \leq 1$ made in this section simplifies few estimates, and is not really useful.*

We prove below the main result of this section:

Lemma 13 *For $1/3 \leq \delta \leq 1$, and ε small enough, the 3-dimensional unstable manifold $\mathcal{W}^{(u)}_{\varepsilon,\delta}$ of M_- exists for*

$$0 \leq B_0(x) \leq (1-\eta_0^2\delta^2)^{1/2} = \left(\frac{1-\alpha_-^2\delta^2}{1+\delta^2}\right)^{1/2}, \ x \in (-\infty, -x_*],$$

where $x_* \geq 0$ is arbitrary, and there exists $\nu_- > 0$ such that η_0 and α_- satisfy

$$(1+\delta^2)\eta_0^2 = 1+\alpha_-^2, \ \varepsilon = \nu_-\alpha_-^{5/2}.$$

The manifold $\mathcal{W}_{\varepsilon,\delta}^{(u)}$ *sits in* $\mathcal{W}_{\varepsilon,\delta}$*, is analytic in* (ε,δ)*, parametrized by* $(X(-x_*), B_0(-x_*))$ *where* $X(x)$ *is a 2-dimensional coordinate on the* B_0-*dependent unstable directions defined by (13). Moreover, for any* $\delta^* < \delta$*, there exist a number* $k_0 > 0$ *independent of* ε, ν_- *such that for*

$$B_0(-x_*) = \sqrt{1-\eta_0^2\delta^2},\ \widetilde{A_*}(x) \geq \alpha\delta,\ |X(-x_*)| \leq k_0\delta\alpha_-^{3/2} = \frac{k_0}{\nu_-^{3/5}}\delta\varepsilon^{3/5},$$

we have

$$\begin{aligned} A_0(x) &= \widetilde{A_*}(x) + B_0(x)\mathcal{O}\left(\frac{|X(-x_*)|}{\widetilde{A_*}(x)^{1/2}}e^{\varepsilon\delta^* x}\right) \\ A_1(x) &= B_0(x)e^{\varepsilon\delta^* x}\mathcal{O}(\nu^{2/5}\varepsilon^{3/5} + |X(-x_*)|) \\ A_2(x) &= B_0(x)\widetilde{A_*}(x)^{1/2}\mathcal{O}(|X(-x_*)|e^{\varepsilon\delta^* x}) \\ A_3(x) &= B_0(x)\widetilde{A_*}(x)\mathcal{O}(|X(-x_*)|e^{\varepsilon\delta^* x}), \end{aligned}$$

$$0 \leq 1 - \widetilde{A_*} \leq cB_0^2,\ \widetilde{A_*}|_{B_0=0} = 1.$$

Remark 14 *We observe that when* $x \to -x_*$*,* A_0 *reaches a value close to* 0 *since* $\widetilde{A_*}$ *reaches* $\mathcal{O}(\varepsilon^{2/5})$ *which is close to* 0*, while* B_0 *reaches* $(1-\eta_0^2\delta^2)^{1/2}$ *which is expected as close as possible to* $1/(1+\delta^2)^{1/2} = 1/\sqrt{g}$*. The numbers* k_0 *and* ν_- *will later be imposed, small enough, independently of* ε*. Later* x_* *will be chosen of order* $\varepsilon^{-1/5}$ *(see section 4.2).*

Strategy in section 3.

The strategy is first in sections 3.1, 3.2, 3.3 to write the system (1) in adapted coordinates, in a neighborhood of $(\widetilde{A_0}, A_1, A_2, A_3, B_1) = 0$, which are B_0- dependent. Then, in sections 3.4, 3.5 we eliminate B_1 (coordinate z_1) in using the first integral (6).We now look for a 3-dimensional unstable manifold lying in the 5-dimensional manifold $\mathcal{W}_{\varepsilon,\delta}$. In sections 3.6, 3.7, 3.8 we solve the system for the unstable manifold, the function $B_0(x)$ taken as a parameter, still unknown. Then, in section 3.9 we solve the remaining differential equation for $B_0(x)$. In section 3.10 we give a quite explicit form for the 2-dimensional intersection of the 3-dimensional plane tangent to the unstable manifold $\mathcal{W}_{\varepsilon,\delta}^{(u)}$ with the hyperplane H_0 defined by $B_0 = \sqrt{1-\eta_0^2\delta^2}$.

With the system of coordinates (11), the system (1) becomes

$$\begin{aligned} \widetilde{A_0}' &= A_1 + \frac{(1+\delta^2)B_0}{\widetilde{A_*}}B_1 \\ A_1' &= A_2 \\ A_2' &= A_3 \\ A_3' &= -2\widetilde{A_*}^2\widetilde{A_0} - 3\widetilde{A_*}\widetilde{A_0}^2 - \widetilde{A_0}^3 \\ B_0' &= B_1 \\ B_1' &= \varepsilon^2\delta^2 B_0(\widetilde{A_*}^2 - B_0^2) + 2\varepsilon^2(1+\delta^2)\widetilde{A_*}B_0\widetilde{A_0} + \varepsilon^2(1+\delta^2)B_0\widetilde{A_0}^2. \end{aligned} \tag{12}$$

We expect that $\widetilde{A_0}, A_1, A_2, A_3, B_1$ stay small enough for $x \in (-\infty, 0]$, so we introduce the B_0-dependent linear operator (not really a linearization at some equilibrium)

$$\boldsymbol{L}_\delta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \frac{(1+\delta^2)B_0}{\widetilde{A_*}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2\widetilde{A_*}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2\varepsilon^2(1+\delta^2)\widetilde{A_*}B_0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (13)$$

The idea is to find new coordinates such that we are able to give nice estimates of the monodromy operator not forgetting that the coefficients of $\boldsymbol{L}_\delta$ are functions of B_0.

The operator $\boldsymbol{L}_\delta$ has a double eigenvalue 0, and is such that the non zero eigenvalues satisfy

$$\lambda^4 - 2\varepsilon^2 B_0^2(1+\delta^2)^2\lambda^2 + 2\widetilde{A_*}^2 = 0, \quad (14)$$

with a discriminant such as

$$\Delta' = \varepsilon^4 B_0^4(1+\delta^2)^4 - 2\widetilde{A_*}^2.$$

We have the following

Lemma 15 *For $B_0 \leq \sqrt{1-\eta_0^2\delta^2}$, $\alpha \geq \frac{10}{3}\varepsilon^2$, and ε small enough, we have*

$$-\Delta' \geq \widetilde{A_*}^2.$$

Proof. $-\Delta' \geq \widetilde{A_*}^2$ is equivalent to

$$\varepsilon^2 B_0^2(1+\delta^2)^2 \leq \widetilde{A_*}, \quad (15)$$

hence

$$B_0^2(1+\delta^2) \leq \frac{[1+4\varepsilon^4(1+\delta^2)^2]^{1/2}-1}{2\varepsilon^4(1+\delta^2)^2},$$

which is satisfied when

$$B_0^2(1+\delta^2) \leq 1 - \varepsilon^4(1+\delta^2)^2,$$

provided that $\varepsilon^4(1+\delta^2)^2 < 2$ which is true for ε small enough. Now, since $B_0 \leq \sqrt{1-\eta_0^2\delta^2}$, the above inequality is satisfied a soon as we have

$$1 - \delta^2\alpha^2 < 1 - \varepsilon^4(1+\delta^2)^2,$$

which is realized when $\alpha \geq \frac{10}{3}\varepsilon^2$. ■

Then we have two pairs of complex eigenvalues

$$\lambda_\pm^2 = \varepsilon^2 B_0^2 (1+\delta^2)^2 \pm i\sqrt{-\Delta'}.$$

We intend to find new coordinates able to manage a new linear operator in the form of two independent blocs

$$\begin{pmatrix} \pm\lambda_r & \lambda_i \\ -\lambda_i & \pm\lambda_r \end{pmatrix} \tag{16}$$

for which the eigenvalues are

$$\pm\lambda_r \pm i\lambda_i,$$

where

$$\begin{aligned} 2\lambda_r^2 &= \sqrt{2}\widetilde{A_*} + \varepsilon^2 B_0^2(1+\delta^2)^2 \\ 2\lambda_i^2 &= \sqrt{2}\widetilde{A_*} - \varepsilon^2 B_0^2(1+\delta^2)^2 \\ \lambda_r^2 - \lambda_i^2 &= \varepsilon^2 B_0^2 (1+\delta^2)^2 \\ \lambda_r^2 + \lambda_i^2 &= \sqrt{2}\widetilde{A_*} \\ 4\lambda_r^2\lambda_i^2 &= -\Delta'. \end{aligned} \tag{17}$$

A form of the linear operator as (16) is such that we are able to have good estimates for the monodromy operator associated with the linear operator $\boldsymbol{L}_\delta$, the coefficients of which are functions of $B_0 \in [0, \sqrt{1-\eta_0^2\delta^2}]$ (see Appendix A.1). Then we have the following

Lemma 16 *For $B_0 \leq \sqrt{1-\eta_0^2\delta^2}$, $\alpha \geq \frac{10}{3}\varepsilon^2$, and for ε small enough, we have*

$$\lambda_r\lambda_i \geq \frac{\widetilde{A_*}}{2},$$

$$2^{1/4}\widetilde{A_*}^{1/2} \geq \lambda_r \geq \frac{\widetilde{A_*}^{1/2}}{2^{1/4}}, \tag{18}$$

$$\frac{1}{2^{5/4}}\widetilde{A_*}^{1/2} \leq \lambda_i \leq \frac{\widetilde{A_*}^{1/2}}{2^{1/4}}, \tag{19}$$

while $\widetilde{A_*}$ varies from 1 to $\alpha\delta$ and B_0 varies from 0 to $\sqrt{1-\eta_0^2\delta^2}$.

Proof. All these inequalities follow from (17) and (15). ■

3.2 New coordinates

The eigenvector and generalized eigenvector for the eigenvalue 0 of $\boldsymbol{L}_\delta$ are :

$$Z_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{A_*} \\ 0 \end{pmatrix}, \; Z_1 = \begin{pmatrix} 0 \\ -(1+\delta^2)B_0 \\ 0 \\ 0 \\ 0 \\ \widetilde{A_*} \end{pmatrix}.$$

Now we denote by

$$V_r^+ \pm i\lambda_i V_i^+, \; V_r^- \pm i\lambda_i V_i^-$$

the eigenvectors belonging respectively to the eigenvalues

$$\lambda_r \pm i\lambda_i, \quad -\lambda_r \pm i\lambda_i$$

then we define

$$V_r^\pm = \begin{pmatrix} \mp\frac{\lambda_r(\lambda_r^2-3\lambda_i^2)}{2\widetilde{A_*}^2} \\ 1 \\ \pm\lambda_r \\ \lambda_r^2-\lambda_i^2 \\ \mp\frac{\lambda_r(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}} \\ -\frac{(\lambda_r^2-\lambda_i^2)^2}{(1+\delta^2)B_0\widetilde{A_*}} \end{pmatrix}, \quad V_i^\pm = \begin{pmatrix} -\frac{3\lambda_r^2-\lambda_i^2}{2\widetilde{A_*}^2} \\ 0 \\ 1 \\ \pm 2\lambda_r \\ -\frac{(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}} \\ \mp\frac{2\lambda_r(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}} \end{pmatrix},$$

and we define new coordinates in $\mathbb{R}^6$: $(x_1, x_2, y_1, y_2, B_0, z_1)$ such that

$$\begin{pmatrix} \widetilde{A_0} \\ A_1 \\ A_2 \\ A_3 \\ 0 \\ B_1 \end{pmatrix} = B_0(x_1V_r^+ + x_2\lambda_iV_i^+ + y_1V_r^- + y_2\lambda_iV_i^- + z_0Z_0 + z_1Z_1).$$

We observe that after eliminating z_0, we still have 6 coordinates, including B_0 as one of the new coordinates.

Remark 17 *We notice that we put B_0 in front of the new coordinates, as this results from the analysis, and shorten the computations.*

The coordinate change is non linear in B_0, given explicitely by:

$$\begin{aligned}
\widetilde{A_0} &= -B_0\frac{\lambda_r(\lambda_r^2-3\lambda_i^2)}{2\widetilde{A_*}^2}(x_1-y_1)-B_0\frac{\lambda_i(3\lambda_r^2-\lambda_i^2)}{2\widetilde{A_*}^2}(x_2+y_2)\\
A_1 &= B_0(x_1+y_1)-(1+\delta^2)B_0^2z_1\\
A_2 &= \lambda_r B_0(x_1-y_1)+\lambda_i B_0(x_2+y_2)\\
A_3 &= (\lambda_r^2-\lambda_i^2)B_0(x_1+y_1)+2\lambda_r\lambda_i B_0(x_2-y_2)\\
0 &= -\frac{(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}}A_2+\widetilde{A_*}B_0z_0,
\end{aligned} \tag{20}$$

$$B_1=-\varepsilon^2(1+\delta^2)B_0\frac{A_3}{\widetilde{A_*}}+\widetilde{A_*}B_0z_1, \tag{21}$$

which needs to be inverted. We obtain

$$\begin{aligned}
B_0x_1 &= \frac{(\lambda_r^2+\lambda_i^2)}{4\lambda_r}\widetilde{A_0}+\frac{3\lambda_r^2-\lambda_i^2}{4\lambda_r(\lambda_r^2+\lambda_i^2)}A_2\\
&\quad +\frac{A_1}{2}+\frac{(1+\delta^2)B_0}{2\widetilde{A_*}}B_1+\frac{(\lambda_r^2-\lambda_i^2)}{2\widetilde{A_*}^2}A_3,
\end{aligned} \tag{22}$$

$$\begin{aligned}
\lambda_iB_0x_2 &= -\frac{(\lambda_r^2+\lambda_i^2)}{4}\widetilde{A_0}-\frac{\lambda_r^2-3\lambda_i^2}{4(\lambda_r^2+\lambda_i^2)}A_2\\
&\quad -\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r}\left(A_1+\frac{(1+\delta^2)B_0}{\widetilde{A_*}}B_1\right)+\frac{1}{4\lambda_r}\left(1-\frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right)A_3,
\end{aligned} \tag{23}$$

$$\begin{aligned}
B_0y_1 &= -\frac{(\lambda_r^2+\lambda_i^2)}{4\lambda_r}\widetilde{A_0}-\frac{3\lambda_r^2-\lambda_i^2}{4\lambda_r(\lambda_r^2+\lambda_i^2)}A_2\\
&\quad +\frac{A_1}{2}+\frac{(1+\delta^2)B_0}{2\widetilde{A_*}}B_1+\frac{(\lambda_r^2-\lambda_i^2)}{2\widetilde{A_*}^2}A_3,
\end{aligned} \tag{24}$$

$$\begin{aligned}
\lambda_iB_0y_2 &= -\frac{(\lambda_r^2+\lambda_i^2)}{4}\widetilde{A_0}-\frac{\lambda_r^2-3\lambda_i^2}{4(\lambda_r^2+\lambda_i^2)}A_2\\
&\quad +\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r}\left(A_1+\frac{(1+\delta^2)B_0}{\widetilde{A_*}}B_1\right)-\frac{1}{4\lambda_r}\left(1-\frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right)A_3,
\end{aligned} \tag{25}$$

$$B_0z_1=\frac{(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}^2}A_3+\frac{1}{\widetilde{A_*}}B_1=\varepsilon^2B_0(1+\delta^2)\frac{A_3}{\widetilde{A_*}^2}+\frac{1}{\widetilde{A_*}}B_1.$$

Let us now define

$$
\begin{aligned}
X &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \\
|X| &= \sqrt{x_1^2+x_2^2},\ |Y| = \sqrt{y_1^2+y_2^2} \text{ (norms in } \mathbb{R}^2).
\end{aligned}
$$

Then, for ε small enough, and using (17), we obtain the following useful estimates

Lemma 18 *For $B_0 \le \sqrt{1-\eta_0^2\delta^2}$, $\varepsilon^2 \le \frac{3\alpha}{10}$, ε small enough we have*

$$
\begin{aligned}
\frac{\widetilde{A_*}^{1/2}}{2^{5/4}} &\le \lambda_r, \lambda_i < 2^{1/4}\widetilde{A_*}^{1/2}, \quad \widetilde{A_*} \ge \alpha\delta, \\
|\widetilde{A_0}| &\le 3\frac{B_0}{\widetilde{A_*}^{1/2}}(|X|+|Y|), \\
|A_1| &\le B_0(|X|+|Y|) + (1+\delta^2)B_0^2|z_1|, \\
|A_2| &\le 2B_0\widetilde{A_*}^{1/2}(|X|+|Y|), \\
|A_3| &\le 2B_0\widetilde{A_*}(|X|+|Y|), \\
|B_1| &\le 2\varepsilon^2(1+\delta^2)B_0^2(|X|+|Y|) + \widetilde{A_*}B_0|z_1|.
\end{aligned} \tag{26}
$$

3.3 System with new coordinates

The system (12) writen in the new coordinates is computed in Appendix A.2. It takes the following form (quadratic and higher order terms are not explicited)

$$
\begin{aligned}
&x_1' = f_1 + \lambda_r x_1 + \lambda_i x_2 \\
&+B_1\left[a_1\widetilde{A_0} + c_1A_2 + d_1A_3 + e_1\frac{B_1}{B_0} - \frac{1}{B_0}x_1\right] \\
&-\varepsilon^2\frac{(1+\delta^2)(2-\delta^2)B_0}{2\widetilde{A_*}}\widetilde{A_0}^2 - \varepsilon^2\frac{(1+\delta^2)B_0}{2\widetilde{A_*}^2}\widetilde{A_0}^3,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
x_2' &= f_2 - \lambda_i x_1 + \lambda_r x_2 + B_1\left[-a_2\widetilde{A_0} + b_2A_1 + c_2A_2 + d_2A_3 + e_2B_1 - \frac{1}{B_0}x_2\right] \\
&-\frac{1}{4\lambda_r\lambda_i\widetilde{A_*}B_0}\left(3\widetilde{A_*}^2 - 2\varepsilon^4B_0^4(1+\delta^2)^4\right)\widetilde{A_0}^2 - \frac{1}{4\lambda_r\lambda_iB_0}\left(1-\frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right)\widetilde{A_0}^3.
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
y_1' &= f_1 - \lambda_r y_1 + \lambda_i y_2 + \\
&+B_1\left[-a_1\widetilde{A_0} - c_1A_2 + d_1A_3 + e_1\frac{B_1}{B_0} - \frac{1}{B_0}y_1\right] \\
&-\varepsilon^2\frac{(1+\delta^2)(2-\delta^2)B_0}{2\widetilde{A_*}}\widetilde{A_0}^2 - \varepsilon^2\frac{(1+\delta^2)B_0}{2\widetilde{A_*}^2}\widetilde{A_0}^3,
\end{aligned} \tag{29}
$$

$$y_2' = -f_2 - \lambda_i y_1 - \lambda_r y_2 + B_1\left[-a_2\widetilde{A_0} - b_2 A_1 + c_2 A_2 - d_2 A_3 + e_2 B_1 - \frac{1}{B_0}y_2\right] \quad (30)$$
$$+\frac{1}{4\lambda_r\lambda_i\widetilde{A_*}B_0}\left(3\widetilde{A_*}^2 - 2\varepsilon^4 B_0^4(1+\delta^2)^4\right)\widetilde{A_0}^2 + \frac{1}{4\lambda_r\lambda_i B_0}\left(1 - \frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right)\widetilde{A_0}^3,$$

with

$$f_1 = \frac{\varepsilon^2\delta^2 B_0(1+\delta^2)(\widetilde{A_*}^2 - B_0^2)}{2\widetilde{A_*}},$$
$$f_2 = -\frac{\varepsilon^2\delta^2 B_0(1+\delta^2)(\lambda_r^2-\lambda_i^2)(\widetilde{A_*}^2 - B_0^2)}{4\lambda_r\lambda_i\widetilde{A_*}}.$$

Using Lemma 16 and (15) we see that we have the estimates

$$|f_j| \leq \frac{B_0\varepsilon^2\delta}{\alpha},\ j=1,2.$$

The coefficients a_j, b_j, c_j, d_j, e_j are defined and estimated in Appendix A.2 in (96,97), (98,99,100), (101,102), (103,104). Here $\widetilde{A_0}, A_1, A_2, A_3, B_1$ should be replaced by their (linear) expressions (20) in coordinates $(x_1, x_2, y_1, y_2, z_1)$ with coefficients functions of B_0. The system above should be completed by the differential equations for z_1' and $B_0'(=B_1)$. In fact we replace the equation for z_1' by the direct resolution of the first integral (6) with respect to z_1 using the expression of B_1 in (21).

