Gravity and capillary-gravity periodic travelling waves for two superposed fluid layers, one being of infinite depth.

Gérard Iooss
Institut Universitaire de France
INLN UMR CNRS-UNSA 6618
1361 route des Lucioles, F-06560 Valbonne
e-mail: iooss@inln.cnrs.fr

Abstract

The mathematical study of 2D travelling waves in the potential flow of two superposed layers of perfect fluid, with free surface and interfaces (with or without surface tensions) and with the bottom layer of infinite depth, is set as an *ill-posed reversible evolution problem*, where the horizontal space variable plays the role of a "time". We give the structure of the spectrum of the linearized operator near equilibrium. This spectrum contains a set of isolated eigenvalues of finite multiplicities, a small number of which lie near or on the imaginary axis, and the entire real axis constitutes the *essential spectrum* where there is no eigenvalue, except 0 in some cases. We give a general constructive proof of bifurcating periodic waves, adapting the Lyapunov-Schmidt method to the present (reversible) case where 0 (which is "resonant") belongs to the continuous spectrum. In particular we give the results for the generic case and for the 1:1 resonance case.

1 Introduction

We consider superposed layers of perfect fluids (potential flow), the bottom layer being of infinite depth. There are many papers dealing with 2D travelling waves in potential flows. Here we restrict our attention to papers where the problem is treated in a mathematically complete way on the Euler equations. This type of study goes back to Nekrasov [17] and Levi-Civita [15]. They both solved the problem of periodic waves for one infinite layer, with no surface tension. Taking advantage of the fact that the one layer problem reduces to one scalar integro-differential equation, Toland and Jones (see [21],[11],[10]) considered the problem with surface tension and studied in particular the periodic waves in weakly resonant cases, i.e. when at criticality two wave numbers have a rational ratio p/q, such that $p+q \geq 5$. Also for the one layer problem with surface tension, it is proved by Iooss and Kirrmann in [9] that there exist (in the 1:1 resonance case) two symmetric solitary waves with an algebraically bounded decay at infinity (decay as $1/x^2$ at infinity as shown by Sun in [20]). Recently, still for the same problem, but by using variational methods Buffoni et al [2] obtained global results for bifurcations of periodic sub-harmonic travelling waves. For several layers, one being infinite, the existing rigorous works only deal with one interface, no surface tension, and a rigid bottom at finite distance. In particular, it is shown in the works by Amick [1] and Sun [19], that there exists a solitary wave with an algebraic decay at infinity, asymptotically identical to the solitary wave obtained in the Benjamin-Ono model ([5],[18]). Finally notice that the paper by Dias & Iooss [4] deals with the problem of two infinite layers with interfacial tension, and gives various solutions like periodic, quasi-periodic, solitary, or "generalized" solitary waves (tending to periodic waves at infinity), depending on the parameters, but the mathematical proof is not complete.

To our knowledge, there is no rigorous treatment of the problem of 2D travelling waves in several layers with more than one free surface or interface, and with one layer infinitely deep. We show below that this adds several mathematical difficulties to the previous problems, for instance the problem does not reduce any more into an integro-differential equation, and techniques developed for finite depth problems cannot be applied because of general (and bad) properties of the spectrum of the linearized problem.

We intend in the present work to give a quite general setting allowing to solve not only the problems cited above, but also new problems with as many layers as desired, with or without surface or interfacial tensions. In this paper, we only consider periodic travelling waves. We give results for the simple case with one critical wave number as well as for the most strongly resonant case: the 1:1 resonance. In a forthcoming paper (with F.Dias) we deal with solitary waves for all these problems. It is clear that weak resonances can be dealt with in a similar way and would fall, after the reduction to the bifurcation equations (in the way we explain below), into the study made in [10].

More precisely, this paper revisits the problem (denoted *Problem 0* hereafter) of one layer of infinite depth with surface tension, and considers the three following 2-dimensional travelling waves problems for two layers:

Problem 1: gravity waves, with only two parameters: Froude number and density ratio;

Problem 2: capillary-gravity waves, with surface tension acting on the upper free surface. This adds a third parameter, the free surface Weber number;

Problem 3: capillary-gravity waves, with interfacial tension as well leads to a fourth parameter.

Assuming infinite depth for the bottom layer leads to major mathematical difficulties. The idea is to treat the problem the same way as finite depth problems: a dynamical system formulation is used, following the method introduced by K.Kirchgässner [13]. After several transformations, in the frame moving with the waves, the problem can be put into the following (*reversible*) form:

$$\frac{dU}{dx} = L_\mu U + N(\mu, U) \tag{1}$$

in a suitable Banach space, where $U = 0$ represents the rest state (flat free surface and interface), x is the unbounded coordinate in the horizontal direction, μ represents the set of parameters, and $L_\mu U$ is the linear part. We only study solutions of (1) near 0. In all the problems we consider, the spectrum of L_μ contains a small number of non zero eigenvalues *near or on the imaginary axis* (2 to 8) and contains the entire real axis (*essential spectrum*), with no eigenvalue in it, except at 0 for problems 1 and 2, due to the presence of at least one of the Bernoulli first integrals which needs to be differentiated for being included into (1). Note that in problems 2 and 3, it is possible to have more than four eigenvalues along the imaginary axis. A consequence of the above structure of the spectrum is that one cannot apply a *Center Manifold reduction*, to obtain a small dimensional ODE describing the "small" bounded solutions, contrary to the finite depth case (see [7],[6]). In addition, it is necessary in this study to "extract" the eigenvalue 0 from the continuous spectrum which crosses the imaginary axis (for Pbs 1 and 2). At this point we should emphasize that the physical relevance of our infinite thickness assumption is linked with the fact that for layers with (finite) physically realistic depths, the linearized problem (as considered in section 3) gives many eigenvalues which might be very close to the imaginary axis, so that a theory which uses reduction of the system into a finite dimensional one, like the Center Manifold reduction (see [7]) would have a very tiny domain of validity in the parameter space. We deliberately choose to consider the "worst case" and then need to find other methods for solving the problem, depending of the seeked solutions.

Using an adapted *Lyapunov-Schmidt technique* we prove below the existence and we compute *periodic travelling waves* for Pbs 0,1,2,3 in various situations *far from resonances and near the strongest resonance (1:1 resonance).* A forthcoming second part (work with F.Dias) which constitutes another work, is concerned with solitary waves for all these problems, these waves *decaying polynomially at infinity*, contrary to the finite depth case (where the decay is exponential). The study of the critical sets in the parameter space is not essential for the study of periodic waves. A detailed study will be presented in the second part.

2 Formulation as a reversible dynamical system

2.1 Case of one layer with surface tension at the free surface

The aim of this section is to recall the formulation which was used in particular by Levi-Civita [15] and which will be systematically used in all considered cases. Let us then first consider the case of one infinitely thick layer of an inviscid fluid, assuming the flow to be potential and driven by gravity g and surface tension T acting at the free surface (see figure 1). We are interested in steady waves of permanent form, i.e. travelling waves with constant velocity c. Formulating the problem in a moving reference frame, our solutions are steady in time, and we intend to consider the unbounded horizontal coordinate ξ as a "time". Let us denote by ρ the fluid density, then we choose c as the velocity scale, $l = T/\rho c^2$ as the length scale and ρc^2 as the pressure scale. The dimensionless parameter occuring in the equations is:

$$\mu = \frac{gT}{\rho c^4}.$$

The complex potential is denoted by $w(\xi + i\eta)$ and the complex velocity

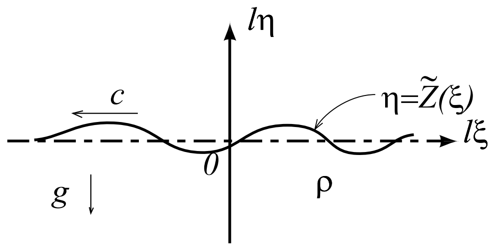

Figure 1: One layer of infinite depth

(in dimensionless form) $w'(\xi + i\eta) = u - iv$. The free surface is denoted by $\eta = \widetilde{Z}(\xi)$. The Euler equations are expressed here by the fact that w is analytic in $\zeta = \xi + i\eta$ and by the following boundary conditions

$$(u, v) \to (1, 0) \quad \text{as } \eta \to -\infty,$$

the kinematic condition at the free surface

$$u\widetilde{Z}'(\xi) - v = 0 \text{ at } \eta = \widetilde{Z}(\xi) \text{ (free surface)},$$

and the Bernoulli first integral at the free surface, expressing the condition on the pressure jump which is proportional to the curvature

$$\frac{1}{2}(u^2 + v^2) + \mu\widetilde{Z} - \frac{\widetilde{Z}''}{(1 + \widetilde{Z}'^2)^{3/2}} = \frac{1}{2} \text{ at } \eta = \widetilde{Z}(\xi) \text{ (free surface)}.$$

For formulating our problem into a dynamical system, we first transform the unknown domain into a strip. There are different ways for such a change of coordinates. We choose the one used by Levi-Civita [15]. Its advantage is that it leads to a weakly nonlinear problem, instead of a true quasi-linear PDE problem as it is the case for the choice of variables introduced in [14]. The new unknown is $\alpha + i\beta$ as an analytic function of $w = x + iy$, where

$$w'(\xi + i\eta) = e^{\beta - i\alpha},$$

the free surface is given by $y = 0$, the region of the flow is $y < 0$. In our formulation, the unknown is denoted by U where

$$[U(x)]\,(y) = (Z(x), \alpha_0(x), \alpha(x,y), \beta(x,y))^t$$

and it is straightforward to obtain the system under the form

$$\frac{dU}{dx} = F(\mu, U) \tag{2}$$

where

$$F(\mu, U) = \left\{ \begin{array}{l} e^{-\beta_0} \sin \alpha_0 \\ \sinh \beta_0 + \mu e^{-\beta_0} Z \\ \left. \begin{array}{l} \frac{\partial \beta}{\partial y}, \\ -\frac{\partial \alpha}{\partial y} \end{array} \right\} - \infty < y < 0. \end{array} \right. \tag{3}$$

Equation (2) has to be understood in the space $\mathbb{H} = \mathbb{R}^2 \times \{L^1(\mathbb{R}^-)\}^2$, and $U(x)$ lies in $\mathbb{D} = \mathbb{R}^2 \times \{W^{1,1}(\mathbb{R}^-)\}^2 \cap \{\alpha_0 = \alpha|_{y=0}\}$, where we denote by β_0 the trace $\beta|_{y=0}$ and by $W^{1,1}(\mathbb{R}^-)$ the space of integrable functions with an integrable first derivative on $(-\infty, 0)$. A solution of our water-wave problem is any $U \in \mathcal{C}^0(\mathbb{D}) \cap \mathcal{C}^1(\mathbb{H})$ which is solution of (2), where (e.g.) $\mathcal{C}^0$ means continuous and bounded for $x \in \mathbb{R}$.

It is clear that $U = 0$ is a particular solution of (2), which corresponds to the flat free surface state. A very important property of (2) is its *reversibility*: indeed let us define the symmetry:

$$SU = (Z, -\alpha_0, -\alpha, \beta)^t$$

then it is easy to see that the linear operator S *anticommutes with* $F(\mu, \cdot)$. This reflects the invariance under reflexion symmetry $x \to -x$ of our problem. Notice that in the unbounded strip $y < 0$ the system (2) is linear, this is very helpful for the study of solitary waves (see [9]) because it simplifies a lot the difficult choice of function space when the behavior at infinity in x coordinate has to be mixed with the behavior at infinity in y coordinate. Another remark for the one layer case, is that it can be reformulated into one integro-differential equation (after solving the linear problem in the region $y < 0$. However, this needs to specify from the beginning what type of solution we would like to study, and the choice would be limited. For instance, for periodic solutions we might use Fourier series spaces (see for instance [10]), and for solitary waves we might use Hilbert transform in the half space (see a remark in [9]). Finally, this possible reduction is no longer tractable for problems with several layers, which is precisely the problem of interest here.

2.2 Case of two layers without surface tension: problem 1

Let us now consider the two layers case (densities ρ_1, ρ_2), assuming that there is no surface tension, neither at the free surface nor at the interface. The thickness at rest of the upper layer is h while the bottom one has infinite thickness (see figure 2). The dimensionless parameters here are $\rho = \rho_2/\rho_1 < 1$, and $\lambda = \frac{gh}{c^2}$ (inverse of (Froude number)^2). The domain can be transformed into two superposed horizontal strips and we may use the same type of variables as above. One difficulty is that the x coordinate is not the same in each strip! In fact we have

$$\frac{dx_2}{dx_1} = e^{\beta_{20}-\beta_{10}}$$

where $\beta_{20} - \beta_{10}$ is the value of $\beta_2 - \beta_1$ taken at the interface $y = 0$.

