The Bénard-Rayleigh convection problem. Mathematical aspects

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Contents

Foreword

In 1901, Henri Bénard (1879 - 1934), a young student at the Ecole Normale Supérieure, presented at the Collège de France, his experimental thesis on the organised movement of a liquid heated from below. Well-versed in optical methods, he observed and characterised the fluid's movement, which mainly took the form of hexagonal cells. His results were spectacular, due to their novelty, revealing physical phenomena exhibiting periodicity and symmetries that had previously been the preserve of crystal studies.

In 1916, the theoretical physicist Lord Rayleigh (J. W. Strutt, 1842 - 1919) applied his earlier developed theory of linear stability, to the case of thermal convection movement caused by heating a fluid from below, after becoming aware of Bénard's findings. To do so, Rayleigh took into account Archimedes's buoyancy and viscous resistance to movement. Thus, he managed to calculate the threshold temperature difference required to initiate movement, as well as the size of the periodic structures, for the ideal case of zero friction at the two horizontal walls ("free - free" boundary conditions).

Rayleigh's early description of this phenomenon, which had been discovered, shortly before, by Bénard, led to it being named the "Bénard - Rayleigh problem".

In 1923, Geoffrey Taylor discovered that centrifugal instability in annular flow between two coaxial rotating cylinders, described by Maurice Couette in 1892, exhibits periodic vortex patterns, called the Couette-Taylor rolls.

These two problems became the prototype for studies of hydrodynamic stability.

For many years, after these works had been published, only few theoretical and experimental studies were conducted in this field, primarily by fluid mechanics specialists. However, following Chandrasekhar's 1961 treatise on hydrodynamic stability, an increasing interest became apparent and the number of annual publications, gradually, increased. Over the past 50 years, this subject has become closely associated with significant progress in the study of turbulence, chaos, instabilities, patterns, nonlinearities and bifurcation theory. Current research includes hundreds of new studies that cover broader subfields, such as magnetoconvection, developed turbulence, and geophysical and astrophysical convection.

This evolution has been accompanied by an intense collaboration between researchers from different fields. One of the defining features of this research is the connections it establishes between various branches of engineering, physics, and mathematics.

Physicists later took up the subject, when they identified analogies with out-of-equilibrium phase transitions. This favoured the phenomenological construction of ad hoc models, such as non-linear amplitude equations, which are also obtained, independently, as a result of asymptotic expansions of the Navier - Stokes and Boussinesq equations.

This book presents rigorous mathematical results based on the theoretical foundations of the Bénard-Rayleigh problem. In particular, chapter 1 provides full mathematical and rigorous justification for that amplitude equations (normal forms).

The book focuses on exact results, derived from the equations of the problem and often relies on detailed analyses of situations observed in experiments. This is evident in chapter 3, which explains the symmetric grain boundaries observed experimentally in roll patterns, and in chapter 4 which discusses observations of non-symmetric grain boundaries. Chapter 2 employs the same rigorous mathematical treatment with an in-depth analysis inspired by observations of quasi-patterns.

As a physicist who has conducted convection experiments, I am aware of the relationship between physics and mathematics and recognize how thermal convection patterns can be thoroughly described through mathematical results. These results not only explain observations but also inspire new ones and open new avenues of research.

The work of Gérard Iooss exemplifies this approach. Together with earlier contributions, such as his book on the Couette - Taylor problem co-authored with Pascal Chossat, this volume highlights his lasting influence on the international community studying instabilities. His predictions have inspired, both, theoretical studies and experimental validation.

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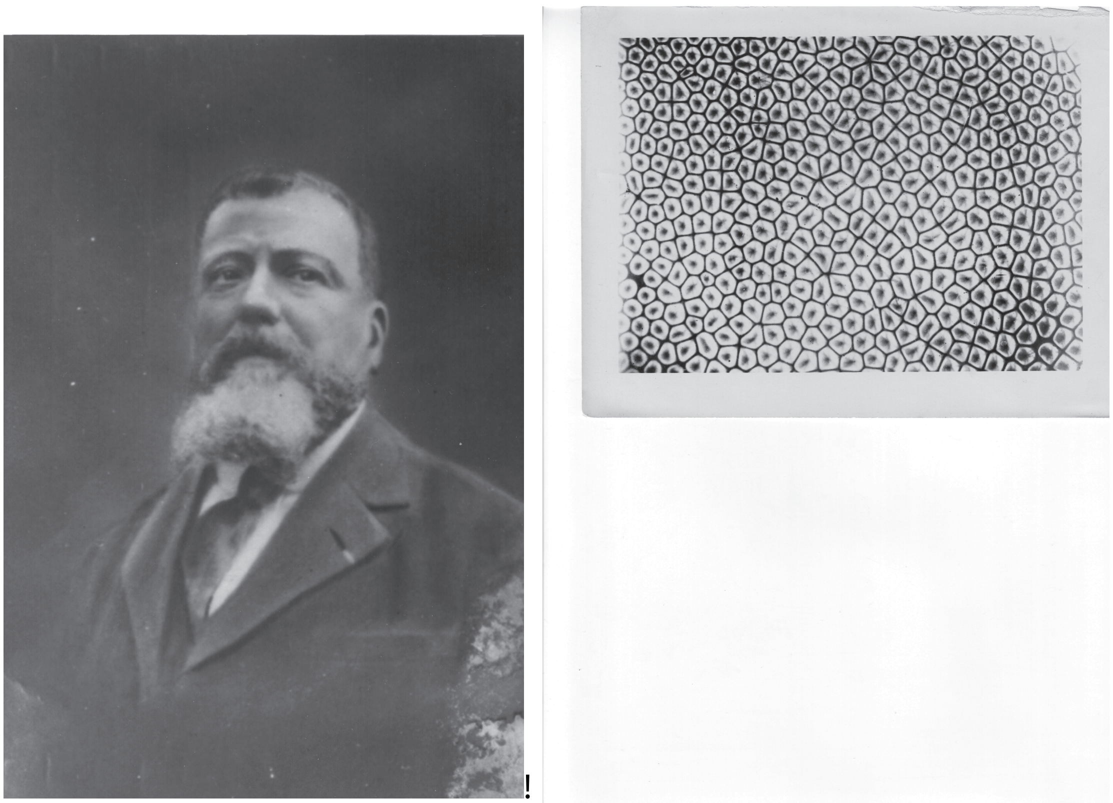

Figure 0.1: Henri Bénard and experiment of Henri Bénard

Introduction

First I acknowledge my adviser Jean Pierre Guiraud who introduced me, during his lectures for graduate studies in 1968, to the Bénard-Rayleigh convection, seen as a bifurcation problem. He directed my PHD thesis (1971). I also warmly acknowledge Mariana Haragus for her collaboration during about 15 years and her constant interest.

This book collects a series of quite recent papers written by the author mainly with Boele Braaksma for Chapter 2, and with Mariana Haragus for Chapters 3, 4 and 5. Chapters 4 and 5 rely on some results obtained via a specific variational method applied on a 4 or 6-dimensional reversible ODE, which method is not presented here. For the details, we refer to the works of van den Berg and van der Vorst [68] used in Chapter 4, and to B.Buffoni et al [12] used in Chapter 5. It should be noticed that at various places in Chapters 1, 4 and 5 we use the reduction methods (center manifold and normal forms) developed in the book [28]. Finally, due to specific technicalities, Chapter 2 is harder to read than next Chapters which are independent of Chapter 2. So (depending on the interest of the reader) it might be wise to start by reading first Chapter 1, and then pass to Chapters 3 and 4 or 5 (independent of Chap.4).

In Chapter one, we present in a modern way the classical Bénard-Rayleigh convection problem between horizontal plane boundaries, inducing convective patterns as rolls and hexagons (see Figure 0.2), steady solutions of a 6-dimensional amplitudes system. We give the results about their stability with respect to perturbations possessing a symmetry built on a periodic lattice invariant under rotations of angle $\pi/3$. An Appendix gives useful details about the computation of bifurcating solutions. These results are based in particular on the book with Mariana Haragus [28] and on [38].

Next Chapters deal with bifurcations of steady convective patterns. Chapters 3-4-5 are independent of Chapter 2. Chapters 4 and 5 use extensively the results of Chapter 3.

Chapter two, based on [7] is devoted to the bifurcation of steady quasipatterns, using the basic formulation presented in Chapter one. This Chapter is quite technical due to the small divisor difficulties.

Chapter three presents the spatial dynamics formulation for steady convection, allowing

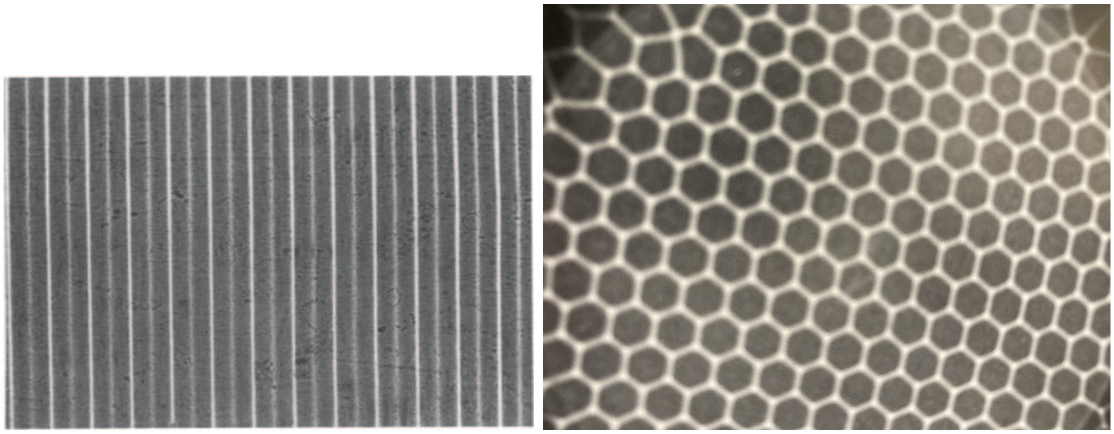

Figure 0.2: Rolls from Joets - Ribotta in [72] p.295, and Hexagons from Koschmieder [51]

to reach solutions with "defects" in domain walls ("grain boundaries" in physical litterature). The structure of the linearized operator is studied in two cases. The first case corresponds to symmetric domain walls where the convective rolls at infinities are symmetric with respect to a mid plane with which they make a certain angle (see Figure 0.3). The second case corresponds to orthogonal domain walls where rolls at infinity on the left side are orthogonal to the mid plane (see Figure 0.4), while rolls at infinity on the right side are parallel to it (partly based on [29] and [12]). Chapter four solves the nonlinear problem

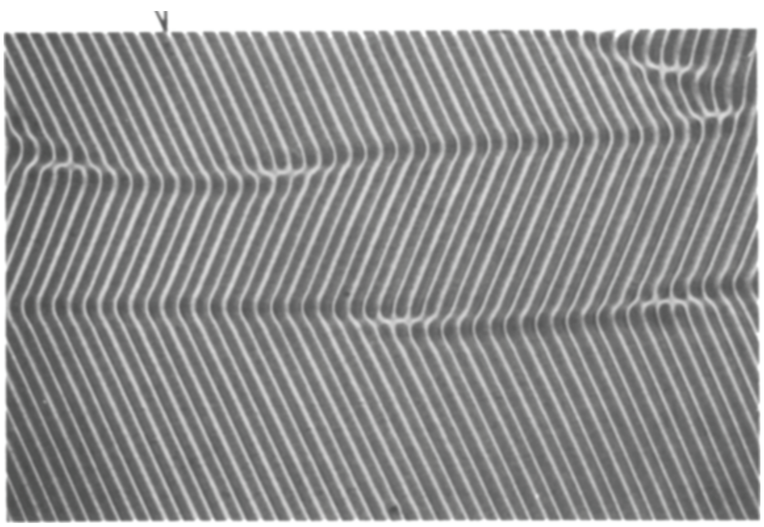

Figure 0.3: Symmetric domain walls, from Joets, Ribotta in [72] p.298

for symmetric domain walls (based on [29] and [30]), while Chapter five solves the case of orthogonal domain walls (based on [12], [39], and [40]).

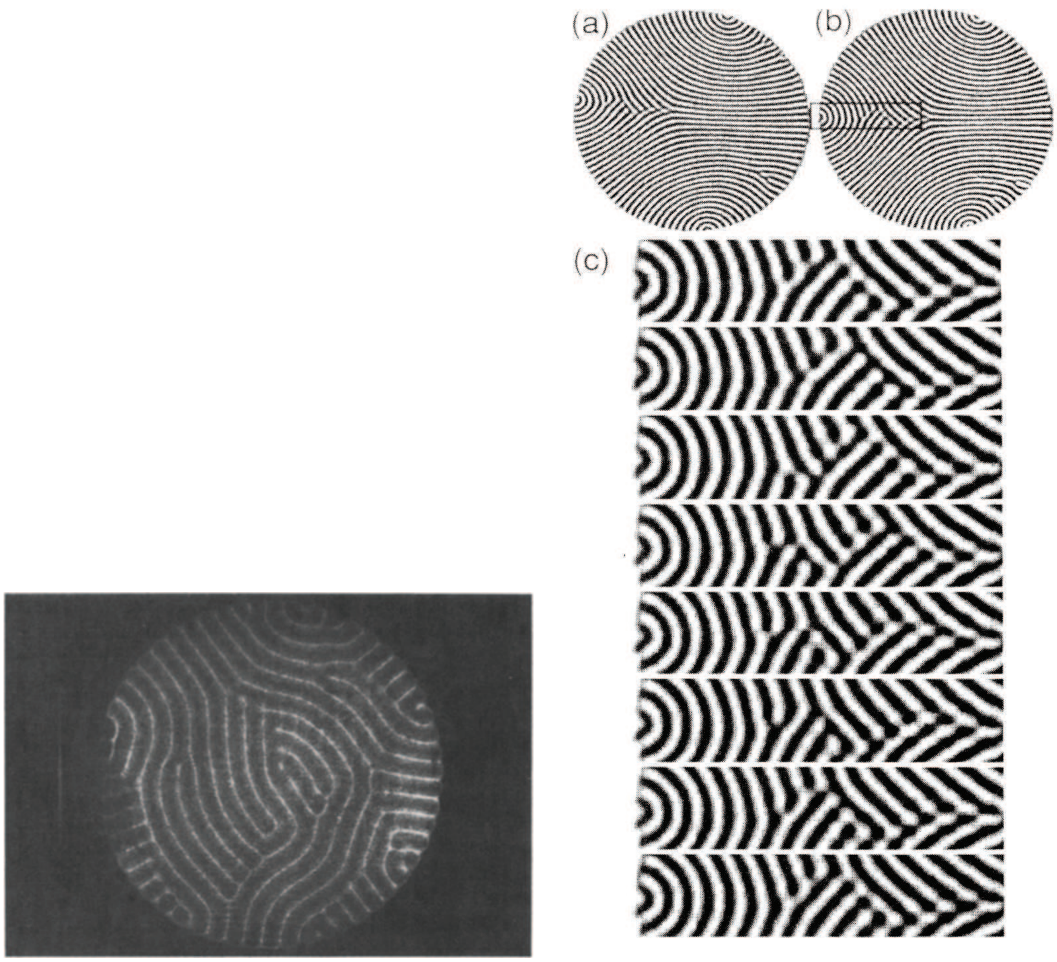

Figure 0.4: Orthogonal domain walls, from Croquette, Pocheau in [72] p.105 and Hu et al [36]

Chapter 1

Classical solutions

1.1 Introduction

In fluid mechanics, the Bénard-Rayleigh convection problem is concerned with the flow of a viscous fluid filling the region between two horizontal planes and heated from below. The governing equations are the Navier-Stokes equations in the Boussinesq approximation completed with an energy conservation equation (see the system (2.4)). Each of the two horizontal planar boundaries may be a rigid plane or a free boundary, hence leading to different possible types of boundary conditions: rigid-rigid, free-free, and free-rigid (see remark (2.1)). Together with these boundary conditions, the equations are invariant under horizontal translations and rotations. In the cases of rigid-rigid and free-free boundary conditions, they have an additional vertical reflection symmetry. In dimensionless variables, the different physical parameters are reduced to two parameters which are the Rayleigh number $\mathcal{R}$ and the Prandtl number $\mathcal{P}$. We refer to [51] for a review on experimental results, starting from the beginning of 20th century with H.Bénard, and [46, Vol. II] for a very complete discussion and bibliography on this problem, and in particular on the various geometries and boundary conditions.

The Bénard-Rayleigh convection is one of the most studied, both analytically and experimentally, and perhaps best underdstood, pattern-forming system. In the hydrodynamic problem, the difference of temperature between the two horizontal planes modifies the fluid density, tending to place the lighter fluid below the heavier one. Having an opposite effect, gravity induces, through the Archimedian force, an instability of the simple "conduction regime" leading to a "convective regime". While the fluid is at rest and the temperature depends linearly on the vertical coordinate in the conduction regime, various steady regular patterns, such as rolls, hexagons, or squares, are formed in the convective regime. The fluid viscosity prevents this instability up to a certain level, and there is a critical value of the temperature difference, below which nothing happens and above which a steady convective regime bifurcates. In dimensionless variables, this bifurcation occurs at a critical value of the Rayleigh number $\mathcal{R}_c$. The numerical value of $\mathcal{R}_c$ depends on the chosen boundary conditions and for the ones mentioned above it has already been computed in the forties by Pellew and Southwell [59]. Starting from the sixties, there has been extensive study of regular convective patterns and numerous mathematical existence results have been obtained. Without being exhaustive, we refer to the first works by Yudovich et al [67],[73],[75],[77], Rabinowitz [60], Görtler et al [26], see also Kirchgässner et al [50], and Sattinger [64], the monograph [51] for further references, and recent works [7] on existence of quasipatterns and [29], [30], [12], [39], [40] on existence of grain boundaries.

1.2 Statement and formulation

Consider a viscous incompressible fluid filling the region between two horizontal planes. The velocity V of fluid particles and the pressure p are functions of $(x,t) \in \Omega \times \mathbb{R}^+$ where $\Omega = \mathbb{R}^2 \times I$, where $I = (0,d)$ is a bounded interval in $\mathbb{R}$, and satisfy the Navier–Stokes equations

$$\begin{aligned} \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} + \frac{1}{\rho}\nabla p &= \nu\Delta\mathbf{V} + f(x), \\ \nabla\cdot\mathbf{V} &= 0. \end{aligned} \tag{2.1}$$

In this system $\mathbf{V}(x,t)$ has three components, the volumic mass ρ is constant, ∇, $\nabla\cdot$, and Δ denote the gradient, divergence and Laplace operators, respectively, ν is the kinematic viscosity, and f represents an external massic force, independent of t. The first equation represents the momentum balance, while the second is the incompressibility condition.

Each planar boundary may be a rigid plane, or a “free” boundary in the sense explained in Remark 2.1 below. In addition, we assume that the lower and upper planes are at temperatures T_0 and T_1, respectively, with $T_0 > T_1$ (see Figure 2.1(i)). The difference of temperature between the two planes modifies the fluid density, tending to place the lighter fluid below the heavier one. The gravity then induces, through the Archimedian force, an instability of the “conduction regime” where the fluid is at rest, while the temperature depends linearly on the vertical coordinate z. This instability is prevented up to a certain level by viscosity, so that there is a critical value of the temperature difference, below which nothing happens and above which a “convective regime” appears.

Remark 2.1 (Free boundaries) *Sometimes the boundary, or part of the boundary, of the domain Ω is “free,” which means that the fluid is in contact with another fluid, the* *common boundary being unknown. Here, we only mention the simplified situation in which one assumes that the part of the boundary $\partial\Omega_1$, say, where the fluid is in contact with another fluid, is fixed. (This is acceptable for instance if the external fluid is mercury and the internal one is water.) Then, on this part of the boundary one has the following conditions, replacing the canceling of velocity field* $\mathbf{V}$*:*

$$\mathbf{V} \cdot n|_{\partial\Omega_1} = 0, \tag{2.2}$$

showing that no fluid crosses the boundary, and

$$(\nabla\mathbf{V} + \nabla^t\mathbf{V}) \cdot n|_{\partial\Omega_1} \times n = 0, \tag{2.3}$$

showing that the tangent stresses cancel.