3.4 Resolution of (6) with respect of $z_1(X, Y, B_0)$

For extending the validity (as a graph with respect to B_0) for the existence of the unstable manifold of M_- we need to replace the differential equation for z_1' by the expression of z_1 given by the first integral (6). This leads to the following

Lemma 19 *For $B_0 \leq \sqrt{1-\eta_0^2\delta^2}$, $\varepsilon^2 \leq \frac{3\alpha}{10}$. Then choose $\rho > 0$ such that for ε and α small enough, and for*

$$|\overline{X}| + |\overline{Y}| \leq \rho,\ \alpha^3\rho^4 << 1, \quad (31)$$

with the scaling

$$(X, Y) = \alpha^{3/2}\delta(\overline{X}, \overline{Y}),\ z_1 = \varepsilon\delta\overline{z_1}, \quad (32)$$

we have

$$\overline{z_1} = \overline{z_{10}}(B_0, \delta)[1 + \mathcal{Z}(\overline{X}, \overline{Y}, B_0, \varepsilon, \alpha, \delta)] \quad (33)$$

where the function $\mathcal{Z}(\overline{X}, \overline{Y}, B_0, \varepsilon, \delta)$ is analytic in its arguments, and with

$$\overline{z_{10}}(B_0, \delta) \stackrel{def}{=} (1 + \frac{\delta^2 B_0^2}{2\widetilde{A_*}^2})^{1/2} \leq \frac{1}{\alpha}, \quad (34)$$

and

$$|\mathcal{Z}(\overline{X}, \overline{Y}, B_0, \varepsilon, \alpha, \delta)| \leq c\alpha^3(1+\rho^2)(|\overline{X}| + |\overline{Y}|)^2, \quad (35)$$

with c independent of $(\varepsilon, \alpha, \rho)$.

Remark 20 *In the Lemma above, we introduce the number ρ which may be large. Precise constraints are given later.*

Proof. Using (21) and (6) we have

$$\begin{aligned} B_1^2 &= \{\widetilde{A_*} B_0 z_1 - \varepsilon^2 \frac{B_0(1+\delta^2)}{\widetilde{A_*}} A_3\}^2 = 2\varepsilon^2 A_1 A_3 - \varepsilon^2 A_2^2 + \\ & \frac{\varepsilon^2}{2}(-\delta^2 B_0^2 + 2\widetilde{A_*}\widetilde{A_0} + \widetilde{A_0}^2)^2 + \varepsilon^2\delta^2(\widetilde{A_*} + \widetilde{A_0})^2 B_0^2, \end{aligned}$$

hence

$$\begin{aligned} \widetilde{A_*}^2 z_1^2 &= \varepsilon^2\delta^2\widetilde{A_*}^2(1+\frac{\delta^2 B_0^2}{2\widetilde{A_*}^2}) + \frac{2\varepsilon^2}{B_0}A_3(x_1+y_1) - \frac{\varepsilon^4(1+\delta^2)^2}{\widetilde{A_*}^2}A_3^2 - \frac{\varepsilon^2}{B_0^2}A_2^2 + \\ & + \frac{2\varepsilon^2\widetilde{A_*}^2}{B_0^2}\widetilde{A_0}^2 + \frac{2\varepsilon^2\widetilde{A_*}}{B_0^2}\widetilde{A_0}^3 + \frac{\varepsilon^2}{2B_0^2}\widetilde{A_0}^4, \end{aligned} \quad (36)$$

where we may observe on the r.h.s., that

$$\varepsilon^2\delta^2 \leq \varepsilon^2\delta^2(1+\frac{\delta^2 B_0^2}{2\widetilde{A_*}^2}) \leq \varepsilon^2\delta^2(1+\frac{1}{2\alpha^2}),$$

which is independent of (X,Y). Moreover there is no linear part in (X,Y) in (36). The scaling (32), Lemma 15 and Lemma 18 imply (c is a generic constant, independent of ε)

$$|\frac{2\varepsilon^2}{B_0}A_3(x_1+y_1)| \leq c\varepsilon^2\alpha^3\widetilde{A_*}(|\overline{X}|+|\overline{Y}|)^2$$

$$|\frac{\varepsilon^4(1+\delta^2)^2}{\widetilde{A_*}^2}A_3^2| \leq c\varepsilon^4\alpha^3(|\overline{X}|+|\overline{Y}|)^2 \leq c\varepsilon^2\alpha^3\widetilde{A_*}(|\overline{X}|+|\overline{Y}|)^2,$$

$$\frac{\varepsilon^2}{B_0^2}A_2^2 \leq c\varepsilon^2\alpha^3\widetilde{A_*}(|\overline{X}|+|\overline{Y}|)^2)$$

$$\frac{2\varepsilon^2\widetilde{A_*}^2}{B_0^2}\widetilde{A_0}^2 \leq c\varepsilon^2\alpha^3\widetilde{A_*}(|\overline{X}|+|\overline{Y}|)^2$$

$$|\frac{2\varepsilon^2\widetilde{A_*}}{B_0^2}\widetilde{A_0}^3| \leq c\varepsilon^2\alpha^3\widetilde{A_*}(|\overline{X}|+|\overline{Y}|)^3$$

$$\frac{\varepsilon^2}{2B_0^2}\widetilde{A_0}^4 \leq c\varepsilon^2\alpha^3\widetilde{A_*}(|\overline{X}|+|\overline{Y}|)^4,$$

so that the factors in the estimates are such that

$$\frac{c\varepsilon^2\alpha^3\widetilde{A_*}}{\varepsilon^2\delta^2\widetilde{A_*}^2} \leq c\frac{\alpha^3}{\widetilde{A_*}},$$

c being independent of ε, α and $\delta \in [1/3, 1]$. Now defining $\overline{z_{10}}$ such that

$$1 \leq \overline{z_{10}}(B_0, \delta) \stackrel{def}{=} (1 + \frac{\delta^2 B_0^2}{2\widetilde{A_*}^2})^{1/2} \leq \frac{1}{\alpha}, \text{ for } \alpha \leq 1/\sqrt{2}, \tag{37}$$

we notice that we have

$$\frac{1}{\overline{z_{10}}^2} = \frac{2\widetilde{A_*}^2}{2\widetilde{A_*}^2 + \delta^2 B_0^2} \leq \frac{2(1+\delta^2)\widetilde{A_*}^2}{\delta^2} \leq 20\widetilde{A_*}^2.$$

It results that, for $|\overline{X}| + |\overline{Y}| \leq \rho$

$$\overline{z_1}^2 = \overline{z_{10}}^2 + \mathcal{O}\left(\frac{\alpha^3(1+\rho^2)}{\widetilde{A_*}}(|\overline{X}| + |\overline{Y}|)^2\right).$$

It is shown in [1] that $0 \leq B_0 \leq 1$ on the heteroclinic, so that $B_1(x) = B_0'(x)$ is positive at least at the starting point $B_0 = 0$. It results that we need to take the positive square root for B_1 (as for (6)), hence also for z_1 (see (21)):

$$\overline{z_1} = \overline{z_{10}}(B_0)\left\{1 + \mathcal{O}[\alpha^3(1+\rho^2)(|\overline{X}| + |\overline{Y}|)^2]\right\}^{1/2}, \text{ for } |\overline{X}| + |\overline{Y}| \leq \rho,$$

and taking the square root, we obtain (33) with estimate (35), $\mathcal{Z}(\overline{X}, \overline{Y}, B_0, \varepsilon, \alpha, \delta)$ being defined in the ball $|\overline{X}| + |\overline{Y}| \leq \rho$, c independent of ε, α provided that ε, α are small enough and ρ satisfies (31). Moreover $\mathcal{Z}$ is analytic in its arguments and is at least quadratic in $(\overline{X}, \overline{Y})$. Notice that, in using (37), we also have

$$\overline{z_{10}}|\mathcal{Z}(\overline{X}, \overline{Y}, B_0, \varepsilon, \delta)| \leq c(1+\rho^2)\alpha^2(|\overline{X}| + |\overline{Y}|)^2. \tag{38}$$

■

Since z_1 contains $\overline{z_{10}}$ which is independent of $(\overline{X}, \overline{Y})$, the new system for $(\overline{X}, \overline{Y})$ has new "constant terms" and "linear terms", appearing as perturbations of the former ones.

3.5 System where z_1 is eliminated

Now we stay on the 5-dimensional invariant manifold (6) and we need to express the new differential system in terms of the 5 coordinates $(\overline{X}, \overline{Y}, B_0)$. The new system is computed in Appendix A.3. We obtain (notice that B_0 is in factor of the "constant" terms, and all operators are B_0- dependent)

$$\begin{aligned} \overline{X}' &= \mathbf{L}_0\overline{X} + B_0\mathcal{F}_0 + \mathcal{L}_{01}(\overline{X}, \overline{Y}) + \mathcal{B}_{01}(\overline{X}, \overline{Y}), \\ \overline{Y}' &= \mathbf{L}_1\overline{Y} + B_0\mathcal{F}_1 + \mathcal{L}_{11}(\overline{X}, \overline{Y}) + \mathcal{B}_{11}(\overline{X}, \overline{Y}), \end{aligned} \tag{39}$$

which should be completed by an equation for B_0' (see (21) in terms of $(\overline{X}, \overline{Y}, B_0)$), and where

$$\mathbf{L}_0 = \begin{pmatrix} \lambda_r & \lambda_i \\ -\lambda_i & \lambda_r \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} -\lambda_r & \lambda_i \\ -\lambda_i & -\lambda_r \end{pmatrix},$$

with the following estimates, for terms independent of $(\overline{X},\overline{Y})$

$$|\mathcal{F}_0|+|\mathcal{F}_1|\leq \frac{c\varepsilon^2}{\alpha^{9/2}}, \tag{40}$$

for terms which are linear in $(\overline{X},\overline{Y})$

$$|\mathcal{L}_{01}(\overline{X},\overline{Y})|+|\mathcal{L}_{11}(\overline{X},\overline{Y})|\leq c\frac{\varepsilon}{\alpha^2}(|\overline{X}|+|\overline{Y}|), \tag{41}$$

and for terms at least quadratic in $(\overline{X},\overline{Y})$, and for

$$|\overline{X}|+|\overline{Y}|\leq\rho,\ \rho<<\alpha^{-3/4},$$

we obtain

$$\begin{aligned} |\mathcal{B}_{01}(\overline{X},\overline{Y})|+|\mathcal{B}_{11}(\overline{X},\overline{Y})| &\leq \alpha^{1/2}(9/2+c\frac{\varepsilon^2}{\alpha^{7/2}})(|\overline{X}|+|\overline{Y}|)^2 \\ &\quad +\alpha^{1/2}(\frac{27}{2}+c\frac{\varepsilon}{\alpha})(|\overline{X}|+|\overline{Y}|)^3+\alpha^{1/2}(c\frac{\varepsilon^2}{\alpha^2})(|\overline{X}|+|\overline{Y}|)^4. \end{aligned} \tag{42}$$

We are now ready to formulate the search for the unstable manifold of M_-.

3.6 Integral formulation for solutions bounded as $x\to-\infty$

Let us introduce the monodromy operators associated with the linear operators $\mathbf{L}_0,\mathbf{L}_1$ which have non constant coefficients:

$$\begin{aligned} \frac{\partial}{\partial x}S_0(x,s) &= \mathbf{L}_0S_0(x,s),\ S_0(x,s_1)S_0(s_1,s_2)=S_0(x,s_2),\ S_0(x,x)=\mathbb{I}, \\ \frac{\partial}{\partial x}S_1(x,s) &= \mathbf{L}_1S_1(x,s),\ S_1(x,s_1)S_1(s_1,s_2)=S_1(x,s_2),\ S_1(x,x)=\mathbb{I}. \end{aligned}$$

The coefficients of operators $\mathbf{L}_0,\mathbf{L}_1$ are functions of B_0, so we need Lemma 40 in Appendix A.1, with the following estimates, valid for $0\leq B_0\leq\sqrt{1-\eta_0^2\delta^2}$, $\varepsilon^2\leq\frac{3\alpha}{10}$, ε small enough:

$$\begin{aligned} ||\boldsymbol{S}_0(x,s)|| &\leq e^{\sigma(x-s)},\ -\infty<x<s\leq -x_*\leq 0, & (43)\\ ||\boldsymbol{S}_1(x,s)|| &\leq e^{-\sigma(x-s)},\ -\infty<s<x\leq -x_*\leq 0, & (44) \end{aligned}$$

with

$$\sigma=\frac{\alpha^{1/2}\delta^{1/2}}{2^{1/4}}.$$

We are looking for solutions of (39) which stay bounded for $x\to-\infty$. Then, thanks to estimates (43) (44), the system (39) may be formulated for $-\infty<x\leq -x_*\leq 0$ as

$$\begin{aligned} \overline{X}(x) &= \mathbf{S}_0(x,-x_*)\overline{X}_0-\int_x^{-x_*}\mathbf{S}_0(x,s)G_0(s)ds & (45)\\ \overline{Y}(x) &= \int_{-\infty}^x\mathbf{S}_1(x,s)G_1(s)ds \end{aligned}$$

$$G_0(s) \stackrel{def}{=} B_0\mathcal{F}_0 + \mathcal{L}_{01}(\overline{X},\overline{Y}) + \mathcal{B}_{01}(\overline{X},\overline{Y}),$$
$$G_1(s) \stackrel{def}{=} B_0\mathcal{F}_1 + \mathcal{L}_{11}(\overline{X},\overline{Y}) + \mathcal{B}_{11}(\overline{X},\overline{Y})$$

where $\overline{X},\overline{Y}$ and B_0 are bounded and continuous functions of s, B_0 tending towards 0 as $s \to -\infty$.

3.7 Strategy (continued)

The 3-dimensional unstable manifold of M_- is such that $(\overline{X}(x), \overline{Y}(x), B_0(x))$ should be expressed in terms of $\overline{X}_0, B_0(-x_*)$. The idea is then

i) solve (45) with respect to $(\overline{X},\overline{Y})$ in function of $(\overline{X}_0, B_0)$;

ii) solve the differential equation for $B_0(x)$ satisfying $B_0(-x_*) = B_{00}$, $B_0(-\infty) = 0$.

The result will be valid for $B_0(x)$, and B_{00} in the interval $[0, \sqrt{1-\eta_0^2\delta^2}]$ and it appears that $A_0(x)$ is then very close to 0 at the end point $x = -x_*$. The hope is that this should allow to obtain an intersection with the 3-dim stable manifold of M_+ which computation should be valid for $B_0(x)$ in the interval $[\sqrt{1-\eta_0^2\delta^2}, 1]$.

3.8 Resolution with respect to $(\overline{X},\overline{Y})$

Let us define, for $\kappa > 0$ the function space

$$C_\kappa^0 = \{\overline{X} \in C^0(-\infty, -x_*]; \overline{X}(x)e^{-\kappa x} \text{ is bounded}\}$$

equiped with the norm

$$||\overline{X}||_\kappa = \sup_{(-\infty,-x_*)} |\overline{X}(x)e^{-\kappa x}|.$$

In this subsection we prove the following

Lemma 21 *Given $M > 0$, for $\varepsilon = \nu\alpha^{5/2}$, with ν small enough, there exists $k_0 > 0$, independent of ν, such that for $||\overline{X}_0|| \leq k_0$, there is a unique solution $(\overline{X},\overline{Y})$ in $(C_\kappa^0)^2$ such that, for $||B||_\kappa \leq M$, we have*

$$\begin{aligned} ||\overline{X}||_\kappa &\leq |\overline{X}_0|(1+c\nu) + c\nu,\ \overline{X}(-x_*) = \overline{X}_0, \\ ||\overline{Y}||_\kappa &\leq c\nu + c(\nu|\overline{X}_0| + |\overline{X}_0|^2), \end{aligned}$$

where c is independent of ε, ν, k_0.