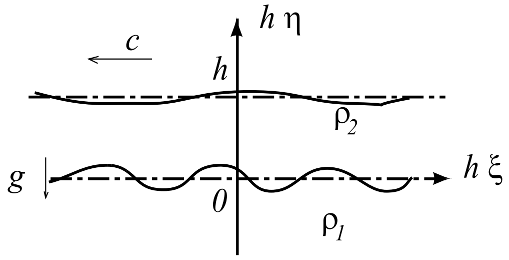

Figure 2: Two layers, the bottom one being of infinite depth

We have to choose as the basic x coordinate the one given by the bottom layer (x_1) which then introduces a factor in the Cauchy-Riemann equations of the upper layer. In such a formulation, the unknown is defined by

$$[U(x)](y) = (\beta_{10}(x), \beta_{21}(x), \alpha_1(x,y), \beta_1(x,y), \alpha_2(x,y), \beta_2(x,y))^t$$

and the system has the form (2) with $\mu = (\rho, \lambda)$ and

$$F(\mu, U) = \begin{cases} -\lambda(1-\rho)e^{-3\beta_{10}}\sin\alpha_{10} - \rho e^{3(\beta_{20}-\beta_{10})}\frac{\partial\alpha_2}{\partial y}|_{y=0} \\ -\lambda e^{-3\beta_{21}+\beta_{20}-\beta_{10}}\sin\alpha_{21} \\ \left.\begin{array}{l} \frac{\partial\beta_1}{\partial y} \\ -\frac{\partial\alpha_1}{\partial y} \end{array}\right\} \qquad y \in (-\infty, 0) \\ \left.\begin{array}{l} \frac{\partial\beta_2}{\partial y}e^{\beta_{20}-\beta_{10}} \\ -\frac{\partial\alpha_2}{\partial y}e^{\beta_{20}-\beta_{10}} \end{array}\right\} \qquad y \in (0,1) \end{cases} \tag{4}$$

where we denote by α_{10}, β_{10} and β_{20} the traces of (resp.) $\alpha_1, \beta_1, \beta_2$ at $y = 0$, and α_{21}, β_{21} the traces of α_2 and β_2 at $y = 1$. Here the basic space $\mathbb{H} = \mathbb{R}^2 \times \{W^{1,1}(\mathbb{R}^-)\}^2 \times \{W^{1,1}(0,1)\}^2$ needs more regularity than in the previous case, to be able to define the term $\frac{\partial\alpha_2}{\partial y}|_{y=0}$ in the first component and the domain of the operator F is now: $\mathbb{D} = \mathbb{R}^2 \times \{W^{2,1}(\mathbb{R}^-)\}^2 \times \{W^{2,1}(0,1)\}^2 \cap \{\alpha_{10} = \alpha_{20}, \beta_{10} = \beta_1|_{y=0}, \beta_{21} = \beta_2|_{y=1}\}$, while the reversibility symmetry reads:

$$SU = (\beta_{10}, \beta_{21}, -\alpha_1, \beta_1, -\alpha_2, \beta_2)^t.$$

The new system (2-4) should be completed by the following two Bernoulli first integrals (interface and free surface):

$$\lambda(1-\rho)\int_{-\infty}^{0}(e^{-\beta_1}\cos\alpha_1 - 1)dy + \frac{1}{2}(e^{2\beta_{10}} - 1) - \frac{\rho}{2}(e^{2\beta_{20}} - 1) = 0, \tag{5}$$

$$\int_{-\infty}^{0}(e^{-\beta_1}\cos\alpha_1 - 1)dy + \int_0^1(e^{-\beta_2}\cos\alpha_2 - 1)dy + \frac{1}{2\lambda}(e^{2\beta_{21}} - 1) = 0, \tag{6}$$

which give the two first components of (2-4) after differentiation. Notice that the interface and free surface, expressed in the new coordinate have the following expressions:

$$Z_I(x) = \int_{-\infty}^{0}(e^{-\beta_1}\cos\alpha_1 - 1)dy,$$

$$1 + Z(x) = 1 + \int_{-\infty}^{0}(e^{-\beta_1}\cos\alpha_1 - 1)dy + \int_0^1(e^{-\beta_2}\cos\alpha_2 - 1)dy.$$

In principle we might choose to treat this problem on a codimension-2 manifold, instead of expressing the two first components of (4) above. It appears that it is easier to work as we do at present, just keeping in mind that there are two constants which should be fixed, because of these known first integrals. In the present paper, we notice that the system is still linear in the unbounded strip $y < 0$. It is no longer linear in the bounded strip $y \in (0,1)$, but the dependancy in y is still occuring in similar "linear terms", the multiplicator being only function of x.

2.3 Case of two layers with surface tension: problem 2

Assuming that we have the same geometry (see figure 2) as in the previous section, but assuming now that there is surface tension T at the upper free surface $\eta = 1 + \widetilde{Z}(\xi)$, we introduce a new dimensionless parameter

$$b = \frac{T}{\rho_2 h c^2} \text{ (Weber number)}.$$

The same choice of coordinates as above leads to the following unknown

$$[U(x)](y) = (\beta_{10}(x), Z(x), \alpha_{21}(x), \alpha_1(x,y), \beta_1(x,y), \alpha_2(x,y), \beta_2(x,y))^t$$

and the right hand side of (2) with $\mu = (\rho, \lambda, b)$ reads

$$F(\mu, U) = \begin{cases} -\lambda(1-\rho)\sin\alpha_{10}e^{-3\beta_{10}} - \rho e^{3(\beta_{20}-\beta_{10})}\frac{\partial\alpha_2}{\partial y}|_{y=0} \\ e^{-\beta_{21}+\beta_{20}-\beta_{10}}\sin\alpha_{21} \\ 1/b(\sinh\beta_{21} + \lambda Z e^{-\beta_{21}})e^{\beta_{20}-\beta_{10}} \\ \left.\begin{array}{l} \frac{\partial\beta_1}{\partial y} \\ -\frac{\partial\alpha_1}{\partial y} \end{array}\right\} \quad y \in (-\infty, 0) \\ \left.\begin{array}{l} \frac{\partial\beta_2}{\partial y}e^{\beta_{20}-\beta_{10}} \\ -\frac{\partial\alpha_2}{\partial y}e^{\beta_{20}-\beta_{10}} \end{array}\right\} \quad y \in (0,1) \end{cases} \tag{7}$$

with the same notations as in previous section. Here, the basic space for (2,7) is $\mathbb{H} = \mathbb{R}^3 \times \{W^{1,1}(\mathbb{R}^-)\}^2 \times \{W^{1,1}(0,1)\}^2$ to be able to define the term $\frac{\partial\alpha_2}{\partial y}|_{y=0}$ in the first component and the domain of the operator F is now: $\mathbb{D} = \mathbb{R}^3 \times \{W^{2,1}(\mathbb{R}^-)\}^2 \times \{W^{2,1}(0,1)\}^2 \cap \{\alpha_{10} = \alpha_{20}, \beta_{10} = \beta_1|_{y=0}, \alpha_{21} = \alpha_2|_{y=1}\}$,while the reversibility symmetry reads:

$$SU = (\beta_{10}, Z, -\alpha_{21}, -\alpha_1, \beta_1, -\alpha_2, \beta_2)^t.$$

Notice that the new system (2,7) should be completed by the following Bernoulli first integral (interface):

$$\lambda(1-\rho)\left\{Z - \int_0^1 (e^{-\beta_2}\cos\alpha_2 - 1)dy\right\} + 1/2(e^{2\beta_{10}} - 1) - \rho/2(e^{2\beta_{20}} - 1) = 0, \tag{8}$$

which gives the first component of (2-7) after differentiation.

2.4 Case of two layers with surface and interfacial tensions: problem 3

Assuming again that we have the same geometry (see figure 2) as in previous section, but assuming now that there are both surface tension T at the upper free surface $\eta = 1+\widetilde{Z}(\xi)$, and interfacial tension T_I at the interface $\eta = \widetilde{Z}_I(\xi)$ we introduce a new dimensionless parameter

$$b_I = \frac{T_I}{\rho_1 hc^2} \text{ (interfacial Weber number).}$$

The same choice of coordinates as above leads to the following unknown

$$[U(x)](y) = (Z_I(x), \alpha_{10}(x), \alpha_{21}(x), \alpha_1(x,y), \beta_1(x,y), \alpha_2(x,y), \beta_2(x,y))^t$$

and the right hand side of (2) with $\mu = (\rho, \lambda, b, b_I)$ reads

$$F(\mu, U) = \begin{cases} e^{-\beta_{10}} \sin \alpha_{10} \\ 1/b_I \{\sinh \beta_{10} - \rho e^{\beta_{20}-\beta_{10}} \sinh \beta_{20} + \lambda(1-\rho) Z_I e^{-\beta_{10}}\} \\ 1/b\{\sinh \beta_{21} + \lambda Z_I e^{-\beta_{21}} + \lambda e^{-\beta_{21}} \int_0^1 (e^{-\beta_2} \cos \alpha_2 - 1) dy\} e^{\beta_{20}-\beta_{10}} \\ \left.\begin{array}{l} \frac{\partial \beta_1}{\partial y} \\ -\frac{\partial \alpha_1}{\partial y} \end{array}\right\} \qquad y \in (-\infty, 0) \\ \left.\begin{array}{l} \frac{\partial \beta_2}{\partial y} e^{\beta_{20}-\beta_{10}} \\ -\frac{\partial \alpha_2}{\partial y} e^{\beta_{20}-\beta_{10}} \end{array}\right\} \qquad y \in (0,1) \end{cases} \tag{9}$$

with the same notations as in previous section. Here, the basic space for (2-9) is $\mathbb{H} = \mathbb{R}^3 \times \{L^1(\mathbb{R}^-)\}^2 \times \{L^1(0,1)\}^2$ and the domain of the operator F is now: $\mathbb{D} = \mathbb{R}^3 \times \{W^{1,1}(\mathbb{R}^-)\}^2 \times \{W^{1,1}(0,1)\}^2 \cap \{\alpha_{10} = \alpha_1|_{y=0} = \alpha_2|_{y=0}, \alpha_{21} = \alpha_2|_{y=1}\}$,while the reversibility symmetry reads:

$$SU = (Z_{I,} - \alpha_{10}, -\alpha_{21}, -\alpha_1, \beta_1, -\alpha_2, \beta_2)^t.$$

Notice that the new system (2-9) should not be completed by any Bernoulli first integral since they are entirely in the first components of the system. Let us also observe that in the present case the function space $\mathbb{H}$ is simpler than before. This is due to the fact that there is no trace term as $\frac{\partial \alpha_2}{\partial y}|_{y=0}$ in the right hand side, so it should be clear that this last problem is *mathematically simpler* than the two previous ones.

Finally, it has to be understood that problem (2) is not an usual evolution problem: the initial value problem is ill-posed! This is in fact an elliptic problem in the half plane $\mathbb{R} \times \mathbb{R}^-$ for problem (2-3), and $\mathbb{R} \times (-\infty, 1)$ for problems (2-4,7,9). The study of an elliptic problem in a strip as an ill-posed evolution problem was introduced by [13], but here the difficulty is that the diameter of the strip is infinite. However, this difficulty was overcome for the (simpler) one layer problem in [9].

The aim of next section is to show the properties of the linearized operator near 0, motivated by the fact that we want to study solutions (periodic here) near equilibrium.

3 The linearized Problem

The linearized system reads

$$\frac{dU}{dx} = L_\mu U \tag{10}$$

in $\mathbb{H}$. In all problems, for layers with finite depth , it can be shown that the spectrum of L_μ which is *symmetric with respect to both axis* of the complex plane, because of reversibility, is only composed of isolated eigenvalues of finite multiplicities, only accumulating at infinity. More precisely, denoting by ik these eigenvalues (not necessarily pure imaginary), then one has the classical "dispersion relation" for solving the eigenvalues, under the form of a complex equation $f(\mu, k) = 0$. When the thickness of the bottom layer tends to infinity, there is an *accumulation of eigenvalues* filling the whole real axis, and at the limit of infinite thickness, all real eigenvalues disappear, except 0 in some cases, as we shall see. The dispersion relation only gives here the eigenvalues, but not the whole spectrum of the linear operator L_μ which contains also the full real axis! Below, we first give the various dispersion relations related with our problems, and we study the *resolvent operator*, specially its behavior near 0 and near the real axis.

3.1 Eigenvalues

Let us look for eigenvalues denoted by ik where k is complex, for the linear operator L_μ in each problem mentioned above.

3.1.1 Problem 0

The linearized problem for system (2-3) is given by the linear operator L_μ acting in $\mathbb{H}$, with domain of definition $\mathbb{D}$, and defined by

$$L_\mu U = \begin{pmatrix} \alpha_0 \\ \beta_0 + \mu Z \\ \frac{\partial \beta}{\partial y} \\ -\frac{\partial \alpha}{\partial y} \end{pmatrix}. \tag{11}$$

Looking at the eigenvalues ik such that $\operatorname{Re} k > 0$ leads to eigenvectors of the form

$$\zeta = (1, ik, ike^{ky}, -ke^{ky})^t,$$

and the following dispersion relation (for $\operatorname{Re} k > 0$):

$$\Delta_0(k, \mu) \equiv k^2 - k + \mu = 0,$$

which should be completed by the symmetric relationship for $\operatorname{Re} k < 0$. Notice that for $\mu < 1/4$, there are two pairs of pure imaginary eigenvalues $\pm ik_1, \pm ik_2$, while for $\mu > 1/4$ there is no eigenvalue on the imaginary axis. For $\mu = 1/4$ there are two double non semi-simple eigenvalues $\pm ik_c = \pm i/2$. Notice that in this problem there are no other eigenvalues than the four eigenvalues above mentioned.