Remark 2.2 *We do not consider here the cases of convection between two concentric spheres, referring for example to [15],[16].[17],[19],[18].*

The Navier–Stokes system (2.1) is not sufficient to describe this situation. An additional equation for energy conservation is needed, where the internal energy is proportional to temperature. In the Boussinesq approximation, the dependency of the density ρ in function of the temperature T,

$$\rho = \rho_0 \left(1 - \alpha(T - T_0)\right),$$

where α is the volume expansion coefficient, is taken into account in the momentum equation, only in the external volumic gravity force $-\rho g e_z$, introducing the coupling between $(\mathbf{V}, p)$ and T. We refer to [46, Vol. II] for a very complete discussion and bibliography on various geometries and boundary conditions in this problem.

Several different scalings are used in literature. For convenience, and for consistency with next chapters, we modify the classical scaling used in [51], and which consists now in choosing the length, time, velocity, and temperature scales respectively as d, d^2/κ, $\mathcal{R}^{1/2}\kappa/d$, $\mathcal{R}\nu\kappa/\alpha g d^3$, where d is the distance between the planes, κ is the thermal diffusivity, and ν, α, and g are as above. There are two dimensionless numbers in this problem: the Prandtl number $\mathcal{P}$ and the Rayleigh number $\mathcal{R}$ defined respectively as

$$\mathcal{P} = \frac{\nu}{\kappa}, \quad \mathcal{R} = \frac{\alpha g d^3 (T_0 - T_1)}{\nu\kappa}.$$

This leads to the system

$$\begin{aligned} \frac{\partial \mathbf{V}}{\partial t} + \mu\mathbf{V} \cdot \nabla\mathbf{V} + \nabla p &= \mathcal{P}(\mu\theta e_z + \Delta\mathbf{V}) \\ \nabla \cdot \mathbf{V} &= 0 \\ \frac{\partial \theta}{\partial t} + \mu\mathbf{V} \cdot \nabla\theta &= \Delta\theta + \mu V_z, \end{aligned} \tag{2.4}$$

replacing (2.1) and where

$$\mu = \mathcal{R}^{1/2}.$$

Here θ is the deviation of the temperature from the conduction profile, which satisfies the boundary conditions, and $\mathbf{V} = (V_1, V_2, V_z)$, p, and θ are functions of (x,t), $x = (X,z)$, with $X = (x_1, x_2) \in \mathbb{R}^2$ the horizontal coordinates and $z \in (0,1)$ the vertical coordinate, e_z being the unitary ascendent vector. The system (2.4) is completed by the boundary conditions

$$V_z = \theta = 0, \quad z = 0, 1,$$

together with either a "rigid surface" condition

$$V_1 = V_2 = 0, \tag{2.5}$$

or a "free surface" condition

$$\frac{\partial V_1}{\partial z} = \frac{\partial V_2}{\partial z} = 0 \tag{2.6}$$

on the planes $z = 0$ or $z = 1$. Notice that here the kinematic viscosity is independent of the temperature T. If this is not the case, some qualitative results change. Also, adding a solute with a certain concentration, satisfying an equation and boundary conditions of the same form as θ, gives richer results [46, Vol. II].

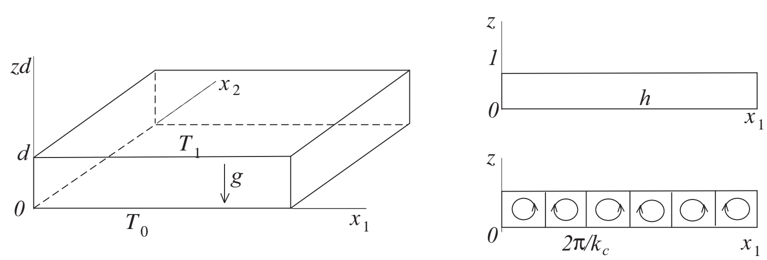

Figure 2.1: (i) Bénard–Rayleigh problem. (ii) Domain of periodicity for bidimensional convection (above) and convection rolls (below).

1.3 Classical bifurcating solutions and their stability

Here we assume a biperiodicity property for (V, θ) :

$$(\mathbf{V}, \theta)(x,t) = (\mathbf{V}, \theta)(x + n_1 e_1 + n_2 e_2, t), \quad \nabla p(x,t) = \nabla p(x + n_1 e_1 + n_2 e_2, t) \tag{3.1}$$

for all $x = (X, z) \in \mathbb{R}^2 \times (0, 1)$, where $(n_1, n_2) \in \mathbb{Z}^2$, and the lattice of periods is generated by two noncolinear vectors e_1 and e_2 in $\mathbb{R}^2$. To these conditions we add two conditions on the flux of the velocity in the directions of two vectors k_1 and k_2 in the X-plane,

$$\int_{\Sigma_1} \mathbf{V} \cdot k_2 \, dS = 0, \quad \int_{\Sigma_2} \mathbf{V} \cdot k_1 \, dS = 0. \tag{3.2}$$

The vectors k_1 and k_2 are such that

$$\langle e_j, k_l \rangle = 2\pi \delta_{jl}, \tag{3.3}$$

where $\delta_{jl} = 0$ for $i \neq j$, and $= 1$ for $i = j$, and Σ_1 (resp., Σ_2) is the face orthogonal to k_2 (resp., to k_1) of the parallelepiped built with vectors e_1, e_2 and the interval $(0, 1)$ orthogonally to the X-plane, which constitutes the domain of periodicity. Below we denote by Γ the lattice spanned by $n_1 e_1 + n_2 e_2, (n_1, n_2) \in \mathbb{Z}^2$, and by Γ' the dual lattice spanned by $n_1 k_1 + n_2 k_2, (n_1, n_2) \in \mathbb{Z}^2$.

Since experimental evidence mostly show convection in rolls and convection in hexagonal cells, we choose a lattice compatible with both patterns, as initiated in [64].

We choose

$$e_1 = h\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad e_2 = h(0, 1), \quad k_1 = k_c(1, 0), \quad k_2 = k_c\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

where h is determined later by the critical wavelength k_c such that

$$hk_c = \frac{4\pi}{\sqrt{3}}.$$

It is not difficult to check that this lattice is invariant under rotations of angle $\pi/3$ (see Figure 3.1(i)).

1.3.1 Formulation as a First Order System

We set $\mathbf{U} = (\mathbf{V}, \theta)$, then the system may be written on the form

$$\frac{d\mathbf{U}}{dt} = \mathbf{L}_\mu \mathbf{U} + \mu \mathbf{R}(\mathbf{U}, \mathbf{U}) \tag{3.4}$$

posed in $\mathcal{X}$ for $\mathbf{U}(\cdot, t) \in \mathcal{Z}$. According to the flux conditions (3.2), we choose the Hilbert spaces

$$\begin{aligned} \mathcal{X} &= \Big\{ \mathbf{U} \in \left(L^2(\mathbb{R}^2/\Gamma \times (0,1))\right)^4 \; ; \; \nabla \cdot \mathbf{V} = 0, \; V_z|_{z=0,1} = 0, \\ \int_{\Sigma_1} \mathbf{V} \cdot k_2 dS &= \int_{\Sigma_2} \mathbf{V} \cdot k_1 dS = 0 \Big\}, \end{aligned}$$

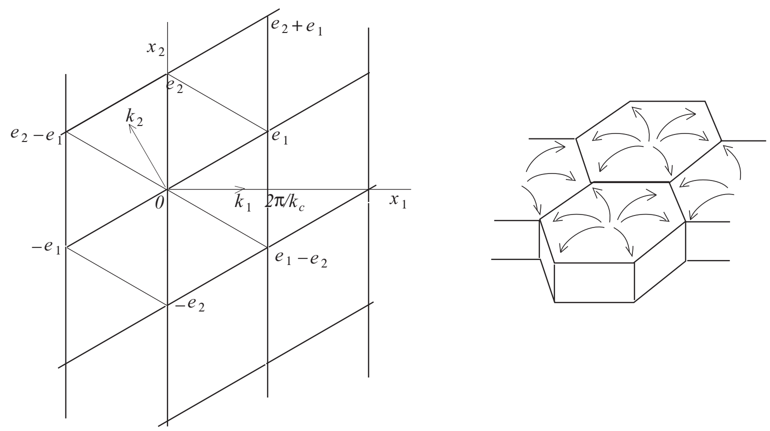

Figure 3.1: (i) Lattice Γ in the X-plane (ii) Convective flow in a hexagonal cell.

and in the case of "rigid-rigid" boundary conditions

$$\begin{aligned} \mathcal{Z}_{(r,r)} &= \Big\{ \mathbf{U} \in \left(H^2(\mathbb{R}^2/\Gamma \times (0,1))\right)^4 \ ; \ \nabla \cdot \mathbf{V} = 0, \ \mathbf{V}|_{z=0,1} = \theta|_{z=0,1} = 0, \\ \int_{\Sigma_1} \mathbf{V} \cdot k_2 dS &= \int_{\Sigma_2} \mathbf{V} \cdot k_1 dS = 0 \Big\}, \end{aligned}$$

and similarly $\mathcal{Z}_{(r,f)}$, $\mathcal{Z}_{(f,r)}$, and $\mathcal{Z}_{(f,f)}$, by replacing the rigid boundary conditions $V_1 = V_2 = 0$ by the free boundary conditions $\partial V_1/\partial z = \partial V_2/\partial z = 0$ on $z = 1$, $z = 0$, and $z = 0, 1$, respectively. Then the system is of the form (3.4). The linear operator $\mathbf{L}_\mu$ and the quadratic map $\mathbf{R}$ are defined as

$$\mathbf{L}_\mu \mathbf{U} = (\mathbf{\Pi}_0 \mathcal{P}(\Delta \mathbf{V} + \mu\theta e_z), \Delta\theta + \mu V_z), \quad \mathbf{R}(\mathbf{U}, \mathbf{U}) = (-\mathbf{\Pi}_0(\mathbf{V} \cdot \nabla \mathbf{V}), -\mathbf{V} \cdot \nabla\theta),$$

with $\mathbf{R} : \mathcal{Z} \to \mathcal{Y} = \mathcal{X} \cap \left(H^1(\mathbb{R}/\Gamma \times (0,1))\right)^4$ quadratic and continuous. Here $\mathcal{Z}$ represents one of the spaces $\mathcal{Z}_{(r,r)}$, $\mathcal{Z}_{(r,f)}$, $\mathcal{Z}_{(f,r)}$, and $\mathcal{Z}_{(f,f)}$ above, depending upon the choice of boundary conditions. A key property of the Hilbert space $\mathcal{X}$ is that the kernel of the orthogonal projection $\mathbf{\Pi}_0$ in $\left(L^2(\Omega)\right)^4$ (where $\Omega = \mathbb{R}/\Gamma \times (0,1)$) on the subspace $\mathcal{X}$ can be identified with the space $\{(\nabla\phi, 0) \ ; \ \phi \in H^1(\Omega)\}$ (e.g., see [76, 53, 66]). Notice that the pressure p is not necessarily periodic and that the orthogonal projection $\mathbf{\Pi}_0$ in $\left(L^2(\mathbb{R}/\Gamma \times (0,1))\right)^4$ on the subspace $\mathcal{X}$ eliminates the periodic gradient ∇p in (2.4).

A specific property of $\mathbf{L}_\mu$ in this case is that the scalar product in the Hilbert space $\mathcal{X}$, with corresponding norm equivalent to the usual one, is such that $\mathbf{L}_\mu$ is *self-adjoint*. This scalar product is defined by [1]

$$\langle \mathbf{U}^{(1)}, \mathbf{U}^{(2)} \rangle = \langle \mathbf{V}^{(1)}, \mathbf{V}^{(2)} \rangle|_{(L^2(\mathbb{R}/\Gamma\times(0,1)))^3} + \mathcal{P}\langle \theta^{(1)}, \theta^{(2)} \rangle|_{L^2(\mathbb{R}/\Gamma\times(0,1))}.$$

As a consequence, the spectrum of $\mathbf{L}_\mu$ is now located on the real axis. Notice that $\mathbf{L}_\mu$ is a relatively compact perturbation of the uncoupled self-adjoint negative operator

$$\mathbf{L}'U = (\mathbf{\Pi}_0\mathcal{P}\Delta\mathbf{V}, \Delta\theta),$$

and that it has a compact resolvent, since its domain is compactly embedded in $\mathcal{X}$ (see [48]). The spectrum of $\mathbf{L}_\mu$ consists then of isolated semisimple real eigenvalues of finite multiplicities, accumulating at $-\infty$, only. Furthermore, there exist positive constants $\omega_0 > 0$, $c > 0$, and $\alpha \in [0,1)$ such that for all $\omega \in \mathbb{R}$, with $|\omega| \geq \omega_0$, we have that $i\omega$ belongs to the resolvent set of $\mathbf{L}_\mu$, and

$$\|(i\omega\mathbb{I} - \mathbf{L}_\mu)^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq \frac{c}{|\omega|}, \tag{3.5}$$

$$\|(i\omega\mathbb{I} - \mathbf{L}_\mu)^{-1}\|_{\mathcal{L}(\mathcal{Y},\mathcal{Z})} \leq \frac{c}{|\omega|^{1-\alpha}} \tag{3.6}$$

(here with $\alpha = 3/4$ (see [37])). Then, the hypotheses required for applying the center manifold theorem as in Chapter 2 of [28] are all satisfied.

1.3.2 Symmetries

This problem is invariant under horizontal translations and under the reflection $x_1 \mapsto -x_1$. Then the system (3.4) possesses a symmetry group represented by $\boldsymbol{\tau}_a$ and $\mathbf{S}$ defined through

$$\begin{aligned}(\boldsymbol{\tau}_a\mathbf{U})(X,z) &= \mathbf{U}(X+a,z), \quad a \in \mathbb{R}^2/\Gamma \\ (\mathbf{S}\mathbf{U})(x_1,x_2,z) &= (-V_1(-x_1,x_2,z), V_2(-x_1,x_2,z), V_z(-x_1,x_2,z), \theta(-x_1,x_2,z)),\end{aligned} \tag{3.7}$$

where $\boldsymbol{\tau}_h = \mathbb{I}$, for $h \in \Gamma$ because of the periodicity assumption. In addition, in the cases of "rigid-rigid" and "free-free" boundary conditions, i.e., with $\mathcal{Z}_{(r,r)}$ and $\mathcal{Z}_{(f,f)}$, respectively, there is the additional symmetry with respect to the half-plane $z = 1/2$,

$$(\mathbf{S}_z\mathbf{U})(X,z) = (V_1(X,1-z), V_2(X,1-z), -V_z(X,1-z), -\theta(X,1-z)). \tag{3.8}$$

The system is also invariant under the representant of the horizontal rotation of angle $2\pi/3$:

$$(\mathbf{R}_{2\pi/3}\mathbf{U})(X,z) = \left(R_{2\pi/3}(\mathbf{V}(R_{-2\pi/3}X,z)), \theta(R_{-2\pi/3}X,z)\right), \tag{3.9}$$

[1] The choice of the scaling we made for obtaining the system (3.4) allows to work with a scalar product in $\mathcal{X}$ independent of $\mathcal{R}$ which is the bifurcation parameter.

where $R_{2\pi/3}$ is the horizontal rotation, in the X-plane, of angle $2\pi/3$. The group generated by $\mathbf{S}$ and $\mathbf{R}_{2\pi/3}$ is denoted by D_6, consisting of rotations on a circle of angle $\pi/3$ together with the symmetries through a diameter. In the cases of "rigid-rigid" and "free-free" boundary conditions, we have in addition the symmetry $\mathbf{S}_z$, defined by (3.8).

All these symmetry operators $\boldsymbol{\tau}_a, \mathbf{S}, \mathbf{R}_{2\pi/3}$ and $\mathbf{S}_z$ when it is relevant, commute with operators $\mathbf{L}_\mu$ and $\mathbf{R}$.

1.3.3 Eigenvalues of $\mathbf{L}_\mu$

Let us look for eigenvalues λ such that

$$\mathbf{L}_\mu \mathbf{U} = \lambda \mathbf{U}, \ \mathbf{U} \in \mathcal{Z}.$$

Classical results may be found in [59]. Periodicity Γ implies that we look for U under the form

$$\mathbf{U} = e^{ik\cdot X}\widehat{U}(z), \ k \in \Gamma' \subset \mathbb{R}^2.$$

Then we find

$$\widehat{U}(z) = (\widehat{V}_\perp(z), \widehat{V_z}(z), \widehat{\theta}(z))$$

$$ik \cdot \widehat{V}_\perp + D\widehat{V_z} = 0,$$

where $V_\perp = (V_1, V_2)$,

$$\begin{aligned} (D^2 - k^2)(D^2 - k^2 - \frac{\lambda}{\mathcal{P}})\widehat{V_z} &= \mu k^2 \widehat{\theta}, \\ (D^2 - k^2 - \lambda)\widehat{\theta} &= -\mu \widehat{V_z}, \end{aligned} \tag{3.10}$$

with the boundary conditions

$$\begin{aligned} \widehat{V_z} &= \widehat{\theta} = 0 \text{ for } z = 0, 1, \\ D^2\widehat{V_z} &= 0 \text{ for a "free" boundary } z = 0 \text{ or } 1, \\ D\widehat{V_z} &= 0 \text{ for a "rigid" boundary } z = 0 \text{ or } 1. \end{aligned} \tag{3.11}$$

The invariance under horizontal rotations of the system implies that the wave vector k only occurs by its length, still denoted by k. Yudovich [73] showed that, for $\lambda = 0$, and for any fixed $k > 0$, there is a countable sequence of parameter values $\mu_0(k) < \mu_1(k) < \mu_2(k) < \ldots$ for which the boundary value problem (3.10)-(3.11) has a unique, up to a multiplicative constant, nontrivial solution $(\widehat{V_z}, \widehat{\theta})$, and that the function $\widehat{V_z}$ is positive for $\mu = \mu_0(k)$. The functions $\mu_j(k)$ are analytic in k and in an analogous case Yudovich [74] showed that they tend to ∞ as k tends to 0 or ∞. Of particular interest for the classical bifurcation problem, and also in our context, is the global minimum of $\mu_0(k)$. Combining analytical arguments and numerical calculations, Pellew and Southwell [59] computed a unique global minimum $\mu_c = \mu_0(k_c)$, for some $k = k_c$, but a complete analytical proof of this property is not available, so far - See Figure 3.2. They also showed that the positive function V is symmetric with respect to $z = 1/2$. A specific property of the free-free boundary value

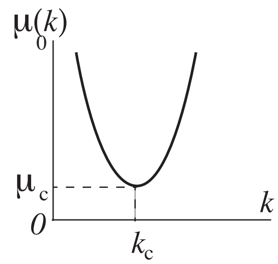

Figure 3.2: Neutral stability curve $\mu_0(k)$.

case, is that we have explicit results for $\widehat{U}(z), \mathcal{R}_c, k_c$. Indeed we obtain in this case (for $\lambda \neq 0$)

$$\widehat{V_z}(z) = \sin n\pi z,\ \ n > 0 \in \mathbb{N},$$

and for $n = 1$

$$(\pi^2 + k^2 + \frac{\lambda}{\mathcal{P}})(\pi^2 + k^2 + \lambda) - \frac{\mu^2 k^2}{(\pi^2 + k^2)} = 0,$$

which is of second degree in λ, with a positive discriminant:

$$(\pi^2 + k^2)^2(1 - \frac{1}{\mathcal{P}})^2 + \frac{4k^2\mu^2}{\mathcal{P}(\pi^2 + k^2)} > 0$$

It then results for all eigenvalues (they are real), that if

$$(\pi^2 + k^2)^2 - \frac{k^2\mu^2}{(\pi^2 + k^2)} > 0$$

(which occurs when $\mathcal{R} = \mu^2$ is small enough) then λ is real < 0. We then need to study the function

$$\mu^2(k^2) = \frac{(\pi^2 + k^2)^3}{k^2},$$

which is convex and reaches its minimum for

$$k_c = \frac{\pi}{\sqrt{2}},$$

with

$$\mu_c = \frac{3\sqrt{3}\pi^2}{2},$$

and tends towards ∞ as $|k| \to 0$ and as $|k| \to \infty$. For other values of n, it is easy to see that the minimum value of μ is larger than μ_c . It results that, in the wave vector plane, for $k \in$ circle of radius k_c the linear operator $\mathbf{L}_{\mu_c}$ possesses a 0 eigenvalue, other eigenvalues staying negative. Fixing the Prandtl number $\mathcal{P}$ and taking μ as a parameter which grows, there is a critical value μ_c for which the largest eigenvalue of $\mathbf{L}_\mu$ crosses the imaginary axis from the left to the right, passing through 0.