Remark 22 *The choice of κ will be in agreement with the behavior of $B_0(x)$ as $x \to -\infty$, which is studied at next subsection.*

Proof. First we observe that, provided that $\kappa < \sigma$

$$|\int_{-\infty}^{x} \boldsymbol{S}_1(x,s)e^{\kappa s}ds| \leq \frac{e^{\kappa x}}{\kappa+\sigma}, \quad x \leq -x_*,$$

$$|\boldsymbol{S}_0(x,-x_*)e^{-\kappa(x+x_*)})| \leq e^{(\sigma-\kappa)(x+x_*)}, \quad x \leq -x_*,$$

$$|\int_{-x_*}^{x} \mathbf{S}_0(x,s)e^{\kappa s}ds| \leq \frac{e^{\kappa x}}{\sigma-\kappa}, \quad x \leq -x_*.$$

Let us choose

$$\kappa \leq \frac{\sigma}{2},$$

then

$$|\int_{-\infty}^{x} \boldsymbol{S}_1(x,s)e^{\kappa s}ds| \leq \frac{e^{\kappa x}}{\sigma} = 2^{1/4}\frac{e^{\kappa x}}{\alpha^{1/2}\delta^{1/2}},$$

$$|\int_{-x_*}^{x} \mathbf{S}_0(x,s)e^{\kappa s}ds| \leq 2^{5/4}\frac{e^{\kappa x}}{\alpha^{1/2}\delta^{1/2}}, \quad x \leq -x_*.$$

Let us assume that

$$||B_0||_\kappa \leq M \tag{46}$$

holds (needs to be proved at next subsection). We wish to use the analytic implicit function theorem (see [3] section X.2) for $(\overline{X},\overline{Y})$ in a neighborhood of $(\overline{X}_0,0)$ in the function space $(C_\kappa^0)^2$, provided that we can choose $\kappa \leq \frac{\sigma}{2}$ and $||\overline{X}||_\kappa + ||\overline{Y}||_\kappa \leq \rho$. Indeed, using the above estimates for coefficients, we obtain for $x \in (-\infty, -x_*]$

$$|\overline{X}(x)e^{-\kappa x}| \leq |\overline{X}(-x_*)|e^{\kappa x_*} + \frac{2^{5/4}}{\alpha^{1/2}\delta^{1/2}}||B_0\mathcal{F}_0 + \mathcal{L}_{01}(\overline{X},\overline{Y}) + \mathcal{B}_{01}(\overline{X},\overline{Y})||_\kappa,$$

hence

$$||\overline{X}||_\kappa \leq |\overline{X}(-x_*)|e^{\kappa x_*} + \frac{2^{5/4}}{\alpha^{1/2}\delta^{1/2}}||B_0\mathcal{F}_0 + \mathcal{L}_{01}(\overline{X},\overline{Y}) + \mathcal{B}_{01}(\overline{X},\overline{Y})||_\kappa, \tag{47}$$

and in the same way

$$||\overline{Y}||_\kappa \leq \frac{2^{1/4}}{\alpha^{1/2}\delta^{1/2}}||B_0\mathcal{F}_1 + \mathcal{L}_{11}(\overline{X},\overline{Y}) + \mathcal{B}_{11}(\overline{X},\overline{Y})||_\kappa. \tag{48}$$

Using estimates (40) for $\mathcal{F}_j$, (41) for $\mathcal{L}_{j1}$, (42) for $\mathcal{B}_{01}$, $\mathcal{B}_{11}$, (47), (48), we obtain, for $S \stackrel{def}{=} ||\overline{X}||_\kappa + ||\overline{Y}||_\kappa \leq \rho$

$$\begin{aligned} S \quad &\leq \quad |\overline{X}(-x_*)|e^{\kappa x_*} + c\frac{\varepsilon^2 M}{\alpha^5} + c\frac{S\varepsilon}{\alpha^{5/2}} + \frac{27}{2^{3/4}\delta^{1/2}}(1 + c\frac{\varepsilon^2}{\alpha^{7/2}})S^2 \\ &+(\frac{81}{2^{3/4}\delta^{1/2}} + c\frac{\varepsilon}{\alpha})S^3 + c\frac{\varepsilon^2}{\alpha^2}S^4, \end{aligned}$$

so that choosing

$$\varepsilon = \nu\alpha^{5/2},\ \nu < 1/M, \tag{49}$$

$$\begin{aligned} S(1-c\nu) &\leq |\overline{X}(-x_*)| + c\nu + \frac{27}{2^{3/4}\delta^{1/2}}(1+c\nu^2\alpha^{3/2})S^2 \\ &+ \frac{81}{2^{3/4}\delta^{1/2}}(1+c\nu\alpha^{3/2})S^3 + c\nu^2\alpha^3 S^4. \end{aligned}$$

Let us choose k_0 such that

$$27k_0 + 81k_0^2 < 2^{3/4}\delta^{1/2}(1-c\nu),$$

which is satisfied for

$$k_0 < 0.13\delta^{1/2}(1-c\nu).$$

Then the estimate above shows that for $0 < k_0$ small enough, we can apply the implicit function theorem (its analytic version) with respect to$(\overline{X},\overline{Y})$ (for $|\overline{X}(-x_*)| \leq k_0(1-c_1\nu)$, and for ε and ν small enough). We find a unique $(\overline{X},\overline{Y}) \in (C^0_\kappa)^2$ such that $||\overline{X}||_\kappa + ||\overline{Y}||_\kappa$ is close to $|\overline{X}(-x_*)|$. Moreover for ε and ν small enough

$$S \leq |\overline{X}(-x_*)|(1+c\nu) + c|\overline{X}(-x_*)|^2 + c\nu,$$

with c independent of (ε,ν,k_0). This leads finally to $\overline{X}$ and $\overline{Y}$ in C^0_κ, depending analytically on $(\overline{X}_0, B_0) \in \mathbb{R}^2 \times C^0_\kappa$, and such that

$$||\overline{Y}||_\kappa \leq c\nu + c(\nu|\overline{X}(-x_*)| + |\overline{X}(-x_*)|^2), \tag{50}$$

$$||\overline{X}||_\kappa \leq |\overline{X}(-x_*)|(1+c\nu) + c\nu, \tag{51}$$

where c is a number independent of (ε,ν,k_0), ε small enough, $S \leq k_0$, and $k_0 < \rho$, which is compatible with (31). Lemma 21 is proved. ■

3.8.1 Estimate of ν

From the proof of Lemma 21, the constraint on ν is

$$\nu < \min\{1/||B_0||_\kappa, 1/c\},$$

where

$$c = \frac{3.2^{1/4}}{\sqrt{\delta}}c'$$

and c' comes from the estimate (41) appearing on the linear term in Appendix A.3. From Appendix A.3 and Lemma 18, we see that c' is given at main order by

$$c' = 3c(a_j) + 2c(c_j) + 2c(d_j)\ ,\ j = 1, 2,$$

where $c(a_j)$ is defined in (96,97), $c(c_j)$ defined in (99,100), $c(d_j)$ defined in (101,102). A careful checking leads to

$$\begin{aligned} c(a_j) &= 3.2^{-1/4}(1+\delta^2), \\ c(c_j) &= 2.2^{1/4}(1+\delta^2), \\ c(d_j) &= 4\sqrt{2}(1+\delta^2), \end{aligned}$$

so that

$$2^{1/4}c' = 28.11(1+\delta^2).$$

Hence

$$c = 84.33\frac{1+\delta^2}{\sqrt{\delta}},$$

which needs to be compared with $||B_0||_\kappa$.

We see at subsection 3.9 (see (58))that

$$||B_0||_\kappa = \sup_{(-\infty,-x_*]} B_0(x)e^{-\kappa x} \leq B_0(-x_*) = \left(\frac{1-\alpha^2\delta^2}{1+\delta^2}\right)^{1/2} < (1+\delta^2)^{-1/2}.$$

Finally the restriction on $(\nu)_{\max}$ is

$$\frac{\nu}{\sqrt{\delta}} \leq \frac{(1+\delta^2)^{-1}}{84.33}. \tag{52}$$

3.9 Resolution for B_0

In this subsection we finish the proof of Lemma 13. It remains to solve the last part of the system (39) for B_0 with $B_0(-x_*) = B_{00}$.

We notice from (21), (33) and (26) that

$$\begin{aligned} B_1 &= \varepsilon\delta\widetilde{A_*}B_0\overline{z_{10}}(B_0)\left(1+\mathcal{Z}(\overline{X},\overline{Y},B_0,\varepsilon,\delta) - \frac{\varepsilon^{3/2}}{\overline{z_{10}}}\frac{(1+\delta^2)}{\widetilde{A_*}^2}\overline{A_3}\right) \\ \overline{A_3} &= B_0[\varepsilon^2B_0^2(1+\delta^2)^2(\overline{x_1}+\overline{y_1}) + 2\lambda_r\lambda_i(\overline{x_2}-\overline{y_2})], \\ \frac{\varepsilon^{3/2}}{\overline{z_{10}}}(1+\delta^2)\frac{\overline{A_3}}{\widetilde{A_*}^2} &\leq 4\varepsilon^{7/6}(|\overline{X}|+|\overline{Y}|), \end{aligned}$$

so that it is clear that (see above estimates for $\mathcal{Z}$)

$$B_1 > 0 \text{ for } B_0 \in (0, \sqrt{1-\eta_0^2\delta^2}), |\overline{X}|+|\overline{Y}| \leq \rho, \tag{53}$$

This is coherent with the study of the linearized system near M_- : Indeed the principal part of the differential equation for B_0 is

$$B_0' = \varepsilon\delta B_0\widetilde{A_*}\overline{z_{10}}(B_0)$$

which may be integrated as

$$
\begin{aligned}
B_0^2(x) &= \frac{1}{(1+\frac{\delta^2}{2})\cosh^2(x_0 - \varepsilon\delta(x+x_*))}, \\
\cosh x_0 &= \frac{1}{B_0(-x_*)(1+\frac{\delta^2}{2})^{1/2}},
\end{aligned} \tag{54}
$$

which satisfies $B_0 = 0$ for $x = -\infty$. More precisely the differential equation for B_0 is now, after replacing $(\overline{X}, \overline{Y})$ by the expression found at previous subsection,

$$
B_0' = \varepsilon\delta\widetilde{A_*}B_0\overline{z_{10}}(B_0)[1 + f(B_0)] \tag{55}
$$

where $f(B_0)$ is a non local analytic function of B_0 in C_κ^0, such that, for ε small enough, and using $||\overline{X}||_\kappa + ||\overline{Y}||_\kappa \leq k_0$, (31) and (49),

$$
||f(B_0)||_\kappa \leq c(\alpha^3(1+\rho^2)S^2 + \varepsilon^{7/6}S) \leq c\varepsilon k_0.
$$

Remark 23 *We may notice that we might replace $c\varepsilon k_0$ in the estimate above, by*

$$
\varepsilon k_0 e^{\kappa x} \to 0 \text{ as } x \to -\infty,
$$

since X and $Y \in C_\kappa^0$.

We are looking for the solution such that $B_0 = 0$ for $x = -\infty$, and $B_0(-x_*) \leq \sqrt{1 - \eta_0^2\delta^2}$ for $x = 0$. We can rewrite (55) as

$$
\frac{2B_0B_0'}{B_0^2\widetilde{A_*}\overline{z_{10}}(B_0)} = 2\varepsilon\delta[1 + f(B_0)]. \tag{56}
$$

We now introduce the variable v :

$$
v = \frac{1 - \sqrt{1 - (1+\frac{\delta^2}{2})B_0^2}}{1 + \sqrt{1 - (1+\frac{\delta^2}{2})B_0^2}}, \; B_0^2 = \frac{1}{1+\frac{\delta^2}{2}}\frac{4v}{(1+v)^2},
$$

so that

$$
(\ln v)' = 2\varepsilon\delta[1 + f(B_0)].
$$

We observe that for x running from $-\infty$ to 0,

$$
w = \ln v \text{ is increasing from } -\infty \text{ to } w_0 = \ln v_0 < 0.
$$

Now let us define h continuous in its argument and such that

$$
\begin{aligned}
h(w) &= f(B_0), \\
B_0 &= \frac{1}{\left(1+\frac{\delta^2}{2}\right)^{1/2}}\frac{2e^{w/2}}{(1+e^w)},
\end{aligned}
$$

and let us find an a priori estimate for the solution $B_0(x)$, for $x \in (-\infty, -x_*]$. We obtain by simple integration

$$\int_{-x_*}^{x} \frac{w'(s)}{1+h(w)(s)} ds = 2\varepsilon\delta(x+x_*).$$

For α small enough we have

$$1 - c\varepsilon k_0 \leq \frac{1}{1+h(w)} \leq 1 + c\varepsilon k_0,$$

hence (since $w < w_0$, and $x < 0$)

$$(w_0 - w)(1 - c\varepsilon k_0) \leq -2\varepsilon\delta(x+x_*) \leq (w_0 - w)(1 + c\varepsilon k_0)$$

so that

$$\exp(\frac{-2\varepsilon\delta(x+x_*)}{1+c\varepsilon k_0}) \leq e^{w_0 - w} \leq \exp(\frac{-2\varepsilon\delta(x+x_*)}{1-c\varepsilon k_0})$$

and

$$v_0 \exp(\frac{2\varepsilon\delta}{1-c\varepsilon k_0}(x+x_*)) \leq v(x) \leq v_0 \exp(\frac{2\varepsilon\delta}{1+c\varepsilon k_0}(x+x_*)).$$

It finally results that we obtain an a priori estimate for

$$B_0(x) = \mathcal{B}_0(\overline{X}_0, B_0(-x_*))(x) \in C^0_\kappa, \tag{57}$$

$$\begin{aligned} \mathcal{B}_0(\overline{X}_0, B_0(-x_*))(x) &= \frac{1}{\left(1+\frac{\delta^2}{2}\right)^{1/2}} \frac{2\sqrt{v(x)}}{(1+v(x))}, \; x \in (-\infty, -x_*), \\ \frac{2\sqrt{v_0}\exp(\frac{\varepsilon\delta}{1-c\varepsilon k_0}(x+x_*))}{1+v_0\exp(\frac{2\varepsilon\delta}{1-c\varepsilon k_0}(x+x_*))} &\leq \left(1+\frac{\delta^2}{2}\right)^{1/2} \mathcal{B}_0 \\ &\leq \frac{2\sqrt{v_0}\exp(\frac{\varepsilon\delta}{1+c\varepsilon k_0}(x+x_*))}{1+v_0\exp(\frac{2\varepsilon\delta}{1+c\varepsilon k_0}(x+x_*))}, \end{aligned} \tag{58}$$

$$v_0 = \frac{1-\sqrt{1-(1+\frac{\delta^2}{2})B_0^2(-x_*)}}{1+\sqrt{1-(1+\frac{\delta^2}{2})B_0^2(-x_*)}} < 1, \; \left(1+\frac{\delta^2}{2}\right)^{1/2} B_0(-x_*) = \frac{2\sqrt{v_0}}{1+v_0}.$$

It remains to notice that we can choose in the proof for $(\overline{X}, \overline{Y})$

$$\kappa = \frac{\varepsilon\delta}{1+c\varepsilon k_0}, \tag{59}$$

which needs to satisfy

$$\kappa \leq \frac{\sigma}{2} = \frac{\alpha^{1/2}\sqrt{\delta}}{2^{5/4}}. \tag{60}$$

We have already chosen $\varepsilon = \nu\alpha^{\frac{5}{2}}$ hence, for α small enough, the choice (59) leads to

$$\kappa < \varepsilon\delta = \delta\nu\alpha^{\frac{5}{2}} \leq \frac{\alpha^{1/2}\sqrt{\delta}}{2^{5/4}}$$

and (60) is satisfied. The *a priori estimate* (58) for B_0 allows to prove that there is a unique solution of the differential equation (56) which may be extended on the whole interval $(-\infty, -x_*]$, and which satisfies the estimate (58) (see for example [5]). Since B_0 is in factor in $\widetilde{A_0}, A_1, A_2, A_3, B_1$ the behavior for $x \to -\infty$ of the coordinates of the unstable manifold, is governed by the behavior of B_0. The estimates indicated in Lemma 13 results from (26), (32), (37), (51), (50), (60) with $\kappa = \varepsilon\delta^*$. This ends the proof of Lemma 13. The part of Corollary 2 corresponding to $x \in (-\infty, -x_*]$ follows from the estimates found at Lemma 13.

For making a difference with the side treated for the stable manifold, from now on we write α_- and ν_- in place of α, ν. Let us define the hyperplane H_0

$$B_0 = B_{00} = \left(\frac{1 - \alpha_-^2\delta^2}{1 + \delta^2}\right)^{1/2}.$$

3.10 Intersection of the unstable manifold with H_0

We need to give precisely the intersection of the unstable manifold with the hyperplane $B_0 = B_{00}$. This gives a two-dimensional manifold lying in the 4-dimensional manifold $\mathcal{W}_{\varepsilon,\delta} \cap H_0$. Taking into account of

$$\begin{aligned}
\widetilde{A_*} &= \delta\alpha_- \\
\lambda_r, \lambda_i &\sim \frac{\delta^{1/2}\alpha_-^{1/2}}{2^{1/4}},\ \varepsilon = \nu_-\alpha_-^{\frac{5}{2}},\ \nu_- < \frac{1}{B_{00}}, \\
\overline{z_{10}} &\sim \frac{B_{00}}{\alpha_-\sqrt{2}},\ B_{00} \sim \frac{1}{\sqrt{1+\delta^2}}, \\
|\overline{Y}(-x_*)| &= \mathcal{O}(|\overline{X_0}|^2 + \nu_-|\overline{X_0}| + \nu_-^2 B_{00}), \quad |\overline{X_0}| \leq k_0, , \overline{X_0} = \overline{X(-x_*)},
\end{aligned}$$

we obtain a two-dimensional intersection which is tangent to a plane (parameters $\overline{x_1}, \overline{x_2}$), given by the following (using (20) and Lemma 19)

Lemma 24 *For $1/3 \leq \delta \leq 1$, ε small enough, $\varepsilon = \nu_-\alpha_-^{5/2}$, the 2-dimensional intersection of the 3-dimensional plane tangent to the unstable manifold, with the 5-dimensional hyperplane H_0, satisfies the system (rescaled parameters are $(\overline{x_1}, \overline{x_2}) = \frac{\delta^{1/2}}{B_{00}}\overline{X}_0$ with $|\overline{X}_0| \leq k_0$)*

$$
\begin{aligned}
A_0 &= \delta\alpha_- + \frac{\delta\alpha_-}{2^{3/4}}(\overline{x_1} - \overline{x_2}) + \mathcal{O}(\alpha_-\nu_-|(\overline{x_1}, \overline{x_2})| + \alpha_-\nu_-^2) \\
A_1 &= (\delta\alpha_-)^{3/2}\overline{x_1} - \frac{\alpha_-^2\delta}{\sqrt{2}}B_{00} + \mathcal{O}(\alpha_-^{3/2}\nu_-|(\overline{x_1}, \overline{x_2})| + \alpha_-^{3/2}\nu_-^2) \\
A_2 &= \frac{(\delta\alpha_-)^2}{2^{1/4}}(\overline{x_1} + \overline{x_2}) + \mathcal{O}(\alpha_-^2\nu_-|(\overline{x_1}, \overline{x_2})| + \alpha_-^2\nu_-^2) \qquad (61) \\
A_3 &= \sqrt{2}(\delta\alpha_-)^{5/2}\overline{x_2} + \mathcal{O}(\alpha_-^{5/2}\nu_-|(\overline{x_1}, \overline{x_2})| + \alpha_-^{5/2}\nu_-^2) \\
B_{00} &\sim (1+\delta^2)^{-1/2},
\end{aligned}
$$

with

$$
\begin{aligned}
(|\overline{x_1}|^2 + |\overline{x_2}|^2)^{1/2} &\leq k_0[\delta(1+\delta^2)]^{1/2}, \quad 1/3 \leq \delta \leq 1, \ \varepsilon = \nu_-\alpha_-^{5/2}, \\
\alpha_-^2\delta^2 &= 1 - B_{00}^2(1+\delta^2) > 0,
\end{aligned}
$$

and where we do not write B_1 since we know that we sit on the 5 dimensional manifold $\mathcal{W}_{\varepsilon,\delta}$.