3.1.2 Problem 1

The linearized problem for system (2-4) is given by the linear operator L_μ acting in $\mathbb{H}$, with domain $\mathbb{D}$, and defined by

$$L_\mu U = \begin{pmatrix} -\lambda(1-\rho)\alpha_{10} - \rho\frac{\partial\alpha_2}{\partial y}|_{y=0} \\ -\lambda\alpha_{21} \\ \frac{\partial\beta_1}{\partial y} \\ -\frac{\partial\alpha_1}{\partial y} \\ \frac{\partial\beta_2}{\partial y} \\ -\frac{\partial\alpha_2}{\partial y} \end{pmatrix} \tag{12}$$

Looking at the eigenvalues ik such that $\operatorname{Re} k > 0$ leads to eigenvectors of the form

$$\zeta = (1, \frac{\lambda}{k}e^k, -ie^{ky}, e^{ky}, -i\frac{\lambda}{k}e^{ky}, \frac{\lambda}{k}e^{ky})^t + \\ \frac{k-\lambda}{2\rho k}\left[(1+\rho)(0, e^k, 0, 0, -ie^{ky}, e^{ky})^t + (1-\rho)(0, e^{-k}, 0, 0, ie^{-ky}, e^{-ky})^t\right],$$

and the following dispersion relation (for $\operatorname{Re} k > 0$):

$$\Delta_1(k,\mu) \equiv (\lambda - k)\left\{[\rho k - (1-\rho)\lambda]\sinh k + k\cosh k\right\} = 0, \tag{13}$$

which should be completed by the symmetric relationship for $\operatorname{Re} k < 0$. An interesting property of problem 1 is that there is an explicit pair of simple eigenvalues $\pm i\lambda$ with eigenvectors ζ_0 and $\overline{\zeta}_0$ defined by

$$\zeta_0 = (1, e^\lambda, -ie^{\lambda y}, e^{\lambda y}, -ie^{\lambda y}, e^{\lambda y})^t.$$

The study of (13) shows that there is another pair of simple eigenvalues on the imaginary axis $\pm ik_1$ *if and only if* $\lambda > (1-\rho)^{-1}$, moreover, we always have $k_1 < \lambda$ and, in playing on ρ and λ we can reach strong resonances such as $k_1/\lambda = 1/2$ and $1/3$.

Remark 1 *It is of physical importance to notice that if k is* real, *then for the eigenvectors, the interface component $Z_I \sim -\int_{-\infty}^0 \beta_1 dy$ and the free surface component $Z \sim Z_I - \int_0^1 \beta_2 dy$ are in phase. This is due to the fact that β_1 and β_2 are both real >0 in the eigenvector ζ. Notice moreover that for $k = \lambda$, $Z \sim e^\lambda Z_I$ which gives an idea of the ratio between amplitudes of the oscillations of the two free surfaces.*

Notice that 0 is always an eigenvalue here, with the eigenvector (and no generalized eigenvector)

$$\xi_0 = (0,1,0,0,0,1)^t, \tag{14}$$

which corresponds to the *existence of the one parameter family of solutions* $\beta_{20}(0,1,0,0,0,1)^t$ of the nonlinear system (2,4). In fact, these solutions would be avoided by the consideration of the Bernoulli first integrals.

In all what follows, we assume that $\lambda \neq (1-\rho)^{-1}$.

3.1.3 Problem 2

The linearized problem for system (2,7) is given by the linear operator L_μ acting in $\mathbb{H}$, with domain $\mathbb{D}$, and defined by

$$L_\mu U = \begin{pmatrix} -\lambda(1-\rho)\alpha_{10} - \rho\frac{\partial \alpha_2}{\partial y}|_{y=0} \\ \alpha_{21} \\ 1/b(\beta_{21} + \lambda Z) \\ \frac{\partial \beta_1}{\partial y} \\ -\frac{\partial \alpha_1}{\partial y} \\ \frac{\partial \beta_2}{\partial y} \\ -\frac{\partial \alpha_2}{\partial y} \end{pmatrix} \tag{15}$$

Looking at the eigenvalues ik such that $\mathrm{Re}\, k > 0$ leads to eigenvectors of the form

$$\begin{aligned}\zeta = \big(&1, -k^{-1}(\cosh k + B\sinh k), -i(\cosh k + B\sinh k),\\ &-ie^{ky}, e^{ky}, -i(\cosh ky + B\sinh ky), \sinh ky + B\cosh ky\big)^t,\end{aligned}$$

with $B = (\rho k)^{-1}[k - \lambda(1-\rho)]$ and the following dispersion relation (for $\mathrm{Re}\, k > 0$):

$$\Delta_2(k,\mu) \equiv k\cosh k(\rho b k^2 + \lambda - k) - \sinh k\left\{(\lambda + bk^2)[\lambda(1-\rho) - k] + \rho k^2\right\} = 0, \tag{16}$$

which should be completed by the symmetric relationship for $\mathrm{Re}\, k < 0$. The study of (16) shows that there are much more possibilities to have eigenvalues on the imaginary axis than for problem 1, since we may have more than two pairs of eigenvalues on this axis, and all possibilities of resonances between two pairs.

Remark 2 *It is of physical importance to notice that if k is* real, *then for the eigenvectors, the interface and the free surface components Z_I and Z are in phase, or out of phase by half of a wave length. This is due to the fact that Z is real in the eigenvector ζ, as well as β_1 occuring in Z_I,but their signs depend on the parameters.*

Notice that 0 is always an eigenvalue here, with the eigenvector (and no generalized eigenvector)

$$\xi_0 = (0, 1, 0, 0, 0, 0, -\lambda)^t, \tag{17}$$

which corresponds to the *existence of the one parameter family of solutions* $(0, 1/2\lambda(1-e^{2\beta_{20}}), 0, 0, 0, \beta_{20})^t$ of the nonlinear system (2,7). In fact, these solutions would be avoided by the consideration of the Bernoulli first integral.

3.1.4 Problem 3

The linearized problem for system (2,9) is given by the linear operator L_μ acting in $\mathbb{H}$,with domain $\mathbb{D}$, and defined by

$$L_\mu U = \begin{pmatrix} \alpha_{10} \\ 1/b_I\,(\beta_{10} - \rho\beta_{20} + \lambda(1-\rho)Z_I) \\ 1/b\left(\beta_{21} + \lambda Z_I - \lambda \int_0^1 \beta_2 dy\right) \\ \frac{\partial \beta_1}{\partial y} \\ -\frac{\partial \alpha_1}{\partial y} \\ \frac{\partial \beta_2}{\partial y} \\ -\frac{\partial \alpha_2}{\partial y} \end{pmatrix} \tag{18}$$

Looking at the eigenvalues ik such that $\mathrm{Re}\,k > 0$ leads to eigenvectors of the form

$$\begin{aligned}\zeta = \big(&k^{-1}, i, i(\cosh k + B\sinh k), \\ &ie^{ky}, -e^{ky}, i(\cosh ky + B\sinh ky), -(\sinh ky + B\cosh ky)\big)^t,\end{aligned}$$

with $B = -(\rho k)^{-1}[k - \lambda(1-\rho) - b_I k^2]$ and the following dispersion relation (for $\mathrm{Re}\,k > 0$):

$$\Delta_3(k,\mu) \equiv k\cosh k[(\rho b + b_I)k^2 + \lambda - k] -$$

$$-\sinh k\left\{(\lambda + bk^2)[\lambda(1-\rho) - k + k^2 b_I] + \rho k^2\right\} = 0, \tag{19}$$

which should be completed by the symmetric relationship for $\mathrm{Re}\,k < 0$. The study of (19) shows that there are even more possibilities than in previous problems 1 and 2 for having several pairs of eigenvalues on the imaginary axis.

Remark 3 *As in previous remark, we see on the eigenvector ζ that if k is real, Z_I is real and $Z - Z_I \sim -\int_0^1 \beta_2 dy$ is real too. So, the interface and the free surface components Z_I and Z are in phase or out of phase by half of a wave length (depending on the parameters).*

In the present problem, 0 is no longer an eigenvalue, except if $\lambda = (1-\rho)^{-1}$. In such a case the eigenvector has the form

$$\xi_0 = (\rho, 0, 0, 0, 0, 0, 1)^t$$

and there is no generalized eigenvector. Let us recall that in all what follows, we assume that $\lambda \neq (1-\rho)^{-1}$.

3.2 Study of the resolvent operator

The roots k of the dispersion relations are the poles of the resolvent operator $(ik\mathbb{I} - L_\mu)^{-1}$. In addition for all our problems, we show below that we have an estimate of the form

$$||(ik\mathbb{I} - L_\mu)^{-1}||_{\mathcal{L}(\mathbb{H})} \leq C/|k|, \tag{20}$$

for k real and large enough $|k|$, where $\mathcal{L}(\mathbb{H})$ is the space of bounded linear operators in $\mathbb{H}$. It is fortunate that, for all these problems, the resolvent $(ik\mathbb{I} - L_\mu)^{-1}$ can be obtained explicitly, specially in problems 1-2, because the "bad" trace term in the first component of L_μ would give difficulties for obtaining such an estimate (20) with another choice of basic space than $\mathbb{H}$. One of the classical consequences is that the spectrum of L_μ is located in a sector $(ik \in \mathbb{C}; |k_r| < p|k_i| + r)$ of the complex plane. Moreover, this estimate appears to be essential in our method of computation of periodic solutions and later for the computation of solutions homoclinic to 0.

3.2.1 Problem 0

Let us solve the resolvent operator for problem 0, i.e. where L_μ is given by (11). The resolvent equation $(ik - L_\mu)U = V$, where V is given in $\mathbb{H}$, and where we look for U in $\mathbb{D}$, leads to the following system

$$\begin{array}{l} ikZ - \alpha_0 = a \\ ik\alpha_0 - (\beta_0 + \mu Z) = b \\ \left.\begin{array}{l} ik\alpha - \beta' = f \\ ik\beta + \alpha' = g \end{array}\right\} \; y \in (-\infty, 0) \end{array} \tag{21}$$

where $V = (a, b, f, g) \in \mathbb{H}$. For $\operatorname{Re} k > 0$ and $\Delta_0(k, \mu) \neq 0$ we have

$$\Delta_0(k,\mu)Z = -i(k-1)a - b + \int_{-\infty}^0 [f(s) + ig(s)]e^{ks}ds, \tag{22}$$

$$\Delta_0(k,\mu)\alpha_0 = -\mu a - ikb + ik\int_{-\infty}^0 [f(s) + ig(s)]e^{ks}ds,$$

$$\begin{aligned} \alpha(y) &= \alpha_0 e^{ky} + H_1[f,g](k,y), \\ \beta(y) &= i\alpha_0 e^{ky} + K_1[f,g](k,y), \end{aligned} \tag{23}$$

with

$$H_1[f,g](k,y) = \frac{i}{2}\int_{-\infty}^{0}[f(s)+ig(s)]e^{k(s+y)}ds - \frac{i}{2}\int_{-\infty}^{0}[f(s)+ig(s)sgn(y-s)]e^{-k|y-s|}ds$$

$$K_1[f,g](k,y) = -\frac{1}{2}\int_{-\infty}^{0}[f(s)+ig(s)]e^{k(s+y)}ds - \frac{1}{2}\int_{-\infty}^{0}[f(s)sgn(y-s)+ig(s)]e^{-k|y-s|}ds.$$

It is then easy to check that for $k \neq 0$ real, we have the following estimate:

$$||H_1[f,g](k,\cdot)||_{L^1} \leq \frac{3}{2|k|}(||f||_{L^1} + ||g||_{L^1})$$

and the same estimate for K_1 holds. Now, we have $|\Delta_0(k,\mu)|^{-1} \leq 2/k^2$ for k real and $|k|$ large enough, hence it is clear that there exists a positive number C such that the following estimate

$$|Z| + |\alpha_0| + ||\alpha||_{L^1} + ||\beta||_{L^1} \leq \frac{C}{|k|}(|a|+|b|+||f||_{L^1}+||g||_{L^1})$$

holds. This is precisely (20).