In Appendix 6.1.1 we give relations between the neutral stability curve of Figure 3.2 and the eigenvectors given by (3.10,3.11).

1.3.4 Bifurcations - Rolls - Hexagons - Triangles

The eigenvalue 0 is of multiplicity six. The associated eigenvectors are of the form

$$\zeta_j = e^{ik_j \cdot X} \widehat{U}_j(z), \quad j = 1, \dots, 6,$$

and satisfy

$$\zeta_2 = \mathbf{R}_{2\pi/3}\zeta_1, \quad \zeta_3 = \mathbf{R}_{-2\pi/3}\zeta_1, \quad \zeta_{j+3} = \mathbf{S}\zeta_j = \overline{\zeta_j}, \quad j = 1, 2, 3,$$

where

$$k_3 = -(k_1 + k_2), \quad k_{j+3} = -k_j, \quad j = 1, 2, 3.$$

Furthermore

$$\boldsymbol{\tau}_a \zeta_j = e^{ik_j \cdot a}\zeta_j, \quad e^{ik_3 \cdot a} = e^{-i(k_1+k_2)\cdot a},$$

and the action of the symmetry $\mathbf{S}_z$ is either the identity $\mathbb{I}$ or $-\mathbb{I}$, when it is relevant.

Applying the center manifold Theorems 3.23 and 3.13 in Chapter 2 of [28], we find a six-dimensional center manifold. For $\mathbf{U}_0 \in \mathcal{E}_0$, the eigenspace associated to the eigenvalue 0 of $\mathbf{L}_{\mu_c}$, we set

$$\mathbf{U}_0 = A\zeta_1 + B\zeta_2 + C\zeta_3 + \overline{A\zeta_1} + \overline{B\zeta_2} + \overline{C\zeta_3}, \tag{3.12}$$

and then we have the induced symmetries

$$\begin{aligned}
\boldsymbol{\tau}_a(A,B,C) &= (Ae^{ik_1\cdot a}, Be^{ik_2\cdot a}, Ce^{ik_3\cdot a}) \text{ for all } a \in \mathbb{R}^2/\Gamma, \\
\mathbf{S}(A,B,C) &= (\overline{A}, \overline{C}, \overline{B}), \quad \mathbf{R}_{2\pi/3}(A,B,C) = (C,A,B), \\
\mathbf{R}_\pi(A,B,C) &= (\overline{A}, \overline{B}, \overline{C})
\end{aligned}$$

and when $\mathbf{S}_z$ is relevant,

$$\mathbf{S}_z(A,B,C) = \pm(A,B,C).$$

The general form of vector fields commuting with these symmetries is given in Appendix 6.1.3, see also [27, Chap. XIII]. Let us first consider the six-dimensional system truncated at order 3, of the form

$$\begin{aligned}
\frac{dA}{dt} &= a\widetilde{\mu}A + c\overline{BC} + bA|A|^2 + dA(|B|^2 + |C|^2) \\
\frac{dB}{dt} &= a\widetilde{\mu}B + c\overline{CA} + bB|B|^2 + dB(|C|^2 + |A|^2) \\
\frac{dC}{dt} &= a\widetilde{\mu}C + c\overline{AB} + bC|C|^2 + dC(|A|^2 + |B|^2).
\end{aligned} \tag{3.13}$$

Here $\widetilde{\mu} = \mu - \mu_c = \mathcal{R}^{1/2} - \mathcal{R}_c^{1/2}$, $a > 0$ (see Appendix 6.1.4), and other coefficients are all real. The coefficient b is the only one occurring in the two-dimensional case. We can prove that $b < 0$ (see Appendix 6.1.4). In general the presence of quadratic terms changes drastically the stability of the steady solutions of (3.13) (see [27, Chap. XIII]). However in the present case, *a specific property of the Navier–Stokes equation implies that* $c = 0$. This comes from the fact that for any real vector field U in the domain of $\mathbf{L}_\mu$, we have

$$\langle \mathbf{R}(\mathbf{U}, \mathbf{U}), \mathbf{U} \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $(L^2)^4$ (see Appendix 6.1.4). This scalar product arises in the computation of c, with $\mathbf{U} = \mathbf{U}_0$ given by (3.12).

Convections rolls

When $B = C = 0$ we recover the Landau equation for A:

$$\frac{dA}{dt} = a\widetilde{\mu}A + bA|A|^2$$

which gives the circle of steady solutions (the phase of A is arbitrary)

$$a\widetilde{\mu} + b|A|^2 = 0, \quad B = C = 0, \tag{3.14}$$

corresponding to the steady two-dimensional convection rolls (see Figure 2.1 (ii)) defined up to a shift along the x_1 axis. See a detailed expression in Appendix 6.1.2 where (1.10) shows the relationship between the amplitude δ of rolls and the eigenvalue crossing the imaginary axis at $\mu = \mu_0(k)$ when k is different from k_c. These solutions have the period $2\pi/k_c$ in the direction x_1 and, by using the periodicity and the symmetry $\mathbf{S}$, we can show that the velocity is tangent to the surfaces $x_1 = n\pi/k_c, n \in \mathbb{Z}$, meaning that particles of fluid stay inside cells. Moreover, a shift by π/k_c of the structure gives symmetric rolls.

In addition, we have here the solutions obtained through the actions of $\mathbf{R}_{2\pi/3}$ which correspond to convection rolls obtained by $2\pi/3$-rotations of the two-dimensional rolls above, so altogether we have *three "circles" of rolls*. The action of symmetry $\mathbf{S}$ corresponds to a shift by half of the period of each system of rolls.

These rolls are stable in the two-dimensional case, when forgetting B and C. Here, taking into account of B and C, the rolls may be unstable. Indeed, since we have a "circle" of bifurcating solutions, one eigenvalue of the linearized operator is 0, and the other eigenvalues are now $2b|A|^2$, and a quadruple eigenvalue $(d-b)|A|^2$. Consequently, the condition for stability of these rolls is

$$d < b < 0.$$

Convective hexagonal structure

Another class of steady solutions of the system (3.13), with $c = 0$, is

$$A = re^{i\theta_1}, \quad B = re^{i\theta_2}, \quad C = re^{i\theta_3},$$

where $r > 0$ satisfies

$$a\widetilde{\mu} + (b + 2d)r^2 = 0, \tag{3.15}$$

and the phases θ_j are arbitrary. For $\theta_j = 0$, this solution is invariant under the actions of $\mathbf{R}_{2\pi/3}$ and $\mathbf{S}$, and corresponds to *hexagonal convection cells* (see Figure 3.1(ii)). All these solutions have the period $2\pi/k_c$ in the direction x_1 and, by using the periodicity and the symmetry $\mathbf{S}$, we can show that the velocity is tangent to the surfaces $x_1 = n\pi/k_c, n \in \mathbb{Z}$. Hence, by the D_6 rotational invariance, the velocity field is tangent to all the vertical planes deduced from this family, by rotations of angles $\pi/3$ and $2\pi/3$. This means that the fluid particles are confined in vertical triangular prisms, and a basic hexagonal prism for the pattern is formed with six of these triangular prisms (see more details in [27, Chap. XIII]). The linearized operator at these hexagonal convection cells has a triple eigenvalue 0, a simple eigenvalue $2(b+2d)r^2$, and a double eigenvalue $2(b-d)r^2$. This latter eigenvalue implies that the hexagonal convection cells and the convection rolls cannot be both stable. In the case of "rigid-rigid" boundary conditions it is shown in [75] that $b+2d < 0$. Actually, the result in [75] shows that hexagonal cells are stable under perturbations with hexagonal symmetry, in which case only the simple eigenvalue $2(b+2d)r^2$ is present. For our 6-dimensional system we need to study more precisely the steady solutions of the full (not truncated system) since the invariance of the system (3.4) under translations of the (x_1, x_2) plane, implies only a double 0 eigenvalue.

Remark 3.1 *Let us point out that if $c \neq 0$ in system (3.13), then the phases of the steady solutions above loose one degree of freedom, and the bifurcation is two-sided. In particular,* *the hexagonal cells are then unstable [64, 27], but this might only apply to a different physical situation, since here $c = 0$.*

In the absence of the symmetry $\mathbf{S}_z$ we need to include the fourth order terms in (3.13), in order to avoid the occurrence of a three-parameter family of hexagonal cells. The 6-dimensional system, truncated at order 4 becomes

$$\begin{aligned}
\frac{dA}{dt} &= a\widetilde{\mu}A + bA|A|^2 + (dA + g\overline{BC})(|B|^2 + |C|^2) + eA^2BC + f|A|^2\overline{BC} \\
\frac{dB}{dt} &= a\widetilde{\mu}B + bB|B|^2 + (dB + g\overline{AC})(|C|^2 + |A|^2) + eB^2AC + f|B|^2\overline{AC} \\
\frac{dC}{dt} &= a\widetilde{\mu}C + bC|C|^2 + (dC + g\overline{AB})(|A|^2 + |B|^2) + eC^2AB + f|C|^2\overline{AB}.
\end{aligned} \tag{3.16}$$

Setting

$$A = r_1 e^{i\theta_1}, B = r_2 e^{i\theta_2}, C = r_3 e^{i\theta_3},\ \theta_1 + \theta_2 + \theta_3 = \Theta,$$

leads to

$$\begin{aligned}
\frac{dr_1}{dt} &= r_1[a\widetilde{\mu} + br_1^2 + d(r_2^2 + r_3^2)] + (e+f)r_1^2 r_2 r_3 \cos\Theta + \\
&\quad + r_2 r_3(r_2^2 + r_3^2) g \cos\Theta,
\end{aligned}$$

$$\begin{aligned}
\frac{dr_2}{dt} &= r_2[a\mu + br_2^2 + d(r_1^2 + r_3^2)] + (e+f)r_1 r_2^2 r_3 \cos\Theta + \\
&\quad + r_1 r_3(r_1^2 + r_3^2) g \cos\Theta,
\end{aligned}$$

$$\begin{aligned}
\frac{dr_3}{dt} &= r_3[a\widetilde{\mu} + br_3^2 + d(r_1^2 + r_2^2)] + (e+f)r_1 r_2 r_3^2 \cos\Theta + \\
&\quad + r_2 r_1(r_2^2 + r_1^2) g \cos\Theta,
\end{aligned}$$

$$\begin{aligned}
\frac{d\theta_1}{dt} &= (e-f)r_1 r_2 r_3 \sin\Theta - \frac{r_2 r_3}{r_1}(r_2^2 + r_3^2) g \sin\Theta, \\
\frac{d\theta_2}{dt} &= (e-f)r_1 r_2 r_3 \sin\Theta - \frac{r_1 r_3}{r_2}(r_1^2 + r_3^2) g \sin\Theta, \\
\frac{d\theta_3}{dt} &= (e-f)r_1 r_2 r_3 \sin\Theta - \frac{r_2 r_1}{r_3}(r_2^2 + r_1^2) g \sin\Theta.
\end{aligned}$$

We observe that it is a 4-dimensional system in (r_1, r_2, r_3, Θ), and two phases are arbitrary (because of the action of $\boldsymbol{\tau}_a$). Equilibrium solutions (valid at any orders) correspond to

$$r_1 = r_2 = r_3 \stackrel{def}{=} \delta \neq 0,\ \Theta = k\pi, \tag{3.17}$$

which leads to

$$a\widetilde{\mu} + (b + 2d)\delta^2 + (-1)^k \delta^3 (e + f + 2g) = 0$$

giving two slightly different hexagonal convective patterns, depending on $k = 0$ or 1. Both solutions still depend on two parameters (θ_1, θ_2 for example) corresponding to horizontal shifts. The study of the linearized operator at the solutions (3.17) give a double 0 eigenvalue, as expected and

$$\begin{aligned}
&2(b+2d)\delta^2 \text{ simple}\\
&2(b-d)\delta^2 \text{ double}\\
&3\delta^3[(-1)^k e - f - 2g] \text{ simple.}
\end{aligned}$$

The corresponding hexagonal convective cells are stable only if these 4 eigenvalues are < 0.

In addition to the above calculations we need to prove that the full (not truncated) system possesses steady solutions with the above principal parts (3.14), and (3.17). This results from a standard application of the implicit function theorem[2], due to the smoothness and the symmetries of the vector field, as shown in [38].

It appears that the symmetry $\mathbf{S}_z$ acts as $-\mathbb{I}$ on $\mathcal{E}_0$ in the case of the "free-free" boundary conditions, because of a factor $\sin(\pi z)$ in the components V_z and θ, and of a factor $\cos(\pi z)$ in the components $V_\perp = (V_1, V_2)$ of $\widehat{U}_j(z)$, in the formula of the eigenvector ζ_j. It is a priori not automatic, but it is shown numerically that it is also the case for "rigid-rigid" boundary conditions, since for $\mathcal{R} = \mathcal{R}_c$ the components V_z and θ in $\widehat{U}_j(z)$ are invariant under the symmetry $z \mapsto 1 - z$ (see [14]). With such a symmetry, the vector field in (3.13) is odd in $(A, B, C, \overline{A}, \overline{B}, \overline{C})$, so that there are no terms of even orders. Consequently, one has to consider the fifth order terms in order to solve the degenerescence and find all steady solutions. For further details we refer to [27, Chap. XIII], where the problem is treated using the Lyapunov–Schmidt method. These results can be obtained with the present approach, as seen in [38] (see the Remark 1.1 in Appendix 6.1.3). It is shown that there are four types of steady solutions: *rolls, hexagons, regular triangles*, and *patchwork quilts* (correspond to $C = 0, r_1 = r_2 = \delta$ which is a new steady solution in this case), which all may be stable, depending on the coefficients, but not simultaneously. This confirms the prediction in [50], though only the first two types of solutions are usually observed. Moreover it is shown in [38] that in the free-free case where explicit computations may be done, only rolls or triangles may be stable, depending on the Prandtl number $\mathcal{P}$.

Remark 3.2 *It should be noticed that all steady solutions found here may be computed in replacing* $\widetilde{\mu} = \mu - \mu_c$ *by*

$$\widetilde{\mu} = \mu - \mu_0(k) \overset{def}{=} \mathcal{R}^{1/2} - \mathcal{R}_0(k)^{1/2},$$

[2]Each time we refer to implicit function theorem in this book, we may send the reader to the book [23] Section X.2 as well in the C^k case as in the analytic case.

since we only use that the largest eigenvalue of $\mathbf{L}_\mu$ *crosses the imaginary axis through* 0 *when* $\mu = \mu_0(k)$*. This implies an adaptation of the periodic lattice where* $2\pi/k$ *replaces* $2\pi/k_c$*. See Appendix 6.1.1 for a detailed expression of this eigenvalue* $\lambda(\mu, k)$*.*

Let us sum up the results above in the following

Proposition 3.3 *Criticality being defined by* $\mathcal{R}_c$ *and* k_c *we look for bifurcating solutions of the N-S-B system (2.4), in a horizontal bi-periodic frame, as defined in section 1.3. For* $\mathcal{R}$ *close to* $\mathcal{R}_c$*, the dynamical system reduces to a six-dimensional system of 3 complex amplitudes* A, B, C *with principal part (3.13) where coefficient* $c = 0$*. All coefficients are functions of the Prandtl number and* $a > 0, b < 0$*.*

Convective rolls (2-dimensional) of wave number k *close to* k_c *bifurcate for* $\mathcal{R} > \mathcal{R}_0$*, with amplitude of order* $\sqrt{\mathcal{R} - \mathcal{R}_0(k)}$*, and are stable provided that* $d < b < 0$*.*

Two types of convective hexagonal structures of wave number k_c*, of amplitude of order* $\sqrt{\mathcal{R} - \mathcal{R}_c}$ *bifurcate for* $b+2d < 0$*. The difference between these two solutions is of high order and originates from the 4th order coefficients in (3.16), or from the 5th order coefficients if symmetry* S_z *acts non trivially. They are stable if* $b - d < 0$ *and if another linear condition on higher order coefficients holds.*

Tridimensional Convection in an Elongated Cylindrical Domain

Finally, we briefly discuss the case of a long horizontal cylindrical container, with rectangular section in the (x_2, z)-plane, and small sides compared to the length of the cylinder along the x_1-axis. Physically, to satisfy the a priori periodicity in x_1 which we impose to the solutions, it might be convenient to take a thin ring-shaped container (a torus) having a radius large with respect to the sides of the rectangular meridian section. This problem possesses an $O(2)$ symmetry, and it turns out to be similar to the case of two-dimensional convection [46, Vol. II]. The same approach as above can be used, showing the existence of only one "circle" of *stable convection rolls*, bifurcating for $\mu > \mu_c$, which are periodic in x_1, the cells being parallel to the x_2-axis.

Second Bifurcation

Let be interested here in the next bifurcation, when μ crosses a second critical value μ_2, at which the stable convection rolls for $\mu > \mu_c$ become unstable.

The "circle" of convection rolls is given by $\boldsymbol{\tau}_a \mathbf{U}_*$, $a \in \mathbb{R}$, where $\mathbf{U}_*$ is a symmetric solution, $\mathbf{S}\mathbf{U}_* = \mathbf{U}_*$. Notice that there are two such symmetric solutions on the "circle," (with $a = 0$ and $a = \pi/k_c$) and that all these solutions are of class $\mathcal{C}^\infty$. The generator of the group $(\boldsymbol{\tau}_a)_{a\in\mathbb{R}}$ is the derivative $\partial_{x_1} \in \mathcal{L}(\mathcal{Z},\mathcal{Y})$, and then $\partial_{x_1}\mathbf{U}_*$, (called the Goldstone mode), satisfies

$$\partial_{x_1}\mathbf{U}_* \in \mathcal{Z}, \quad (\mathbf{L}_\mu + 2\mu\mathbf{R}(\mathbf{U}_*,\cdot))(\partial_{x_1}\mathbf{U}_*) = 0, \quad \mathbf{S}(\partial_{x_1}\mathbf{U}_*) = -\partial_{x_1}\mathbf{U}_*.$$

In particular, this shows that the operator $\mathbf{L}_\mu + 2\mu\mathbf{R}(\mathbf{U}_*,\cdot)$ has an eigenvalue 0 with eigenvector $\partial_{x_1}\mathbf{U}_*$.