4 Stable manifold of M_+

Assuming some estimates which need to be checked at the end, the strategy here is to first solve with respect to B_0 in using the first integral (6), and an implicit function argument. Hence B_0 becomes function of $(A_j(x), B_0(-x_*))$, $j = 0, 1, 2, 3$ defined on $(-x_*, +\infty)$. Afterwards for the search of the stable manifold of M_+ (with only 2 remaining dimensions), we need to solve a 4-dimensional system, with variable coefficients on $(-x_*, +\infty)$.

4.1 Using the first integral (6)

The first integral (6) is solved with respect to $B_0(x)$. Since it is shown in [1] that $B_0(x) \in [0, 1]$ and $B_0(+\infty) = 1$ for the heteroclinic solution of (1) we take the positive square-root:

$$
B_0' = \frac{\varepsilon}{\sqrt{2}}[(1 - B_0^2)^2 + A_0^2(A_0^2 + 2\delta^2 B_0^2 + 2(B_0^2 - 1)) - 2A_2^2 + 4A_1A_3]^{1/2},
$$

which shows that B_0 is growing as soon as the bracket does not cancel. Let us define

$$
-1 \leq v = B_0 - 1 \leq 0, \qquad (62)
$$

then we obtain

$$
v' = \frac{-\varepsilon v}{\sqrt{2}}(2+v)\left(1 + \frac{A_0^2(A_0^2 + 2\delta^2 B_0^2 + 2(B_0^2 - 1)) - 2A_2^2 + 4A_1A_3}{(1 - B_0^2)^2}\right)^{1/2}. \qquad (63)
$$

For any $\kappa \geq 0$ let us introduce a function space adapted for this section

$$C^0_\kappa = \{X \in C^0(-x_*, +\infty); X(x)e^{\kappa x} \text{ bounded}\},$$

equiped with the norm

$$||X||_\kappa = \sup_{(-x_*,\infty)} |X(x)e^{\kappa x}|.$$

For the connection with previous section, we take

$$B_0^2(-x_*) = \frac{1-\alpha_-^2\delta^2}{1+\delta^2},\ \varepsilon = \nu_-\alpha_-^{5/2}. \tag{64}$$

Then we prove the following

Lemma 25 *For $1/3 \leq \delta \leq 1$, $\kappa > \varepsilon$, assume (64) holds and assume that there exists γ such that*

$$\begin{aligned} |A_j(x)| &\leq \gamma|v(x)|,\ j = 0,1,2,3, \\ x &\in (-x_*, +\infty),\ \gamma \leq 1/5. \end{aligned} \tag{65}$$

Then there exists a unique solution $v = \mathcal{V}(A_0, A_1, A_2, A_3) \in C^1(-x_*, +\infty)$ of (63). $\mathcal{V}$ depends analytically on $A_j \in C^0_\kappa$, $j = 0,1,2,3$, with $B_0 = 1 + \mathcal{V}$,

$$\begin{aligned} \mathcal{V}(A_0, A_1, A_2, A_3) &= \mathcal{V}_0 + \mathcal{H}(A_0, A_1, A_2, A_3), \\ ||\mathcal{H}(A_0, A_1, A_2, A_3)||_\kappa &\leq c\frac{\varepsilon}{\kappa}||(A_0, A_1, A_2, A_3)||^2_\kappa, \\ \mathcal{V}_0(x) &= \frac{(1-\sqrt{1+\delta^2})[1-\tanh(\varepsilon\sqrt{2}x)]}{\sqrt{1+\delta^2}+\tanh(\varepsilon\sqrt{2}x)} < 0, \\ 1 + \mathcal{V}_0(0) &= \frac{1}{\sqrt{1+\delta^2}},\ \mathcal{V}_0(-x_*) = B_0(-x_*) - 1 < 0, \end{aligned} \tag{66}$$

and x_* defined in (70), is such that $B_0|_{x=0} = (1+\delta^2)^{-1/2}$. Moreover we have

$$\frac{v(-x_*)[1-\tanh(\frac{3\varepsilon(x+x_*)}{4\sqrt{2}})]}{1+B_0(-x_*)\tanh(\frac{3\varepsilon(x+x_*)}{4\sqrt{2}})} \leq v(x) \leq \frac{v(-x_*)[1-\tanh(\frac{5\varepsilon(x+x_*)}{4\sqrt{2}})]}{1+B_0(-x_*)\tanh(\frac{5\varepsilon(x+x_*)}{4\sqrt{2}})}. \tag{67}$$

Remark 26 *Later we need to check that (65) is indeed satisfied for the stable manifold.*

Proof. Let us assume that (65) holds, and using (64) for ε small enough, we have

$$A_0^2 + 2\delta^2 B_0^2 + 2(B_0^2 - 1) < 3,$$

so that

$$|A_0^2(A_0^2 + 2\delta^2 B_0^2 + 2(B_0^2-1)) - 2A_2^2 + 4A_1A_3| \leq 9\gamma^2|v|^2.$$

Now, by (62)

$$(1-B_0^2) = |v|(2+v) > |v|,$$

hence

$$\left|\frac{A_0^2(A_0^2+2\delta^2B_0^2+2(B_0^2-1))-2A_2^2+4A_1A_3}{(1-B_0^2)^2}\right| \leq 9\gamma^2 < \frac{1}{2},$$

and the square root is analytic in (v, A_0, A_1, A_2, A_3) leading to

$$v' = -\varepsilon\sqrt{2}v(1+\frac{1}{2}v)[1+\mathcal{Z}(v, A_0, A_1, A_2, A_3)], \quad ||\mathcal{Z}||_\kappa \leq 1/4, \tag{68}$$

with

$$||\mathcal{Z}(v, A_0, A_1, A_2, A_3)||_\kappa \leq c||(A_0, A_1, A_2, A_3)||_\kappa^2. \tag{69}$$

Then we can integrate the differential equation, as in section 3.9. We introduce the new variable w as

$$w' = \frac{2v'}{v(2+v)}, \ w = \ln\left(\frac{-v}{1+v/2}\right), \ v = -\frac{e^w}{1+\frac{1}{2}e^w},$$

w decreases from w_0 to $-\infty$ for $x \in (-x_*, \infty)$, while v grows from $v_0 = v(-x_*) < 0$ to 0. Then, defining $h(w, A_0, A_1, A_2, A_3) \overset{def}{=} \mathcal{Z}(v, A_0, A_1, A_2, A_3)$, we obtain, by simple integration

$$\varepsilon\sqrt{2}(x+x_*)(1-1/4) \leq w_0 - w(x) \leq \varepsilon\sqrt{2}(x+x_*)(1+1/4),$$

from which we deduce the estimate (67). This a priori estimate for v allows to prove (see for example [5]) the existence and uniqueness of a solution for (68) on the whole interval $x \in [-x_*, \infty)$. We define x_* in choosing to satisfy

$$B_0(0) = \frac{1}{\sqrt{1+\delta^2}},$$

which gives the expression of $\mathcal{V}_0(x)$ given in (66) and

$$x_* \sim \frac{\sqrt{1+\delta^2}}{2\sqrt{2}}\frac{\alpha_-^2}{\varepsilon} = \frac{\sqrt{1+\delta^2}}{2\nu_-^{4/5}\sqrt{2}}\varepsilon^{-1/5}. \tag{70}$$

The estimate in (66) results from (69). Lemma 25 is proved. ■

4.2 About the interval $(-x_*, x_*^+)$

For the first part of the proof for the stable manifold, we do not start from $x = -x_*$, but from $x = x_*^+$ for which $B_0(x_*^+)$ satisfies

$$B_0^2(x_*^+) = \frac{1+\alpha_+^2\delta^2}{1+\delta^2}, \tag{71}$$

where α_+ is defined at next section, with $\nu_+ \neq \nu_-$ and

$$\varepsilon = \nu_+ \alpha_+^{5/2}.$$

In fact we obtain below an estimate on ν_+ which is much better than for ν_- and this will allow to extend later the existence of the stable manifold on the interval

$$\left(\frac{1-\alpha_-^2\delta^2}{1+\delta^2}\right)^{1/2} \leq B_0 \leq \left(\frac{1+\alpha_+^2\delta^2}{1+\delta^2}\right)^{1/2}.$$

For finding the heteroclinic we need to connect $B_0(x)$ found for the unstable manifold until its upper limit

$$B_0(-x_*) = \left(\frac{1-\alpha_-^2\delta^2}{1+\delta^2}\right)^{1/2},$$

with $B_0(x)$ found at Lemma 25 valid until the same limit of $B_0(-x_*)$ (now the lower limit). Then we have

$$B_0(x_*^+) \sim \frac{1+\sqrt{1+\delta^2}\tanh(\varepsilon\sqrt{2}x_*^+)}{\sqrt{1+\delta^2}+\tanh(\varepsilon\sqrt{2}x_*^+)}$$

which gives

$$x_*^+ \sim \frac{\sqrt{(1+\delta^2)}}{2\sqrt{2}}\frac{\alpha_+^2}{\varepsilon} \sim \frac{\sqrt{1+\delta^2}}{2\nu_+^{4/5}\sqrt{2}}\varepsilon^{-1/5}, \tag{72}$$

where x_*^+ is a priori different from x_* since $\nu_+ \neq \nu_-$ and $B_0(x)$ is not invariant under the change $x \mapsto -x$.

4.3 Formulation with new coordinates

Here again, the strategy is first in section 4.3, to write the system (1) in adapted coordinates, still B_0- dependent, where B_0 satisfies Lemma 25. The remaining two stable directions lead, for the search of the stable manifold, to a system solved for $x \in [x_*^+, +\infty)$ in section 4.4. Finally we give nearly explicitely the 2-dimensional intersection of the tangent plane of the stable manifold built on $x \in [x_*^+, +\infty)$, with the hyperplane $B_0 = \left(\frac{1+\alpha_+^2\delta^2}{1+\delta^2}\right)^{1/2}$.

As in section 3, the new coordinates are such that we are able to use monodromy operators with easy estimates in the formulation of the search for the 3-dimensional stable manifold of M_+.

For

$$(1+\delta^2)B_0^2 - 1 \geq \alpha_+^2\delta^2,\ \varepsilon = \nu_+\alpha_+^{5/2},\ \nu_+ > 0,$$

let us define

$$\widetilde{\delta} = \{(1+\delta^2)B_0^2 - 1\}^{1/4} \geq (\alpha_+\delta)^{1/2}, \tag{73}$$

and choose a new basis (function of B_0)

$$V_r^\pm = \begin{pmatrix} 1 \\ \pm\frac{\widetilde{\delta}}{\sqrt{2}} \\ 0 \\ \mp\frac{\widetilde{\delta}^3}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \ V_i^\pm = \begin{pmatrix} 0 \\ \pm\frac{\widetilde{\delta}}{\sqrt{2}} \\ \widetilde{\delta}^2 \\ \pm\frac{\widetilde{\delta}^3}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix},$$

$$W_1^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -\varepsilon\sqrt{2} \end{pmatrix}, \ W_1^+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \varepsilon\sqrt{2} \end{pmatrix},$$

for defining new coordinates $(x_1, x_2, y_1, y_2, z_0, z_1)$ such that

$$\begin{aligned} Z &= (0,0,0,0,1,0)^t \\ &\quad +x_1 V_r^- + x_2 V_i^- + y_1 V_r^+ + y_2 V_i^+ + z_0 W_1^- + z_1 W_1^+ \end{aligned} \tag{74}$$

hence

$$\begin{aligned} A_0 &= x_1 + y_1 \\ A_1 &= -\frac{\widetilde{\delta}}{\sqrt{2}}(x_1 - y_1 + x_2 - y_2) \\ A_2 &= \widetilde{\delta}^2(x_2 + y_2) \\ A_3 &= \frac{\widetilde{\delta}^3}{\sqrt{2}}(x_1 - y_1 - x_2 + y_2) \\ B_0 &= 1 + v \end{aligned} \tag{75}$$

which is easy to invert, where v is given by Lemma 25, and coordinates z_0, z_1 are not used, replaced by the use of v. Now the system (1) reads as

$$\begin{aligned} A_0' &= A_1, \\ A_1' &= A_2, \\ A_2' &= A_3, \\ A_3' &= -A_0[A_0^2 + \widetilde{\delta}^4(v)], \end{aligned} \tag{76}$$

where we know (and use) that

$$v' = -\varepsilon\sqrt{2}v(1 + \frac{1}{2}v)[1 + \mathcal{Z}(v, A_0, A_1, A_2, A_3)].$$

With new variables defined in (75) this leads to the new 4-dimensional system

$$\begin{aligned} X' &= -\boldsymbol{L}X + G_0(\varepsilon, v, X, Y), \\ Y' &= \boldsymbol{L}Y + G_1(\varepsilon, v, X, Y), \end{aligned}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ \boldsymbol{L} = \frac{\widetilde{\delta}(v)}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\begin{aligned} G_0(\varepsilon, v, X, Y) &= \frac{(x_1+y_1)^3}{2\sqrt{2}\widetilde{\delta}^3} V_0 + \frac{\varepsilon}{\widetilde{\delta}^4} \frac{(1+v)(\delta^2 - \widetilde{\delta}^4)}{4\sqrt{2}} [1 + \mathcal{Z}(v, X, Y)](\boldsymbol{M}_1 X + \boldsymbol{M}_2 Y), \\ G_1(\varepsilon, v, X, Y) &= -\frac{(x_1+y_1)^3}{2\sqrt{2}\widetilde{\delta}^3} V_0 + \frac{\varepsilon}{\widetilde{\delta}^4} \frac{(1+v)(\delta^2 - \widetilde{\delta}^4)}{4\sqrt{2}} [1 + \mathcal{Z}(v, X, Y)](\boldsymbol{M}_1 Y + \boldsymbol{M}_2 X), \end{aligned}$$

$$V_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \ \boldsymbol{M}_1 = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix}, \ \boldsymbol{M}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix},$$

where $\mathcal{Z}(v, X, Y)$ is defined in (68), and where we used

$$\widetilde{\delta}' = \frac{(1+v)(\delta^2 - \widetilde{\delta}^4)}{2\widetilde{\delta}^3} \frac{\varepsilon}{\sqrt{2}} (1 + \mathcal{Z}).$$

The above system is completed by the expression of v given by Lemma 25. Notice that the coefficients of the linear part in (X, Y) are functions of v, where the expected part, which has the factor $\frac{\widetilde{\delta}(v)}{\sqrt{2}}$, is perturbed by a linear part bounded by $\mathcal{O}(\frac{\varepsilon}{\alpha_+^2}) = \mathcal{O}(\nu_+ \alpha_+^{1/2})$. Below, we choose ν_+ such that the perturbed part is really a perturbation of the first part.

For finding the stable manifold of M_+ we put the system in an integral form, looking for solutions tending to 0 as $x \to +\infty$. With $x \geq x_*^+$, we obtain the system

$$\begin{aligned} X(x) &= S_0(x, x_*^+) X_0 + \int_{x_*^+}^{x} S_0(x, s) G_0[\varepsilon, v(s), X(s), Y(s)] ds, \\ Y(x) &= -\int_x^{+\infty} S_1(x, s) G_1[\varepsilon, v(s), X(s), Y(s)] ds, \end{aligned} \tag{77}$$

where we notice that

$$S_0(x, s) = e^{-\int_s^x \frac{\widetilde{\delta}(v(\tau))d\tau}{\sqrt{2}}} \begin{pmatrix} \cos \int_s^x \frac{\widetilde{\delta} d\tau}{\sqrt{2}} & -\sin \int_s^x \frac{\widetilde{\delta} d\tau}{\sqrt{2}} \\ \sin \int_s^x \frac{\widetilde{\delta} d\tau}{\sqrt{2}} & \cos \int_s^x \frac{\widetilde{\delta} d\tau}{\sqrt{2}} \end{pmatrix}, \tag{78}$$

$$S_1(x, s) = e^{\int_s^x \frac{\widetilde{\delta}(v(\tau))d\tau}{\sqrt{2}}} \begin{pmatrix} \cos \int_s^x \frac{\widetilde{\delta} d\tau}{\sqrt{2}} & \sin \int_s^x \frac{\widetilde{\delta} d\tau}{\sqrt{2}} \\ -\sin \int_s^x \frac{\widetilde{\delta} d\tau}{\sqrt{2}} & \cos \int_s^x \frac{\widetilde{\delta} d\tau}{\sqrt{2}} \end{pmatrix}, \tag{79}$$

and, using (73)

$$
\begin{aligned}
||S_0(x,s)|| &\leq e^{-\sqrt{\frac{\alpha_+\delta}{2}}(x-s)},\ x_*^+ \leq s \leq x < \infty, \\
||S_1(x,s)|| &\leq e^{-\sqrt{\frac{\alpha_+\delta}{2}}(s-x)},\ \ x_*^+ \leq x \leq s < \infty.
\end{aligned}
$$

The 3-dimensional stable manifold is obtained in expressing $(X(x), Y(x), B_0(x))$ as a function of $(X_0, B_0(x_*^+))$ solving (77), where $B_0 = 1+v$ is given by Lemma 25 and $B_0(x_*^+)$ has the lower bound (71).