3.2.2 Problem 1

Let us now consider the linear operator given by (12), and solve the resolvent equation as above. This leads to the following system

$$\begin{array}{l} ik\beta_{10} + \lambda(1-\rho)\alpha_{10} + \rho\alpha_2'(0) = a \\ ik\beta_{21} + \lambda\alpha_{21} = b \\ \left.\begin{array}{l} ik\alpha_1 - \beta_1' = f_1 \\ ik\beta_1 + \alpha_1' = g_1 \end{array}\right\} \quad y \in (-\infty, 0) \\ \left.\begin{array}{l} ik\alpha_2 - \beta_2' = f_2 \\ ik\beta_2 + \alpha_2' = g_2 \end{array}\right\} \quad y \in (0, 1) \end{array} \tag{24}$$

where $V = (a, b, f_1, g_1, f_2, g_2) \in \mathbb{H}$. For $\mathrm{Re}\, k > 0$ and $\Delta_1(k,\mu) \neq 0$ we have

$$\begin{aligned} \alpha_1(y) &= Ae^{ky} + H_1[f_1,g_1](k,y), \\ \beta_1(y) &= iAe^{ky} + K_1[f_1,g_1](k,y), \\ \alpha_2(y) &= A\cosh ky + B\sinh ky + H_2[f_2,g_2](k,y), \\ \beta_2(y) &= iA\sinh ky + iB\cosh ky + iK_2[f_2,g_2](k,y), \end{aligned} \tag{25}$$

where

$$\begin{aligned} H_2[f_2,g_2](k,y) &= \int_0^1 [if_2(s)H_{21}(y,s) - g_2(s)H_{22}(y,s)]ds, \\ K_2[f_2,g_2](k,y) &= \int_0^1 [if_2(s)K_{21}(y,s) - g_2(s)K_{22}(y,s)]ds, \end{aligned} \tag{26}$$

with

$$H_{21}(y,s) = H_{21}(s,y) = (\sinh k)^{-1}\sinh k(y-1)\sinh(ks), \text{ for } 0<s<y<1,$$
$$H_{22}(y,s) = K_{21}(s,y) = (\sinh k)^{-1}\begin{cases}\sinh k(y-1)\cosh(ks), \text{ for } 0<s<y<1,\\ \sinh(ky)\cosh k(s-1), \text{ for } 0<y<s<1,\end{cases}$$
$$K_{22}(y,s) = K_{22}(s,y) = (\sinh k)^{-1}\cosh k(y-1)\cosh(ks), \text{ for } 0<s<y<1.$$

$$\Delta_1(k,\mu)A = (k\cosh k - \lambda\sinh k)a_1 + \rho k b_1,$$
$$\Delta_1(k,\mu)B = (\lambda\cosh k - k\sinh k)a_1 + [k - \lambda(1-\rho)]b_1,$$

$$a_1 = a + ik\int_{-\infty}^{0}[f_1(s)+ig_1(s)]e^{ks}ds - \rho g_2(0) - \tag{27}$$
$$- \rho k(\sinh k)^{-1}\int_0^1[if_2(s)\sinh k(s-1) - g_2(s)\cosh k(s-1)]ds$$
$$b_1 = b + k(\sinh k)^{-1}\int_0^1[if_2(s)\sinh(ks) - g_2(s)\cosh(ks)]ds. \tag{28}$$

It is then easy to state that for $k>0$, large enough, we have a constant c such that

$$|A|+|B| \le c/k(|a_1| + e^{-k}|b_1|),$$
$$|a_1|+|b_1| \le c(|a|+|b|+||f_1||_{L^\infty}+||g_1||_{L^\infty}+||f_2||_{L^\infty}+||g_2||_{L^\infty}).$$

Now, we also have the following identities

$$\frac{\partial H_1[f,g]}{\partial y}(k,y) = if(0)e^{ky} + H_1[f',g'](k,y) + i\int_{-\infty}^{0}[f'(s)+ig'(s)]e^{k(s+y)}ds,$$
$$\tag{29}$$
$$\frac{\partial K_1[f,g]}{\partial y}(k,y) = -f(0)e^{ky} + K_1[f',g'](k,y) - \int_{-\infty}^{0}[f'(s)+ig'(s)]e^{k(s+y)}ds,$$

which give good estimates on $H_1[f_1,g_1]$ and $K_1[f_1,g_1]$ in $W^{1,1}(\mathbb{R}^-)$. Observing that

$$\beta_{10} = iA - \int_{-\infty}^{0}[f_1(s)+ig_1(s)]e^{ks}ds$$

holds, and knowing that the embedding $L^\infty \hookrightarrow W^{1,1}$ is continuous, for the interval $(0,1)$ as well as $(-\infty,0)$, the estimates above and estimates on kernels H_1 and K_1 already give for $k>0$ large enough

$$||\alpha_1||_{W^{1,1}} + ||\beta_1||_{W^{1,1}} + |\beta_{10}| \le C/k(|a|+|b|+||f_1||_{W^{1,1}}+||g_1||_{W^{1,1}}+||f_2||_{L^\infty}+||g_2||_{L^\infty}),$$

which concerns half of the components of U in $\mathbb{H}$. For the last three components, we observe that α_2 and β_2 may be rewritten as follows

$$\alpha_2(y) = \frac{a_1}{\Delta_1(k,\mu)}[\lambda \sinh k(y-1) + k\cosh k(y-1)] + \frac{b_1}{\Delta_1(k,\mu)}\{[k-\lambda(1-\rho)]\sinh ky + \rho k \cosh ky\} + H_2[f_2, g_2](k,y), \quad (30)$$

$$\beta_2(y) = \frac{ia_1}{\Delta_1(k,\mu)}[\lambda \cosh k(y-1) + k\sinh k(y-1)] + \frac{ib_1}{\Delta_1(k,\mu)}\{[k-\lambda(1-\rho)]\cosh ky + \rho k \sinh ky\} + iK_2[f_2, g_2](k,y). \quad (31)$$

For $k \to \infty$ we have $\Delta_1(k,\mu) \sim -1/2(1+\rho)k^2e^k$, $||\sinh k(y-1)||_{W^{1,1}} \leq e^k$, $||\cosh k(y-1)||_{W^{1,1}} \leq e^k$, and

$$\sup_{s\in(0,1)} \max\left(||H_{21}(\cdot,s)||_{L^1}, ||H_{22}(\cdot,s)||_{L^1}, ||K_{21}(\cdot,s)||_{L^1}, ||K_{22}(\cdot,s)||_{L^1}\right) \leq 1/k.$$

Now, we also have, after integration by parts

$$\frac{\partial H_2[f,g]}{\partial y}(k,y) = \frac{i}{\sinh k}[f(1)\cosh ky - f(0)\cosh k(y-1)] + iK_2[-g', f'](k,y), \quad (32)$$

$$\frac{\partial K_2[f,g]}{\partial y}(k,y) = \frac{i}{\sinh k}[f(1)\sinh ky - f(0)\sinh k(y-1)] + iH_2[-g', f'](k,y),$$

which allows to easily obtain the following estimate for the last three components of U

$$||\alpha_2||_{W^{1,1}} + ||\beta_2||_{W^{1,1}} + |\beta_{21}| \leq C/k(|a| + |b| + ||f_1||_{L^\infty} + ||g_1||_{L^\infty} + ||f_2||_{W^{1,1}} + ||g_2||_{W^{1,1}}).$$

Hence (20) is proved for problem 1.

The computation and estimates on the resolvent for problems 2 and 3 are done in Appendices 1.and 2.

Remark 4 *If we build a basic space $\mathbb{H}$ with $L^2(\mathbb{R}^-)$ instead of $L^1(\mathbb{R}^-)$, it can be checked that one obtains the same estimate (20) for the resolvent related with problems 0 and 3. This is different for problems 1 and 2, since one cannot obtain such an estimate if we replace $W^{1,1}(\mathbb{R}^-)$ by $W^{2,1}(\mathbb{R}^-)$ (i.e. $H^2(\mathbb{R}^-)$). For instance, the term Ae^{ky} has only an estimate in $1/k^{1/2}$. This shows the bad influence of the trace term in the first component of the linear operators (12), (15).*

3.3 Structure of the spectrum of L_μ

We show in this section the following (again assuming that $\lambda \neq (1-\rho)^{-1}$ for problems 1,2,3):

Theorem 1 *The spectrum of the linear operator L_μ is formed i) with a discrete set of isolated eigenvalues σ of finite multiplicities, these ones being the roots of the dispersion relation $\Delta_j(-i\sigma,\mu)=0, j=0,1,2,3$; ii) with the entire real axis which constitutes the* "essential spectrum"*. 0 is an eigenvalue embedded in this continuous part of the spectrum for problems 1 and 2. In such cases, it is (however) possible to find a closed subspace of $\mathbb{H}$ of codimension 1, invariant under L_μ, where the restriction of L_μ has the same spectrum as L_μ except the 0 eigenvalue which disappears.*

We gave above expressions of the resolvent operator $(ik\mathbb{I}-L_\mu)^{-1}$ for $\operatorname{Re} k>0$. For $\operatorname{Re} k<0$, we just need to observe that

$$(ik\mathbb{I}-L_\mu)^{-1}=\overline{(-i\overline{k}\mathbb{I}-L_\mu)^{-1}}.$$

Now, it is clear from the study of eigenvalues and from the analyticity of functions $\Delta_j(k,\mu), j=0,1,2,3$ that *outside the real axis*, the spectrum of L_μ is formed with *isolated eigenvalues of finite multiplicities*, given by the roots of $\Delta_j(k,\mu)=0$. Let us assume that ik_0, with $\operatorname{Im} k_0\neq 0$, is an eigenvalue of L_μ, then it will be useful in next sections to be able to express the projection on the invariant subspace, which commutes with L_μ. There is here a new difficulty, for problems 1 and 2 especially, because the usual way for expressing these projections would be to compute a formal adjoint operator for L_μ and to compute corresponding eigenvectors and generalized eigenvectors of this adjoint operator for building the projections. For problems 1 and 2, the adjoints, even formally are hard to obtain, because the choice of our basic spaces involves first order derivatives in L^1. We need to use another way to get these projections. Let us assume, for instance, that k_0 is a double root of $\Delta_j(k,\mu)=0$. Then, using the classical theory (see Kato [12]) we have, for k near k_0, the following Laurent expansion in $\mathcal{L}(\mathbb{H},\mathbb{D})$:

$$(ik\mathbb{I}-L_\mu)^{-1}=\frac{D_\mu}{(k-k_0)^2}-i\frac{P_\mu}{k-k_0}+S_\mu-i(k-k_0)S_\mu^2+... \qquad (33)$$

where P_μ is the projection, on the finite dimensional invariant subspace belonging to the eigenvalue ik_0, commuting with L_μ, and $D_\mu=(ik_0\mathbb{I}-L_\mu)P_\mu$ is nilpotent, and $S_\mu=(ik_0\mathbb{I}-L_\mu)^{-1}(\mathbb{I}-P_\mu)$ is the "pseudo-inverse" of $(ik_0\mathbb{I}-L_\mu)$ in the complementary invariant subspace $(\mathbb{I}-P_\mu)\mathbb{H}$. Using the explicit form of the resolvent, it is then easy to obtain directly the projection P_μ.

3.3.1 Study of the resolvent operator on the real axis

Let us now study the resolvent equation $(\sigma\mathbb{I} - L_\mu)U = V$ given in $\mathbb{H}$, for σ *real*.

Problem 0 We have in this case

$$\begin{aligned} Z &= \mu^{-1}[\sigma\alpha_0 - b - \beta(0)], \\ \alpha_0 &= \int_{-\infty}^{0} [g(y)\cos\sigma y - f(y)\sin\sigma y]dy, \\ \alpha(y) &= \widetilde{H}_1[f,g](\sigma,y), \\ \beta(y) &= \widetilde{K}_1[f,g](\sigma,y), \end{aligned}$$

where

$$\begin{aligned} \widetilde{H}_1[f,g](\sigma,y) &= \int_{-\infty}^{y} [f(s)\sin\sigma(y-s) + g(s)\cos\sigma(y-s)]ds, \\ \widetilde{K}_1[f,g](\sigma,y) &= \int_{-\infty}^{y} [g(s)\sin\sigma(y-s) - f(s)\cos\sigma(y-s)]ds. \end{aligned}$$

In addition, we have a *compatibility condition*

$$\mu a + \sigma b = (\sigma^2 - \mu)\alpha_0 - \sigma\beta_0,$$

and conditions on f and g to ensure that $\widetilde{H}_1$ and $\widetilde{K}_1 \in L^1(\mathbb{R}^-)$. For instance, for $\sigma = 0$, this gives

$$a + \int_{-\infty}^{0} g(y)dy = 0, \quad \int_{-\infty}^{y} f(s)ds \text{ and } \int_{-\infty}^{y} g(s)ds \in L^1(\mathbb{R}^-). \tag{34}$$

It results that for any σ real, σ is not an eigenvalue, the range of $(\sigma\mathbb{I} - L_\mu)$ is not closed, and its closure has codimension one. So, the entire real axis constitutes the *"essential spectrum"* of L_μ.

Problem 1 We have in this case

$$\begin{aligned} \beta_{10} &= -\int_{-\infty}^{0} [f_1(y)\cos\sigma y + g_1(y)\sin\sigma y]dy, \\ \beta_{21} &= \alpha_{10}\sin\sigma + \beta_{20}\cos\sigma + \widetilde{K}_2[f_2,g_2](\sigma,1) \\ \alpha_1 &= \widetilde{H}_1[f_1,g_1](\sigma,y) \\ \beta_1 &= \widetilde{K}_1[f_1,g_1](\sigma,y) \\ \alpha_2 &= \alpha_{10}\cos\sigma y - \beta_{20}\sin\sigma y + \widetilde{H}_2[f_2,g_2](\sigma,y), \\ \beta_2 &= \alpha_{10}\sin\sigma y + \beta_{20}\cos\sigma y + \widetilde{K}_2[f_2,g_2](\sigma,y), \end{aligned} \tag{35}$$

where

$$\widetilde{H}_2[f_2,g_2](\sigma,y) = \int_0^y [f_2(s)\sin\sigma(y-s) + g_2(s)\cos\sigma(y-s)]ds,$$
$$\widetilde{K}_2[f_2,g_2](\sigma,y) = \int_0^y [g_2(s)\sin\sigma(y-s) - f_2(s)\cos\sigma(y-s)]ds,$$
$$\alpha_{10} = \int_{-\infty}^0 [g_1(y)\cos\sigma y - f_1(y)\sin\sigma y]dy,$$
$$\beta_{20} = \sigma^{-1}[g_2(0) - \rho^{-1}a + \lambda(\rho^{-1}-1)\alpha_{10}] + \rho^{-1}\beta_{10}$$

holds for $\sigma \neq 0$. In addition, we have a compatibility condition

$$\sigma\beta_{21} + \lambda\alpha_2(1) - b = 0$$

and conditions on f_1 and g_1 to ensure that $\widetilde{H}_1$ and $\widetilde{K}_1 \in L^1(\mathbb{R}^-)$. Now, for $\sigma = 0$, β_{20} is undetermined and this gives the conditions

$$a - \lambda(1-\rho)\int_{-\infty}^0 g_1(y)dy - \rho g_2(0) = 0, \tag{36}$$
$$\lambda\int_{-\infty}^0 g_1(y)dy + \lambda\int_0^1 g_2(y)dy = b, \tag{37}$$
$$\int_{-\infty}^y f_1(s)ds \quad \text{and} \quad \int_{-\infty}^y g_1(s)ds \in L^1(\mathbb{R}^-). \tag{38}$$

Notice that the two linear conditions (36,37) correspond to the linearization of the two Bernoulli first integrals (5,6). The undetermination on β_{20} corresponds to the fact that $\sigma = 0$ is an eigenvalue with eigenvector (14). Finally for problem 1, as well as for problem 0, the *entire real axis* constitutes the *essential spectrum*, with the additional fact that 0 is an eigenvalue embedded in the middle of this continuous spectrum. We refer to Appendices 1 and 2 for the cases of problems 2 and 3. Roughly speaking, problem 2 is like problem 1 and problem 3 is like problem 0.