Following the method of construction of center manifolds near a line of equilibria as in Section 2.3.3 of Chapter 2 in [28], we consider the new coordinates (α, V) defined through

$$\mathbf{U} = \boldsymbol{\tau}_\alpha(\mathbf{U}_* + V), \quad \langle V, \partial_{x_1}\mathbf{U}_*\rangle = 0,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $(L^2)^4$. Then (3.4) becomes

$$\begin{aligned}
\boldsymbol{\tau}_\alpha(\partial_{x_1}\mathbf{U}_* + \partial_{x_1}V)\frac{d\alpha}{dt} + \boldsymbol{\tau}_\alpha\frac{dV}{dt} &= \mathbf{L}_\mu\boldsymbol{\tau}_\alpha V + \mu\mathbf{R}(\boldsymbol{\tau}_\alpha(\mathbf{U}_*+V), \boldsymbol{\tau}_\alpha(\mathbf{U}_*+V)) \\
&\quad -\mu\mathbf{R}(\boldsymbol{\tau}_\alpha\mathbf{U}_*, \boldsymbol{\tau}_\alpha\mathbf{U}_*) \\
&= \mathbf{A}_\mu\boldsymbol{\tau}_\alpha V + \mu\mathbf{R}(\boldsymbol{\tau}_\alpha V, \boldsymbol{\tau}_\alpha V),
\end{aligned}$$

where

$$\mathbf{A}_\mu = \mathbf{L}_\mu + 2\mu\mathbf{R}(\mathbf{U}_*,\cdot).$$

Factoring out the translation operator $\boldsymbol{\tau}_\alpha$ leads to

$$(\partial_{x_1}\mathbf{U}_* + \partial_{x_1}V)\frac{d\alpha}{dt} + \frac{dV}{dt} = \mathbf{A}_\mu V + \mu\mathbf{R}(V,V).$$

Now defining by Π the orthogonal projection in $\mathcal{X}$ on $\{\partial_{x_1}\mathbf{U}_*\}^\perp$, we split the system above in two equations

$$\frac{d\alpha}{dt} = g_\mu(V), \tag{3.18}$$

with

$$g_\mu(V) = [\langle\partial_{x_1}U_*, \partial_{x_1}U_*\rangle + \langle\partial_{x_1}V, \partial_{x_1}U_*\rangle]^{-1}\langle\mathbf{A}_\mu V + \mu\mathbf{R}(V,V), \partial_{x_1}U_*\rangle,$$

and

$$\frac{dv}{dt} = \Pi(\mathbf{A}_\mu V + \mu\mathbf{R}(V,V)) - g_\mu(V)\Pi(\partial_{x_1}V). \tag{3.19}$$

The linear operator defined by

$$\Pi\mathbf{A}_\mu = \Pi[\mathbf{L}_\mu + 2\mu\mathbf{R}(\mathbf{U}_*,\cdot)]$$

as $\mathbf{L}'$ in section 2.3.3 in Chapter 2 of [28] acting on V, which commutes with $\mathbf{S}$ due to the choice of U_*, has a simple eigenvalue crossing the imaginary axis through 0, when μ crosses μ_2. Then there exists an eigenvector ξ_0 in $\mathcal{Z}$ such that

$$\Pi\mathbf{A}_{\mu_2}\xi_0 = 0,$$

which implies that, with a suitable scaling of ξ_0, we have

$$\mathbf{A}_{\mu_2}\xi_0 = \partial_{x_1}\mathbf{U}_*,$$

with (because of the antisymmetry of $\partial_{x_1}\mathbf{U}_*$ and the commutativity of $\mathbf{S}$ with $\mathbf{A}_\mu$)

$$\mathbf{S}\xi_0 = -\xi_0.$$

Applying again the center manifold Theorems 3.19 and 3.13 in Chapter 2 of [28], we conclude that a *pitchfork bifurcation* occurs in the equation for V when $\mu = \mu_2$ (see also the general study of the ten possible solutions generically bifurcating from a one-dimensional periodic pattern in [21]). Since $\alpha(t)$ has a small constant derivative (see (3.18)), the bifurcating solutions are *traveling waves with speeds close to* 0, which arise in pairs exchanged by the symmetry $\mathbf{S}$, i.e., traveling in opposite directions. This type of flow is indeed observed in experiments [1] (see also [20, p. 102], for an analogue for the Couette–Taylor problem).

Chapter 2

Quasipatterns

We extend the results obtained in Chapter 1 for bifurcating periodic patterns. Existence of bifurcating quasipatterns in the steady Bénard-Rayleigh convection problem is proved. These are two-dimensional patterns, quasiperiodic in any horizontal direction, invariant under horizontal rotations of angle π/q. There is a small divisor problem for $q \geq 4$. This Chapter is based on the paper [7]. These results may be extended as in [45] to quasipatterns resulting from the "superposition" of two convective hexagonal patterns.

Using the results of Berti-Bolle-Procesi in 2010, we adapt the method to a Navier-Stokes-Boussinesq system ruling the Bénard-Rayleigh convection problem. Our solution is approximated by the truncated power series which is formally obtained by Iooss in [43], but which is divergent in general (Gevrey series).

First, we formulate the problem in introducing a suitable parameter, able to move the spectrum of the linearized operator, as a whole, as for the Swift-Hohenberg PDE model (see [6]). For using the Nash-Moser process, we are faced with the problem of inverting a linear operator which is the differential at a non zero point.

There are two specific difficulties :

i) first, the extra dimension leading to a more complicated spectrum of the linear operator. This first difficulty leads to use specific projections for reducing the spectrum of the studied operator, which we want to invert, to a finite set very close to 0.

ii) The second difficulty is the fact that the linearization $L^{(N)}$ at a non zero point leads to a non selfadjoint operator, contrary to what occurs in previous works. This is more serious, and leads to use the spectrum of $L^{(N)}L^{(N)*}$ which depends mainly quadratically on the main parameter.

A careful study of the "bad set" of parameters, with an assumption on the convexity of the eigenvalues of this operator, allows to obtain a good estimate, as it is necessary for using the results of Berti et al for solving "the range equation". We use again separation properties

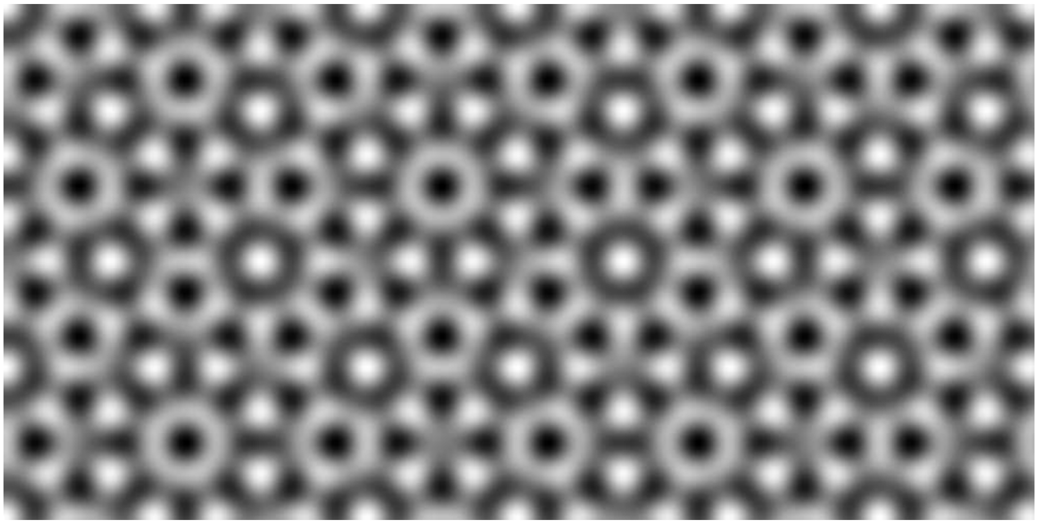

Figure 1.1: Example 8-fold quasipattern. This is an approximate solution of the Swift–Hohenberg equation, see [44].

of the Fourier spectrum (see Bourgain and Craig results) for obtaining an estimate in high Sobolev norms.
It then remains to solve the one-dimensional "bifurcation equation". For any $q \geq 4$ and provided that a weak transversality conjecture is realized, we prove the existence of a bifurcating convective quasipattern of order $2q$, above the critical Rayleigh number.

2.1 Introduction

We are studying the same problem as in Chapter 1, but looking for a different type of steady convective patterns. Quasipatterns are two-dimensional patterns which have no translation symmetries and are quasiperiodic in any spatial direction (see figure 1.1). The spatial Fourier transforms of quasipatterns have discrete rotational order (most often, 8, 10 or 12-fold) and were first discovered in nonlinear pattern-forming systems in the Faraday wave experiment [9, 11], in which a layer of fluid is subjected to vertical oscillation. Since their discovery, they have also been in particular observed, in shaken convection [70, 61].

In many of these experiments, the domain is large compared to the size of the pattern, and the boundaries appear to have little effect. Furthermore, the pattern is usually formed in two directions (x_1 and x_2), while the third direction (z) plays little role. Mathematical models of the experiments are therefore often posed with two unbounded directions, and the basic symmetry of the problem is the Euclidean group of rotations, translations and reflections of the (x_1, x_2) plane. This is in particular the case for the studies made in the works [62], [63], [44] and [6].

Quasipatterns do not fit into any spatially periodic domain and have Fourier expansions with wavevectors that live on a *quasilattice* (defined below). At the onset of pattern formation, the primary modes have zero growth rate, and there are other modes on the quasilattice which have negative growth rates arbitrarily close to zero, and techniques (like Lyapunov-Schmidt reduction, or center manifold reduction) which are used for periodic patterns cannot be applied. These small growth rates appear as *small divisors*, as seen below.

This Chapter is in the spirit on the paper [6] dealing with the Swift-Hohenberg PDE. It is known that this PDE is a simple model of Bénard-Rayleigh convection for the bifurcation to a steady spatially periodic convective regime. Here we solve the same problem but ruled by the full Navier-Stokes-Boussinesq equations (2.4) in Chapter 1, which are usually taken for the study of Bénard-Rayleigh convection between two horizontal planes. This problem was studied in [43], where Gevrey estimates are given for the formal series solution of the problem. Summing this series by an incomplete Borel resummation, provides a solution of our problem, *only up to an exponentially small term* (as the Rayleigh number tends towards its critical value).

In this Chapter, we first define the functional setting in sections 2.2, 2.3 and 2.4 for our unknown u. In section 2.5 we formulate the problem in suitable form. In section 2.6 we study in details the linearized operator $\mathcal{A}$, and the criticality conditions. This determines the critical value λ_0 of the vbifurcation parameter λ, linked to the Rayleigh number by $\lambda = \mathcal{R}^{-1/2}$, and the critical wave number k_c. We then give the formal series for (u, λ) in powers of the amplitude ε of the bifurcating solution. We use the truncated series as the center of the neighborhood where one applies later the Nash-Moser process. Section 2.7 reformulates the problem for adapting it to the method used in [5] and [6] which exploits the fact that the parameter $\mu = \lambda_0 - \lambda$ appears[1] in a way which moves the spectrum of the linearized operator, as a whole This introduces finally parameters ε, μ', where μ' is a scaling of μ (see (7.1)). We are now faced with new difficulties: the problem is no longer in 2 dimensions, since we have now the vertical coordinate z introducing a dependency of Fourier coefficients in z. This leads to an infinite dimensional system, even when we truncate the Fourier modes at a finite number N (as in [6]). This needs the use of a new projection, complicating the operator to be inverted (see section 2.7.4, Lemma 7.2 and section 2.7.6).

[1]In this Chapter μ is different from the one defined in Chapter 1

The second specific difficulty is that the linear operator which we have to invert in the Nash-Moser process is *no longer selfadjoint*. This serious complication is treated in section 2.8. In particular, this needs the use of singular values of the truncated operator, instead of its eigenvalues as in [6]. The square of these singular values mainly behave quadratically in the parameter. We need an assumption on the convexity of these singular values for being able to bound suitably the "bad set" of parameters and obtain directly a good estimate for the inverse of the linearized operator in the basic space with small Sobolev norm (denoted $\mathcal{K}_{0,s_0}$). We then need to use separation properties of the eigenvalues $\lambda_0(|\mathbf{k}|^2)$ of the unperturbed operator, near λ_0, where the wave vectors $\mathbf{k}$ of the Fourier modes, are restricted to $N_{\mathbf{k}} \leq N$ ($N_{\mathbf{k}}$ is the $\mathbb{Z}^d$ norm in the quasilattice). This tool, introduced by Bourgain [8] and Craig [10], was already used on simpler systems in [4] and [6] and is necessary for obtaining good estimates in high Sobolev norms.

We show in section 2.9 that we can adapt the method developed in [5] by Berti, Bolle, Procesi.

The existence of bifurcating convective quasipatterns, is proved in section 2.10 . It results from the non empty intersection of a curve (H) defined by the bifurcation equation in the plane of parameters (ε, μ), and the complement of the "bad set" of parameters. This needs a transversality assumption depending on q.

We sum up our result in the following

Theorem 1.1 *Let $q \geq 4$ be an integer and let $d \leq q$ be the dimension of the $\mathbb{Q}$-vector space spanned by the wave vectors $\mathbf{k}_j$, $j = 1, ..., 2q$ in $\mathbb{R}^2$ equally spaced on a circle centered at the origin (see the definition (3.1)). Assume that the neutral stability curve $\mathcal{R}(|\mathbf{k}|^2)$ leading to the critical Rayleigh number $\mathcal{R}_c = 1/\lambda_0^2$ for $|\mathbf{k}| = k_c$, has a unique minimum, and such that $\mathcal{R}''(k_c^2) > 0$ (see Figure 6.1 and Condition 6.2). We assume a convexity condition 8.7 and we assume that transversality Conjecture 10.2 is verified. Then, there exists $s_0 > d/2$, $\varepsilon_0 > 0$, such that, for $\varepsilon < \varepsilon_0$, there exists a 1-dimensional set $\overline{\Lambda}_\varepsilon$ centered on μ_4, with the following property: for any $\varepsilon < \varepsilon_0$, belonging to a set, of asymptotically full measure as $\varepsilon \to 0$, there exists $\overline{\mu}_\varepsilon \in \overline{\Lambda}_\varepsilon$ such that the steady Bénard - Rayleigh system (5.17) admits a quasipattern solution $(u(\varepsilon), \lambda(\varepsilon))$, C^1 in the parameter ε, $u(\varepsilon) \in \mathcal{K}_{0,s_0}$ (see Definition 5.1), invariant under rotations of angle π/q, of the form*

$$
\begin{aligned}
u &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4 + \mathcal{O}(\varepsilon^5), \\
\lambda &= \lambda_0 - \mu_2 \varepsilon^2 - \mu_3 \varepsilon^3 - \varepsilon^4 \overline{\mu}_\varepsilon
\end{aligned}
$$

where $\mu_2 > 0$, $\overline{\mu}_\varepsilon = \mu_4 + O(\varepsilon)$. The quasiperiodic function u_1 spans the kernel of $\lambda_0 - \mathcal{A}$, and coefficients μ_j, u_j occurring in formulae above, are the ones defined in the truncated asymptotic expansion of the solution (see section 2.6.3).

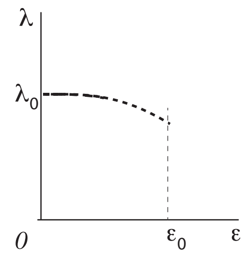

Figure 1.2: Bifurcation curve. The set of "good" ε's is of asymptotically full measure

Remark 1.2 *Condition 6.2 is "generic" and can be checked numerically, while Transversality Conjecture 10.2 depends on q. This one is hard to check but maybe weakened as indicated in Remarks 10.4 and 10.5. This is then probably valid for all q. Notice that for any $s_0' > s_0$, the result of the Theorem above is still valid, maybe for a smaller ε_0.*

Remark 1.3 *Hypothesis 8.7 is used for bounding the measure of the bad set of parameters. The quadratic dependence on $\tilde{\mu} = \mathcal{O}(\varepsilon^4)$ of the truncated selfadjoint linear operator, needs to control the convexity of its eigenvalues, while we have no means to provide a reasonable bound for their second derivative. This is an open question here.*

Remark 1.4 *The expression that we obtain for the bifurcating set, solution of (5.17), is under parametric form. The bifurcating set (u, λ) lies on a C^1 curve. At Figure 1.2, we sketch the projection of this curve in the (ε, λ) plane.*

Remark 1.5 *It should be noticed that the results presented in this Chapter may be extended as in [45] for obtaining quasipatterns resulting from the "superposition" of two hexagonal convective patterns making a certain angle between themselves.*

2.2 The Bénard - Rayleigh convection problem

We use the formulation obtained with (2.4) in Chapter 1

$$\begin{aligned} V \cdot \nabla V + \nabla p &= \mathcal{P}(\theta e_z + \mathcal{R}^{-1/2} \Delta V), \\ V \cdot \nabla \theta &= \mathcal{R}^{-1/2} \Delta \theta + V \cdot e_z, \\ \nabla \cdot V &= 0. \end{aligned} \tag{2.1}$$

Here $V = (V^{(H)}, v^{(z)})$, $V^{(H)} = (v_1, v_2)$, p, and θ are functions of $X = (\mathbf{x}, z)$, with $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ the horizontal coordinates and $z \in (0, 1)$ the vertical coordinate, e_z being the unitary ascendent vector. The system (2.1) is completed by the boundary conditions

$$v^{(z)} = \theta = 0, \quad z = 0, 1,$$

together with either a "rigid surface" condition

$$v_1 = v_2 = 0, \tag{2.2}$$

or a "free surface" condition (in fact no tangential stress condition)

$$\frac{\partial v_1}{\partial z} = \frac{\partial v_2}{\partial z} = 0, \tag{2.3}$$

on the planes $z = 0$ or $z = 1$. Notice that we shall not consider here the case of free surface condition on both planes $z = 0$ and 1, since this case induces an additional (little) difficulty, which is exposed below.

Our next task is to formulate the problem ruled by the system (2.1) in a suitable function space, and find critical values of the parameters, for being able to use a method similar to the one in [6].

2.3 Quasilattices and Diophantine bounds

Consider an integer $q \geq 4$, where $2q$ is the *order of a quasipattern*, and define equally spaced wavevectors in $\mathbb{R}^2$

$$\mathbf{k}_j = k_c \left(\cos\left(\pi \frac{j-1}{q}\right), \sin\left(\pi \frac{j-1}{q}\right) \right) = R_{(j-1)\pi/q}\mathbf{k}_1, \qquad j = 1, 2, \ldots, 2q \tag{3.1}$$

where k_c is a positive number which is defined later, and R_θ is the rotation of angle θ around the vertical axis (see figure 3.1a). We define the *quasilattice* $\Gamma \subset \mathbb{R}^2$ to be the set of points spanned by integer combinations $\mathbf{k_m}$ of the form

$$\mathbf{k_m} = \sum_{j=1}^{2q} m_j \mathbf{k}_j, \qquad \text{where} \qquad \mathbf{m} = (m_1, m_2, \ldots, m_{2q}) \in \mathbb{N}^{2q}. \tag{3.2}$$

The set Γ is dense in $\mathbb{R}^2$. Since $\mathbf{k}_j$ and $-\mathbf{k}_j = \mathbf{k}_{j+q}$ belong to Γ, then $\mathbf{k_m}$ and $-\mathbf{k_m}$ are both in Γ. This allows to obtain real quantities of the form

$$\sum_{\mathbf{k}\in\Gamma} u_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^2, \ u_\mathbf{k} \in \mathbb{C}$$

provided that

$$\overline{u_{-\mathbf{k}}} = u_\mathbf{k}.$$

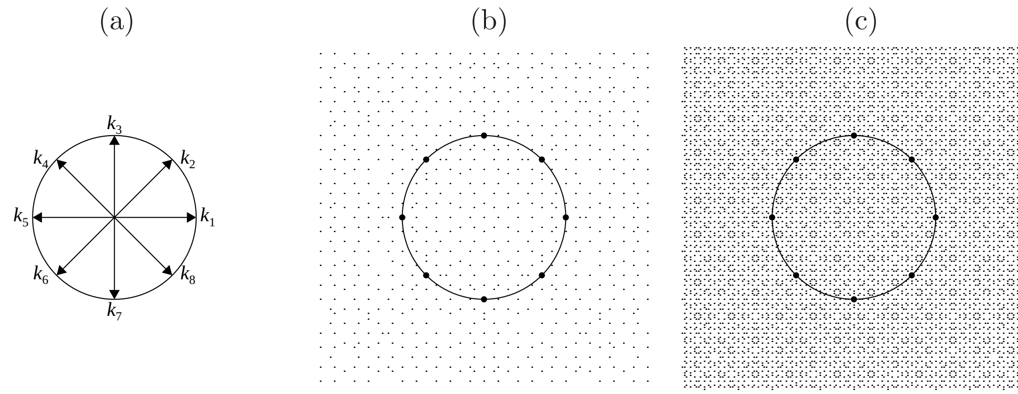

Figure 3.1: Example of quasilattice with $2q = 8$, after [62]. (a) The 8 wavevectors with $|\mathbf{k}| = 1$ which form the basis of the quasilattice. (b,c) The truncated quasilattices Γ_9 and Γ_{27}. The small dots mark the positions of combinations of up to 9 or 27 of the 8 basis vectors on the unit circle.

We know (see [71]) that the $\mathbb{Q}-$ vector space spanned by $\{\mathbf{k}_j,\ j = 1, 2, .., 2q\}$ has dimension $d = \varphi(2q) = 2(l_0 + 1)$ where φ is the Euler totient function, and $l_0 + 1$ is the order of the algebraic integer $\omega := 2\cos\pi/q$ ($l_0 = 1$ for $q = 4, 5, 6,\ l_0 = 2$ for $q = 7...$) with $2(l_0 + 1) \leq q$. Let us define the subset of the d vectors $\{\mathbf{k}_j^*,\ j = 1, 2, .., d\}$ of $\{\mathbf{k}_j,\ j = 1, 2, .., 2q\}$ which forms a basis. Then

$$\mathbf{k}_j = \sum_{s=1}^{d} \alpha_{js}\mathbf{k}_s^*,\ \ \alpha_{js} \in \mathbb{Q}.$$

and any $\mathbf{k} \in \Gamma$ may be written in two different ways

$$\mathbf{k} = \sum_{j=1}^{2q} m_j\mathbf{k}_j = \sum_{s=1}^{d} r_s\mathbf{k}_s^*,\ \ m_j \in \mathbb{N}, r_s \in \mathbb{Q}$$

where $r_s = \sum_{j=1}^{2q} m_j\alpha_{js}$.