4.4 The stable manifold for $x \in [x_*^+, +\infty)$

We show the following

Lemma 27 *For $1/3 \leq \delta \leq 1$, $0 < k_1 < \frac{1}{\sqrt{2}}$, $\nu_+ > 0$ small enough, and for ε small enough, the 3-dimensional stable manifold $\mathcal{W}_{\varepsilon,\delta}^{(s)}$ of M_+ exists for $x \in [x_*^+, +\infty)$, is included in the 5-dimensional manifold $\mathcal{W}_{\varepsilon,\delta}$, is analytic in (ε, δ), parameterized by $(X_0, B_0(x_*^+))$ where $X(x)$ is a 2-dimensional coordinate defined in (74), and $X_0 = X(x_*^+)$. Moreover choosing δ_*such that*

$$
\delta_* = \frac{1}{10}\delta^{2/5}
$$

we have

$$
\begin{aligned}
\left(\frac{1+\alpha_+^2\delta^2}{1+\delta^2}\right)^{1/2} &\leq B_0(x_*^+) \leq B_0(x) \leq 1,\ B_0'(x) > 0,\ x \in [x_*^+, +\infty), \\
A_j(x) &= \mathcal{O}(|X_0|e^{-\delta_*\varepsilon^{1/5}x}),\ j = 0,1,2,3,\ \varepsilon = \nu_+\alpha_+^{5/2},
\end{aligned}
$$

where $|X_0| \leq k_1\alpha_+\delta$. As $x \to +\infty$, $(A_0, A_1, A_2, A_3) \to 0$ as $exp(-\sqrt{\frac{\delta}{2}}x)$, $(1 - B_0, B_1) \to 0$ as $\exp(-\sqrt{2}\varepsilon x)$.

Proof. Let us solve (77) with respect to $(X, Y) \in C_\kappa^0$ for $|X_0|$ small enough and choose

$$
\kappa = \frac{1}{10}\delta^{2/5}\varepsilon^{1/5} = \overline{\kappa}\sqrt{\frac{\alpha_+\delta}{2}},\ \ \overline{\kappa} = \frac{\sqrt{2}}{10}\left(\frac{\nu_+}{\sqrt{\delta}}\right)^{1/5}
$$

then ν_+ is choosen such that $\overline{\kappa} < 1$. Then (77) implies (below, the space C_κ^0 is built on $[x_*^+, +\infty)$ instead of $[-x_*, +\infty)$)

$$
\begin{aligned}
||X||_\kappa &\leq |X_0| + \frac{1}{1-\overline{\kappa}}\sqrt{\frac{2}{\alpha_+\delta}}||G_0||_\kappa, \\
||Y||_\kappa &\leq \frac{1}{1+\overline{\kappa}}\sqrt{\frac{2}{\alpha_+\delta}}||G_1||_\kappa.
\end{aligned}
$$

Moreover we have for $j = 0, 1$ (see the expressions of G_j), using $||M_j|| \leq \sqrt{18}$,

$$||G_j||_\kappa \leq \frac{(1+c\alpha_+^2)}{2\sqrt{2}\delta^{3/2}\alpha_+^{3/2}}(||X||_\kappa + ||Y||_\kappa)^3 + \frac{3\nu_+\delta^2\alpha_+^{1/2}}{4}(||X||_\kappa + ||Y||_\kappa)$$

with c independent of ε. Now making the scaling

$$(X, Y) = \alpha_+(\overline{X}, \overline{Y}),$$

we obtain

$$\begin{aligned} ||\overline{X}||_\kappa &\leq |\overline{X_0}| + \frac{3}{2\sqrt{2}(1-\overline{\kappa})}\frac{\nu_+\delta^2}{\sqrt{\delta}}(||\overline{X}||_\kappa + ||\overline{Y}||_\kappa) + \frac{(1+c\alpha_+^2)}{(1-\overline{\kappa})\delta^2}(||\overline{X}||_\kappa + ||\overline{Y}||_\kappa)^3, \\ ||\overline{Y}||_\kappa &\leq \frac{3}{2\sqrt{2}(1+\overline{\kappa})}\frac{\nu_+\delta^2}{\sqrt{\delta}}(||\overline{X}||_\kappa + ||\overline{Y}||_\kappa) + \frac{(1+c\alpha_+^2)}{(1+\overline{\kappa})\delta^2}(||\overline{X}||_\kappa + ||\overline{Y}||_\kappa)^3. \end{aligned}$$

It is then clear that, for

$$\frac{\nu_+\delta^2}{\sqrt{\delta}} < \frac{\sqrt{2}}{3}(1-\overline{\kappa}^2),$$

i.e. (using the definition of $\overline{\kappa}$)

$$\nu_+\delta^{3/2} \leq 0.46, \;\; \overline{\kappa} \leq 0.1645 \tag{80}$$

we can appy the implicit function theorem (in the analytic frame as in [3] section X.2) for $|\overline{X_0}| \leq k_1$ with $0 < k_1$ small enough and for ε small enough, so that we obtain a unique solution $(\overline{X}, \overline{Y})$ in C_κ^0 satisfying

$$\begin{aligned} ||\overline{X}||_\kappa &\leq (1 + c\nu_+)|\overline{X_0}|, \\ ||\overline{Y}||_\kappa &\leq c\nu_+|\overline{X_0}| + c|\overline{X_0}|^3, \end{aligned}$$

with c independent of ε, ν_+ small enough, and provided that

$$|\overline{X_0}| \leq k_1, \quad k_1 \text{ small enough.}$$

By construction (see (75)) of (X, Y) we obtain the estimates on $A_j(x)$ indicated at Lemma 27. It then remains to check the validity of condition (65). Indeed estimates (67) of v imply

$$|v_0|e^{\frac{-5\varepsilon\sqrt{2}x}{4}} \leq |v(x)| \leq \frac{|v_0|}{1 - \frac{|v_0|}{2}}e^{\frac{-3\varepsilon\sqrt{2}x}{4}},$$

where

$$|v_0| = 1 - B_0(-x_*) \sim 1 - \frac{1}{\sqrt{1+\delta^2}}.$$

Moreover, for $j = 0, 1, 2, 3$ and ε small enough

$$|A_j(x)| \leq 2|X_0|e^{-\kappa x} \leq 2k_1\alpha_+e^{-\kappa x}, \;\; x > x_*^+.$$

Since we have $\kappa = \mathcal{O}(\varepsilon^{1/5})$, then for ε small enough

$$e^{-\kappa x} \leq e^{\frac{-5\varepsilon\sqrt{2}x}{4}}, \; x > x_*^+ > 0,$$

the required condition (65) is realized as soon as

$$2k_1\alpha_+ \leq 1 - B_0(-x_*)$$

which holds true for ε small enough. The exponential behavior declared in Lemma 27 follows from the linear study of section 2.2 as $x \to +\infty$. This ends the proof of Lemma 27, and of Corollary 3 for the part $x \in [x_*^+, +\infty)$. ■

4.4.1 Estimate of ν_+

From the proof of Lemma 27, and from the fact that $\overline{\kappa}$ may be stated as small as we need, the restriction on ν_+ is

$$\nu_+\delta^{3/2} \leq \frac{\sqrt{2}}{3}. \tag{81}$$

4.5 Intersection of the stable manifold with H_1

We need to compute the intersection of the 3-dimensional stable manifold $\mathcal{W}_{\varepsilon,\delta}^{(s)}$ of M_+ with the hyperplane H_1 defined by

$$B_0 = B_{01} \stackrel{def}{=} \sqrt{\frac{1+\alpha_+^2\delta^2}{1+\delta^2}}. \tag{82}$$

We then obtain a 2-dimensional sub-manifold living in the 4-dimensional manifold $\mathcal{W}_{\varepsilon,\delta} \cap H_1$. We have the following

Lemma 28 *For $1/3 \leq \delta \leq 1$, $\varepsilon = \nu_+\alpha_+^{5/2}$, ε and ν_+ small enough, the two-dimensional intersection of the 3-dimensional plane, tangent to the stable manifold of M_+, with the 5-dimensional hyperplane H_1, satisfies a linear system with rescaled parameters $(\overline{x_{10}}, \overline{x_{20}}) = \frac{1}{\delta}\overline{X_0}$ and, by construction $\widetilde{\delta}|_{B_{01}} = (\alpha_+\delta)^{1/2}$,*

$$\begin{aligned}
A_0 &= \delta\alpha_+(\overline{x_{10}} + \overline{y_{10}}), \\
A_1 &= -\frac{(\delta\alpha_+)^{3/2}}{\sqrt{2}}(\overline{x_{10}} + \overline{x_{20}} - \overline{y_{10}} - \overline{y_{20}}) \\
A_2 &= (\alpha_+\delta)^2(\overline{x_{20}} + \overline{y_{20}}) \\
A_3 &= \frac{(\alpha_+\delta)^{5/2}}{\sqrt{2}}(\overline{x_{10}} - \overline{x_{20}} - \overline{y_{10}} + \overline{y_{20}}),
\end{aligned} \tag{83}$$

where $\overline{Y_0}$ is a linear function of $\overline{X_0}$ such that $|\overline{Y_0}| \leq c\nu_+|\overline{X_0}|$, with the restriction

$$(|\overline{x_{10}}|^2 + |\overline{x_{20}}|^2)^{1/2} \leq k_1.$$

5 Extension and Intersection of the two manifolds

5.1 Extension of the stable manifold of M_+ for $x \in [-x_*, x_*^+]$

We need to extend the definition of the stable manifold of M_+ to the region where

$$\frac{1-\alpha_-^2\delta^2}{1+\delta^2} \leq B_0^2(x) \leq \frac{1+\alpha_+^2\delta^2}{1+\delta^2},$$

i.e where

$$-x_* \leq x \leq x_*^+,$$

and where

$$A_0^{(4)} = -A_0[A_0^2 + (1+\delta^2)B_0^2 - 1], \tag{84}$$

B_0 satisfying Lemma 25. We prove the following

Lemma 29 *For $1/3 \leq \delta \leq 1$, and ε small enough, the 3-dimensional stable manifold $\mathcal{W}_{\varepsilon,\delta}^{(s)}$ of M_+ still exists for $B_0^2 \in [\frac{1-\alpha_-^2\delta^2}{1+\delta^2}, \frac{1+\alpha_+^2\delta^2}{1+\delta^2}]$, is analytic in ε, δ, parameterized by $(X_0, B_0(x_*^+))$, where $|X_0| \leq k_1\alpha_+$ with k_1 small enough, as in Lemma 27. Moreover $B_0(-x_*) \leq B_0(x) \leq B_0(x_*^+)$, $B_0'(x) > 0$ where B_0 is given by Lemma 25, and A_0 and it derivatives solve the differential equation (84) on the interval $-x_* \leq x \leq x_*^+$.*

We observe that Lemma 25 is valid, provided that (65) holds, hence

$$\begin{aligned} (1+\delta^2)B_0^2 - 1 &= \frac{2\sqrt{2}\delta^2}{\sqrt{1+\delta^2}}[\varepsilon x + \mathcal{O}(\varepsilon x)^2 + \frac{\varepsilon}{\kappa}\mathcal{O}(|(A_0, A_1, A_2, A_3)|^2)], \\ -\alpha_-^2\frac{\sqrt{1+\delta^2}}{2\sqrt{2}} &\sim -\varepsilon x_* \leq \varepsilon x \leq \varepsilon x_*^+ \sim \alpha_+^2\frac{\sqrt{1+\delta^2}}{2\sqrt{2}}, \end{aligned}$$

where, from lemma 27

$$|A_j(x_*^+)| \leq ck_1\alpha_+^{(1+j/2)},\ j = 0,1,2,3,$$

with k_1 small enough and $\frac{\varepsilon}{\kappa} = \frac{\varepsilon}{\delta_*\varepsilon^{1/5}} = \mathcal{O}(\varepsilon^{4/5})$.

The strategy here consists to solve backwards the principal part of the differential equation (84) on $[-x_*, x_*^+]$, where A_0 satisfies the boundary conditions in x_*^+. In fact we only consider the boundary conditions coming from the tangent plane of the stable manifold for $x = x_*^+$ (role of hyperplanr H_1), since, for A_0 we stay in a neighborhoof of 0 and we only need at the end the trace of the tangent plane to the manifold at the extreme point for$x = -x_*$. It is a sort of managing the transport of the tangent plane to the manifold on the interval $[-x_*, x_*^+]$. Then we need to study the intersection of the 2-dimensional tangent plane along the intersection $\mathcal{W}_{\varepsilon,\delta}^{(u)} \cap H_0$ with the 2-dimensional tangent plane along the intersection $\mathcal{W}_{\varepsilon,\delta}^{(s)} \cap H_0$, all this lying in the 4-dimensional tangent plane to the manifold $H_0 \cap \mathcal{W}_{\varepsilon,\delta}$.

Let us first consider the principal part of (84) and rescale as

$$\begin{aligned} z &= K\varepsilon^{1/5}x, \; A_0(x) = K^2\varepsilon^{2/5}\overline{A_0}(z) \\ K &= \left(\frac{2\sqrt{2}\delta^2}{\sqrt{1+\delta^2}}\right)^{1/5}, \end{aligned}$$

Now

$$z \in [-a_-, a_+] \;\text{ with } a_\pm = \left(\frac{(1+\delta^2)\delta}{8\nu_\pm^2}\right)^{2/5} \text{ independent of } \varepsilon, \tag{85}$$

with

$$\delta^2 = (K\nu_\pm^{1/5})^4 a_\pm.$$

The principal part of the differential equation for A_0 reads now as

$$\frac{d^4\overline{A_0}}{dz^4} = -\overline{A_0}(\overline{A_0}^2 + z), \tag{86}$$

with boundary conditions coming from the intersection of the unstable manifold of M_- with H_0 for $z = -a_-$, and from the intersection of the stable manifold of M_+ with H_1 for $z = +a_+$.

Remark 30 *From the definition of* $a_\pm$ *and from estimates (52) for* ν_- *and (81) for* ν_+*, we obtain*

$$(a_-)_{\min} = 34.74 \left(\frac{1+\delta^2}{2}\right)^{6/5}, \tag{87}$$

$$(a_+)_{\min} = 1.05 \left(\frac{1+\delta^2}{2}\right)^{2/5} \delta^{8/5}. \tag{88}$$

For the principal part of the intersection with H_0, for $z = -a_-$ (see (61))

$$\begin{aligned} \overline{A_0} &= a_-^{1/2}[1 + \frac{1}{2^{3/4}}(\overline{x_1}^u - \overline{x_2}^u)] \\ \frac{d\overline{A_0}}{dz} &= a_-^{3/4}\overline{x_1}^u \\ \frac{d^2\overline{A_0}}{dz^2} &= \frac{a_-}{2^{1/4}}(\overline{x_1}^u + \overline{x_2}^u) \\ \frac{d^3\overline{A_0}}{dz^3} &= \sqrt{2}a_-^{5/4}\overline{x_2}^u \end{aligned} \tag{89}$$

where $(\overline{x_1}^u, \overline{x_2}^u)$ is a 2-dimensional parameter of size k_0 assumed to be small enough and ν_- is also small enough, independent of ε, k_0.

We also obtain the principal part of the intersection with H_1, for $z = +a$ (see (83)) as

$$\begin{aligned} \overline{A_0} &= a_+^{1/2}\overline{x_{10}}^s, \\ \frac{d\overline{A_0}}{dz} &= -\frac{a_+^{3/4}}{\sqrt{2}}(\overline{x_{10}}^s + \overline{x_{20}}^s) \\ \frac{d^2\overline{A_0}}{dz^2} &= a_+\overline{x_{20}}^s \\ \frac{d^3\overline{A_0}}{dz^3} &= \frac{a_+^{5/4}}{\sqrt{2}}(\overline{x_{10}}^s - \overline{x_{20}}^s), \end{aligned} \tag{90}$$

where $(\overline{x_{10}}^s, \overline{x_{20}}^s)$ is a 2-dimensional parameter assumed to be bounded in $\mathbb{R}^2$ by k_1. For (86), the heteroclinic curve is obtained when we find a solution $\overline{A_0}$ satisfying the boundary conditions with principal parts given above above in $z = \pm a_\pm$. We observe that ε has completely disappeared from this formulation, and that we have 4 parameters for this 4th order differential equation on the interval $[-a_-, +a_+]$. It is clear that in satisfying the boundary conditions at $z = +a_+$, we obtain a two-parameter family of solutions of (86) which needs to exist until $z = -a_-$, where ε does not play any role. In Appendix A.4 we prove the following

Lemma 31 *For the initial conditions (90) in $z = +a_+$, and for k_1 small enough, and $|(\overline{x_{10}}^s, \overline{x_{20}}^s)| \leq k_1$ the differential equation (86) has a 2-dimensional (parameter $(\overline{x_{10}}^s, \overline{x_{20}}^s)$) family of solutions for $z \in [-a_-, +a_+]$. These solutions depend analytically of $\delta \in [1/3, 1]$.*

Proof. The proof made in Appendix A.4 uses the standard way starting from a_+, with the fixed point theorem in a function space of bounded continuous functions. The only restriction for its use is that

$$a_+^5 < 3/2.$$

This is verified with our suitable choice of a_+ subjected to the restriction (88). ■

Now, the complete system (84), which depends on ε, is a small regular perturbation of size $\mathcal{O}(\varepsilon^{4/5})$ (after scaling) of (86), as well for the perturbed boundary conditions, so that the solution of (84) with its given boundary conditions at $x = x_*^+$, exists until $x = -x_*$, and corresponds to a small perturbation of the solution of (86) satisfying the principal part of the boundary condition at $z = a_+$.

Remark 32 *The differential equation (86) is mentionned in [15], where a matching asymptotic method is used, it also appears in [2] where a variational method is used. Both studies are for an infinite interval. Here we have the big advantage to deal with a finite interval, allowing to prove Lemma 31.*

5.2 Intersection of the two manifolds

In this section we prove the following:

Lemma 33 *For ε small enough, and for $1/3 \le \delta \le 1$, then, except maybe for a finite number of values, the unstable manifold $\mathcal{W}^{(u)}_{\varepsilon,\delta}$ of M_- intersects transversally the stable manifold $\mathcal{W}^{(s)}_{\varepsilon,\delta}$ of M_+ along the heteroclinic solution. Moreover, for $x = 0$, we have the estimates $A_j(0) = \mathcal{O}(\varepsilon^{(2+j)/5})$, $j = 0, 1, 2, 3$.*

Proof. For proving the intersection it is sufficient to prove that the manifolds intersect in the hyperplane H_0.

For the differential equation (86), $\overline{A_0}(-a_-)$ and its 3 first derivatives should satisfy (89). Considering the 2-dimensional tangent plane to the manifold formed by the 2-parameter family of solutions coming from the solution satisfying the conditions in $z = a_+$, we then obtain a linear system in $\mathbb{R}^4$ for the 4 unknowns $(\overline{x_{10}}^s, \overline{x_{20}}^s, \overline{x_1}^u, \overline{x_2}^u)$. Looking at the system, which is independent of ε, we see that the solution is independent of ε.