3.3.2 Study of the resolvent near 0

In this section we study the behavior of the resolvent $(ik\mathbb{I} - L_\mu)^{-1}$ for k real, tending towards 0. Since 0 belongs to the spectrum, we need to understand the singularity of the resolvent, which should not be a pole, since this is specific to isolated eigenvalues.

Problem 0 Let us define the space $\widetilde{W}^1 = \{u \in C_b^0(\mathbb{R}^-); u' \in L^1(\mathbb{R}^-)\}$ with the norm $||u||_{\widetilde{W}^1} = ||u||_{L^\infty} + ||u'||_{L^1}$. We observe that $W^{1,1}(\mathbb{R}^-) \subset \widetilde{W}^1$ with continuous embedding, but $W^{1,1}$ is not dense in $\widetilde{W}^1$. Let us introduce now

$$\widetilde{\mathbb{D}} = \mathbb{R}^2 \times \{\widetilde{W}^1\}^2 \cap \{\alpha_0 = \alpha|_{y=0}\},$$

then, for any $V \in \mathbb{H}$, it is easy to check that we have the following limits in $\widetilde{\mathbb{D}}$:

$$\lim_{k\to 0^{\pm}} (ik\mathbb{I} - L_\mu)^{-1}V = (sgnk)B_\mu V + C_\mu V \tag{39}$$

where B_μ and $C_\mu \in \mathcal{L}(\mathbb{H}, \widetilde{\mathbb{D}})$ are defined by

$$B_\mu V = [a + \int_{-\infty}^{0} g(y)dy](i\mu^{-1}, 0, 0, i)^t \in \widetilde{\mathbb{D}},$$

$$C_\mu V = \left(-\mu^{-1}b, -a, -[a + \int_{-\infty}^{0} g(y)dy] + \int_{-\infty}^{y} g(s)ds, -\int_{-\infty}^{y} f(s)ds\right)^t \in \widetilde{\mathbb{D}}.$$

It results that, even after extending $\mathbb{D}$ into $\widetilde{\mathbb{D}}$, we still have a *jump* for the resolvent operator, when crossing the real axis at 0. Finally, notice that when conditions (34) are satisfied, the singular behavior disappears, because $B_\mu V = 0$, and $C_\mu V \in \mathbb{D}$.

Problem 1 We now introduce, in a way analogous than above $\widetilde{W}^2 = \{u \in \widetilde{W}^1; u'' \in L^1(\mathbb{R}^-)\}$, and

$$\widetilde{\mathbb{D}} = \mathbb{R}^2 \times \{\widetilde{W}^2\}^2 \times \{W^{2,1}(0,1)\}^2 \cap \{\alpha_{10} = \alpha_{20}, \beta_{10} = \beta_1|_{y=0}, \beta_{21} = \beta_2|_{y=1}\}.$$

We have indeed $\mathbb{D} \subset \widetilde{\mathbb{D}}$ with continuous embedding, but $\mathbb{D}$ is not dense in $\widetilde{\mathbb{D}}$. Then, for any $V \in \mathbb{H}$, it is easy to check that we have the following limits in $\widetilde{\mathbb{D}}$:

$$(ik\mathbb{I} - L_\mu)^{-1}V \underset{k\to 0^{\pm}}{=} \frac{1}{ik}A_\mu V + (sgnk)B_\mu V + C_\mu V + O(k||V||), \tag{40}$$

where A_μ, B_μ and $C_\mu \in \mathcal{L}(\mathbb{H}, \widetilde{\mathbb{D}})$ are defined below. We observe that the behavior is more singular than for problem 0, and this is due to the occurence of eigenvalue 0. For any $V \in \mathbb{H}$, and $\lambda \neq (1-\rho)^{-1}$, we have

$$A_\mu V = \xi_\mu^*(V)\xi_0, \text{ with } \xi_\mu^* \in \mathbb{H}^* \text{(linear continuous forms on } \mathbb{H}\text{) and } \xi_0 \text{ given by (14)},$$

$$\xi_\mu^*(V) = [1 - \lambda(1-\rho)]^{-1}\left\{-a + \rho g_2(0) + (1-\rho)b - \lambda(1-\rho)\int_0^1 g_2(s)ds\right\},$$

$$B_\mu V = \zeta_\mu^*(V)\left(i, i[1-\lambda(1-\rho)]^{-1}, 0, i, 0, i[1-\lambda(1-\rho)]^{-1}\right)^t, \text{ with } \zeta_\mu^* \in \mathbb{H}^*,$$

$$\zeta_\mu^*(V) = \lambda^{-1}[1-\lambda(1-\rho)]^{-1}\left\{\rho\left[b - \lambda\int_{-\infty}^0 g_1(y)dy - \lambda\int_0^1 g_2(y)dy\right] + (1-\lambda)\left[a - \lambda(1-\rho)\int_{-\infty}^0 g_1(y)dy - \rho g_2(0)\right]\right\},$$

$$C_\mu V = \begin{pmatrix} -\int_{-\infty}^{0} f_1(y)dy \\ -[1-\lambda(1-\rho)]^{-1}\left[\int_{-\infty}^{0} f_1(y)dy + \int_0^1[\rho-\lambda(1-\rho)s]f_2(s)ds\right] \\ \zeta_\mu^*(V) + \int_{-\infty}^{y} g_1(s)ds \\ -\int_{-\infty}^{y} f_1(s)ds \\ \lambda^{-1}\xi_\mu^*(V) + \lambda^{-1}b + \int_0^y g_2(s)ds - y\int_0^1 g_2(s)ds \\ \int_y^1 f_2(s)ds - [1-\lambda(1-\rho)]^{-1}\left[\int_{-\infty}^{0} f_1(y)dy + \int_0^1[\rho-\lambda(1-\rho)s]f_2(s)ds\right] \end{pmatrix}.$$

We notice that $\xi_\mu^*(L_\mu U) = 0, \forall U \in \mathbb{D}$, and that $\xi_\mu^*(\xi_0) = 1$. This means that the linear operator A_μ is a projection on ξ_0 and, as expected, that it commutes with L_μ. This shows that we might consider the restriction of L_μ on the closed invariant subspace $\xi_\mu^*(V) = 0$ of $\mathbb{H}$, and that on this subspace the behavior of the resolvent is the same as for problem 0, i.e. this reduction eliminates the eigenvalue 0 of L_μ. This remark is useful in the further study, specially for homoclinic solutions (forthcoming paper) because it allows to split the space as if 0 were a simple isolated eigenvalue!

As for problem 0 above, the conditions $A_\mu V = 0$, and $B_\mu V = 0$ are equivalent to conditions (36-37).

For the study of the resolvent operator near 0, in problems 2 and 3, we refer to Appendices 1 and 2. It appears that problem 2 is analogous to problem 1 and problem 3 is analogous to problem 0.

4 Periodic waves in the simple cases

In this section, we consider the cases when there is a pair of simple eigenvalues $\pm ik_0$ of the operator L_μ, on the imaginary axis, such that other pairs $\pm ik_1$ which might be on this axis, satisfy $k_1 \neq nk_0$ for $n = 1, 2, ...$This condition is in general satisfied because there is only a finite number of eigenvalues on the imaginary axis, whose positions depend on the parameter set μ.

It results that in general the only point in the spectrum of L_μ in resonance with our pair is 0. The "exotic" character of this point of the spectrum leads to a specific difficulty we are dealing with below, by adapting the classical Lyapunov-Schmidt method for periodic orbits.

Let us set $s = (k_0+\gamma)x$, where γ is close to 0, and $k_0+\gamma$ is the wave number of the periodic solution we are looking for. We denote by $H_\sharp^p(E)$ the space of ($2\pi-$periodic) functions of s, such that their derivatives up to order p are in $L^2(\mathbb{R}/2\pi\mathbb{Z})$, taking values in the Banach space E. For such a space we use the norm defined by

$$||u||_{H_\sharp^p}^2 = \sum_{n\in\mathbb{Z}}(1+n^{2p})||u_n||_E^2$$

which gives a Banach space structure. Let us define, in the space $\mathbb{H}_\sharp = H_\sharp^1(\mathbb{H})$, the linear operator $T_\mu = k_0\frac{d}{ds} - L_\mu$, with domain $\mathbb{D}_\sharp = H_\sharp^2(\mathbb{H}) \cap$

$H^1_\sharp(\mathbb{D})$. The basic tool is the inversion of $T_\mu U = V$ where V is given in $\mathbb{H}_\sharp$, and where we look for $U \in \mathbb{D}_\sharp$. Now expanding in Fourier series V and U we have for $n \neq 0, 1, -1$

$$U_n = (ink_0\mathbb{I} - L_\mu)^{-1}V_n,$$

which, with the resolvent estimate (20), insures that if we define

$$U' = \sum_{n\in\mathbb{Z}\backslash\{0,1,-1\}} U_n e^{nis} \in \mathbb{D}_\sharp, \text{ then } ||U'||_{\mathbb{D}_\sharp} \leq C_1||V||_{\mathbb{H}_\sharp} \text{ holds.} \tag{41}$$

It then remains to study the equations $(ink_0\mathbb{I} - L_\mu)U_n = V_n$ for $n = 0, 1, -1$. For $n = 0$ we gave the conditions {(34) for problem 0,(36,37,38) for problem 1} on V_0 for being in the range of L_μ. For $n = 1$ we need to give explicitly the linear condition $\zeta^*(V_1) = 0$, which is necessary and sufficient for finding U_1 (non unique). This is not an obvious task, since the adjoint operator L^*_μ is not explicit here. Instead, we use the resolvent expansion near the pole ik_0 for obtaining the linear form $\zeta^* \in \mathbb{H}^*$ [see (33)]:

Linear form giving the compatibility condition for problem 0

$$\zeta^*(V) = -\mu a - ik_0 b + ik_0 \int_{-\infty}^0 [f(s) + ig(s)]e^{k_0 s}ds,$$

we notice that $\zeta^*(SV) = \overline{\zeta^*(V)}$.

Linear form giving the compatibility condition for problem 1

$$\text{for } k_0 = \lambda,\ \zeta^*(V) = a + \rho b e^\lambda - \rho g_2(0) + i\lambda \int_{-\infty}^0 (f_1 + ig_1)e^{\lambda s}ds + i\rho\lambda \int_0^1 (f_2 + ig_2)e^{\lambda s}ds,$$

$$\text{for } k_0 \neq \lambda, \quad \zeta^*(V) = (k_0 \cosh k_0 - \lambda \sinh k_0)a_1 + \rho k_0 b_1$$

where a_1 and b_1 are given by (27-28). Here again, we notice that $\zeta^*(SV) = \overline{\zeta^*(V)}$. See Appendices 1 and 2 for ζ^* in problems 2 and 3.

It is easy to check that $\zeta^*(\zeta) \neq 0$, where ζ is the eigenvector belonging to ik_0. The necessary and sufficient conditions for finding U_1 and U_{-1} may be written as

$$\zeta^*(V_1) = \overline{\zeta}^*(V_{-1}) = 0,$$

and U_1 and U_{-1} are respectively defined up to arbitrary multiples of the eigenvectors ζ and $\overline{\zeta}$.

Let us now look for solutions in $\mathbb{D}_\sharp$ of the following equation in $\mathbb{H}_\sharp$:

$$T_\mu U = G(\mu, \gamma, U) \tag{42}$$

where

$$G(\mu, \gamma, U) = -\gamma \frac{dU}{ds} + N(\mu, U), \tag{43}$$

and where N is defined by $F(\mu, U) = L_\mu U + N(\mu, U)$.

We observe that $G(\mu, \cdot, \cdot)$ is analytic: $\mathbb{R} \times \mathbb{D}_\sharp \to \mathbb{H}_\sharp$ in the neighborhood of 0, and is such that

$$G(\mu, \gamma, 0) = 0, \qquad D_{(\gamma,U)} G(\mu, 0, 0) = 0.$$

Now, let us define the symmetry $\widehat{S}$ in $\mathcal{L}(\mathbb{H}_\sharp) \cap \mathcal{L}(\mathbb{D}_\sharp)$ by

$$(\widehat{S}U)(s) = SU(-s).$$

It is then easy to check that $\widehat{S}T_\mu = -T_\mu \widehat{S}$, $\widehat{S}G(\mu, \gamma, U) = -G(\mu, \gamma, \widehat{S}U)$ holds.

For problems 1 and 2, we have to add one condition resulting from the Bernoulli first integrals, written below as $h(\mu, U) = 0$, in order to kill the arbitrariness (arbitrary constant z) of the solutions, corresponding to the eigenvector ξ_0 belonging to the 0 eigenvalue. We notice that $h(\mu, SU) = h(\mu, U)$ because $S\xi_0 = \xi_0$.