Let us define $\alpha_{js} := \frac{n_{js}}{d_{js}}$ with irreducible fractions and

$$\mathfrak{d} = l.c.m_{\substack{j=1,..2q\\ s=1,..d}}\{d_{js}\},\ \text{ then } \mathfrak{d}\alpha_{js} = \beta_{js} \in \mathbb{Z}.$$

Remark 3.1 *Notice that we have* $\mathfrak{d} = 1$ *for example for* $q = 4, 5, 6, 7, 8, 9, 10, 11, 12$ *where we can choose* $\mathbf{k}_s^* = \mathbf{k}_s, s = 1, .., d$ *(see [6])*

Then $m_s^* := \mathfrak{d}r_s = \sum_{j=1}^{2q} m_j\beta_{js} \in \mathbb{Z}$ and

$$\mathbf{k} = \mathfrak{d}^{-1}\sum_{s=1}^{d} m_s^*\mathbf{k}_s^* =: \mathbf{k}(\mathbf{m}^*) \tag{3.3}$$

where $\mathbf{m}^* := (m_1^*, \ldots, m_d^*)$ and we define the following *norm in the lattice* Γ, identified with a subset of $\mathbb{Z}^d$:

$$\sum_{s=1}^{d} |m_s^*| =: N_{\mathbf{k}}.$$

Remark 3.2 *If $\mathfrak{d} = 1$ we can identify Γ with $\mathbb{Z}^d$. If $\mathfrak{d} > 1$, for an arbitrary $\mathbf{m}^* \in \mathbb{Z}^d \backslash \{0\}$, we don't know a priori if there exists $\mathbf{k} \in \Gamma$ such that $\mathbf{k}(\mathbf{m}^*) = \mathbf{k}$.*

Remark 3.3 *Whenever solutions are computed numerically, it is necessary to use only a finite number of Fourier modes, so we define the* truncated quasilattice Γ_N *to be:*

$$\Gamma_N = \{\mathbf{k} \in \Gamma : N_{\mathbf{k}} \leq N\}. \tag{3.4}$$

Figure 3.1(b,c) shows the truncated quasilattices Γ_9 and Γ_{27} in the case $q = 4$.

In what follows we need a lower bound of quantities as

$$(k_c^2 - |\mathbf{k}|^2)^2, \mathbf{k} \in \Gamma$$

which occur in the denominator of the inverse of the linear operator, when they are not 0. We show in [6] (after a trivial scaling)

Lemma 3.4 *Assume $q \geq 4$, then for any $\mathbf{k} \in \Gamma$ such that $|\mathbf{k}| \neq k_c$, i.e. $\mathbf{k} \neq \mathbf{k}_j, j = 1, ..., 2q$ the following estimate holds true*

$$||\mathbf{k}|^2 - k_c^2| \geq \frac{c}{(1 + N_{\mathbf{k}}^2)^{l_0}}, \tag{3.5}$$

for a certain $c > 0$ only depending on q.

2.4 Function spaces and operators

We characterise the functions of interest by their Fourier coefficients on the quasilattice Γ generated by the $2q$ basic vectors $\mathbf{k}_j$:

$$u(\mathbf{x}) = \sum_{\mathbf{k} \in \Gamma} u_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^2.$$

Define now the (Sobolev) space of scalar functions

$$\mathcal{H}_s = \left\{ u = \sum_{\mathbf{k} \in \Gamma} u_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} : \; ||u||_s^2 = \sum_{\mathbf{k} \in \Gamma} (1 + N_{\mathbf{k}}{}^2)^s |u_{\mathbf{k}}|^2 < \infty \right\}, \tag{4.1}$$

which becomes a Hilbert space with the scalar product

$$\langle w, v\rangle_s = \sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}{}^2)^s w_{\mathbf{k}}\overline{v}_{\mathbf{k}}. \tag{4.2}$$

The two following Lemmas are classical results on Sobolev spaces.

Lemma 4.1 *Assume $q \geq 4$, then for $s > d/2$, for any $u \in \mathcal{H}_s$ and any $v \in \mathcal{H}_0$, we have*

$$||uv||_0 \leq c_s||u||_s||v||_0$$

for a certain constant $c_s > 0$.

Lemma 4.2 *(Moser-Nirenberg inequality) Assume $q \geq 4$, and let $s \geq s_0 > d/2$ and let $u, v \in \mathcal{H}_s$. Then,*

$$\|uv\|_s \leq C(s, s_0)(\|u\|_s\|v\|_{s_0} + \|u\|_{s_0}\|v\|_s)$$

for some positive constant $C(s, s_0)$ that depends only on s and s_0. For $\ell \geq 0$ and $s > \ell+d/2$, $\mathcal{H}_s$ is continuously embedded into $\mathcal{C}^\ell$.

In fact we need more complicate function spaces for our system (2.4). This is due to the necessity to control in terms of $|\mathbf{k}|$ (instead of $N_{\mathbf{k}}$) the gain of regularity provided by the inverse of the linear operator on the complementary space of its kernel (here, contrary to [44] and [6], the nonlinear term looses one derivative), hence the inverse of the linear operator is used to regain this loss (for large $|\mathbf{k}|$), while the loss due to the small divisor problem (for $|\mathbf{k}|$ close to k_c) is in terms of $N_{\mathbf{k}}$.

2.4.1 Projection $\mathfrak{P}$

First, as it is a "rule" for Navier-Stokes systems, we define a projection operator $\mathfrak{P}$ on divergence free vector fields. Let consider a vector field $V(\mathbf{x}, z)$ under the form

$$V(\mathbf{x}, z) = \sum_{\mathbf{k}\in\Gamma} V_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}},$$

which, for a fixed z belongs to $(\mathcal{H}_s)^3$. We would like to decompose V as follows:

$$V = W + \nabla\phi, \quad \nabla \cdot W = 0, \quad w^{(z)}|_{z=0,1} = 0.$$

Then we consider the system

$$\begin{aligned} W_{\mathbf{k}}^{(H)} + i\mathbf{k}\phi_{\mathbf{k}} &= V_{\mathbf{k}}^{(H)}, \\ w_{\mathbf{k}}^{(z)} + \frac{d\phi_{\mathbf{k}}}{dz} &= v_{\mathbf{k}}^{(z)}, \\ i\mathbf{k}\cdot W_{\mathbf{k}}^{(H)} + \frac{dw_{\mathbf{k}}^{(z)}}{dz} &= 0, \end{aligned} \tag{4.3}$$

where $V_{\mathbf{k}} = (V_{\mathbf{k}}^{(H)}, v_{\mathbf{k}}^{(z)})$, $V_{\mathbf{k}}^{(H)}$and $v_{\mathbf{k}}^{(z)}$ being respectively the horizontal and vertical components of $V_{\mathbf{k}}$, and where we want to satisfy the boundary condition

$$w_{\mathbf{k}}^{(z)}|_{z=0,1} = 0, \tag{4.4}$$

for the unknown vector field $W_{\mathbf{k}} = (W_{\mathbf{k}}^{(H)}, w_{\mathbf{k}}^{(z)})$. We then obtain the following equation for $\phi_{\mathbf{k}}$:

$$\begin{aligned} \frac{d^2\phi_{\mathbf{k}}}{dz^2} - |\mathbf{k}|^2\phi_{\mathbf{k}} &= i\mathbf{k} \cdot V_{\mathbf{k}}^{(H)} + \frac{dv_{\mathbf{k}}^{(z)}}{dz}, \\ \frac{d\phi_{\mathbf{k}}}{dz}|_{z=0,1} &= v_{\mathbf{k}}^{(z)}|_{z=0,1}. \end{aligned} \tag{4.5}$$

For $\mathbf{k} \neq \mathbf{0}$, it is well known that, if $V_{\mathbf{k}}^{(H)} \in \{L^2(0,1)\}^2$, $v_{\mathbf{k}}^{(z)} \in H^1(0,1)$, then there is a unique solution $\phi_{\mathbf{k}} \in H^2(0,1)$ of this Neumann problem, which satisfies the estimates

$$|\mathbf{k}|^2||\phi_{\mathbf{k}}||^2 + \left\|\frac{d\phi_{\mathbf{k}}}{dz}\right\|^2 \leq ||V_{\mathbf{k}}||^2, \tag{4.6}$$

and there exists a constant $c_1 > 0$ ($c_1 = 7$) such that

$$|\mathbf{k}|^4||\phi_{\mathbf{k}}||^2 + |\mathbf{k}|^2\left\|\frac{d\phi_{\mathbf{k}}}{dz}\right\|^2 + \left\|\frac{d^2\phi_{\mathbf{k}}}{dz^2}\right\|^2 \leq c_1\left\{\left\|\frac{dv_{\mathbf{k}}^{(z)}}{dz}\right\|^2 + |\mathbf{k}|^2||V_{\mathbf{k}}||^2\right\}. \tag{4.7}$$

In the case when $\mathbf{k} = \mathbf{0}$, we have $w_{\mathbf{0}}^{(z)} = 0$, $W_{\mathbf{0}}^{(H)} = V_{\mathbf{0}}^{(H)}$, and $\frac{d\phi_{\mathbf{0}}}{dz} = v_{\mathbf{0}}^{(z)}$ defines $\phi_{\mathbf{0}}$ up to a constant. Hence, this remark, with (4.6) and (4.7) and the identity

$$\int_0^1 \{i\mathbf{k}\phi_{\mathbf{k}} \cdot \overline{W_{\mathbf{k}}^{(H)}} + \frac{d\phi_{\mathbf{k}}}{dz}\overline{w_{\mathbf{k}}^{(z)}}\}dz = 0, \tag{4.8}$$

lead to

$$||W_{\mathbf{k}}||_{L^2}^2 = \langle V_{\mathbf{k}}, W_{\mathbf{k}}\rangle_{L^2},$$

hence

$$\begin{aligned} ||W_{\mathbf{k}}||_{L^2} &\leq ||V_{\mathbf{k}}||_{L^2}, \\ |\mathbf{k}|^2||W_{\mathbf{k}}||_{L^2}^2 + \left\|\frac{dW_{\mathbf{k}}}{dz}\right\|_{L^2}^2 &\leq c^2\left\{|\mathbf{k}|^2||V_{\mathbf{k}}||_{L^2}^2 + \left\|\frac{dV_{\mathbf{k}}}{dz}\right\|_{L^2}^2\right\}, \end{aligned} \tag{4.9}$$

for a constant c^2 independent of $\mathbf{k} \in \Gamma$.

Definition 4.3 *The operator $\mathfrak{P}$ is the linear operator defined as*

$$V = \sum_{\mathbf{k}\in\Gamma} V_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}} \overset{\mathfrak{P}}{\mapsto} W = \sum_{\mathbf{k}\in\Gamma} W_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}},$$

where $W_{\mathbf{k}}$ is solution of (4.3).

We notice that if V is divergence free and satisfies $v^{(z)}|_{z=0,1} = 0$ then $\mathfrak{P}$ acts as the identity. Hence *the operator* $\mathfrak{P}$ *is a projection.*

Remark 4.4 *Notice that for $V_{\mathbf{k}} \in \{L^2(0,1)\}^3$ such that $v_{\mathbf{k}}^{(z)} \in H^1(0,1)$, $\mathbf{k} \in \Gamma$, (which is the case when V is divergence free), the boundary values $v_{\mathbf{k}}^{(z)}|_{z=0,1}$ have a meaning, then we still have $||W_{\mathbf{k}}||_{L^2} \leq ||V_{\mathbf{k}}||_{L^2}$.*

2.4.2 Function spaces

Let us define function spaces for the 4-components vector field $U = (V, \theta)$:

$$\mathcal{H}_{r,s} = \left\{ U = (V,\theta)(\mathbf{x}, z) = \sum_{\mathbf{k}\in\Gamma} U_{\mathbf{k}}(z) e^{i\mathbf{k}\cdot\mathbf{x}}; \sum_{\mathbf{k}\in\Gamma} \left((1+N_{\mathbf{k}}^2)^s ||U_{\mathbf{k}}||_r^2 \right) < \infty \right\} \tag{4.10}$$

where

$$||U_{\mathbf{k}}||_r^2 = \sum_{0\leq l\leq r} |\mathbf{k}|^{2(r-l)} ||U_{\mathbf{k}}||_{H^l}^2.$$

Notice the following equivalence between (squared) norms in (4.10)

$$\sum_{0\leq l\leq r} |\mathbf{k}|^{2(r-l)} ||U_{\mathbf{k}}||_{H^l}^2 \sim \sum_{0\leq l\leq r} (1+|\mathbf{k}|^2)^{(r-l)} ||\frac{d^l U_{\mathbf{k}}}{dz^l}||_{L^2}^2.$$

The space $\mathcal{H}_{r,s}$ has a natural Hilbertian structure. For example, for U, $U' \in \mathcal{H}_{0,s}$, the scalar product reads

$$\langle U, U' \rangle_{0,s} = \sum_{\mathbf{k}\in\Gamma} \left((1+N_{\mathbf{k}}^2)^s \int_0^1 U_{\mathbf{k}} \cdot \overline{U'_{\mathbf{k}}} dz \right),$$

where $U_{\mathbf{k}} \cdot \overline{U'_{\mathbf{k}}}$ is the usual hermitian scalar product in $\mathbb{C}^4$.

Now denoting $\mathfrak{P}U = (\mathfrak{P}V, \theta)$, we have the following

Proposition 4.5 *The projection $\mathfrak{P}$ is bounded in $\mathcal{H}_{r,s}$ for $r \geq 1$, and bounded in the subspace $\mathcal{H}'_{0,s}$ of $\mathcal{H}_{0,s}$ such that $v_{\mathbf{k}}^{(z)} \in H^1(0,1)$, $\mathbf{k} \in \Gamma$. For any U, $U' \in \mathcal{H}_{1,s}$, or $\mathcal{H}'_{0,s}$, we have*

$$\langle U, \mathfrak{P}U' \rangle_{0,s} = \langle \mathfrak{P}U, \mathfrak{P}U' \rangle_{0,s}.$$

Remark 4.6 *The above Proposition means that $(\mathbb{I} - \mathfrak{P})\mathcal{H}_{1,s}$ is orthogonal to $\mathfrak{P}\mathcal{H}_{1,s}$ with the scalar product of $\mathcal{H}_{0,s}$. In other words, $\mathfrak{P}$ is an orthogonal projection in $\mathcal{H}_{0,s}$ restricted to subspaces $\mathcal{H}_{1,s}$ and $\mathcal{H}'_{0,s}$. Moreover, for $U \in \mathcal{H}'_{0,s}$, then $\mathfrak{P}U \in \mathcal{H}_{0,s}$ is orthogonal to any $(\nabla\phi, 0) \in \mathcal{H}_{0,s}$, and $||\mathfrak{P}U||_{0,s} \leq ||U||_{0,s}$ (see (4.8)).*

Proof. The boundedness of $\mathfrak{P}$ in $\mathcal{H}_{1,s}$ results immediately from (4.9), and in $\mathcal{H}'_{0,s}$ from Remark 4.4. For the boundedness in $\mathcal{H}_{r,s}$ for $r > 1$, this follows easily after differentiating (4.3) and (4.5). Now assume U, $U' \in \mathcal{H}_{1,s}$ or $\mathcal{H}'_{0,s}$, and define $\mathfrak{P}U' = (V', \theta')$, then from the form of $V_{\mathbf{k}} - W_{\mathbf{k}} = (\nabla\phi, 0)_{\mathbf{k}}$ indicated in (4.3), we have (notice that V' satisfies de conditions required on W in (4.3))

$$\begin{aligned}
\langle(\mathbb{I} - \mathfrak{P})U, \mathfrak{P}U'\rangle_{0,s} &= \sum_{\mathbf{k}\in\Gamma} \left((1 + N_{\mathbf{k}}^2)^s \int_0^1 \left(i\mathbf{k}\phi_{\mathbf{k}} \cdot \overline{V_{\mathbf{k}}'^{(H)}} + \frac{d\phi_{\mathbf{k}}}{dz} \overline{v_{\mathbf{k}}'^{(z)}} + 0 \right) dz \right) \\
&= \sum_{\mathbf{k}\in\Gamma} \left((1 + N_{\mathbf{k}}^2)^s \int_0^1 \phi_{\mathbf{k}} \left(i\mathbf{k} \cdot \overline{V_{\mathbf{k}}'^{(H)}} - \frac{\overline{dv_{\mathbf{k}}'^{(z)}}}{dz} \right) dz \right) \\
&= 0.
\end{aligned}$$

■

Now we need to extend the definition of the orthogonal projector $\mathfrak{P}$ in all $\mathcal{H}_{0,s}$. Let us consider the orthogonal projection $\mathfrak{P}_0$ in $\mathcal{H}_{0,s}$ on the orthogonal complement of the subspace

$$\mathcal{G}_{0,s} = \{U = (\nabla\phi, 0); \phi \in \mathcal{H}_{1,s}^{(1)}\} \subset \mathcal{H}_{0,s},$$

where we denote by an upper index $^{(1)}$ a space of scalar functions. Then, $\mathfrak{P}_0$ is an extension of $\mathfrak{P}$ obtained by density of $H^1(0,1)$ in $L^2(0,1)$ for all $v_{\mathbf{k}}^{(z)}, \mathbf{k} \in \mathbf{\Gamma}$. It then results

Lemma 4.7 *The projection $\mathfrak{P}$ is bounded in $\mathcal{H}_{r,s}$ for $r \geq 0$. It is an orthogonal projection in $\mathcal{H}_{0,s}$, orthogonal to elements of $\mathcal{G}_{0,s}$.*

In the following we need to use analogues of Lemma 4.2.

Lemma 4.8 *Let $u, v \in \mathcal{H}_{1,s}^{(1)}$ (scalar functions) with $s \geq s_0 > d/2$. Then $uv \in \mathcal{H}_{1,s}^{(1)}$ and there exists $c(s, s_0) > 0$ such that*

$$||uv||_{1,s} \leq c(s, s_0)(||u||_{1,s}||v||_{1,s_0} + ||u||_{1,s_0}||v||_{1,s}).$$

Lemma 4.9 *Let u, v be scalar functions respectively in $\mathcal{H}_{1,s}^{(1)}$ and $\mathcal{H}_{0,s}^{(1)}$ with $s \geq s_0 > d/2$. Then $uv \in \mathcal{H}_{0,s}^{(1)}$ and there exists $c(s, s_0) > 0$ such that*

$$||uv||_{0,s} \leq c(s, s_0)(||u||_{1,s}||v||_{0,s_0} + ||u||_{1,s_0}||v||_{0,s}).$$

Lemma 4.10 *Let u, v be scalar functions respectively in $\mathcal{H}_{1,s}^{(1)}$ and $\mathcal{H}_{0,0}^{(1)}$ with $s \geq s_0 > d/2$. Then $uv \in \mathcal{H}_{0,0}^{(1)}$ and there exists $c(s) > 0$ such that*

$$||uv||_{0,0} \leq c(s)||u||_{1,s}||v||_{0,0}.$$

Lemma 4.11 *Let u, v be scalar functions respectively in $\mathcal{H}^{(1)}_{1,0}$ and $\mathcal{H}^{(1)}_{0,s}$ with $s \geq s_0 > d/2$. Then $uv \in \mathcal{H}^{(1)}_{0,0}$ and there exists $c(s) > 0$ such that*

$$||uv||_{0,0} \leq c(s)||u||_{1,0}||v||_{0,s}.$$

The proofs of these Lemmas are made in Appendix 6.2.2.

2.5 Formulation of the convection problem

2.5.1 Operators L, A and B

Definition 5.1 *We say that U satisfies* Condition ***b.c.*** *if one of the following boundary conditions are satisfied*
(i) $V^{(H)}|_{z=0,1} = 0$ (rigid-rigid),
(ii) $V^{(H)}|_{z=0} = \frac{dV^{(H)}}{dz}|_{z=1} = 0$ (rigid - free),
(iii) $\frac{dV^{(H)}}{dz}|_{z=0} = V^{(H)}|_{z=1} = 0$ (free - rigid).