If we ignore the relationship coming from the differential equation (86) and just equate both sides, we obtain the unique solution

$$\begin{aligned}
\overline{x_1}^u &= -2\rho\frac{(2^{1/4}\rho - 1)}{2\rho^4 - 1},\ \rho = \left(\frac{a_-}{a_+}\right)^{1/4}, \qquad (91)\\
\overline{x_2}^u &= 2^{3/4}\frac{(2^{1/4}\rho - 1)}{2\rho^4 - 1},\\
\overline{x_{10}}^s &= \frac{\sqrt{2}\rho(\sqrt{2}\rho^3 - 1)}{2\rho^4 - 1},\\
\overline{x_{20}}^s &= -\rho^4\frac{\sqrt{2}(2^{1/4}\rho - 1)^2}{2\rho^4 - 1}.
\end{aligned}$$

Now the action of the differential equation (86) on $[-a_-, +a_+]$ "rotates" the 2-dimensional tangent plane, starting from the plane (90) in $z = +a_+$, and arriving for $z = -a_-$ to a system different from the above, hence giving a solution different from (91). The 2-parameter family of solutions depends analytically of δ, so that if the solution is not degenerated, the solution of the 4-dimensional linear system is unique, except maybe for a finite number of values of δ. The solution (91) which is obtained above, shows that the linear system should need to "rotate" suitably for becoming degenerate, where the "rotation" is function of δ. This can only happen for a discrete set of values of δ.

It remains to consider the full equation (84) which is a regular perturbation of (86), of order $\varepsilon^{4/5}$ including the bounds of the finite interval. It is then clear that the same result holds true for ε small enough, i.e the existence of a unique solution, meaning the transversality of the two manifolds, except maybe for a finite number of values of δ. However we observe that the conditions on the norms of $\overline{x_{10}}^s, \overline{x_{20}}^s$ and on $\overline{x_1}^u, \overline{x_2}^u$ cannot be checked directly, even though they all are of order 1, as expected. Here, we need to use the result of [1] which asserts the existence of a heteroclinic where B_0 starts from 0 for $x=-\infty$, arriving at $B_0=1$ for $x=+\infty$, and where $B_0(x)\in[0,1]$. Indeed, such a heteroclinic necessarily belongs to the stable manifold of M_- which is constructed here for $x\in(-\infty,-x_*]$, and also belongs to the stable manifold of M_+ which we have contructed for $x\in[-x_*,+\infty)$. This means that the intersection of tangent planes for $x=-x_*$, given, for the principal part, by the above mentioned 4-dimensional system has a solution. Such a solution is unique, from the proof made above, except maybe for a finite number of values of δ. Theorem 1 is proved. ■

Remark 34 *Since the size of $A_j(x)$ with respect to ε is not modified on $[-x_*,x_*^+]$, the result above implies the validity of the parts of Corollaries 2 and 3 for $-x_*\leq x\leq 0$ and $0\leq x\leq x_*^+$ respectively.*

6 Study of the linearized operator

Let us redefine the heteroclinic connection we found at Theorem 1 as

$$(A_*(x),B_*(x))\subset\mathbb{R}^2$$

with

$$1+1/9\leq g=1+\delta^2\leq 2,$$

and where we know that, for ε small enough

$$\begin{array}{rcl} B_*(x) & > & 0,\ B_*'(x)>0 \\ (A_*(x),B_*(x)) & \rightarrow & \left\{ \begin{array}{l} (1,0) \text{ as } x\rightarrow-\infty \\ (0,1) \text{ as } x\rightarrow+\infty \end{array}\right. , \end{array}$$

at least as $e^{\varepsilon\delta x}$ for $x\rightarrow-\infty$, and at least as $e^{-\sqrt{2}\varepsilon x}$ for $x\rightarrow+\infty$.

At section 1.1 we show that the perturbed system (2) leading to system (1) is now considered with B_0 complex valued, so in (1) B^2 is replaced by $|B|^2$.

For being able to prove any persistence result under reversible perturbations of system (1) in $\mathbb{R}^4\times\mathbb{C}^2$, as it appears in (2), we need to study the linearized operator at the above heteroclinic solution. We follow the lines of [6].

The linearized operator is given by

$$\begin{array}{rcl} A^{(4)} & = & (1-3A_*^2-gB_*^2)A-gA_*B_*(B+\overline{B}), \\ B'' & = & \varepsilon^2(-1+gA_*^2+2B_*^2)B+2\varepsilon^2 gA_*B_*A+\varepsilon^2 B_*^2\overline{B}. \end{array}$$

Taking real and imaginary parts for B :

$$B=C+iD,$$

we then obtain the linearized system

$$\begin{array}{rcl} -A^{(4)}+(1-3A_*^2-gB_*^2)A-2gA_*B_*C & = & 0, \\ \frac{1}{\varepsilon^2}C''+(1-gA_*^2-3B_*^2)C-2gA_*B_*A & = & 0, \\ \frac{1}{\varepsilon^2}D''+(1-gA_*^2-B_*^2)D & = & 0. \end{array}$$

Notice that the equation for D decouples, so that we can split the linear operator in an operator $\mathcal{M}_g$ acting on (A, C) and an operator $\mathcal{L}_g$ acting on D :

$$\mathcal{M}_g \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} + (1 - 3A_*^2 - gB_*^2)A - 2gA_*B_*C \\ \frac{1}{\varepsilon^2}C'' + (1 - gA_*^2 - 3B_*^2)C - 2gA_*B_*A \end{pmatrix},$$

$$\mathcal{L}_g D = \frac{1}{\varepsilon^2}D'' + (1 - gA_*^2 - B_*^2)D.$$

Let us define the Hilbert spaces

$$L_\eta^2 = \{u; u(x)e^{\eta|x|} \in L^2(\mathbb{R})\},$$

$$\begin{aligned} \mathcal{D}_0 &= \{(A, C) \in H_\eta^4 \times H_\eta^2; A \in H_\eta^4, C \in \mathcal{D}_1\} \\ \mathcal{D}_1 &= \{C \in H_\eta^2; \varepsilon^{-2}||C''||_{L_\eta^2} + \varepsilon^{-1}||C'||_{L_\eta^2} + ||C||_{L_\eta^2} \stackrel{def}{=} ||C||_{\mathcal{D}_1} < \infty\} \end{aligned}$$

equiped with natural scalar products. Below, we prove the following

Lemma 35 *Except maybe for a set of isolated values of g, the kernel of $\mathcal{M}_g$ in L_η^2 is one dimensional, span by (A'_*, B'_*), and its range has codimension 1, L^2-orthogonal to (A'_*, B'_*). $\mathcal{M}_g$ has a pseudo-inverse acting from L_η^2 to $\mathcal{D}_0$ for any $\eta > 0$ small enough, with bound independent of ε.*

The operator $\mathcal{L}_g$ has a trivial kernel, and its range which has codimension 1, is L^2- orthogonal to B_* ($B_* \notin L^2$). $\mathcal{L}_g$ has a pseudo-inverse acting from L_η^2 to $\mathcal{D}_1$ for $\eta > 0$ small enough, with bound independent of ε.

Remark 36 *For proving the above Lemma, we use the uniqueness (resulting from the transversality of manifolds $\mathcal{W}_{\varepsilon,\delta}^{(u)}$ and $\mathcal{W}_{\varepsilon,\delta}^{(s)}$) and analyticity in δ (i.e. g) of the heteroclinic, proved in previous section (see subsection 6.3.1).*

Remark 37 *The above Lemma is useful for proving the persistence under reversible perturbations, as indicated in (2), of our heteroclinic. This is done in [9] and appears to be more difficult than the symmetric case solved in [6]. Indeed, it is needed to introduce two different wave numbers for the two systems of convective rolls at $\pm\infty$. In [9] it is shown that the component on the kernel of $\mathcal{M}_g$ corresponds to a sort of adapted phase shift of rolls parallel to the wall, while the codimension 2 of the range implies that each wave number is function not only of the amplitude of rolls but also of the above shift. This then leads to a one parameter family of domain walls, for any fixed small amplitude ε^2.*

6.1 Asymptotic operators

Let us define the operators obtained when $x = \pm\infty$:

$$\mathcal{M}_\infty^- \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} - 2A \\ \varepsilon^{-2}C'' - (g-1)C \end{pmatrix},$$

$$\mathcal{M}_\infty^+ \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} - (g-1)A \\ \varepsilon^{-2}C'' - 2C \end{pmatrix},$$

$$\begin{aligned} \mathcal{L}_\infty^- D &= \varepsilon^{-2}D'' - (g-1)D, \\ \mathcal{L}_\infty^+ D &= \varepsilon^{-2}D''. \end{aligned}$$

Notice that all these operators are negative. Furthermore, their spectra in $L^2(\mathbb{R})$ are such that

$$\begin{aligned} \sigma(\mathcal{M}_\infty^-) &= (-\infty, -c_-],\ c_- = \max\{2, (g-1)\} > 0, \\ \sigma(\mathcal{M}_\infty^+) &= (-\infty, -c_+],\ c_+ = c_-, \\ \sigma(\mathcal{L}_\infty^-) &= (-\infty, -(g-1)], \\ \sigma(\mathcal{L}_\infty^+) &= (-\infty, 0]. \end{aligned}$$

Operators $\mathcal{M}_g$ and $\mathcal{L}_g$ are respectively relatively compact perturbations of the corresponding asymptotic operators $\mathcal{M}_\infty$ and $\mathcal{L}_\infty$ defined as

$$\mathcal{M}_\infty = \begin{cases} \mathcal{M}_\infty^-, & x < 0 \\ \mathcal{M}_\infty^+, & x > 0 \end{cases}, \quad \mathcal{L}_\infty = \begin{cases} \mathcal{L}_\infty^-, & x < 0 \\ \mathcal{L}_\infty^+, & x > 0 \end{cases},$$

Their essential spectrum, i.e. the set of $\lambda \in \mathbb{C}$ for which $\lambda - \mathcal{M}_g$ (resp. $\lambda - \mathcal{L}_g$) is not Fredholm with index 0, is equal to the essential spectrum of $\mathcal{M}_\infty$ (resp. $\mathcal{L}_\infty$) (see [12]). The latter spectra are found from the spectra of $\mathcal{M}_\infty^\pm$ and $\mathcal{L}_\infty^\pm$:

$$\begin{aligned} \sigma_{ess}(\mathcal{M}_\infty) &= (-\infty, -c_+], \\ \sigma_{ess}(\mathcal{L}_\infty) &= (-\infty, 0]. \end{aligned}$$

In particular, this implies that 0 does not belong to the essential spectrum of $\mathcal{M}_g$, so that the operator $\mathcal{M}_g$ is Fredholm with index 0. Moreover operators $\mathcal{M}_\infty$ and $\mathcal{L}_\infty$ are self adjoint negative operators in L^2, and $\mathcal{M}_\infty$ has a bounded inverse [12].

$$||\mathcal{M}_\infty^{-1}||_{L^2} \leq \frac{1}{c_+}.$$

This last property remains valid in exponentially weighted spaces, with weights $e^{\eta|x|}$, and η sufficiently small, since this acts as a small perturbation of the differential operator (see [11] section 3.1).

6.2 Properties of $\mathcal{L}_g$

Notice that $\mathcal{L}_g$ is self adjoint in $L^2(\mathbb{R})$ and that

$$\mathcal{L}_g B_* = 0, \quad \text{but } B_* \notin L^2(\mathbb{R}).$$

This property allows to solve explicitely the equation $\mathcal{L}_g u = f \in L^2_\eta$ with respect to $u \in L^2_\eta$ (using variation of constants method), and shows that it has a unique solution, provided that

$$\int_{\mathbb{R}} f B_* dx = 0.$$

We obtain

$$\begin{aligned} u(x) &= \int_x^\infty \frac{\varepsilon^2 B_*(x)}{B_*^2(s)} F(s) ds \\ \text{with } F(s) &= \int_s^\infty f(\tau) B_*(\tau) d\tau \text{ for } s \geq 0 \\ &= -\int_{-\infty}^s f(\tau) B_*(\tau) d\tau \text{ for } s \leq 0. \end{aligned}$$

By Fubini's theorem we can write for $x \geq 0$

$$u(x) = \varepsilon^2 B_*(x) \int_x^\infty f(\tau) B_*(\tau) \left(\int_x^\tau \frac{ds}{B_*^2(s)} \right) d\tau$$

and, for $x \leq 0$

$$\begin{aligned} u(x) &= -\varepsilon^2 B_*(x) \int_{-\infty}^x f(\tau) B_*(\tau) \left(\int_x^0 \frac{ds}{B_*^2(s)} \right) d\tau \\ &\quad -\varepsilon^2 B_*(x) \int_x^0 f(\tau) B_*(\tau) \left(\int_\tau^0 \frac{ds}{B_*^2(s)} \right) d\tau. \end{aligned}$$

The asymptotic properties of $B_*(x)$ at $\pm\infty$ imply, for $x \geq 0$

$$|u(x)| e^{\eta x} \leq C \varepsilon^2 \int_x^\infty |f(\tau) e^{\eta \tau}| (\tau - x) e^{-\eta(\tau - x)} d\tau,$$

and for $x \leq 0$

$$\begin{aligned} |u(x)| e^{-\eta x} &\leq \frac{C\varepsilon^2}{2\varepsilon\delta} \int_{-\infty}^x |f(\tau) e^{-\eta\tau}| e^{-(\eta+\varepsilon\delta)(x-\tau)} d\tau \\ &\quad + \frac{C\varepsilon^2}{2\varepsilon\delta} \int_x^0 |f(\tau) e^{-\eta\tau}| e^{(\eta-\varepsilon\delta)(\tau-x)} d\tau. \end{aligned}$$

The bound

$$||u||_{L^2_\eta} \leq c_2 ||f||_{L^2_\eta}$$

follows from classical convolution results between functions in L^2 and functions in L^1, since

$$\begin{aligned} \int_{-\infty}^0 e^{(\eta-\varepsilon\delta)\tau} d\tau &= \frac{1}{\eta - \varepsilon\delta}, \\ \int_0^\infty \tau e^{-\eta\tau} d\tau &= \frac{1}{\eta^2}. \end{aligned}$$

Then, we choose $\eta = \frac{1}{2}\varepsilon\delta$, so that the pseudo-inverse of $\mathcal{L}_g$ has a bounded inverse in L^2_η :

$$||\widetilde{\mathcal{L}_g}^{-1}|| \leq c_2,$$

where c_2 is independent of ε. Using the form of $\mathcal{L}_g$ we obtain easily

$$||u||_{\mathcal{D}_1} \leq c_3 ||f||_{L^2_\eta}$$

with c_3 independent of ε.

Remark 38 *The choice made for η is such that*

$$\eta < \varepsilon\delta, \quad \eta < \varepsilon\sqrt{2},$$

for values of δ for which Theorem 1 is valid. This means that as $x \to -\infty$ $(A_* - 1,\, B_*)$, and, as $x \to +\infty$ $(A_*, B_* - 1)$ tend exponentially to 0 *faster than* $e^{-\eta|x|}$.

In fact, $\mathcal{L}_g$ has the same properties as the operator $\mathcal{M}_i$ in the proof of Lemma 7.3 in [6], see also [8]: $\mathcal{L}_g$ is Fredholm with index -1, when acting in L^2_η, for η small enough. $\mathcal{L}_g$ has a trivial kernel, and its range is orthogonal to B_*, with the scalar product of $L^2(\mathbb{R})$.

6.3 Properties of $\mathcal{M}_g$

We saw that $\mathcal{M}_g$ is Fredholm with index 0. Furthermore the derivative of the heteroclinic solution belongs to its kernel:

$$\begin{aligned} \mathcal{M}_g \begin{pmatrix} A'_* \\ B'_* \end{pmatrix} &= \begin{pmatrix} -A_*^{(5)} + A'_* - (A_*^3)' - gB_*^2 A'_* - gA_*(B_*^2)' \\ \varepsilon^{-2} B'''_* + [B'_* - gA_*^2 B'_* - (B_*^3)' - gB_*(A_*^2)'] \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{92}$$

The part of the proof which differs from the proof made in [6], where the symmetry play an essential role, consists in showing at section 6.3.1 that the kernel of $\mathcal{M}_g$ is one-dimensional (except for a finite set of values of g), spanned by $(A'_*, B'_*) \stackrel{def}{=} U_*$ with a range orthogonal to U_* in L^2. Let us admit this result for the moment, and define the projections Q_0 on $U_*^\perp$ and P_0 on U_* , which are orthogonal projections in L^2, then we need to solve in L^2_η

$$\mathcal{M}_g u = f$$

in decomposing

$$u = zU_* + v, \quad v = Q_0 u,$$

$$\mathcal{M}_g v = (\mathcal{M}_\infty + \mathcal{A}_g) v = Q_0 f$$

and we need to satisfy the compatibility condition

$$\langle f, U_* \rangle = 0,$$

while z is arbitrary and we obtain for v :

$$(\mathbb{I} + \mathcal{M}_\infty^{-1} \mathcal{A}_g) v = \mathcal{M}_\infty^{-1} Q_0 f,$$

where the operator $\mathcal{M}_\infty^{-1}\mathcal{A}_g$ is now a compact operator for which -1 is not an eigenvalue, since $v \in U_*^\perp$. It results that there is a number c independent of ε such that

$$||v||_{L^2_\eta} \leq c||f||_{L^2_\eta}.$$

From the form of operator $\mathcal{M}_g$ and using interpolation properties, we obtain for $v = (A, C)$

$$||(A, C)||_{\mathcal{D}_0} \leq c||f||_{L^2_\eta}$$

with a certain c independent of ε.

We show below (see section 6.3.1) that the kernel of $\mathcal{M}_g$, is one dimensional, then this implies that the range of $\mathcal{M}_g$ needs satisfy the orthogonality with only one element. In fact, because of selfadjointness in L^2, the range of $\mathcal{M}_g$ is orthogonal in $L^2(\mathbb{R})$ to

$$(A'_*, B'_*) \in L^2_\eta.$$

6.3.1 Dimension of $\ker \mathcal{M}_g$

Any element $\zeta(x)$ in the kernel lies, by definition, in L^2_η, hence $\zeta(x)$ tends towards 0 exponentially at $\pm\infty$. Near $x = \pm\infty$ the vector $\zeta(x) \sim \zeta_\pm(x)$ should verify

$$\mathcal{M}_\infty^\pm \zeta_\pm(x) = 0$$

where there are only 2 possible good dimensions (on each side). This gives a bound $= 2$ to the dimension of the kernel of $\mathcal{M}_g$. Let us show that *dimension 2 of* $\ker \mathcal{M}_g$ *implies non uniqueness of the heteroclinic*, which contradicts Theorem 1, hence the only possibility is that the dimension is one.