We now decompose as well the space $\mathbb{H}_\sharp$ as $\mathbb{D}_\sharp$, as follows:

$$U = U_0 + V + W, \tag{44}$$
$$V = Ae^{is}\zeta + \overline{A}e^{-is}\overline{\zeta} \tag{45}$$
$$U_0 = z\xi_0 + U_0', \tag{46}$$
$$W = \Pi U, \tag{47}$$

where $S\zeta = \overline{\zeta}$, $\xi_\mu^*(U_0') = 0$ (this defines the codimension 1 subspace $\widehat{\mathbb{D}}$ of $\mathbb{D}$), and where the projection Π is defined by

$$\int_0^{2\pi} \zeta^*[W(s)]e^{-is} ds = 0, \int_0^{2\pi} \overline{\zeta}^*[W(s)]e^{is} ds = 0, \int_0^{2\pi} [W(s)] ds = 0.$$

Let us denote respectively by $\widehat{\mathbb{H}}_\sharp, \widehat{\mathbb{D}}_\sharp$ the subspaces of $\mathbb{H}_\sharp$ and $\mathbb{D}_\sharp$ where W lives (depending on the regularity of U). Then, thanks to (41) we notice that the unbounded linear operator T_μ has a bounded pseudo-inverse $\widehat{T}_\mu^{-1}$: $\widehat{\mathbb{H}}_\sharp \to \widehat{\mathbb{D}}_\sharp$.

The system (42) now reads

$$T_\mu W = -\gamma \frac{dW}{ds} + \Pi N(\mu, z\xi_0 + U_0' + V + W), \tag{48}$$
$$-L_\mu U_0' = [N(\mu, z\xi_0 + U_0' + V + W)]_0, \tag{49}$$
$$0 = \zeta^* \left(-i\gamma A\zeta + [N(\mu, z\xi_0 + U_0' + V + W)]_1 \right), \tag{50}$$

where indices "0" and "1" refer to the corresponding Fourier components. In the case where 0 is eigenvalue (ξ_0 exists for problems 1 and 2), we need to complete the above system by

$$h(\mu, z\xi_0 + U_0' + V + W) = 0, \tag{51}$$

defined below. Let us now define the translational symmetry by $(\tau_\varphi U)(s) = U(s+\varphi)$, it is clear that if (W, z, U_0', A) is a solution of (48,49,50,51) in $\widehat{\mathbb{D}}_\sharp \times \mathbb{R} \times \widehat{\mathbb{D}} \times \mathbb{C}$, then $(\tau_\varphi W, z, U_0', Ae^{i\varphi})$ is also solution for all φ ($SO(2)$ invariance), and because of the equivariance under the reversibility symmetry $(\widehat{S}W, z, SU_0', \overline{A})$ is solution too.

Notice that, for all our problems the form of the nonlinear term N is such that the components corresponding to the infinite layer cancel. This allows to satisfy the two last conditions (34) for problem 0, conditions (38),(73),(80) for problem 1, 2 and 3. This is a *major advantage of our formulation*, since it remains for N_0 in (49) to only satisfy a true codimension-one condition for problems 0 and 3, and a codimension-two condition for problems 1 and 2. Let us define a projection $\widehat{\pi}$ on the "restricted" range of L_μ, and define the bounded pseudo-inverse operator $\widehat{L}_\mu^{-1}$ on this restricted range (components (f, g) for problem 0, (f_1, g_1) for problems 1,2,3 are identically 0, in addition to the compatibility conditions). We can then replace (49) by the two equations

$$-L_\mu U_0' = \widehat{\pi}[N(\mu, z\xi_0 + U_0' + V + W)]_0, \tag{52}$$
$$0 = (1-\widehat{\pi})[N(\mu, z\xi_0 + U_0' + V + W)]_0. \tag{53}$$

We now recall that for a periodic solution which is globally invariant under the reversibility symmetry S , i.e. such that $\forall s, \exists s'$ such that $SU(s) = U(s')$, there is a choice of the origin of s such that we have $SU(s) = U(-s)$, i.e. $\widehat{S}U = U$. Indeed this is due to the fact that $SU(-s)$ is also a periodic solution of the same problem. So, we restrict our attention to *reversible periodic solutions*. This restricts in fact our study to periodic solutions globally invariant under S. The other case would give a pair of periodic solutions exchanged by S.

We have now the following

Lemma 2 *If $\widehat{S}U = U$, then (for problems 0, 1, 2, 3)*

$$N_0 = \frac{1}{2\pi}\int_0^{2\pi} N[\mu, U(s)]ds \in \text{ range } L_\mu.$$

Proof: $SN_0 = -\int_0^{2\pi} N[\mu, SU(s)]ds = -\int_0^{2\pi} N[\mu, U(-s)]ds = -N_0$. Now we just observe that the compatibility conditions $(34)_a$, (36-37), (71-72), (79) hold automatically when V satisfies $SV = -V$.

It results that if $\widehat{S}U = U$ equation (53) is automatically satisfied, and since this equation is invariant under time shift, it is also automatically satisfied for $U(s)$ globally invariant under S.

We want now to solve our system (48-49-50-51), where (51) and z only occur for problems 1 and 2. Let us start in solving the system (48-52-51) with respect to (W, U_0', z) in $\widehat{\mathbb{D}}_\sharp \times \widehat{\mathbb{D}} \times \mathbb{R}$. The system is analytic with respect to all its arguments in a neighborhood of 0 (trivial solution).We show below, that a suitable choice of function h in (51) gives a particularly simple linearization at 0: we obtain the following linear operator $(W, U_0', z) \to (T_\mu W, -L_\mu U_0', z)$.

Lemma 3 *For problem 1 we choose* $h(\mu, U) = [1-\lambda(1-\rho)]^{-1}\{\lambda(1-\rho)(6)_0 - (5)_0\}$, *where the index "$_0$" means the 0^{th} Fourier component. For problem 2 we choose* $h(\mu, U) = \lambda^{-1}[1-\lambda(1-\rho)]^{-1}(8)_0$. *Then* $D_U h(\mu, 0)V = \xi_\mu^*(V)$ *for any* V *in* $\mathbb{D}$. *Hence,* $D_U h(\mu, 0)U = z$ *for* $U \in \mathbb{D}_\sharp$.

Proof: for problem 1, we have by definition for any U in $\mathbb{D}$
$h(\mu, U) = [1-\lambda(1-\rho)]^{-1}\{\lambda(1-\rho)\int_0^1 (e^{-\beta_2}\cos\alpha_2 - 1)dy + 1/2(1-\rho)e^{2\beta_{21}} - 1/2e^{2\beta_{10}} + \rho/2e^{2\beta_{20}}\}_0$,

so, the result of the lemma is immediate. For problem 2, the lemma comes directly from the linearization of (8) and the definition of $\xi_\mu^*(V)$ given in Appendix 1.

Finally, we already noticed that the equation $L_\mu U_0' = V_0$ has a unique solution U_0' in $\widehat{\mathbb{D}}$ as soon as V_0 has the two components (α_1, β_1) which cancel and that it belongs to the restricted range of L_μ. It results that the analytic implicit function theorem applies, and leads to a unique analytic solution $(W, U_0', z) = (\mathcal{W}, \mathcal{U}_0, \mathcal{Z})(\gamma, A, \overline{A})$ near 0 in $\widehat{\mathbb{D}}_\sharp \times \widehat{\mathbb{D}} \times \mathbb{R}$, for (γ, A) in a neighborhood of 0 in $\mathbb{R} \times \mathbb{C}$.

Moreover, we have

$$(\mathcal{W}, \mathcal{U}_0, \mathcal{Z})(\gamma, 0, 0) = 0$$

$$(\tau_\varphi \mathcal{W}, \mathcal{U}_0, \mathcal{Z})(\gamma, A, \overline{A}) = (\mathcal{W}, \mathcal{U}_0, \mathcal{Z})(\gamma, Ae^{i\varphi}, \overline{A}e^{-i\varphi}),$$
$$(\widehat{S}\mathcal{W}, S\mathcal{U}_0, \mathcal{Z})(\gamma, A, \overline{A}) = (\mathcal{W}, \mathcal{U}_0, \mathcal{Z})(\gamma, \overline{A}, A)$$

This directly leads to

$$\mathcal{W}(\gamma, A, \overline{A}) = \widehat{T}_\mu^{-1}\Pi N_2[\mu; V^{(2)}] + h.o.t.,$$
$$(\mathcal{U}_0, \mathcal{Z})(\gamma, A, \overline{A}) = (\widetilde{\mathcal{U}}_0, \widetilde{\mathcal{Z}})(\gamma, |A|^2) = O(|A|^2),$$

where N_2 defines the quadratic part of N in its argument U.

It remains to solve (50) after replacing (W, U_0', z) by $(\mathcal{W}, \mathcal{U}_0, \mathcal{Z})(\gamma, A, \overline{A})$. This equation takes the form

$$-i\gamma A + g(\gamma, A, \overline{A}) = 0, \tag{54}$$

where g is analytic in its arguments and satisfies $g(\gamma, Ae^{i\varphi}, \overline{A}e^{-i\varphi}) = e^{i\varphi}g(\gamma, A, \overline{A})$ for any $\varphi \in \mathbb{R}$, because of the τ_φ equivariance of $\mathcal{W}$, and $g(\gamma, \overline{A}, A) = \overline{-g(\gamma, A, \overline{A})}$ because of the reversibility symmetry S and the symmetry property of the linear form ζ^*. It results that (54) may be now written as:

$$-i\gamma A + iA\widehat{g}(\gamma, |A|^2) = 0, \tag{55}$$

where $\widehat{g}$ is real and analytic in its arguments near 0. The solution $A = 0$ corresponds to the trivial solution. Now $\widehat{g}$ cancels for $A = 0$ because N is at least quadratic in U, and $(\mathcal{W}, \mathcal{U}_0, \mathcal{Z}) = 0$ for $A = 0$, as well as V. It results that (55) can be solved by the implicit function theorem with respect to γ as an analytic function of $|A|^2$. We then proved the following

Theorem 4 *Assume that μ is such that $\pm ik_0$ are simple eigenvalues of L_μ and that other pure imaginary eigenvalues ik_1 satisfy $k_1 \neq nk_0$ for $n \in \mathbb{N}\backslash\{0, 1, -1\}$, 0 being a simple eigenvalue for problems 1 and 2, embedded in the essential spectrum of L_μ. Assume moreover that μ is not a special value of the parameter set for the point 0 in the spectrum of L_μ. Then, (for μ fixed) a one parameter family of reversible periodic waves bifurcates, with the following power series in A and $\overline{A}$, converging for $|A|$ small enough:*

$$U(s) = Ae^{is}\zeta + \overline{A}e^{-is}\overline{\zeta} + \sum_{p+q\geq 2} A^p\overline{A}^q e^{i(p-q)s}U_{pq} \in \mathbb{D}_\sharp, \tag{56}$$
$$\gamma = \sum_{r\geq 1} \gamma_r |A|^{2r} \in \mathbb{R},$$

where $k_0 + \gamma$ is the wave number defined by $s = (k_0 + \gamma)x$. We have $SU_{pq} = U_{qp} = \overline{U}_{pq}$ in $\mathbb{D}$, and $\tau_\varphi U$ is given by replacing A by $Ae^{i\varphi}$, $\widehat{S}U = U$ for A real and the amplitude $|A|$ is the only parameter of the family of periodic orbits.

Remark 5 *This theorem is the analogous of a classical theorem of Lyapunov [16] for a finite dimensional Hamiltonian system.*

Remark 6 *The free surface and the interface are in phase or in opposition of phases, depending on the parameters (see remarks 1,2,3).*

5 Periodic waves in the 1:1 resonance case

In this section, we consider the cases when for $\mu = \mu_0$ the operator L_{μ_0} has a pair of double eigenvalues $\pm ik_0$, such that the pairs of any other eigenvalues $\pm ik_1$ on the imaginary axis satisfy $k_1 \neq nk_0$ for $n = 1, 2, ...$This is often denoted as the 1:1 resonance case for reversible systems. This codimension one situation occurs in problems 0, 2 and 3 (not in problem 1). In particular, for problem 0, it occurs for $\mu_0 = 1/4$. The study of bifurcating solitary waves in problem 0, is made in [9], but not the easier study of periodic solutions. By "main" periodic solutions, we mean solutions whose wave numbers are close to k_0. Below, we give a way for obtaining main travelling periodic waves in the case of this strong resonance occuring as well in problems 2 and 3. There is numerical evidence for the existence of many other periodic solutions [3], as it is well known in the finite dimensional case, with arbitrary long wave length (easily obtained on the normal form [8]).

In what follows, we choose the eigenvector ζ_0 and generalized eigenvector ζ_1 such that

$$\begin{aligned}
(L_{\mu_0} - ik_0)\zeta_0 &= 0, S\zeta_0 = \overline{\zeta}_0 \\
(L_{\mu_0} - ik_0)\zeta_1 &= \zeta_0, S\zeta_1 = -\overline{\zeta}_1
\end{aligned}$$

which is always possible, due to reversibility symmetry. The projection P_{μ_0} on the 2 dim space spanned by ζ_0 and ζ_1 is obtained via the Laurent expansion (33) of $(ik\mathbb{I} - L_{\mu_0})^{-1}$ for k near k_0. From this expansion we can deduce the two bounded linear forms ζ_0^* and ζ_1^* as follows:

$$\begin{aligned}
&\text{for any } U \in \mathbb{H}, \text{ we have} \\
&\quad P_{\mu_0} U = A\zeta_0 + B\zeta_1, \\
&\quad D_{\mu_0} U = -B\zeta_0, \\
&\quad \zeta_0^*(U) = A, \qquad \zeta_1^*(U) = B,
\end{aligned}$$

and as it is easily seen, we have

$$\begin{aligned}
\zeta_1^*(L_{\mu_0} - ik_0) = 0, \quad \zeta_0^*(L_{\mu_0} - ik_0) = \zeta_1^*, \\
S\zeta_0^* = \overline{\zeta}_0^*, \qquad S\zeta_1^* = -\overline{\zeta}_1^*.
\end{aligned}$$

Let us proceed as in section 4 and concentrate on the differences with the present proof.