Then, we define the following function spaces for r and s non-negative integers

$$\begin{aligned} \mathcal{K}_{r,s} &= \mathfrak{P}\mathcal{H}_{r,s} = \{U = (V, \theta) \in \mathcal{H}_{r,s}; \nabla \cdot V = 0, v^{(z)}|_{z=0,1} = 0\}, \\ D_s(L) &= \mathcal{K}_{2,s} \cap \{U \text{ satisfies Condition } \mathbf{b.c.},\ \theta|_{z=0,1} = 0\}, \end{aligned} \tag{5.1}$$

and we put, respectively on these subspaces, the norms of $\mathcal{H}_{r,s}$ and $\mathcal{H}_{2,s}$. We notice that we do not consider the case of conditions $\frac{dV^{(H)}}{dz}|_{z=0,1} = 0$ (free - free) (see Remark 5.4 below).

Definition 5.2 *For any $U \in D_s(L)$ operators L and A are defined by:*

$$\begin{aligned} LU &= (\mathfrak{P}\Delta V, \Delta\theta), \quad U \in D_s(L) \\ AU &= (\mathfrak{P}(\theta e_z), V \cdot e_z), \ U \in \mathcal{K}_{0,s}, \end{aligned}$$

and the quadratic operator B by

$$B(U, U) = \left(\frac{1}{\mathcal{P}}\mathfrak{P}(V \cdot \nabla V), V \cdot \nabla\theta\right), U \in \mathcal{K}_{1,s}.$$

It is clear that L maps continuously $D_s(L)$ to $\mathcal{K}_{0,s}$. For $s > d/2$ the quadratic operator B maps continuously $D_s(L)$ to $\mathcal{K}_{1,s}$ as this results easily from the fact that $H^1(0,1)$ is an algebra, as well as $\mathcal{H}_s$ for $s > d/2$ (see Lemma 4.8 and see Appendix 6.2.3 for the rest of the proof). This means that there exists $c(s, s_0)$ such that for any $U, U' \in D_s(L)$, and $s \geq s_0 > d/2$ we have

$$||B(U, U')||_{1,s} \leq c(s, s_0)(||U||_{2,s}||U'||_{2,s_0} + ||U||_{2,s_0}||U'||_{2,s}), \tag{5.2}$$

where we define the bilinear symmetric operator $(U,U') \mapsto B(U,U')$ as

$$2B(U,U') =: \left(\frac{1}{\mathcal{P}}\mathfrak{P}(V\cdot\nabla V' + V'\cdot\nabla V), V\cdot\nabla\theta' + V'\cdot\nabla\theta\right).$$

Moreover, we also have easily $B(U,U) \in \mathcal{K}_{0,s}$ for $U \in \mathcal{K}_{1,s}$, as this results from the fact that the product of a function in $H^1(0,1)$ with another in $L^2(0,1)$ lies in $L^2(0,1)$, then $V\cdot\nabla V \in \mathcal{H}_{0,s}$ (see Appendix 6.2.3) and for $U,U' \in \mathcal{K}_{1,s}$ and $s \geq s_0 > d/2$ we have the estimate

$$||B(U,U')||_{0,s} \leq c(s,s_0)(||U||_{1,s}||U'||_{1,s_0} + ||U||_{1,s_0}||U'||_{1,s}). \tag{5.3}$$

Now solving the system (2.1) reduces to solving the equation

$$(\lambda L + A)U - B(U,U) = 0, \quad U \in D_s(L). \tag{5.4}$$

where $\lambda =: \mathcal{R}^{-1/2}$.

Then, we show the following useful basic properties of operators L, A and B:

Lemma 5.3 *For any $s \geq 0$, the unbounded operator L with domain $D_s(L)$ is selfadjoint, definite negative, in the space $\mathcal{K}_{0,s}$. Moreover, for $U \in D_s(L)$, there exits a scalar function $c(\lambda)$ such that*

$$\langle(\lambda L + A)U, U\rangle_{0,s} \leq c(\lambda)||U||^2_{0,s} \tag{5.5}$$

holds, with $c(\lambda) = 1 - 2\lambda < 0$ for $\mathcal{R} < 4$ (in the case of free-free boundary condition, which we exclude, $c(\lambda) = 0$.)

For $s > d/2$, and $U,U' \in \mathcal{K}_{1,s}$ and U,U' real, i.e. $U = \overline{U}$, $U' = \overline{U'}$ we have

$$\langle B(U,U), U\rangle_{0,0} = 0, \tag{5.6}$$

$$\langle 2B(U,U'), U\rangle_{0,0} = -\langle B(U,U), U'\rangle_{0,0}. \tag{5.7}$$

Proof. First we have, by using Lemma 4.7

$$\begin{aligned}
\langle(\lambda L + A)U, U'\rangle_{0,s} &= \langle(\mathfrak{P}(\lambda\Delta V + \theta e_z), \lambda\Delta\theta + V\cdot e_z), (V',\theta')\rangle_{0,s} \\
&= \lambda\langle(\Delta V, \Delta\theta), (V',\theta')\rangle_{0,s} + \langle(\theta e_z, V\cdot e_z), (V',\theta')\rangle_{0,s} \\
&= \lambda\langle\Delta V, V'\rangle_{0,s} + \lambda\langle\Delta\theta,\theta'\rangle_{0,s} + \langle\theta, v'^{(z)}\rangle_{0,s} + \langle v^{(z)},\theta'\rangle_{0,s}.
\end{aligned}$$

Then we observe that $\langle\theta, v'^{(z)}\rangle_{0,s} + \langle v^{(z)},\theta'\rangle_{0,s}$ is symmetric in (U,U'). Moreover by integrating by parts, since $\theta_{\mathbf{k}}|_{z=0,1} = 0$,

$$\begin{aligned}
\langle\Delta\theta,\theta'\rangle_{0,s} &= \sum_{\mathbf{k}\in\Gamma}(1+N^2_{\mathbf{k}})^s\int_0^1\left(\frac{d^2\theta_{\mathbf{k}}}{dz^2} - |k|^2\theta_{\mathbf{k}}\right)\overline{\theta'_{\mathbf{k}}}dz \\
&= -\sum_{\mathbf{k}\in\Gamma}(1+N^2_{\mathbf{k}})^s\int_0^1\left(\frac{d\theta_{\mathbf{k}}}{dz}\overline{\frac{d\theta'_{\mathbf{k}}}{dz}} + |k|^2\theta_{\mathbf{k}}\overline{\theta'_{\mathbf{k}}}\right)dz,
\end{aligned}$$

which is symmetric in (U, U'). The same computation holds by using the boundary conditions satisfied by V for $U \in D_s(L)$, and shows that $\langle \Delta V, V' \rangle_{0,s}$ is symmetric in (U, U'). This proves that

$$\langle (\lambda L + A)U, U' \rangle_{0,s} = \langle U, (\lambda L + A)U' \rangle_{0,s},$$

i.e. the operators L and A are symmetric in $\mathcal{K}_{0,s}$.

The operator L is selfadjoint in $\mathcal{K}_{0,s}$ because it is easy to prove that L^{-1} is symmetric in $\mathcal{K}_{0,s}$, bounded from $\mathcal{K}_{0,s}$ into $D_s(L)$ (with norm of $\mathcal{K}_{2,s}$), see Appendix 6.2.1. The operator A is symmetric and bounded in $\mathcal{K}_{0,s}$. Hence by theorem 4.3 in [48] p.287, the sum $\lambda L + A$ with domain $D_s(L)$ is also selfadjoint in $\mathcal{K}_{0,s}$.

To prove the inequality (5.5), we come back to the computation above, valid for $U \in D_s(L)$:

$$\begin{aligned} \langle (\lambda L + A)U, U \rangle_{0,s} &= -\lambda \langle \nabla V, \nabla V \rangle_{0,s} - \lambda \langle \nabla \theta, \nabla \theta \rangle_{0,s} + 2\mathrm{Re} \langle \theta, v^{(z)} \rangle_{0,s} \\ &\leq 2||V||_{0,s}||\theta||_{0,s} \leq 2||U||^2_{0,s}. \end{aligned} \tag{5.8}$$

For all boundary conditions (see Definition 5.1) we have Poincaré inequalities: $\theta, v^{(z)}$ and $V^{(H)}$ cancel at $z = 0$ or (and) $z = 1$, so, for example

$$|v^{(z)}(z)|^2 = |\int_0^z Dv^{(z)}(s)ds|^2 \leq z \int_0^1 |Dv^{(z)}(s)|^2 ds,$$

and integrating on $(0, 1)$ leads to the Poincaré estimates

$$||V||_{0,s} \leq \frac{1}{\sqrt{2}} ||\nabla V||_{0,s}, \quad ||\theta||_{0,s} \leq \frac{1}{\sqrt{2}} ||\nabla \theta||_{0,s}. \tag{5.9}$$

Hence this leads to

$$|2\mathrm{Re}\langle \theta, v^{(z)} \rangle_{0,s}| \leq ||\nabla V||_{0,s}||\nabla \theta||_{0,s} \leq 1/2||\nabla V||^2_{0,s} + 1/2||\nabla \theta||^2_{0,s},$$

and

$$\langle (\lambda L + A)U, U \rangle_{0,s} \leq (1/2 - \lambda)[||\nabla V||^2_{0,s} + ||\nabla \theta||^2_{0,s}] < 0 \text{ for } \lambda > 1/2, \text{ i.e. } \mathcal{R} < 4.$$

Hence for $\mathcal{R} < 4$ (i.e. $\lambda > 1/2$) we have

$$\langle (\lambda L + A)U, U \rangle_{0,s} \leq -(2\lambda - 1)[||V||^2_{0,s} + ||\theta||^2_{0,s}] = c(\lambda)||U||^2_{0,s},$$

with

$$c(\lambda) = 1 - 2\lambda.$$

Remark 5.4 *In the case of free-free Boundary conditions which we exclude here, we have not* $||V^{(H)}||_{0,s} \leq \frac{1}{\sqrt{2}}||\nabla V^{(H)}||_{0,s}$*, hence we only have*

$$\langle(\lambda L + A)U, U\rangle_{0,s} \leq (1/2 - \lambda)[||\nabla v^{(z)}||_{0,s}^2 + ||\nabla\theta||_{0,s}^2] - \lambda||\nabla V^{(H)}||_{0,s}^2 \leq 0 \text{ for } \lambda \geq 1/2.$$

In such a case, 0 is an eigenvalue of $\lambda L + A$ *corresponding to the eigenvector* $U = (V^{(H)}, 0)$ *where* $V^{(H)} = Const.$

In the same way as above, for $U \in \mathcal{K}_{1,s}$, we have

$$\langle B(U,U), U\rangle_{0,s} = \frac{1}{\mathcal{P}}\langle V \cdot \nabla V, V\rangle_{0,s} + \langle V \cdot \nabla\theta, \theta\rangle_{0,s},$$

and by using $\overline{\theta_{\mathbf{p}+\mathbf{q}}} = \theta_{\mathbf{r}}$, when $\mathbf{p}+\mathbf{q}+\mathbf{r} = \mathbf{0}$, since θ is real,

$$\begin{aligned}
\langle V \cdot \nabla\theta, \theta\rangle_{0,0} &= \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left((i\mathbf{q} \cdot V_{\mathbf{p}}^{(H)})\theta_{\mathbf{q}} + v_{\mathbf{p}}^{(z)}\frac{d\theta_{\mathbf{q}}}{dz}\right)\theta_{\mathbf{r}} dz \\
&= \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left((i\mathbf{r} \cdot V_{\mathbf{p}}^{(H)})\theta_{\mathbf{r}} + v_{\mathbf{p}}^{(z)}\frac{d\theta_{\mathbf{r}}}{dz}\right)\theta_{\mathbf{q}} dz \\
&= \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left((-i\mathbf{p} \cdot V_{\mathbf{p}}^{(H)})\theta_{\mathbf{q}}\theta_{\mathbf{r}} + v_{\mathbf{p}}^{(z)}\frac{d(\theta_{\mathbf{q}}\theta_{\mathbf{r}})}{dz}\right) dz \\
&= \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left(\frac{dv_{\mathbf{p}}^{(z)}}{dz}\theta_{\mathbf{q}}\theta_{\mathbf{r}} + v_{\mathbf{p}}^{(z)}\frac{d(\theta_{\mathbf{q}}\theta_{\mathbf{r}})}{dz}\right) dz = 0.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
\langle V \cdot \nabla V, V\rangle_{0,0} &= \langle V \cdot \nabla V^{(H)}, V^{(H)}\rangle_{0,s} + \langle V \cdot \nabla v^{(z)}, v^{(z)}\rangle_{0,s} \\
&= \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \frac{d(v_{\mathbf{p}}^{(z)}V_{\mathbf{q}} \cdot V_{\mathbf{r}})}{dz} dz = 0,
\end{aligned}$$

which ends the proof of (5.6). Identity (5.7) is a consequence of (5.6): indeed let us consider the identity

$$\langle B(U + tU', U + tU'), U + tU'\rangle_{0,0} = 0$$

which holds for any $t \in \mathbb{R}$. It results that the coefficient of degree 1 in t of this polynomial is zero, which is exactly the property (5.7). ∎

2.5.2 New formulation

For applying a method analogue to the one developed in [5] and [6], we need to control a parameter able to move all the spectrum of the linearized operator. In the present problem, we are lucky enough to have λ in front of an invertible operator, allowing to reformulate suitably the problem.

We know that the operator $-L$ is selfadjoint and positive, so we can define the sefadjoint positive operator $(-L)^{1/2}$ with dense domain (see [48] section V.11 p.281) as the inverse of

$$(-L)^{-1/2} = \frac{1}{\pi}\int_0^\infty \zeta^{-1/2}(\zeta - L)^{-1}d\zeta,$$

which is selfadjoint and bounded, with the following properties. First, for $U \in D_s(L)$ we have

$$(-L)^{1/2}(-L)^{1/2}U = -LU.$$

Let us define the Hilbert space, adapted to boundary conditions **b.c.** (see definition 5.1),

$$\widetilde{\mathcal{K}_{1,s}} = \{U = (V,\theta) \in \mathcal{K}_{1,s}; \theta = v^{(z)}|_{z=0,1} = 0, V^{(H)}|_{z=0} = 0, \text{ or (and) } V^{(H)}|_{z=1} = 0\}.$$

We can take in $\widetilde{\mathcal{K}_{1,s}}$ the norm

$$||U||_{\widetilde{1,s}} := \{||\nabla V||_{0,s}^2 + ||\nabla\theta||_{0,s}^2\}^{1/2}, \tag{5.10}$$

which is equivalent to the usual norm in $\mathcal{K}_{1,s}$, due to Poincaré inequalities (5.9). Then, because of the identity

$$\langle -LU, U\rangle_{0,s} = ||\nabla V||_{0,s}^2 + ||\nabla\theta||_{0,s}^2,$$

valid for any $U \in D_s(L)$, it is clear that the following identity holds

$$||(-L)^{1/2}U||_{0,s} = ||U||_{\widetilde{1,s}}, \tag{5.11}$$

which can be extended to any $U \in D_s[(-L)^{1/2}]$ the domain of $(-L)^{1/2}$ acting in $\mathcal{K}_{0,s}$. This shows that the domain $D_s[(-L)^{1/2}]$ (dense in $\mathcal{K}_{0,s}$) satisfies

$$D_s[(-L)^{1/2}] \subset \widetilde{\mathcal{K}_{1,s}}, \tag{5.12}$$

with a continuous embedding.

Definition 5.5 *We denote by*

$$\mathcal{D}_{1/2,s} := D_s[(-L)^{1/2}].$$

This is an Hilbert subspace of $\mathcal{K}_{1,s}$, with the scalar product associated with the norm (5.10) in $\widetilde{\mathcal{K}_{1,s}}$.

Remark 5.6 *In the sequel, the norm in $\mathcal{D}_{1/2,s}$ is denoted by $||\cdot||_{\widetilde{1,s}}$ or $||\cdot||_{1,s}$ or $||\cdot||_{\mathcal{D}_{1/2,s}}$ as well.*

Now let us consider the following equation in $\mathcal{K}_{0,s}$:

$$\lambda u - \mathcal{A}u + \mathcal{B}(u,u) = 0, \tag{5.13}$$

where operators $\mathcal{A}$ and $\mathcal{B}$ are defined as:

$$\begin{aligned}\mathcal{A} \ &: \ = (-L)^{-1/2}A(-L)^{-1/2},\\ \mathcal{B}(u,u) \ &: \ = (-L)^{-1/2}B((-L)^{-1/2}u,(-L)^{-1/2}u).\end{aligned}$$

Since the operator A is bounded in $\mathcal{K}_{0,s}$ this is also the case for $\mathcal{A}$. Now for the quadratic operator $\mathcal{B}$ we have

Lemma 5.7 *Assume $s > d/2$, then the quadratic operator $\mathcal{B}$ is bounded from $\mathcal{K}_{0,s}$ to $\mathcal{D}_{1/2,s} \hookrightarrow \widetilde{\mathcal{K}_{1,s}} \hookrightarrow \mathcal{K}_{0,s}$. Moreover for $u,u' \in \mathcal{K}_{0,s}$, with $s \geq s_0 > d/2$ we have*

$$||\mathcal{B}(u,u')||_{0,s} \leq ||(-L)^{-1/2}||_{0,s}||\mathcal{B}(u,u')||_{\widetilde{1,s}} \leq c(s,s_0)(||u||_{0,s}||u'||_{0,s_0} + ||u||_{0,s_0}||u'||_{0,s}). \tag{5.14}$$

Moreover for $u \in \mathcal{K}_{0,s}$, with $s > d/2$, the linear operator $v \mapsto \mathcal{B}(u,v)$ is bounded in $\mathcal{K}_{0,0}$ with the estimate

$$||\mathcal{B}(u,v)||_{0,0} \leq c||u||_{0,s}||v||_{0,0}. \tag{5.15}$$

Proof. Using (5.11) and (5.3) we obtain

$$\begin{aligned}||\mathcal{B}(u,u')||_{\widetilde{1,s}} \ &= \ ||B((-L)^{-1/2}u,(-L)^{-1/2}u')||_{0,s}\\ &\leq \ c(s,s_0)(||(-L)^{-1/2}u||_{\widetilde{1,s}}||(-L)^{-1/2}u'||_{\widetilde{1,s_0}} + ||(-L)^{-1/2}u||_{\widetilde{1,s_0}}||(-L)^{-1/2}u'||_{\widetilde{1,s}})\\ &\leq \ c(s,s_0)(||u||_{0,s}||u'||_{0,s_0} + ||u||_{0,s_0}||u'||_{0,s})\end{aligned}$$

and

$$||\mathcal{B}(u,u')||_{0,s} \leq ||(-L)^{-1/2}||_{0,s}|\mathcal{B}(u,u')||_{\widetilde{1,s}} = c_1(s,s_0)(||u||_{0,s}||u'||_{0,s_0} + ||u||_{0,s_0}||u'||_{0,s}). \tag{5.16}$$

For finding estimate (5.15) we just need to prove that for $((-L)^{-1/2}u,(-L)^{-1/2}v) = (U,V) \in \mathcal{K}_{2,s} \times \mathcal{K}_{1,0}$ then $||B(U,V)||_{0,0} \leq c'||U||_{2,s}||V||_{1,0}$. This is proved in Appendix 6.2.3. ■