Let us choose arbitrarily g_0 and assume that the kernel of $\mathcal{M}_{g_0}$ consists in

$$\zeta_0(x), \zeta_1(x)$$

where $\zeta_0 = (A'_*, B'_*)|_{g_0}$ and let us decompose a solution of (1) in the neighborhood of g_0 as

$$U = \mathbf{T}_a(U_*^{(g_0)} + a_1\zeta_1 + Y), \tag{93}$$

where $\mathbf{T}_a$ represents the shift $x \mapsto x + a$, where a, $a_1 \in \mathbb{R}$, and Y belongs to a subspace transverse to $\ker \mathcal{M}_{g_0}$. Let us denote by $\mathbf{Q}_0$ and $\mathbf{P}_0 = \mathbb{I} - \mathbf{Q}_0$, projections, respectively on the range of $\mathcal{M}_{g_0}$, and on a complementary subspace ($\mathbf{Q}_0$ may be built in using the eigenvectors ζ_0^*, ζ_1^* of the adjoint operator $\mathcal{M}_{g_0}^*$). Let us denote by

$$\mathcal{F}(U, g) = 0$$

the system (1) where we look for an heteroclinic U for $g \neq g_0$. Then, we have

$$\begin{aligned} \mathcal{F}(U_*^{(g_0)}, g_0) &= 0, \\ D_U\mathcal{F}(U_*^{(g_0)}, g_0) &= \mathcal{M}_{g_0}, \end{aligned}$$

and since

$$\mathcal{M}_{g_0}\zeta_j = 0, \ j = 0, 1,$$

using the equivariance under operator $\mathbf{T}_a$, we obtain (denoting $\mathcal{F}_0 = \mathcal{F}(U_*^{(g_0)}, g_0)$ and $[..]^{(2)}$ the argument of a quadratic operator)

$$\begin{aligned} 0 &= \mathcal{M}_{g_0} Y + (g-g_0)\partial_g \mathcal{F}_0 + \frac{1}{2} D^2_{UU}\mathcal{F}_0[a_1\zeta_1 + Y]^{(2)} + \\ &\quad + \mathcal{O}(|g-g_0|[|g-g_0| + |a_1| + ||Y||] + ||Y||^3). \end{aligned}$$

The projection Q_0 of this equation allows to use the implicit function theorem to solve with respect to Y and then obtain a unique solution

$$Y = \mathcal{Y}(a_1, g),$$

with

$$\begin{aligned} \mathcal{Y} &= -(g-g_0)\widetilde{\mathcal{M}_{g_0}}^{-1}\mathbf{Q}_0\partial_g\mathcal{F}_0 - \frac{1}{2}\widetilde{\mathcal{M}_{g_0}}^{-1}\mathbf{Q}_0 D^2_{UU}\mathcal{F}_0[a_1\zeta_1]^{(2)} + \\ &\quad + \mathcal{O}(|g-g_0|(|g-g_0| + |a_1|) + |a_1|^3)). \end{aligned}$$

Then projecting on the complementary space, (only one equation since we work in the subspace orthogonal to ζ_0^*), we may observe (see the proof in Appendix A.5) that $\mathbf{P}_0\partial_{g_0}\mathcal{F}_0 = 0$ and then obtain the "bifurcation" equation as

$$q(a_1, g-g_0) = \mathcal{O}((|g-g_0| + |a_1|)^3),$$

where the function q is quadratic in its arguments and

$$q|_{g=g_0}\zeta_1 = \frac{1}{2}\mathbf{P}_0 D^2_{UU}\mathcal{F}_0[a_1\zeta_1]^{(2)}.$$

This equation is just at main order a second degree equation in a_1 depending on $g - g_0$. Provided that the discriminant is not 0, the generic number of solutions is 2 or 0. If the discriminant is 0 for $g = g_0$, we just go a little farther in g, and obtain a non zero discriminant, since the discriminant cannot stay $= 0$. Indeed the heteroclinic is analytic in g and if the discriminant were identically 0, this would mean that we have a double root for any g, contradicting the transversality for all g, except a finite number, of the intersection of the two manifolds (unstable one of M_-, stable one of M_+). Hence, this is true except for a set of isolated values of g. We can then use the implicit function theorem for finding corresponding solutions for the system with higher order terms. In fact we already know a solution, corresponding to $U_*^{(g)} = U_*^{(g_0)} + (g-g_0)\partial_g U_*^{(g_0)} +$ *h.o.t.* which corresponds to specific values for a_1 and Y, of order $\mathcal{O}(g-g_0)$. It then results that there is at least another solution of order $\mathcal{O}(g-g_0)$, so that there exists another heteroclinic, in the neighborhood of the known one (then in contradiction with Theorem 1).

Remark 39 *The above proof with only 1 dimension in the Kernel, provides* $Y = -(g-g_0)\widetilde{\mathcal{M}_{g_0}}^{-1}\partial_g\mathcal{F}_0 + \mathcal{O}((g-g_0)^2)$*, which gives a unique heteroclinic. Since we found only one heteroclinic, this shows that the kernel is of dimension 1.*

A Appendix

A.1 Monodromy operator

Let us prove the estimate for the monodromy operators. We prove the following

Lemma 40 *For $B_0 \leq \sqrt{1-\eta_0^2\delta^2}$, $\alpha \geq \frac{10}{3}\varepsilon^2$, and for ε small enough, the following estimates hold*

$$\begin{aligned} ||\boldsymbol{S}_0(x,s)|| &\leq e^{\sigma(x-s)}, \quad -\infty < x < s \\ ||\boldsymbol{S}_1(x,s)|| &\leq e^{-\sigma(x-s)}, \quad -\infty < s < x \end{aligned}$$

with

$$\sigma = \frac{(\alpha\delta)^{1/2}}{2^{1/4}}.$$

Proof. We start with the system

$$\begin{aligned} x_1' &= \lambda_r x_1 + \lambda_i x_2 \\ x_2' &= -\lambda_i x_1 + \lambda_r x_2 \end{aligned}$$

where λ_r and λ_i are functions of x. From Lemma 16 we have, for ε small enough

$$\lambda_r \geq \frac{(\alpha\delta)^{1/2}}{2^{1/4}} = \sigma.$$

Now we obtain

$$(x_1^2+x_2^2)' = 2\lambda_r(x_1^2+x_2^2)$$

hence

$$(x_1^2+x_2^2)(x) = e^{\int_s^x 2\lambda_r(\tau)d\tau}(x_1^2+x_2^2)(s),$$

which, for $x < s$, leads to

$$\sqrt{(x_1^2+x_2^2)(x)} \leq e^{\sigma(x-s)}\sqrt{(x_1^2+x_2^2)(s)}.$$

The proof is then done for the operator $\boldsymbol{S}_0$. The estimate for $\boldsymbol{S}_1$ is obtained in the same way. ■

Remark 41 *We have*

$$\boldsymbol{S}_0(x,s) = e^{\int_s^x \lambda_r(\tau)d\tau}\begin{pmatrix} \cos(\int_s^x \lambda_i(\tau)d\tau) & \sin(\int_s^x \lambda_i(\tau)d\tau) \\ -\sin(\int_s^x \lambda_i(\tau)d\tau) & \cos(\int_s^x \lambda_i(\tau)d\tau) \end{pmatrix}.$$

A.2 Computation of the system with new coordinates

Let us look for the system (12) writen in the new coordinates, first in forgetting quadratic and higher orders terms

$$\begin{aligned}
B_0x_1' &= \frac{\widetilde{A_*}}{2\sqrt{2}\lambda_r}\left(A_1+\frac{(1+\delta^2)B_0B_1}{\widetilde{A_*}}\right)+\frac{3\lambda_r^2-\lambda_i^2}{4\sqrt{2}\lambda_r\widetilde{A_*}}A_3 \\
&\quad +\frac{A_2}{2}+\frac{(1+\delta^2)}{2\widetilde{A_*}}B_0^2\varepsilon^2\left(\delta^2(\widetilde{A_*}^2-B_0^2)+2(1+\delta^2)\widetilde{A_*}\widetilde{A_0}\right)-(\lambda_r^2-\lambda_i^2)\widetilde{A_0} \\
&= B_0f_1+\frac{\widetilde{A_*}}{2\sqrt{2}\lambda_r}B_0(x_1+y_1)+\frac{A_2}{2}+\frac{1}{4\lambda_r}A_3,
\end{aligned}$$

$$\begin{aligned}
\lambda_iB_0x_2' &= -\frac{\widetilde{A_*}}{2\sqrt{2}}\left(A_1+\frac{(1+\delta^2)B_0B_1}{\widetilde{A_*}}\right)-\frac{\lambda_r^2-3\lambda_i^2}{4(\lambda_r^2-\alpha)}A_3 \\
&\quad -\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r}\left(A_2+\frac{(1+\delta^2)B_0^2\varepsilon^2}{\widetilde{A_*}}\delta^2(\widetilde{A_*}^2-B_0^2)\right) \\
&\quad -\frac{1}{4\lambda_r}2\widetilde{A_*}^2\widetilde{A_0} \\
&= \lambda_iB_0f_2-\frac{\widetilde{A_*}}{2\sqrt{2}}B_0(x_1+y_1)-\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r}A_2 \\
&\quad +\frac{1}{4}A_3-\frac{1}{4\lambda_r}2\widetilde{A_*}^2\widetilde{A_0},
\end{aligned}$$

with

$$f_1=\frac{\varepsilon^2\delta^2B_0(1+\delta^2)(\widetilde{A_*}^2-B_0^2)}{2\widetilde{A_*}},$$

$$f_2=-\frac{\varepsilon^2\delta^2B_0(1+\delta^2)(\lambda_r^2-\lambda_i^2)(\widetilde{A_*}^2-B_*^2)}{4\lambda_r\lambda_i\widetilde{A_*}},$$

hence

$$\begin{aligned}
x_1' &= f_1+\lambda_rx_1+\lambda_ix_2, \\
x_2' &= f_2-\lambda_ix_1+\lambda_rx_2,
\end{aligned} \tag{94}$$

as expected. In the same way we obtain

$$\begin{aligned}
y_1' &= f_1-\lambda_ry_1+\lambda_iy_2, \\
y_2' &= -f_2-\lambda_iy_1-\lambda_ry_2, \\
z_1' &= \frac{2\varepsilon^2\delta^2(\widetilde{A_*}^2-B_0^2)}{\widetilde{A_*}}=\frac{2f_1}{(1+\delta^2)B_0}, \\
B_*' &= -\frac{(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}}A_3+\widetilde{A_*}B_0z_1.
\end{aligned} \tag{95}$$

We notice that the following estimates hold (using (15) and Lemma 16)

$$|f_1|, |f_2| \leq \frac{B_0 \varepsilon^2 \delta^2}{\widetilde{A_*}} \leq \frac{B_0 \varepsilon^2 \delta}{\alpha},$$

A.2.1 Full system in new coordinates

We intend to derive the full system (1) with coordinates $(x_1, x_2, y_1, y_2, B_0, z_1)$. Differentiating (22) and (23) we see that we respectively need to add to the previous expressions (94) for x_1' and x_2'

$$\frac{1}{B_0}\left\{\left(\frac{\widetilde{A_*}}{2\sqrt{2}\lambda_r}\right)'\widetilde{A_0} + \left(\frac{(3\lambda_r^2 - \lambda_i^2)}{4\sqrt{2}\lambda_r\widetilde{A_*}}\right)' A_2 + \varepsilon^2\left(\frac{(1+\delta^2)^2 B_0^2}{2\widetilde{A_*}^2}\right)' A_3 + \left(\frac{(1+\delta^2)B_0}{2\widetilde{A_*}}\right)' B_1\right\}$$
$$-\varepsilon^2 \frac{(1+\delta^2)^2 B_0}{2\widetilde{A_*}^2}[3\widetilde{A_*}\widetilde{A_0}^2 + \widetilde{A_0}^3] + \frac{B_0\varepsilon^2(1+\delta^2)^2\widetilde{A_0}^2}{2\widetilde{A_*}} - \frac{B_1}{B_0}x_1.$$

and

$$\frac{1}{B_0}\left\{-\left(\frac{\widetilde{A_*}}{2\sqrt{2}\lambda_i}\right)'\widetilde{A_0} - \left(\frac{(\lambda_r^2 - \lambda_i^2)}{4\lambda_r\lambda_i}\right)' A_1 - \left(\frac{(\lambda_r^2 - 3\lambda_i^2)}{4\sqrt{2}\lambda_i\widetilde{A_*}}\right)' A_2 + \left(\frac{\varepsilon^2(1+\delta^2)^3 B_0^3}{4\lambda_r\lambda_i\widetilde{A_*}}\right)' B_1\right\}$$
$$+\frac{1}{B_0}\left(\frac{1}{4\lambda_r\lambda_i}\left[1 - \frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right]\right)' A_3 - \frac{1}{4\lambda_r\lambda_i B_0}\left(1 - \frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right)[3\widetilde{A_*}\widetilde{A_0}^2 + \widetilde{A_0}^3]$$
$$-\frac{\varepsilon^4 B_0^3(1+\delta^2)^4}{4\lambda_r\lambda_i\widetilde{A_*}}\widetilde{A_0}^2 - \frac{B_1}{B_0}x_2.$$

We then arrive to the system (27,28,29,30).

Using Lemma 16 and Lemma 15 we obtain

$$\widetilde{A_*}' = -\frac{(1+\delta^2)B_0}{\widetilde{A_*}}B_1$$

$$(\lambda_r^2)' = -\frac{(1+\delta^2)B_0B_1}{\sqrt{2}\widetilde{A_*}}(1 - \varepsilon^2\sqrt{2}(1+\delta^2)\widetilde{A_*})$$

$$(\lambda_i^2)' = -\frac{(1+\delta^2)B_0B_1}{\sqrt{2}\widetilde{A_*}}(1 + \varepsilon^2\sqrt{2}(1+\delta^2)\widetilde{A_*})$$

$$\left(\frac{\widetilde{A_*}}{2\sqrt{2}\lambda_r}\right)' = a_1 B_0 B_1, \quad |a_1| \leq \frac{c}{\widetilde{A_*}^{3/2}}, \tag{96}$$

$$\left(\frac{\widetilde{A_*}}{2\sqrt{2}\lambda_i}\right)' = a_2 B_0 B_1, \quad |a_2| \leq \frac{c}{\widetilde{A_*}^{3/2}}, \tag{97}$$

$$\left(-\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r\lambda_i}\right)' = b_2 B_0 B_1, \quad |b_2| \le \frac{c\varepsilon^2}{\widetilde{A_*}^2}, \tag{98}$$

$$\left(\frac{(3\lambda_r^2-\lambda_i^2)}{4\sqrt{2}\lambda_r\widetilde{A_*}}\right)' = c_1 B_0 B_1, \quad |c_1| \le \frac{c}{\widetilde{A_*}^{5/2}}, \tag{99}$$

$$\left(-\frac{(\lambda_r^2-3\lambda_i^2)}{4\sqrt{2}\lambda_i\widetilde{A_*}}\right)' = c_2 B_0 B_1, \quad |c_2| \le \frac{c}{\widetilde{A_*}^{5/2}}, \tag{100}$$

$$\varepsilon^2\left(\frac{(1+\delta^2)^2 B_0^2}{2\widetilde{A_*}^2}\right)' = d_1 B_0 B_1, \quad |d_1| \le \frac{c}{\widetilde{A_*}^3}, \tag{101}$$

$$\left(\frac{1}{4\lambda_r\lambda_i}\left[1-\frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right]\right)' = d_2 B_0 B_1, \quad |d_2| \le \frac{c}{\widetilde{A_*}^3}, \tag{102}$$

$$\left(\frac{(1+\delta^2)B_0}{2\widetilde{A_*}}\right)' = e_1 B_1, \quad |e_1| \le \frac{c}{\widetilde{A_*}^3} \tag{103}$$

$$\left(\frac{\varepsilon^2(1+\delta^2)^3 B_0^2}{4\lambda_r\lambda_i\widetilde{A_*}}\right)' = e_2 B_0 B_1, \quad |e_2| \le \frac{c}{\widetilde{A_*}^3}, \tag{104}$$

with c independent of ε, α and $\delta \in [1/3, 1]$.