The study of equation $(ink_0\mathbb{I} - L_{\mu_0})U_n = V_n$ for $n = 1, -1$ leads to the necessary conditions $\zeta_1^*(V_1) = 0$, $\overline{\zeta}_1^*(V_{-1}) = 0$, and U_1, U_{-1} are resp. defined up to arbitrary multiples of ζ_0 and $\overline{\zeta}_0$. Finally, let us observe that we have

$$\begin{aligned}
T_{\mu_0}(e^{is}\zeta_0) = 0, T_{\mu_0}(e^{is}\zeta_1) = -e^{is}\zeta_0, \\
\widehat{S}(e^{is}\zeta_0) = (\overline{e^{is}\zeta_0}), \widehat{S}(e^{is}\zeta_1) = -(\overline{e^{is}\zeta_1}).
\end{aligned}$$

In this section, the decomposition of the space $\mathbb{H}_\sharp$ or $\mathbb{D}_\sharp$, is as follows:

$$\begin{aligned}
U &= U_0 + V + W, \\
V &= Ae^{is}\zeta_0 + Be^{is}\zeta_1 + \overline{A}e^{-is}\overline{\zeta}_0 + \overline{B}e^{-is}\overline{\zeta}_1, \qquad (57)\\
U_0 &= z\xi_0 + U_0' \quad (z \text{ exists only for pb 2 here}) \\
W &= \Pi_0 U,
\end{aligned}$$

where $\xi^*_{\mu_0}(U'_0) = 0$, and W satisfies (definition of projection Π_0)

$$\int_0^{2\pi} \zeta_j^*[W(s)]e^{-is}ds = 0, \int_0^{2\pi} \overline{\zeta}_j^*[W(s)]e^{is}ds = 0, \int_0^{2\pi} [W(s)]ds = 0, j = 1, 2.$$

The system (42), written for μ near μ_0, now reads

$$\begin{aligned}
T_{\mu_0}W &= -\gamma\frac{dW}{ds} + \Pi_0\left\{(L_\mu - L_{\mu_0})U + N(\mu, U)\right\}, && (58)\\
-L_{\mu_0}U'_0 &= \widehat{\pi}\left\{(L_\mu - L_{\mu_0})[z\xi_0 + U'_0] + [N(\mu, U)]_0\right\}, && (59)\\
-B &= -i\gamma A + \zeta_0^*\left\{(L_\mu - L_{\mu_0})[A\zeta_0 + B\zeta_1 + W_1] + [N(\mu, U)]_1\right\}, && (60)\\
0 &= -i\gamma B + \zeta_1^*\left\{(L_\mu - L_{\mu_0})[A\zeta_0 + B\zeta_1 + W_1] + [N(\mu, U)]_1\right\}, && (61)\\
0 &= h(\mu, z\xi_0 + U'_0 + V + W), && (62)\\
0 &= (1 - \widehat{\pi})\left\{(L_\mu - L_{\mu_0})[z\xi_0 + U'_0] + [N(\mu, U)]_0\right\}, && (63)
\end{aligned}$$

where (62) only occurs if there exists ξ_0 and z (i.e. in problems 0 and 2). We notice that if $\widehat{S}U = U$, then $SU'_0 = U'_0$ and $(L_\mu - L_{\mu_0})[z\xi_0 + U'_0]$ is antisymmetric, hence belongs to the range of L_{μ_0}. It results that equation (63) is automatically satisfied as soon as $\widehat{S}U = U$, and also for any U obtained from a symmetric one after a time shift.

Let us solve first the system (58-59-60-62) with respect to (W, U'_0, B, z).We note that

$$Dh_U(\mu, 0)U = \xi^*_\mu[z\xi_0 + U'_0] = z + O[|\mu - \mu_0|(|z| + ||U'_0||)],$$

so, the (analytic) implicit function theorem applies as in section 4, and leads to

$$(W, U'_0, B, z) = (\mathcal{W}, \mathcal{U}_0, \mathcal{B}, \mathcal{Z})(\gamma, \mu, A, \overline{A}) \text{ near } 0 \text{ in } \widehat{\mathbb{D}}_\sharp \times \widehat{\mathbb{D}} \times \mathbb{C} \times \mathbb{R}$$

for (γ, μ, A) in a neighborhood of $(0, \mu_0, 0)$ with the properties

$$(\mathcal{W}, \mathcal{U}_0, \mathcal{B}, \mathcal{Z})(\gamma, \mu, 0, 0) = 0,$$

$$\begin{aligned}
(\tau_\varphi\mathcal{W}, \mathcal{U}_0, e^{i\varphi}\mathcal{B}, \mathcal{Z})(\gamma, A, \overline{A}) &= (\mathcal{W}, \mathcal{U}_0, \mathcal{B}, \mathcal{Z})(\gamma, \mu, Ae^{i\varphi}, \overline{A}e^{-i\varphi}),\\
(\widehat{S}\mathcal{W}, S\mathcal{U}_0, -\overline{\mathcal{B}}, \mathcal{Z})(\gamma, A, \overline{A}) &= (\mathcal{W}, \mathcal{U}_0, \mathcal{B}, \mathcal{Z})(\gamma, \mu, \overline{A}, A).
\end{aligned}$$

This directly leads to

$$\begin{aligned}
\mathcal{W}(\gamma, \mu, A, \overline{A}) &= O(|A|^2 + |\mu - \mu_0||A|) \text{ in } \widehat{\mathbb{D}}_\sharp,\\
(\mathcal{U}_0, \mathcal{Z})(\gamma, \mu, A, \overline{A}) &= (\widetilde{\mathcal{U}}_0, \widetilde{\mathcal{Z}})(\gamma, \mu, |A|^2) = O(|A|^2) \text{ in } \widehat{\mathbb{D}} \times \mathbb{R}.
\end{aligned}$$

Let us now define the coefficients (p_0, q_0, p_1, q_1)which are all real (due to reversibility symmetry) and of order $O(\mu - \mu_0)$:

$$\begin{aligned}
\zeta_0^*(L_\mu - L_{\mu_0})\zeta_0 &= ip_0(\mu), \\
\zeta_1^*(L_\mu - L_{\mu_0})\zeta_0 &= q_0(\mu), \\
\zeta_0^*(L_\mu - L_{\mu_0})\zeta_1 &= q_1(\mu), \\
\zeta_1^*(L_\mu - L_{\mu_0})\zeta_1 &= ip_1(\mu).
\end{aligned}$$

We can then write

$$\begin{aligned}
\mathcal{B}(\gamma, A, \overline{A}) &= iA\widetilde{\mathcal{B}}(\gamma, \mu, |A|^2), \text{ with } \widetilde{\mathcal{B}} \text{ real and} \\
\widetilde{\mathcal{B}}(\gamma, \mu, |A|^2) &= \gamma - p_0(\mu) + O(|\gamma||\mu - \mu_0| + |\mu - \mu_0|^2 + |A|^2).
\end{aligned}$$

It remains to solve (61) which takes now the following form

$$\begin{aligned}
&A\widehat{g}(\gamma, \mu, |A|^2) = 0, \quad \text{with} \\
&\widehat{g}(\gamma, \mu, |A|^2) = [\gamma - p_1(\mu)][\gamma - p_0(\mu)] + q_0(\mu) + q_2|A|^2 + \\
&\qquad + O\left\{|\mu - \mu_0|^2 + |A|^2(|\mu - \mu_0| + |\gamma| + |A|^2)\right\}.
\end{aligned}$$

After eliminating the trivial solution $A = 0$, it remains an equation of the form

$$\nu(\mu) + (\gamma - 1/2[p_0(\mu) + p_1(\mu)])^2 + q_2|A|^2 + h.o.t. = 0. \qquad (64)$$

Let us observe that the quantity $\nu(\mu) = q_0(\mu) + O(|\mu - \mu_0|^2)$ is the "bifurcation parameter": when $\nu(\mu) > 0$, the double eigenvalues $\pm ik_0$ split into 4 simple eigenvalues, symmetric with respect to both axis and lying at a distance $[\nu(\mu)]^{1/2}$ from the imaginary axis. For $\nu(\mu) < 0$, the four eigenvalues lie on the imaginary axis.

Let us assume that we fix all components of parameter μ as in μ_0 except one which is equivalent to $\nu(\mu)$, then we proved the following

Theorem 5 *Assume that for a value of the parameter set μ_0 the operator L_{μ_0} has a pair of double eigenvalues $\pm ik_0$, and that other pure imaginary eigenvalues ik_1 satisfy $k_1 \neq nk_0$ for $n \in \mathbb{N}\backslash\{0, 1, -1\}$, 0 being a simple eigenvalue for problems 2, embedded in the essential spectrum of L_{μ_0}. Assume moreover that μ_0 is not a special value of the parameter set for the point 0 in the spectrum of L_{μ_0}. Then, a family of reversible periodic waves bifurcates, such that the solution and the bifurcation parameter ν may be parameterized by γ (where $k_0 + \gamma$ is the wave number defined by $s = (k_0 + \gamma)x$) and the complex amplitude A and $\overline{A}$. The following power series converges* *for $(\gamma, |A|)$ small enough:*

$$U(s) = Ae^{is}\zeta_0 + \overline{A}e^{-is}\overline{\zeta}_0 + i\gamma\left\{Ae^{is}\zeta_1 - \overline{A}e^{-is}\overline{\zeta}_1\right\} + \\ + \sum_{p+q\geq 1, r\geq 0, p+q+r\geq 3} A^p\overline{A}^q\gamma^r e^{i(p-q)s}U_{pqr},$$
$$\nu = \sum_{2p+r\geq 2} \delta_{pr}\gamma^r|A|^{2p} \in \mathbb{R}.$$

We have $SU_{pqr} = U_{qpr} = \overline{U}_{pqr}$ in $\mathbb{D}$, and $\tau_\varphi U$ is given by replacing A by $Ae^{i\varphi}$, $\widehat{S}U = U$ for A real.

Remark 7 *It is easy to visualize the existence region of these periodic waves in the plane (wave-number $k_0 + \gamma$, bifurcation parameter ν). The sign of the coefficient q_2 in equation (64) plays an essential role. For $q_2 > 0$ there exist periodic waves only inside the parabolic region, while for $q_2 < 0$ they exist only outside this region (see figure 3).*

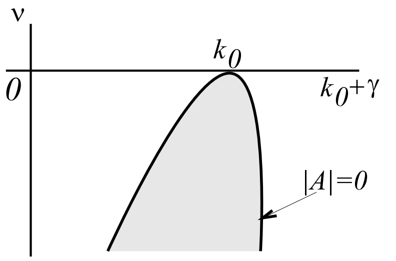

Figure 3: Hatched region of existence of periodic waves for $q_2 > 0$; complementary region for $q_2 < 0$ (case of the water-wave problem 0).

6 Appendix 1.Resolvent for problems 2

The linear operator is given by (15), and the resolvent equation reads

$$\begin{array}{l} ik\beta_{10} + \lambda(1-\rho)\alpha_{10} + \rho\alpha_2'(0) = a_1 \\ ikZ - \alpha_{21} = a_2 \\ ik\alpha_{21} - 1/b(\beta_{21} + \lambda Z) = a_3 \\ \quad \left.\begin{array}{l} ik\alpha_1 - \beta_1' = f_1 \\ ik\beta_1 + \alpha_1' = g_1 \end{array}\right\} \; y \in (-\infty, 0) \\ \quad \left.\begin{array}{l} ik\alpha_2 - \beta_2' = f_2 \\ ik\beta_2 + \alpha_2' = g_2 \end{array}\right\} \; y \in (0, 1) \end{array} \tag{65}$$

with $\alpha_{10} = \alpha_{20}, \beta_{10} = \beta_1|_{y=0}, \alpha_{21} = \alpha_2|_{y=1}$ and where we denoted $V = (a_1, a_2, a_3, f_1, g_1, f_2, g_2) \in \mathbb{H}$. For $\operatorname{Re} k > 0$ and $\Delta_2(k, \mu) \neq 0$ we have again the solution under the form (25), (26) as for problem 1, but now

$$\Delta_2(k,\mu)A = [k\cosh k - (bk^2+\lambda)\sinh k]a_1' + \rho k a_3',$$
$$\Delta_2(k,\mu)B = [(bk^2+\lambda)\cosh k - k\sinh k]a_1' + [k-\lambda(1-\rho)]a_3',$$

$$a_1' = a_1 + ik\int_{-\infty}^{0}[f_1(s)+ig_1(s)]e^{ks}ds - \rho g_2(0) - \tag{66}$$
$$- \rho k(\sinh k)^{-1}\int_0^1 [if_2(s)\sinh k(s-1) - g_2(s)\cosh k(s-1)]ds$$
$$a_3' = -ikba_3 - \lambda a_2 + k(\sinh k)^{-1}\int_0^1 [if_2(s)\sinh(ks) - g_2(s)\cosh(ks)]ds. \tag{67}$$

We may also write

$$\alpha_2(y) = \frac{a_1'}{\Delta_2(k,\mu)}[(bk^2+\lambda)\sinh k(y-1) + k\cosh k(y-1)] +$$
$$\frac{a_3'}{\Delta_2(k,\mu)}\{[k-\lambda(1-\rho)]\sinh ky + \rho k\cosh ky\} + H_2[f_2,g_2](k,y), \tag{68}$$
$$\beta_2(y) = \frac{ia_1'}{\Delta_2(k,\mu)}[(bk^2+\lambda)\cosh k(y-1) + k\sinh k(y-1)] +$$
$$\frac{ia_3'}{\Delta_2(k,\mu)}\{[k-\lambda(1-\rho)]\cosh ky + \rho k\sinh ky\} + iK_2[f_2,g_2](k,y). \tag{69}$$

This has to be completed by β_{10}, α_{21} which are easily deduced from formulas above, and by

$$Z = \frac{a_2}{ik} + \frac{\alpha_{21}}{ik}.$$

Now, for large $|k|$, k real, we have $\Delta_2(k,\mu) \sim (\rho-1)bk^3e^k$. It is clear that a_1' and a_3'/k satisfy the same estimates as a_1 and a_2 for problem 1, so A and B finally satisfy the same estimates for large real k as for problem 1. Following the same way as for problem 1, it is then clear that we obtain here again the required estimate (20).