Then we have the following

Lemma 5.8 *Assuming $s > d/2$ and $\lambda > 0$, then finding a solution $u \in \mathcal{K}_{0,s}$ of*

$$\lambda u - \mathcal{A}u + \mathcal{B}(u,u) = 0, \tag{5.17}$$

where the linear operator $\mathcal{A}$ is bounded and selfadjoint in $\mathcal{K}_{0,s}$, implies $u \in \mathcal{D}_{1/2,s}$, and is equivalent to finding a solution $U = (-L)^{-1/2}u \in D_s(L)$ of

$$\lambda LU + AU - B(U,U) = 0. \tag{5.18}$$

Proof. Indeed, we notice that for $u \in \mathcal{K}_{0,s}$ solution of (1.9), then $(-L)^{-1/2}u \in \mathcal{D}_{1/2,s} \subset \mathcal{K}_{1,s}$, hence $B((-L)^{-1/2}u, (-L)^{-1/2}u) \in \mathcal{K}_{0,s}$ (see (5.3)) and finally $\mathcal{B}(u,u) \in \mathcal{D}_{1/2,s}$. It is also clear that $\mathcal{A}u \in \mathcal{D}_{1/2,s}$. For $\lambda \neq 0$ this last property and (1.9) show that $u \in \mathcal{D}_{1/2,s}$, and we can apply the operator $(-L)^{1/2}$ to (1.9). Then defining $U = (-L)^{-1/2}u$ gives U in $D_s(L)$ verifying (5.18). Conversely, the knowledge of a solution U of (5.18) gives a solution $u = (-L)^{1/2}U$ of (1.9). We may observe that the quadratic operator $\mathcal{B}$ is bounded in $\mathcal{K}_{0,s}$ (see (5.16)). Now due to the selfadjointness of operators A and $(-L)^{-1/2}$ in $\mathcal{K}_{0,s}$, the operator $\mathcal{A}$ *is also selfadjoint in* $\mathcal{K}_{0,s}$. ■

Remark 5.9 *We might think that it would be advantageous to work in $\mathcal{D}_{1/2,s}$ instead of $\mathcal{K}_{0,s}$. However for the method we are using in the following, it is necessary that $\mathcal{A}$ be selfadjoint. If we consider this operator in $\mathcal{D}_{1/2,s}$, then it can be shown that this is not true for boundary conditions (ii) and (iii) in Definition 5.1.*

2.5.3 Rotationnal Symmetry

The system (2.1), completed with the boundary conditions included in the definition of $D_s(L)$, is invariant under horizontal rotations of angle π/q. To make this precise, let us define the linear operator $\mathbf{R}_{\pi/q}$, by

$$\mathbf{R}_{\pi/q}U(\mathbf{x},z) = \left(R_{\pi/q}V(R_{-\pi/q}\mathbf{x},z), \theta(R_{-\pi/q}\mathbf{x},z)\right),$$

where R_ϕ is the horizontal rotation of angle ϕ. More precisely, by using the identity $\mathbf{k} \cdot R_{-\phi}\mathbf{x} = R_\phi \mathbf{k} \cdot \mathbf{x}$, we have

$$\mathbf{R}_{\pi/q} \sum_{\mathbf{k}\in\Gamma} U_{\mathbf{k}}(z) e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\mathbf{k}\in\Gamma} \left(R_{\pi/q}V_{\mathbf{k}}(z), \theta_{\mathbf{k}}(z)\right) e^{iR_{\pi/q}\mathbf{k}\cdot\mathbf{x}}. \tag{5.19}$$

Definition 5.10 *We say that $U = (V,\theta)$ is invariant under $\mathbf{R}_{\pi/q}$ if the following holds*

$$R_{\pi/q}V_{\mathbf{k}}(z) = V_{R_{\pi/q}\mathbf{k}}(z), \quad \theta_{\mathbf{k}}(z) = \theta_{R_{\pi/q}\mathbf{k}}(z).$$

Then, we have the following

Lemma 5.11 *The linear operators $L, A, \mathcal{A}$ and the quadratic operators B and $\mathcal{B}$ commute with $\mathbf{R}_{\pi/q}$: for $U \in D_s(L)$ and $u \in \mathcal{K}_{0,s}$*

$$\begin{aligned} \mathbf{R}_{\pi/q}(\lambda L + A)U &= (\lambda L + A)\mathbf{R}_{\pi/q}U, \; \mathbf{R}_{\pi/q}\mathcal{A}u = \mathcal{A}\mathbf{R}_{\pi/q}u \\ \mathbf{R}_{\pi/q}B(U,U) &= B(\mathbf{R}_{\pi/q}U, \mathbf{R}_{\pi/q}U), \; \mathbf{R}_{\pi/q}\mathcal{B}(u,u) = \mathcal{B}(\mathbf{R}_{\pi/q}u, \mathbf{R}_{\pi/q}u). \end{aligned} \tag{5.20}$$

Proof. This results from the commutation of the original system (2.1) under any horizontal rotations, and from the commutation property

$$\mathbf{R}_{\pi/q}\mathfrak{P} = \mathfrak{P}\mathbf{R}_{\pi/q}$$

which is easy to check from the construction of projection $\mathfrak{P}$. Moreover the operator L commutes with $\mathbf{R}_{\pi/q}$, hence this is also valid for $(-L)^{-1/2}$. ■

2.6 Criticality for $\mathcal{A} - \lambda\mathbb{I}$ and Formal bifurcation

2.6.1 Study of criticality

Let us consider the linear system

$$(\mathcal{A} - \lambda)u = G \in \mathcal{K}_{0,s}, \tag{6.1}$$

where we look for $u \in \mathcal{K}_{0,s}$. This system is equivalent to looking for $U = (-L)^{-1/2}u \in D_{1/2,s}$ such that

$$(\lambda L + A)U = G' = (-L)^{1/2}G = (F, g) \in (D_{1/2,s})^* \tag{6.2}$$

where $G' = (F, g)$ is given in $(D_{1/2,s})^*$ (see the definition and properties of this dual space in Appendix 6.2.1).

Let us define the Fourier components

$$\begin{aligned} U_{\mathbf{k}} &= (V_{\mathbf{k}}^{(H)}, v_{\mathbf{k}}^{(z)}, \theta_{\mathbf{k}}), \\ G'_{\mathbf{k}} &= (F, g)_{\mathbf{k}} = (F_{\mathbf{k}}^{(H)}, f_{\mathbf{k}}^{(z)}, g_{\mathbf{k}}), \end{aligned}$$

then for a fixed $\mathbf{k}$, the system has the form

$$(\lambda L_{\mathbf{k}} + A_{\mathbf{k}})U_{\mathbf{k}} = G'_{\mathbf{k}} \tag{6.3}$$

which is *exactly the same as the one obtained in the periodic case*, described in details for example in Chapter II of [14] and solved in details by V.Yudovich [74]. Notice that for each $|\mathbf{k}|$ the linear operator $\lambda L_{\mathbf{k}} + A_{\mathbf{k}}$ is selfadjoint in the space $\{\mathcal{K}_{0,s}\}_{\mathbf{k}}$ spanned by the $\mathbf{k}$-th Fourier components of elements in $\mathcal{K}_{0,s}$, and operator $L_{\mathbf{k}}$ is studied in particular in Appendix 6.2.1.

Remark 6.1 *Notice that in [14] and in [74] the only case of $\lambda > 0$ is considered, since $\lambda = 1/\sqrt{\mathcal{R}}$ by definition. It results that we don't know anything a priori on the operator, for $\lambda \leq 0$. However we may observe that the homogeneous system associated with (6.3) is invariant when changing λ into $-\lambda$ and θ into $-\theta$ (see also (2.1) in changing $\sqrt{\mathcal{R}}$ into its opposite). It results that the spectrum of $\mathcal{A}$ is symmetric with respect to 0. Moreover $\lambda = 0$ is an eigenvalue with infinite multiplicity.*

Then, it is known (see Yudovich [74]) that for a fixed $|\mathbf{k}|$ there is a denumerable sequence of $\mathcal{R}_j$ $(=1/\lambda_j^2)$ such that the system (6.3) has a non trivial solution for $(F,g)_{\mathbf{k}}=0$, and there is a variational principle for finding $\mathcal{R}_0(|\mathbf{k}|^2)=\min\mathcal{R}_j$ (see Velte [69]). It is also known mathematically (see Yudovich [74]) that the function $\mathcal{R}_0(|\mathbf{k}|^2)$ is analytic, tends towards ∞ as $|\mathbf{k}|^2\to 0$ and as $|\mathbf{k}|^2\to\infty$, and that there is a minimum $\mathcal{R}_c$ obtained for a critical value k_c^2. However, it is *only known numerically* (see [14]), that *this minimum is unique* and the kernel of $\lambda L_{\mathbf{k}}+A_{\mathbf{k}}$ for $\mathbf{k}=\mathbf{k}_1=(k_c,0)$ is one-dimensional ([74]). We now define $\lambda_0=1/\sqrt{\mathcal{R}_c}$.

It results that the kernel of the linear operator $(\mathcal{A}-\lambda_0\mathbb{I})$ is $2q$ - dimensional, spanned by $\boldsymbol{\xi}_j=(-L)^{1/2}\boldsymbol{\xi}_j'$, with

$$\boldsymbol{\xi}_j'=\mathbf{R}_{\frac{\pi(j-1)}{q}}\left(\widehat{U}_{\mathbf{k}_1}(z)e^{i\mathbf{k}_1\cdot\mathbf{x}}\right),\ j=1,2,...,2q,\tag{6.4}$$

in the kernel of $\lambda_0 L+A$, where

$$\widehat{U}_{\mathbf{k}_1}=(V_{\mathbf{k}_1}^{(H)},v_{\mathbf{k}_1}^{(z)},\theta_{\mathbf{k}_1})$$

is solution of the homogeneous system (6.3) for $\mathbf{k}=\mathbf{k}_1$, and with $G'_{\mathbf{k}}=0$, and $\mathcal{R}=\mathcal{R}_c$.

We need now to estimate the inverse of the linear operator defined by the system (6.3) for $\mathcal{R}=\mathcal{R}_c$ and $|\mathbf{k}|\neq k_c$. From the now standard study of the resolvent operator for Navier-Stokes type of system (see [76]), as here, but in a periodic frame, we deduce that there is a function $c(|\mathbf{k}|^2)$ bounded as $|\mathbf{k}|\to\infty$ and $|\mathbf{k}|\to 0$ such that (we notice that $||G'||_{(D_{1/2,s})^*}\leq c||G||_{0,s}$ for a certain $c>0$)

$$||U_{\mathbf{k}}||_1^2=||DU_{\mathbf{k}}||_{L^2}^2+(1+|\mathbf{k}|^2)||U_{\mathbf{k}}||_{L^2}^2\leq[c(|\mathbf{k}|^2)]^2||G_{\mathbf{k}}||_{L^2}^2.\tag{6.5}$$

For $|\mathbf{k}|$ near k_c, we know that $c(|\mathbf{k}|^2)$ diverges as $|\mathbf{k}|^2\to k_c^2$. In fact let consider the dispersion equation, obtained when we look for eigenvectors of the homogeneous system (6.3) which has constant coefficients (see [14]). Then, the modulus of the dispersion equation, which cancels for $|\mathbf{k}|=k_c$, is bounded from below by the inverse $c(|\mathbf{k}|^2)^{-1}$. This dispersion equation depends analytically on $|\mathbf{k}|^2$ ([74]) and we now need

Condition 6.2 *We assume that the second derivative $\mathcal{R}_0''(|\mathbf{k}|^2)\neq 0$ for $|\mathbf{k}|=k_c$ at $\mathcal{R}_0(k_c^2)=\mathcal{R}_c$.*

Notice that we give a formula for $\frac{d^2}{d|\mathbf{k}|^2}(1/\sqrt{\mathcal{R}_c})$ in Appendix 6.2.4. The dispersion relation cancels with a (only) double root for $|\mathbf{k}|^2=k_c^2$. This means that we have in fact

$$c(|\mathbf{k}|^2)=\frac{c_1(|\mathbf{k}|^2)}{(|\mathbf{k}|^2-k_c^2)^2}\quad\text{ for }\ |\mathbf{k}|\neq k_c\tag{6.6}$$

where c_1 is bounded for all bounded $|\mathbf{k}|^2$ and is $O(|\mathbf{k}|^4)$ as $|\mathbf{k}| \to \infty$.

For $|\mathbf{k}| = k_c$ and $\mathbf{k} \in \Gamma$, this implies that $\mathbf{k}$ belongs to the basis of the quasipattern. Then, following [14], [74] and [75] the system (6.3) is solvable provided the compatibility conditions

$$\langle G, \boldsymbol{\xi}_j \rangle_{0,0} = \langle G', \boldsymbol{\xi}'_j \rangle_{0,0} = \int_0^1 G'_{\mathbf{k}_j} \cdot \overline{\widehat{U}_{\mathbf{k}_j}} dz = \int_0^1 (F_{\mathbf{k}_j} \cdot \overline{\widehat{V}_{\mathbf{k}_j}} + g_{\mathbf{k}_j} \cdot \overline{\widehat{\theta}_{\mathbf{k}_j}}) dz = 0, \ j = 1, ..., 2q$$

hold. It results that

$$||u_{\mathbf{k}}||_0 = ||U_{\mathbf{k}}||_1 \leq c(|\mathbf{k}|^2) ||G_{\mathbf{k}}||_0. \tag{6.7}$$

Remark 6.3 *The classical linear stability theory ([14], [77]) says that*

$$\langle (\lambda_0 L + A)U, U \rangle_{0,0} < 0 \text{ for all } U \in D_s(L) \text{ orthogonal to } \ker(\lambda_0 L + A), \tag{6.8}$$

i.e., using that $\mathcal{D}_{1/2,s}$ is dense in $\mathcal{K}_{0,s}$:

$$\langle (\mathcal{A} - \lambda_0)u, u \rangle_{0,0} < 0 \text{ for all } u \in \mathcal{K}_{0,s} \text{ orthogonal to } \ker(\mathcal{A} - \lambda_0). \tag{6.9}$$

We also know, from the discussion above, that for any fixed $|\mathbf{k}|$ we have a decreasing sequence of positive eigenvalues, and a sequence of symmetric ones, for the selfadjoint bounded operator $\mathcal{A}$:

$$\lambda_0(|\mathbf{k}|^2) > \lambda_1(|\mathbf{k}|^2) .... \geq \lambda_n(|\mathbf{k}|^2) \geq ... \geq 0... \geq -\lambda_n(|\mathbf{k}|^2) \geq ... - \lambda_1(|\mathbf{k}|^2) > -\lambda_0(|\mathbf{k}|^2),$$

(see Figure 6.1 for positive eigenvalues) corresponding to eigenvectors, depending on $\mathbf{x}$ as $e^{i\mathbf{k}\cdot\mathbf{x}}$. The largest eigenvalue reaches a maximum λ_0 at k_c^2. Now the lattice Γ is well defined, thanks to (3.1). When $\mathbf{k}$ varies in Γ, the set of values for $|\mathbf{k}|$ is dense on the half positive line. It results that the spectrum (closed in $\mathbb{R}$) of $\mathcal{A}$ is the closed interval $[-\lambda_0, \lambda_0]$. Moreover, as this is useful later, we notice that for all $\mathbf{k} \in \Gamma$,

$$\lambda_0 - \lambda_j(|\mathbf{k}|^2) > \delta_0, \ \lambda_0(|\mathbf{k}|^2) - \lambda_j(|\mathbf{k}|^2) \geq \delta_0(|k|) > 0, \ j = 1, 2, ..., \infty, \tag{6.10}$$

with $\delta_0(|k|) > \delta_0 > 0$ for $|k|$ close to k_c.

2.6.2 Pseudo-inverse of $\mathcal{A} - \lambda_0 \mathbb{I}$

Let us define the orthogonal projection $\mathbf{P}_0$ on the kernel of $\mathcal{A} - \lambda_0 \mathbb{I}$: for any $u \in \mathcal{K}_{0,s}$

$$\mathbf{P}_0 u = \sum_{1 \leq j \leq Q} \gamma_j \boldsymbol{\xi}_j, \quad \gamma_j = \frac{\langle u, \boldsymbol{\xi}_j \rangle_{0,0}}{\langle \boldsymbol{\xi}_1, \boldsymbol{\xi}_1 \rangle_{0,0}}, \tag{6.11}$$

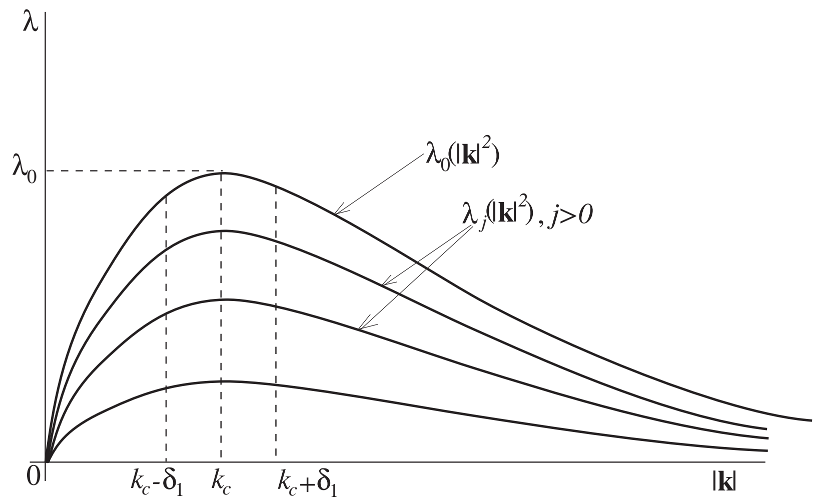

Figure 6.1: Sketch of positive eigenvalues of $\mathcal{A}$ in function of $|\mathbf{k}|$, and definition of critical lambda λ_0. δ_1 is used to define the projection π_0 at section 2.7.4

where we notice that

$$\langle \boldsymbol{\xi}_1, \boldsymbol{\xi}_1 \rangle_{0,0} = \langle \boldsymbol{\xi}_j, \boldsymbol{\xi}_j \rangle_{0,0}, \quad j = 2, .., 2q.$$

We denote by $\mathbf{Q}_0 = \mathbb{I} - \mathbf{P}_0$ the projection on the complementary space (of codimension $2q$). Since the eigenvectors $\boldsymbol{\xi}_j$ belong to $\mathcal{K}_{0,s}$ for any s, the projection $\mathbf{Q}_0$ is bounded in $\mathcal{K}_{0,s}$ for any s. Notice that when u is invariant under $\mathbf{R}_{\pi/q}$, then $\gamma_j = \gamma_1$ for $j = 2, .., 2q$.