A.3 Elimination of z_1

A.3.1 System after scaling

After the scaling (32) our system (27,28,29,30) takes the form

$$\begin{aligned}
\overline{X}' &= \mathbf{L}_0\overline{X} + B_0\overline{F_0} + \mathbf{B}_{01}(\overline{X},\overline{Y}) + \overline{z_1}\boldsymbol{M}_{01}(\overline{X},\overline{Y}) \\
&\quad + \overline{z_1}^2 B_0\boldsymbol{n}_0 + \boldsymbol{C}_{01}(\overline{X},\overline{Y}), \\
\overline{Y}' &= \mathbf{L}_1\overline{Y} + B_0\overline{F_1} + \mathbf{B}_{11}(\overline{X},\overline{Y}) + \overline{z_1}\boldsymbol{M}_{11}(\overline{X},\overline{Y}) \\
&\quad + \overline{z_1}^2 B_0\boldsymbol{n}_1 + \boldsymbol{C}_{11}(\overline{X},\overline{Y}),
\end{aligned}$$

where $\overline{F_0}, \overline{F_1}, \boldsymbol{n}_0, \boldsymbol{n}_1$ are two-dimensional vectors $\mathbf{M}_{01}, \mathbf{M}_{11}$ are linear operators in $(\overline{X},\overline{Y})$, $\mathbf{B}_{01}, \mathbf{B}_{11}$ are quadratic and $\mathbf{C}_{01}, \mathbf{C}_{11}$ are cubic in $(\overline{X},\overline{Y})$, all functions of B_0. More precisely we have

$$\overline{F_0} = \begin{pmatrix} \frac{f_1}{\alpha^{3/2}\delta B_0} \\ \frac{f_2}{\alpha^{3/2}\delta B_0} \end{pmatrix}, \ \overline{F_1} = \begin{pmatrix} \frac{f_1}{\alpha^{3/2}\delta B_0} \\ -\frac{f_2}{\alpha^{3/2}\delta B_0} \end{pmatrix}, \ |\overline{F_j}| \le c\frac{\varepsilon^2}{\alpha^{5/2}},$$

$$\boldsymbol{n}_0 = \frac{\varepsilon^2\delta}{\alpha^{3/2}}\begin{pmatrix} e_1\widetilde{A_*}^2 \\ e_2\widetilde{A_*}^2 B_0 - b_2(1+\delta^2)\widetilde{A_*}B_0^2 \end{pmatrix},$$

$$\boldsymbol{M}_{01}(\overline{X},\overline{Y}) = \varepsilon\delta\begin{pmatrix} m_{01}(\overline{X},\overline{Y}) \\ m_{02}(\overline{X},\overline{Y}) \end{pmatrix},$$

$$\begin{aligned} m_{01}(\overline{X},\overline{Y}) &= \widetilde{A_*}B_0\left(a_1\overline{\widetilde{A_0}}+c_1\overline{A_2}+(d_1-2e_1(1+\delta^2)\varepsilon^2\frac{B_0}{\widetilde{A_*}})\overline{A_3}-\frac{\overline{x_1}}{B_0}\right),\\ m_{02}(\overline{X},\overline{Y}) &= \widetilde{A_*}B_0\left(-a_2\overline{\widetilde{A_0}}+c_2\overline{A_2}+(d_2-2e_2(1+\delta^2)\varepsilon^2\frac{B_0^2}{A_*})\overline{A_3}-\frac{\overline{x_2}}{B_0}\right)\\ &\quad+\widetilde{A_*}B_0^2b_2(\overline{x_1}+\overline{y_1})+(1+\delta^2)^2\varepsilon^2\frac{B_0^3}{\widetilde{A_*}}b_2\overline{A_3}, \end{aligned}$$

$$\mathbf{B}_{01}(\overline{X},\overline{Y})=\alpha^{3/2}\delta\begin{pmatrix} b_{01}(\overline{X},\overline{Y})\\ b_{02}(\overline{X},\overline{Y})\end{pmatrix},$$

$$\begin{aligned} b_{01}(\overline{X},\overline{Y}) &= -\varepsilon^2\frac{(1+\delta^2)(2-\delta^2)B_0}{2\widetilde{A_*}}\overline{\widetilde{A_0}}^2+e_1\frac{\varepsilon^4(1+\delta^2)^2B_0}{\widetilde{A_*}^2}\overline{A_3}^2\\ &\quad-\varepsilon^2\frac{(1+\delta^2)B_0}{\widetilde{A_*}}\overline{A_3}[a_1\overline{\widetilde{A_0}}+c_1\overline{A_2}+d_1\overline{A_3}-\frac{\overline{x_1}}{B_0}], \end{aligned}$$

$$\begin{aligned} b_{02}(\overline{X},\overline{Y}) &= -\frac{1}{4\lambda_r\lambda_i\widetilde{A_*}B_0}\left(3\widetilde{A_*}^2-2\varepsilon^4B_0^4(1+\delta^2)^4\right)\overline{\widetilde{A_0}}^2+e_2\frac{\varepsilon^4(1+\delta^2)B_0^2}{\widetilde{A_*}^2}\overline{A_3}^2\\ &\quad-\varepsilon^2\frac{(1+\delta^2)B_0}{\widetilde{A_*}}\overline{A_3}[-a_2\overline{\widetilde{A_0}}+b_2B_0(\overline{x_1}+\overline{y_1})+c_2\overline{A_2}+d_2\overline{A_3}-\frac{\overline{x_2}}{B_0}], \end{aligned}$$

$$\boldsymbol{C}_{01}(\overline{X},\overline{Y})=\alpha^3\delta^2\overline{\widetilde{A_0}}^3\begin{pmatrix} -\varepsilon^2\frac{(1+\delta^2)B_0}{2\widetilde{A_*}^2}\\ -\frac{1}{4\lambda_r\lambda_iB_0}\left(1-\frac{\varepsilon^4B_0^4(1+\delta^2)^4}{\widetilde{A_*}^2}\right)\end{pmatrix}.$$

$\boldsymbol{n}_1$, $\boldsymbol{M}_{11}$, $\boldsymbol{B}_{11}$, $\boldsymbol{C}_{11}$ are deduced respectively from $\boldsymbol{n}_0$, $\boldsymbol{M}_{01}$, $\boldsymbol{B}_{01}$, $\boldsymbol{C}_{01}$ in changing (a_1,c_1,b_2,d_2,e_2) into their opposite.

A.3.2 System after elimination of z_1

Let us replace $\overline{z_1}$ by $\overline{z_{10}}[1+\mathcal{Z}(\overline{X},\overline{Y},B_0,\varepsilon,\alpha,\delta)]$ in the differential system for $(\overline{X},\overline{Y})$. The new system becomes (notice that B_0 is in factor of the "constant" terms)

$$\begin{aligned} \overline{X}' &= \mathbf{L}_0\overline{X}+B_0\mathcal{F}_0+\mathcal{L}_{01}(\overline{X},\overline{Y})+\mathcal{B}_{01}(\overline{X},\overline{Y}),\\ \overline{Y}' &= \mathbf{L}_1\overline{Y}+B_0\mathcal{F}_1+\mathcal{L}_{11}(\overline{X},\overline{Y})+\mathcal{B}_{11}(\overline{X},\overline{Y}), \end{aligned}$$

which is (39) with

$$\mathcal{F}_0=\overline{F_0}+\overline{z_{10}}^2\boldsymbol{n}_0,$$

$$\mathcal{L}_{01}(\overline{X},\overline{Y})=\overline{z_{10}}\boldsymbol{M}_{01}(\overline{X},\overline{Y}),$$

$$\begin{aligned} \mathcal{B}_{01}(\overline{X},\overline{Y}) &= \mathbf{B}_{01}(\overline{X},\overline{Y})+\overline{z_{10}}\mathcal{Z}(\overline{X},\overline{Y})\boldsymbol{M}_{01}(\overline{X},\overline{Y})+\boldsymbol{C}_{01}(\overline{X},\overline{Y})\\ &\quad+2\overline{z_{10}}^2\mathcal{Z}(\overline{X},\overline{Y})B_0\boldsymbol{n}_0+\overline{z_{10}}^2\mathcal{Z}(\overline{X},\overline{Y})^2B_0\boldsymbol{n}_0. \end{aligned}$$

In using estimates (26), (96) to (104), it is straightforward to check that

$$|\mathcal{F}_0| + |\mathcal{F}_1| \leq \frac{c\varepsilon^2}{\alpha^{9/2}},$$

$$\begin{aligned}
|\boldsymbol{M}_{01}(\overline{X},\overline{Y})| &\leq c\frac{\varepsilon\delta}{\widetilde{A_*}}(|\overline{X}| + |\overline{Y}|), \\
|\boldsymbol{n}_0| &\leq c\frac{\varepsilon^2}{\alpha^{5/2}},\ |b_{01}| \leq c\frac{\varepsilon^2}{\alpha^2},\ |b_{02}| \leq \frac{9}{2\alpha} + c\frac{\varepsilon^2}{\alpha^2},
\end{aligned}$$

hence

$$|\mathcal{L}_{01}(\overline{X},\overline{Y})| + |\mathcal{L}_{11}(\overline{X},\overline{Y})| \leq c\frac{\varepsilon}{\alpha^2}(|\overline{X}| + |\overline{Y}|).$$

For higher order terms we have

$$\begin{aligned}
|\mathbf{B}_{01}(\overline{X},\overline{Y})| &\leq \alpha^{1/2}[\frac{9}{2} + c\frac{\varepsilon^2}{\alpha}](|\overline{X}| + |\overline{Y}|)^2, \\
|2\overline{z_{10}}^2\mathcal{Z}(\overline{X},\overline{Y})\boldsymbol{n}_0| &\leq c\frac{\varepsilon^2(1+\rho^2)}{\alpha^{3/2}}(|\overline{X}| + |\overline{Y}|)^2, \\
|\overline{z_{10}}\mathcal{Z}(\overline{X},\overline{Y})\boldsymbol{M}_{01}(\overline{X},\overline{Y})| &\leq c\alpha\varepsilon(1+\rho^2)(|\overline{X}| + |\overline{Y}|)^3, \\
\overline{z_{10}}^2|\mathcal{Z}(\overline{X},\overline{Y})^2\boldsymbol{n}_0| &\leq c\alpha^{3/2}\varepsilon^2(1+\rho^4)(|\overline{X}| + |\overline{Y}|)^4, \\
|\boldsymbol{C}_{01}(\overline{X},\overline{Y})| &\leq \frac{27\alpha^{1/2}}{2}(|\overline{X}| + |\overline{Y}|)^3,
\end{aligned}$$

hence for

$$|\overline{X}| + |\overline{Y}| \leq \rho,$$

we obtain (with c independent of $\varepsilon, \alpha, \delta$)

$$\begin{aligned}
|\mathcal{B}_{01}(\overline{X},\overline{Y})| + |\mathcal{B}_{11}(\overline{X},\overline{Y})| \leq\ & \alpha^{1/2}[9/2 + c\frac{\varepsilon^2}{\alpha} + c\frac{\varepsilon^2}{\alpha^2}(1+\rho^2)](|\overline{X}| + |\overline{Y}|)^2 \\
& + [\frac{27\alpha^{1/2}}{2} + c\alpha\varepsilon(1+\rho^2)](|\overline{X}| + |\overline{Y}|)^3 + c\alpha^{3/2}\varepsilon^2(1+\rho^4)(|\overline{X}| + |\overline{Y}|)^4,
\end{aligned}$$

and in using the constraint (31)

$$\begin{aligned}
|\mathcal{B}_{01}(\overline{X},\overline{Y})| + |\mathcal{B}_{11}(\overline{X},\overline{Y})| \leq\ & \alpha^{1/2}(9/2 + c\frac{\varepsilon^2}{\alpha^{7/2}})(|\overline{X}| + |\overline{Y}|)^2 \\
& + \alpha^{1/2}(\frac{27}{2} + c\frac{\varepsilon}{\alpha})(|\overline{X}| + |\overline{Y}|)^3 + \alpha^{1/2}(c\frac{\varepsilon^2}{\alpha^2})(|\overline{X}| + |\overline{Y}|)^4.
\end{aligned}$$

A.4 Proof of Lemma 31

A.4.1 First step: integration on $[-a_+, a_+]$

We have a clear control on a_+ while this is more complicate for a_- which is possibly larger. Hence, we consider the 4th-order differential equation (86) with boundary conditions (90) and first find the solution on the interval $[-a_+, a_+]$.

Integrating simply the 4th order ODE (86) leads to

$$\begin{aligned}\overline{A_j}(z) &= \overline{A_{j+}} + \int_{a_+}^{z} \overline{A_{j+1}}(s)ds,\ j=0,1,2,\\ \overline{A_3(z)} &= \overline{A_{3+}} - \int_{a_+}^{z} \overline{A_0}(s)(\overline{A_0}^2(s)+s)ds,\end{aligned}$$

where

$$\overline{A_{j+}} = \frac{d^j\overline{A_0}}{dz^j}(a_+),\ j=0,1,2,3.$$

This gives for $z < a_+$

$$\begin{aligned}\overline{A_0}(z) &= \overline{A_{0+}} + (z-a_+)\overline{A_{1+}} + \frac{(z-a_+)^2}{2}\overline{A_{2+}} + \frac{(z-a_+)^3}{6}\overline{A_{3+}}\\ &\quad + \int_z^{a_+} \frac{(z-s)^3}{6}\overline{A_0}(s)[\overline{A_0}^2(s)+s]ds. \qquad (105)\end{aligned}$$

This leads to the estimate

$$\begin{aligned}|\overline{A_0}(z)| &\leq |\overline{A_{0+}}| + (a_+-z)|\overline{A_{1+}}| + \frac{(z+a_+)^2}{2}|\overline{A_{2+}}| + \frac{(a_+-z)^3}{6}|\overline{A_{3-}}|\\ &\quad + \int_z^{a_+} \frac{(s-z)^3}{6}|\overline{A_0}(s)||\overline{A_0}^2(s)+s|ds, \quad \text{for } z<a_+. \qquad (106)\end{aligned}$$

Let us define

$$||\overline{A_0}||_0 = \sup_{z\in(-a_+,a_+)} |\overline{A_0}(z)|,$$

and look for a solution $\overline{A_0} \in C^0[-a_+,a_+]$ of (105). The estimate (106) leads to

$$||\overline{A_0}||_0 \leq |\overline{A_{0+}}| + 2a_+|\overline{A_{1+}}| + 2a_+^2|\overline{A_{2+}}| + \frac{4a_+^3}{3}|\overline{A_{3+}}| + \frac{2a_+^5}{3}||\overline{A_0}||_0 + \frac{2a_+^4}{3}||\overline{A_0}||_0^3.$$

The fixed point theorem applies in a small ball for $\overline{A_0}$, provided that $\frac{2a_+^5}{3} < 1$. From (85) this condition is equivalent to

$$\frac{\nu_+}{\sqrt{\delta}} > \frac{\sqrt{1+\delta^2}}{2.6^{1/4}}. \qquad (107)$$

We notice that (107) is compatible with (80) since for $\delta < 1$

$$\frac{\sqrt{1+\delta^2}}{2.6^{1/4}} \leq 0.452 < 0.46.$$

Moreover, we have the estimates

$$|\overline{A_{0+}}| \leq a_+^{1/2}k_1,\ |\overline{A_{1+}}| \leq a_+^{3/4}k_1,\ |\overline{A_{2+}}| \leq a_+k_1,\ |\overline{A_{3+}}| \leq a_+^{5/4}k_1,$$

so that

$$|\overline{A_{0+}}| + 2a_+|\overline{A_{1+}}| + 2a_+^2|\overline{A_{2+}}| + \frac{4a_+^3}{3}|\overline{A_{3+}}| \leq k_1a_+^{1/2}(1+2a_+^{5/4}+2a_+^{5/2}+\frac{4a_+^{15/4}}{3}).$$

Choosing k_1 small enough, and assuming that (107) holds, we then find a unique fixed point $\overline{A_0}$ in $C^0[-a_+,a_+]$ function of the two parameters $(\overline{x_{10}}^s, \overline{x_{20}}^s)$.

A.4.2 Second step: integration on $[-a_-, -a_+]$

For extending the solution on $[-a_-, -a_+]$ we need to solve with respect to $\overline{A_0} \in C^0[-a_-, -a_+]$

$$\begin{aligned}\overline{A_0}(z) &= \overline{A_0}(-a_+) + (z+a_+)\overline{A_1}(-a_+) + \frac{(z+a_+)^2}{2}\overline{A_2}(-a_+) + \frac{(z+a_+)^3}{6}\overline{A_3}(-a_+) \\ &\quad + \int_z^{-a_+} \frac{(z-s)^3}{6}\overline{A_0}(s)[\overline{A_0}^2(s)+s]ds,\end{aligned}$$

where $z \in [-a_-, -a_+]$. For applying the fixed point argument for $\overline{A_0}$ close to 0, as above, we need to satisfy

$$\int_{-a_-}^{-a_+} \frac{|(s+a_-)^3 s|}{6} ds < 1,$$

i.e.

$$\frac{(a_- - a_+)^5}{120} + \frac{(a_- - a_+)^4 a_+}{24} < 1,$$

where we notice that

$$\frac{a_+}{a_-} = \left(\frac{\nu_-}{\nu_+}\right)^{4/5}. \tag{108}$$

Using (108) and (107) the above condition holds if the following condition on ν_+/ν_- holds

$$\frac{1}{80}\left((\frac{\nu_+}{\nu_-})^{4/5} - 1\right)^5 + \frac{1}{16}\left((\frac{\nu_+}{\nu_-})^{4/5} - 1\right)^4 < 1.$$

We may check that

$$(1/80)X^5 + (1/16)X^4 < 1$$

holds for

$$X < 1.84$$

hence we need

$$\frac{\nu_+}{\nu_-} < 3.69.$$

If this is realized, we are done! If not, we need to iterate as follows:

i) the first step is as above, then we reach $-a_1$ such that

$$a_1 = a_+(\frac{\nu_+}{\nu_1})^{4/5}, \quad \nu_1 = \frac{\nu_+}{3.69},$$

and the solution is obtained on $[-a_1, +a_+]$ for k_1 small enough.

ii) The second step starts at $z = -a_1$ and proceeds as above. We then reach $-a_2$ such that

$$a_2 = a_1(\frac{\nu_1}{\nu_2})^{4/5}, \ \nu_2 = \frac{\nu_1}{3.69},$$

and the solution is obtained on $[-a_2, +a_+]$ for k_1 small enough.

iii) We iterate the process n times, until

$$\nu_n = \frac{\nu_+}{(3.69)^n} \leq \nu_-.$$

Then, the solution is obtained, for k_1 small enough, on $[-a_n, +a_+]$ where

$$-a_n = -a_+ \left(\frac{\nu_+}{\nu_n}\right)^{4/5} \leq -a_-.$$

The Lemma is proved.

A.5 Proof of $\mathbf{P}_0 \partial_g \mathcal{F}_0 = 0$

Lemma 42 *Any (u, v) in the kernel of $\mathcal{M}_g$ satisfies*

$$\int_{\mathbb{R}} A_* B_* (B_* u + A_* v) dx = 0,$$

and $\partial_g \mathcal{F}_0(U_*, g) = (A_* B_*^2, A_*^2 B_*)$ belongs to the range of $\mathcal{M}_g$, hence $\mathbf{P}_0 \partial_g \mathcal{F}_0 = 0$.

Proof.

Differentiating with respect to g the system (1) verified by the heteroclinic, we obtain

$$\mathcal{M}_g \begin{pmatrix} \partial_g A_* \\ \partial_g B_* \end{pmatrix} = \begin{pmatrix} A_* B_*^2 \\ A_*^2 B_* \end{pmatrix} = \partial_g \mathcal{F}_0(U_*, g),$$

hence $(A_* B_*^2, A_*^2 B_*)$ belongs to the range of $\mathcal{M}_g$. When $(u, v) \in \ker \mathcal{M}_g$, then $(u, v) \in \ker \mathcal{M}_g^*$ where $\mathcal{M}_g = \mathcal{M}_g^*$, when the adjoint is computed with the scalar product of L^2, hence

$$\int_{\mathbb{R}} A_* B_* (B_* u + A_* v) dx = 0. \tag{109}$$

Hence, the eigenvectors ζ_0^*, ζ_1^* of the adjoint $\mathcal{M}_g^*$ (the orthogonal of this 2-dimensional eigenspace is the range of $\mathcal{M}_g$), are orthogonal to $\partial_g \mathcal{F}_0 = (A_* B_*^2, A_*^2 B_*)|_{g_0}$ in L^2.

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