Let us now solve the resolvent equation $(\sigma\mathbb{I} - L_\mu)U = V \in \mathbb{H}$, for σ *real*. Again, formulas (35) hold, where α_{10} and β_{20} are given by the same formulas (for $\sigma \neq 0$) as for problem 1, just replacing a by a_1. In addition, we have $\sigma Z = \alpha_{21} + a_2$ and the compatibility condition (giving a continuous linear form on $\mathbb{H}$ after making explicit α_{21} and β_{21})

$$(\lambda - b\sigma^2)\alpha_{21} + \sigma\beta_{21} + \sigma ba_3 + \lambda a_2 = 0 \tag{70}$$

and conditions on f_1 and g_1 to ensure that $\widetilde{H}_1$ and $\widetilde{K}_1 \in L^1(\mathbb{R}^-)$. Now, for $\sigma = 0$, β_{20} and Z are linked by $\beta_{20} + \lambda Z = \int_0^1 f_2(s)ds - ba_3$ and we have the following compatibility conditions for V

$$a_1 - \lambda(1-\rho)\int_{-\infty}^0 g_1(y)dy - \rho g_2(0) = 0, \tag{71}$$

$$\int_{-\infty}^0 g_1(y)dy + \int_0^1 g_2(y)dy + a_2 = 0, \tag{72}$$

$$\int_{-\infty}^y f_1(s)ds \quad \text{and} \quad \int_{-\infty}^y g_1(s)ds \in L^1(\mathbb{R}^-). \tag{73}$$

close to the ones for problem 1. Despite of the fact that we only have one additional first integral (8) for problem 2, the range of the operator L_μ has *its closure of codimension two.* The undeterminacy on β_{20} and Z corresponds to the unique eigenvector ξ_0 (17).

The study of the resolvent operator near 0 may be made similarly to problem 1. First we define

$$\widetilde{\mathbb{D}} = \mathbb{R}^3 \times \{\widetilde{W}^2\}^2 \times \{W^{2,1}(0,1)\}^2 \cap \{\alpha_{10} = \alpha_{20}, \beta_{10} = \beta_1|_{y=0}, \alpha_{21} = \alpha_2|_{y=1}\},$$

where we have again $\mathbb{D} \subset \widetilde{\mathbb{D}}$ with continuous embedding, but $\mathbb{D}$ is not dense in $\widetilde{\mathbb{D}}$. For any $V \in \mathbb{H}$, we can again check that we have the following limits in $\widetilde{\mathbb{D}}$:

$$(ik\mathbb{I} - L_\mu)^{-1}V \underset{k\to 0^\pm}{=} \frac{1}{ik}A_\mu V + (sgnk)B_\mu V + C_\mu V + O(k||V||), \tag{74}$$

where A_μ, B_μ and $C_\mu \in \mathcal{L}(\mathbb{H}, \widetilde{\mathbb{D}})$, analogously as for problem 1. Let us just give explicitly the term A_μ. For any $V \in \mathbb{H}$, we have for $\lambda \neq (1-\rho)^{-1}$

$$A_\mu V = \xi_\mu^*(V)\xi_0, \text{ with } \xi_\mu^* \in \mathbb{H}^* \text{(linear continuous forms on } \mathbb{H}) \text{ and } \xi_0 \text{ given by (17)},$$

$$\xi_\mu^*(V) = \lambda^{-1}[1-\lambda(1-\rho)]^{-1}\left\{a_1 - \rho g_2(0) + (1-\rho)\lambda a_2 + \lambda(1-\rho)\int_0^1 g_2(s)ds\right\},$$

and we check that $\xi_\mu^*(\xi_0) = 1$.

Let us end this appendix by giving explicitly the linear form $V \longmapsto \zeta^*(V)$ expressing the compatibility condition $\zeta^*(V) = 0$ for solving $(ik_0\mathbb{I} - L_\mu)U = V \in \mathbb{H}$ when ik_0 is a simple eigenvalue of L_μ. It may be expressed as

$$\zeta^*(V) = (k_0 \cosh k_0 - (\lambda + bk_0^2)\sinh k_0)a_1' + \rho k_0 a_3'$$

where a_1' and a_3' are given by (66-67). Here again, we notice that $\zeta^*(SV) = \overline{\zeta^*(V)}$.

7 Appendix 2. Resolvent for problem 3

The linear operator is given by (18), and the resolvent equation reads

$$\begin{array}{l} ikZ_I - \alpha_{10} = a_1 \\ ik\alpha_{10} - 1/b_I(\beta_{10} - \rho\beta_{20} + \lambda(1-\rho)Z_I) = a_2 \\ ik\alpha_{21} - 1/b(\beta_{21} + \lambda Z_I - \lambda \int_0^1 \beta_2 dy) = a_3 \\ \quad \left. \begin{array}{l} ik\alpha_1 - \beta_1' = f_1 \\ ik\beta_1 + \alpha_1' = g_1 \end{array} \right\} \; y \in (-\infty, 0) \\ \quad \left. \begin{array}{l} ik\alpha_2 - \beta_2' = f_2 \\ ik\beta_2 + \alpha_2' = g_2 \end{array} \right\} \; y \in (0, 1) \end{array} \tag{75}$$

with $\alpha_{10} = \alpha_{20}, \alpha_{10} = \alpha_1|_{y=0}, \alpha_{21} = \alpha_2|_{y=1}$ and where we denoted $V = (a_1, a_2, a_3, f_1, g_1, f_2, g_2) \in \mathbb{H}$. For $\mathrm{Re}\, k > 0$ and $\Delta_3(k,\mu) \neq 0$ we have again the solution under the form (25), (26) as for problem 1, but now

$$\Delta_3(k,\mu)A = [(bk^2 + \lambda)\sinh k - k\cosh k]a_1' - \rho k a_3',$$
$$\Delta_3(k,\mu)B = -[(bk^2+\lambda)\cosh k - k\sinh k]a_1' + [b_I k^2 - k + \lambda(1-\rho)]a_3',$$

$$a_1' = \lambda(1-\rho)a_1 + ikb_I a_2 - ik\int_{-\infty}^0 [f_1(s) + ig_1(s)]e^{ks}ds + \tag{76}$$
$$+ \rho k(\sinh k)^{-1}\int_0^1 [if_2(s)\sinh k(s-1) - g_2(s)\cosh k(s-1)]ds$$
$$a_3' = ikba_3 + \lambda a_1 - \lambda\int_0^1 g_2 dy \tag{77}$$
$$- k(\sinh k)^{-1}\int_0^1 [if_2(s)\sinh(ks) - g_2(s)\cosh(ks)]ds.$$

We may also write

$$\alpha_2(y) = \frac{a_1'}{\Delta_3(k,\mu)}[(bk^2+\lambda)\sinh k(1-y) - k\cosh k(1-y)] +$$
$$\frac{a_3'}{\Delta_3(k,\mu)}\{[b_I k^2 - k + \lambda(1-\rho)]\sinh ky - \rho k\cosh ky\} + H_2[f_2, g_2](k,y),$$
$$\beta_2(y) = \frac{ia_1'}{\Delta_3(k,\mu)}[k\sinh k(1-y) - (bk^2+\lambda)\cosh k(1-y)]$$
$$\frac{ia_3'}{\Delta_3(k,\mu)}\{[b_I k^2 - k + \lambda(1-\rho)]\cosh ky - \rho k\sinh ky\} + iK_2[f_2, g_2](k,y).$$

This has to be completed by α_{10}, α_{21} which are easily deduced from formulas above, and by

$$Z = \frac{a_1}{ik} + \frac{\alpha_{10}}{ik}.$$

For large $|k|$, k real, we have $\Delta_3(k,\mu) \sim -\frac{1}{2}bb_I e^k k^4$, so we obtain the following estimate

$$|k^{-1}a_1'| + |k^{-1}a_3'| + |A| + |B| \leq c|k|^{-1}\{|a_1| + |a_2| + |a_3| + ||f_1||_{L^1} + \\ + ||g_1||_{L^1} + ||f_2||_{L^1} + ||g_2||_{L^1}\},$$

and using the same estimates as for problem 0, for $\alpha_1, \beta_1, \alpha_{10}$ and explicit expressions of α_2, α_{21} and β_2 it is easy to finally obtain the required estimate (20).

Let us now solve the resolvent equation $(\sigma\mathbb{I} - L_\mu)U = V \in \mathbb{H}$, for σ *real*. Again, formulas (35) hold, where α_{10} and β_{20} are given by the following formulas (for $\sigma \neq 0$)

$$\alpha_{10} = \int_{-\infty}^{0} [g_1(y)\cos\sigma y - f_1(y)\sin\sigma y]dy,$$
$$\beta_{10} = -\int_{-\infty}^{0} [g_1(y)\sin\sigma y + f_1(y)\cos\sigma y]dy,$$
$$\beta_{20} = (\rho\sigma)^{-1}[\lambda(1-\rho)a_1 + \sigma b_I a_2 + \sigma\beta_{10}[\lambda(1-\rho) - b_I\sigma^2].$$

In addition, we have $\sigma Z_I = \alpha_{10} + a_1$ and the *compatibility condition* (giving a continuous linear form on $\mathbb{H}$ after making explicit α_2 and β_2 in function of V)

$$\beta_{21} - b\sigma\alpha_{21} + \lambda Z_I + ba_3 - \lambda\int_0^1 \beta_2 dy = 0, \tag{78}$$

and conditions on f_1 and g_1 to ensure that $\widetilde{H}_1$ and $\widetilde{K}_1 \in L^1(\mathbb{R}^-)$. Now, for $\sigma = 0$, β_{20} and Z_I are given by

$$-\rho\beta_{20} + \lambda(1-\rho)Z_I = -b_I a_2 + \int_{-\infty}^{0} f_1 dy,$$
$$(1-\lambda)\beta_{20} + \lambda Z_I = -ba_3 + \int_0^1 [(1-\lambda) + \lambda s]f_2 dy,$$

provided that $\lambda \neq (1-\rho)^{-1}$ and we have the following compatibility conditions for V

$$a_1 + \int_{-\infty}^{0} g_1(y)dy = 0, \tag{79}$$

$$\int_{-\infty}^{y} f_1(s)ds \text{ and } \int_{-\infty}^{y} g_1(s)ds \in L^1(\mathbb{R}^-). \tag{80}$$

identical to the ones (34) for problem 0.

The study of the resolvent operator near 0 may be made similarly to problem 0. First we define

$$\widetilde{\mathbb{D}} = \mathbb{R}^3 \times \{\widetilde{W}^1\}^2 \times \{W^{1,1}(0,1)\}^2 \cap \{\alpha_{10} = \alpha_{20}, \alpha_{10} = \alpha_1|_{y=0}, \alpha_{21} = \alpha_2|_{y=1}\},$$

where we have again $\mathbb{D} \subset \widetilde{\mathbb{D}}$ with continuous embedding, but $\mathbb{D}$ is not dense in $\widetilde{\mathbb{D}}$. For any $V \in \mathbb{H}$, we can again check that we have, for $\lambda \neq (1-\rho)^{-1}$, the following limits in $\widetilde{\mathbb{D}}$:

$$(ik\mathbb{I} - L_\mu)^{-1}V \underset{k\to 0^\pm}{=} (sgnk)B_\mu V + C_\mu V + O(k||V||), \tag{81}$$

where B_μ and $C_\mu \in \mathcal{L}(\mathbb{H}, \widetilde{\mathbb{D}})$, analogously as for problem 0. We leave to the reader the computation of the operator B_μ which leads to a linear form (continuous on $\mathbb{H}$), proportional to the form occuring in compatibility condition (79), as in problem 0.

Finally, let us give explicitly the linear form $V \longmapsto \zeta^*(V)$ expressing the compatibility condition $\zeta^*(V) = 0$ for solving $(ik_0\mathbb{I} - L_\mu)U = V \in \mathbb{H}$ when ik_0 is a simple eigenvalue of L_μ. It may be expressed as

$$\zeta^*(V) = (k_0 \cosh k_0 - (\lambda + bk_0^2)\sinh k_0)a_1' + \rho k_0 a_3'$$

where a_1' and a_3' are given by (76-77). Here again, we notice that $\zeta^*(SV) = \overline{\zeta^*(V)}$.

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