Now coming back to the linear system

$$(\mathcal{A} - \lambda_0)u = G,$$

where $G \in \mathcal{K}_{0,s}$ satisfies the compatibility condition $\mathbf{P}_0 G = 0$, the above estimate (6.7), and the form (6.6) of $c(|\mathbf{k}|^2)$ show that there is a unique solution u satisfying $\mathbf{P}_0 u = 0$ and there exists a constant $c > 0$ such that

$$||u_{\mathbf{k}}||_0 \leq c \left[\frac{(1 - \delta_{k_c}(|\mathbf{k}|))(1 + |\mathbf{k}|^2)^2}{(|\mathbf{k}|^2 - k_c^2)^2} + \delta_{k_c}(|\mathbf{k}|) \right] ||G_{\mathbf{k}}||_0,$$

where $\delta_{k_c}(|\mathbf{k}|) = 1$ if $|\mathbf{k}| = k_c$, and $= 0$ otherwise. By using the diophantine inequality (3.5), this leads to the following

Lemma 6.4 *Assuming $\lambda_0''(|\mathbf{k}|^2)|_{|\mathbf{k}|=k_c} \neq 0$ (see (2.25)) for the second derivative of λ_0 at $|\mathbf{k}| = k_c$, then for any $s \geq 0$, the linear operator $(\mathcal{A} - \lambda_0)$ has a bounded inverse from the subspace $\mathbf{Q}_0\mathcal{K}_{0,s}$ to the subspace $\mathbf{Q}_0\mathcal{K}_{0,s-4l_0}$. In other words, for any $\delta_1 > 0$ small enough,* *there exists $c > 0$ such that for u solution in $\mathbf{Q}_0\mathcal{K}_{0,s-4l_0}$ of $(\mathcal{A}-\lambda_0)u = G \in \mathbf{Q}_0\mathcal{K}_{0,s}$, the following estimate holds*

$$\begin{aligned}
||u_{\mathbf{k}}||_0 &\leq c(1+N_{\mathbf{k}}^2)^{2l_0}||G_{\mathbf{k}}||_0, \text{ for } ||\mathbf{k}|-k_c| < \delta_1, \\
||u_{\mathbf{k}}||_0 &\leq \frac{c}{\delta_1^2}||G_{\mathbf{k}}||_0, \text{ for } ||\mathbf{k}|-k_c| \geq \delta_1.
\end{aligned}$$

2.6.3 Formal power series for bifurcating solution

Let us rewrite the system (5.17) as

$$(\mathcal{A}-\lambda_0)u = -\mu u + \mathcal{B}(u,u), \tag{6.12}$$

where

$$\lambda_0 = \frac{1}{\sqrt{\mathcal{R}_c}}, \ \lambda = \lambda_0 - \mu.$$

We are looking for a solution of (6.12) in $\mathcal{K}_{0,s}$, $s > d/2$, which is invariant under $\mathbf{R}_{\pi/q}$ under the form of a formal expansion

$$u = \sum_{n\geq 1}\varepsilon^n u_n, \tag{6.13}$$

$$\mu = \sum_{n\geq 1}\varepsilon^n \mu_n \tag{6.14}$$

where in fact $u_n \in \mathcal{D}_{1/2,s}$ (see Lemma 5.8). Identifying powers of ε at orders ε, ε^2, ε^3, leads to the system

$$(\mathcal{A}-\lambda_0)u_1 = 0, \tag{6.15}$$

$$(\mathcal{A}-\lambda_0)u_2 = -\mu_1 u_1 + \mathcal{B}(u_1,u_1) \tag{6.16}$$

$$(\mathcal{A}-\lambda_0)u_3 = -\mu_1 u_2 - \mu_2 u_1 + 2\mathcal{B}(u_1,u_2). \tag{6.17}$$

Equation (6.15) gives (here we choose the coefficient in front of the eigenvector, which determines the parameter ε)

$$u_1 = \sum_{1\leq j\leq 2q}\boldsymbol{\xi}_j, \tag{6.18}$$

which is invariant under $\mathbf{R}_{\pi/q}$, and we observe, thanks to property (5.6), still valid for $\mathcal{B}$ that

$$\langle \mathcal{B}(u_1,u_1), u_1\rangle_{0,0} = 0,$$

and since

$$\mathbf{R}_{\pi/q}\mathcal{B}(u_1,u_1) = \mathcal{B}(u_1,u_1),$$

this means that (see the definition of projection $\mathbf{P}_0$ in (6.11))

$$\mathbf{P}_0\mathcal{B}(u_1,u_1)=0,$$

hence equation (6.16) is solvable with $\mu_1=0$, and since the Fourier series of $\mathcal{B}(u_1,u_1)$ is finite, we find a unique $u_2\in\mathcal{D}_{1/2,s}$, orthogonal to u_1 in $\mathcal{K}_{0,0}$:

$$u_2=\widetilde{(\mathcal{A}-\lambda_0)}^{-1}\mathcal{B}(u_1,u_1), \tag{6.19}$$

which is invariant under $\mathbf{R}_{\pi/q}$, and where $\widetilde{(\mathcal{A}-\lambda_0)}^{-1}$ is the pseudo-inverse of $(\mathcal{A}-\lambda_0)$ as defined by Lemma 6.4. Now, the compatibility condition for solving (6.17) gives

$$\langle\mu_2u_1-2\mathcal{B}(u_1,u_2),u_1\rangle_{0,0}=0. \tag{6.20}$$

Then we use the identity (5.7) to obtain

$$\begin{aligned}\langle 2\mathcal{B}(u_1,u_2),u_1\rangle_{0,0} &= -\langle\mathcal{B}(u_1,u_1),u_2\rangle_{0,0}\\ &= -\langle(\mathcal{A}-\lambda_0)u_2,u_2\rangle_{0,0}>0.\end{aligned}$$

The result above, in the periodic case, was first obtained by V.Yudovich in [74] (see also Appendix 6.1.4 in Chapter 1). The last inequality results from the fact that $\mathbf{P}_0u_2=0$, and from the property (6.9). It results that μ_2 is positive, determined by

$$\mu_2=\frac{-\langle(\mathcal{A}-\lambda_0)u_2,u_2\rangle_{0,0}}{\langle u_1,u_1\rangle_{0,0}}>0. \tag{6.21}$$

Now the unique solution u_3 of (6.17), orthogonal to u_1 in $\mathcal{K}_{0,0}$, takes the form

$$u_3=2\widetilde{(\mathcal{A}-\lambda_0)}^{-1}\mathbf{Q}_0\mathcal{B}(u_1,u_2), \tag{6.22}$$

and is invariant under $\mathbf{R}_{\pi/q}$ and again lies in $\mathcal{D}_{1/2,s}$ because of the finiteness of its Fourier series. Now, from

$$(\mathcal{A}-\lambda_0)u_4=-\mu_2u_2-\mu_3u_1+2\mathcal{B}(u_1,u_3)+\mathcal{B}(u_2,u_2),$$

we observe that μ_2u_2 is orthogonal to u_1. The factors $e^{i\mathbf{k}\cdot\mathbf{x}}$ in the expression for $2\mathcal{B}(u_1,u_3)+\mathcal{B}(u_2,u_2)$ are such that

$$\mathbf{k}=\sum_{1\leq j\leq 2q}m_j\mathbf{k}_j,\ \text{ and } \sum m_j=4,$$

so that the scalar product with u_1 may be different from 0, as this can be seen in the case when q is a multiple of 3. It results from the compatibility condition, that

$$\mu_3=\frac{\langle 2\mathcal{B}(u_1,u_3)+\mathcal{B}(u_2,u_2),u_1\rangle_{0,0}}{\langle u_1,u_1\rangle_{0,0}}, \tag{6.23}$$

and

$$u_4 = \widetilde{(\mathcal{A}-\lambda_0)}^{-1}\mathbf{Q}_0[-\mu_2 u_2 + 2\mathcal{B}(u_1,u_3) + \mathcal{B}(u_2,u_2)]$$

is invariant under $\mathbf{R}_{\pi/q}$ and still in $\mathcal{D}_{1/2,s}$. Going on the computation, we obtain in particular

$$\mu_4 = \frac{\langle 2\mathcal{B}(u_1,u_4) + 2\mathcal{B}(u_2,u_3), u_1\rangle_{0,0}}{\langle u_1,u_1\rangle_{0,0}}. \tag{6.24}$$

We show in [43] that we can go on in computing the successive terms of the series which appear to be of *Gevrey type*. Making an incomplete Borel resummation of these series, invariant under $\mathbf{R}_{\pi/q}$, provides a solution of (6.12) up to an exponentially small term as ε tends towards 0. Our purpose now is to improve such a result in proving that there exist indeed quasipatterns solutions of (6.12).

2.7 Adapted formulation and Splitting of the space

2.7.1 Decomposition of u

In all what follows, we study functions u, v in $\mathcal{K}_{0,s}$, invariant under rotations $\mathbf{R}_{\pi/q}$. In this frame the kernel of the linear operator $(\mathcal{A}-\lambda_0)$ is one-dimensional. Let us define the new unknown function $\tilde{v}$ in rewriting the solution of (6.12) in $\mathcal{K}_{0,s}$, $s > d/2$ as

$$\begin{aligned} u &= u_\varepsilon + \varepsilon^4\widetilde{v},\ \mu = \mu_\varepsilon + \varepsilon^3\mu', \\ u_\varepsilon &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4, \\ \mu_\varepsilon &= \varepsilon^2\mu_2 + \varepsilon^3\mu_3, \qquad \widetilde{v} \in \{u_1\}^\perp \cap \mathcal{K}_{0,s}, \end{aligned} \tag{7.1}$$

where the coefficients $u_1, u_2, u_3, u_4, \mu_2, \mu_3$ are defined above, and we assume below that $\varepsilon > 0$ (the same proof applies for $\varepsilon < 0$). Then

$$(\mathcal{A}-\lambda_0)u_\varepsilon = -\mu_\varepsilon u_\varepsilon + \mathcal{B}(u_\varepsilon,u_\varepsilon) + \varepsilon^5 f_\varepsilon$$

where f_ε is a known quasiperiodic function with a finite Fourier expansion with $N_{\mathbf{k}} \leq 8$. Now we have by (6.12):

$$(\mathcal{A}-\lambda_0)(u_\varepsilon + \varepsilon^4\widetilde{v}) + (\mu_\varepsilon + \varepsilon^3\mu')(u_\varepsilon + \varepsilon^4\widetilde{v}) - \mathcal{B}(u_\varepsilon + \varepsilon^4\widetilde{v}, u_\varepsilon + \varepsilon^4\widetilde{v}) = 0,$$

which becomes

$$\mathfrak{L}_\varepsilon v + \varepsilon^3\mu'\widetilde{v} + \mu'\varepsilon^{-1}u_\varepsilon + \varepsilon f_\varepsilon - \varepsilon^4\mathcal{B}(\widetilde{v},\widetilde{v}) = 0, \tag{7.2}$$

with

$$\mathfrak{L}_\varepsilon\widetilde{v} = (\mathcal{A}-\lambda_0+\mu_\varepsilon)\widetilde{v} - 2\mathcal{B}(u_\varepsilon,\widetilde{v}). \tag{7.3}$$

2.7.2 Decomposition of the system

Let us use the projection $\mathbf{Q}_0 = \mathbb{I} - \mathbf{P}_0$ on the orthogonal complement of u_1 in the subspace of $\mathcal{K}_{0,0}$ invariant under $\mathbf{R}_{\pi/q}$, defined at subsection 2.6.2. We might notice that the formal computation made at section 2.6.3 gives $\widetilde{v} = \varepsilon u_5 + \mathcal{O}(\varepsilon^2)$ in $\{u_1\}^{\perp}$, with $\mu' = \varepsilon\mu_4 + \mathcal{O}(\varepsilon^2)$.

Equation (7.2) decomposes into the *bifurcation equation*, using the projection $\mathbf{P}_0$ onto the kernel $\{u_1\}$ of $(\mathcal{A} - \lambda_0)$:

$$\mu' u_1 + \varepsilon \mathbf{P}_0 f_\varepsilon - 2\mathbf{P}_0\mathcal{B}(\widetilde{v}, u_\varepsilon) - \varepsilon^4 \mathbf{P}_0\mathcal{B}(\widetilde{v}, \widetilde{v}) = 0, \tag{7.4}$$

with

$$\mathbf{P}_0 f_0 = -\mu_4 u_1,$$

and the *range equation* (projection onto $\{u_1\}^{\perp}$) :

$$\mathbf{Q}_0\mathfrak{L}_\varepsilon\widetilde{v} + \varepsilon^3\mu'\widetilde{v} + \widetilde{g}(\varepsilon, \mu') - \varepsilon^4\mathbf{Q}_0\mathcal{B}(\widetilde{v}, \widetilde{v}) = 0, \tag{7.5}$$

where

$$\mathbf{Q}_0\mathfrak{L}_\varepsilon = \mathbf{Q}_0(\mathcal{A} - \lambda_0 + \mu_\varepsilon) - 2\mathbf{Q}_0\mathcal{B}(u_\varepsilon, \cdot),$$
$$\widetilde{g}(\varepsilon, \mu') := \mu'\varepsilon(u_2 + \varepsilon u_3 + \varepsilon^2 u_4) + \varepsilon\mathbf{Q}_0 f_\varepsilon.$$

2.7.3 Optimization of variables

In what follows, we need to obtain a solution of (7.5) which is C^2- bounded in $\widetilde{\mu}$. So,we need to have operators and functions in (7.5) with bounded first and second derivatives with respect to $\widetilde{\mu} = \varepsilon^3\mu'$. This is not the case for the term $\widetilde{g}(\varepsilon, \mu')$, so we need to slightly modify the definition of $\widetilde{v}$, in such a way that $\widetilde{g}(\varepsilon, \mu')$ has a more suitable form.

Let us define (see Lemma 5.7, using that $u_\varepsilon \in \mathcal{K}_{0,t}$ for all $t > 0$) the linear operator $\mathcal{S}_\varepsilon$ bounded by $c_s\varepsilon$ in $\mathbf{Q}_0\mathcal{K}_{0,s}$ for any $s \geq 0$, as

$$\begin{aligned} \mathbf{Q}_0\mathfrak{L}_\varepsilon &= \mathbf{Q}_0(\mathcal{A} - \lambda_0) + \mathcal{S}_\varepsilon, \\ \mathcal{S}_\varepsilon &:= \mu_\varepsilon - 2\mathbf{Q}_0\mathcal{B}(u_\varepsilon, \cdot). \end{aligned}$$

We notice that $\widetilde{g}(\varepsilon, \mu')$ has a finite Fourier expansion with $N_{\mathbf{k}} \leq 8$ (because of f_ε). Hence $[\mathbf{Q}_0(\mathcal{A} - \lambda_0)]^{-1}\widetilde{g}(\varepsilon, \mu') \in \mathbf{Q}_0\mathcal{K}_{0,s}$ for any $s \geq 0$. In the same way, we can define

$$h(\varepsilon, \mu') = \left\{ \mathbb{I} + \sum_{n=1,2,3,4} (-1)^n \left([\mathbf{Q}_0(\mathcal{A} - \lambda_0)]^{-1}(\mathcal{S}_\varepsilon + \widetilde{\mu}) \right)^n \right\} [\mathbf{Q}_0(\mathcal{A} - \lambda_0)]^{-1}\widetilde{g}(\varepsilon, \mu'), \tag{7.6}$$

which is still well defined in $\mathbf{Q}_0\mathcal{K}_{0,s}$ for $s \geq 0$, and h is analytic in its arguments (ε, μ'). Indeed, the operator $\left([\mathbf{Q}_0(\mathcal{A} - \lambda_0)]^{-1}(\mathcal{S}_\varepsilon + \widetilde{\mu})\right)^n$ is bounded on finite Fourier series in $e^{i\mathbf{k}\cdot\mathbf{x}}$

leading to a finite Fourier series with $N_{\mathbf{k}}$ increased by $4n$. Finally $h(\varepsilon,\mu')$ has a finite Fourier expansion with wave vectors bounded by $N_{\mathbf{k}}=16+8=24$.

We can now check that

$$(\mathbf{Q}_0\mathfrak{L}_\varepsilon+\widetilde{\mu})h(\varepsilon,\mu')=\widetilde{g}(\varepsilon,\mu')+\left((\mathcal{S}_\varepsilon+\widetilde{\mu})[\mathbf{Q}_0(\mathcal{A}-\lambda_0)]^{-1}\right)^5\widetilde{g}(\varepsilon,\mu').$$

We do not use Neumann series for inverting $(\mathbf{Q}_0\mathfrak{L}_\varepsilon+\widetilde{\mu})$ because of the small divisor difficulty. We notice that $\varepsilon^2\widetilde{g}(\varepsilon,\mu')=\widetilde{\mu}(u_2+\varepsilon u_3+\varepsilon^2u_4)+\varepsilon^3\mathbf{Q}_0f_\varepsilon$, hence $\varepsilon^2h(\varepsilon,\mu'):=\widetilde{h}(\varepsilon,\widetilde{\mu})$ is analytic in $(\varepsilon,\widetilde{\mu})$ with

$$||\widetilde{h}(\varepsilon,\widetilde{\mu})||_{0,s}\leq c_s(\varepsilon^3+|\widetilde{\mu}|).$$

Now we define the new v as

$$v=\widetilde{v}+h(\varepsilon,\mu'), \tag{7.7}$$

so that (7.5) becomes

$$\mathfrak{L}_{\varepsilon,\widetilde{\mu}}v+g(\varepsilon,\widetilde{\mu})-\varepsilon^4\mathbf{Q}_0\mathcal{B}(v,v)=0, \tag{7.8}$$

with

$$\begin{aligned}\mathfrak{L}_{\varepsilon,\widetilde{\mu}} &= \mathbf{Q}_0\mathfrak{L}_\varepsilon+\widetilde{\mu}+2\varepsilon^2\mathbf{Q}_0\mathcal{B}(\widetilde{h}(\varepsilon,\widetilde{\mu}),\cdot), \\ g(\varepsilon,\widetilde{\mu}) &= -\left((\mathcal{S}_\varepsilon+\widetilde{\mu})[\mathbf{Q}_0(\mathcal{A}-\lambda_0)]^{-1}\right)^5\widetilde{g}(\varepsilon,\mu')-\mathbf{Q}_0\mathcal{B}(\widetilde{h}(\varepsilon,\widetilde{\mu}),\widetilde{h}(\varepsilon,\widetilde{\mu})).\end{aligned} \tag{7.9}$$

We notice that the first term on the right hand side of $g(\varepsilon,\widetilde{\mu})$ is now C^4- bounded in $\widetilde{\mu}$ since, up to order $\widetilde{\mu}^4$ it is analytic, and the non analyticity only occurs at orders $\varepsilon^2\mu'\widetilde{\mu}^4$ and $\varepsilon\mu'\widetilde{\mu}^5$. Since we restrict to $\widetilde{\mu}\in[-\varepsilon,\varepsilon]$ the values for $\widetilde{\mu}$, we finally obtain in (7.8) the required properties for all terms, with

$$\begin{aligned}||g(\varepsilon,\widetilde{\mu})||_{0,s} &\leq c_s\varepsilon^2, ||\partial_{\varepsilon,\widetilde{\mu}}g(\varepsilon,\widetilde{\mu})||_{0,s}\leq c_s\varepsilon^2, \\ ||\partial^2_{\widetilde{\mu}^2}g(\varepsilon,\widetilde{\mu})||_{0,s} &\leq c_s, ||\partial^2_{\varepsilon^2}g(\varepsilon,\widetilde{\mu})||_{0,s}\leq c_s\varepsilon^2, ||\partial^2_{\varepsilon\widetilde{\mu}}g(\varepsilon,\widetilde{\mu})||_{0,s}\leq c_s\varepsilon^2.\end{aligned} \tag{7.10}$$

Let us define the linearized operator

$$\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}:=\mathfrak{L}_{\varepsilon,\widetilde{\mu}}-2\varepsilon^4\mathbf{Q}_0\mathcal{B}(V,\cdot),$$

for $V\in\mathcal{K}_{0,s}$ for $s>d/2$. Then, we need a careful study of this linearized operator for applying the result of [5].

Lemma 7.1 *The operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}$ is analytic in its arguments for $(\varepsilon,\widetilde{\mu},V)\in(0,\varepsilon_0)\times[-\varepsilon,\varepsilon]\times\mathbf{Q}_0\mathcal{K}_{0,s}, s\geq s_0>d/2$. It is acting in $\mathbf{Q}_0\mathcal{K}_{0,t}$ for $t\in[0,s]$ (see the result of Lemma 5.7),with*

$$\begin{aligned}\mathfrak{L}_{\varepsilon,\widetilde{\mu},V} &: = \mathbf{Q}_0(\mathcal{A}-\lambda_0)+\widetilde{\mu}+\mathcal{R}_{\varepsilon,\widetilde{\mu}}-2\varepsilon^4\mathbf{Q}_0\mathcal{B}(V,\cdot) \\ \mathcal{R}_{\varepsilon,\widetilde{\mu}} &= \mu_\varepsilon-2\mathbf{Q}_0\mathcal{B}(u_\varepsilon-\varepsilon^2\widetilde{h}(\varepsilon,\widetilde{\mu}),\cdot),\end{aligned} \tag{7.11}$$

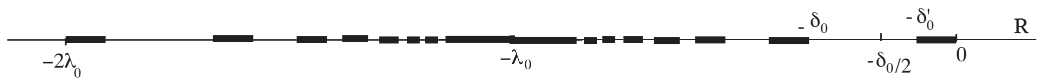

Figure 7.1: Spectrum of $\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0$

and, for $||V||_{0,s_0}\leq 1$, *we have the estimates*

$$
\begin{aligned}
||\mathcal{R}_{\varepsilon,\widetilde{\mu}}v||_{0,s} &\leq c_s\varepsilon||v||_{0,s},\\
||\partial_{\widetilde{\mu}}\mathcal{R}_{\varepsilon,\widetilde{\mu}}v||_{0,s}+||\partial^2_{\widetilde{\mu}^2}\mathcal{R}_{\varepsilon,\widetilde{\mu}}v||_{0,s}+||\partial^2_{\varepsilon\widetilde{\mu}}\mathcal{R}_{\varepsilon,\widetilde{\mu}}v||_{0,s} &\leq c_s\varepsilon^2||v||_{0,s},\\
||\partial_{\varepsilon}\mathcal{R}_{\varepsilon,\widetilde{\mu}}v||_{0,s}+||\partial^2_{\varepsilon^2}\mathcal{R}_{\varepsilon,\widetilde{\mu}}v||_{0,s} &\leq c_s||v||_{0,s},\\
||2\varepsilon^4\mathbf{Q}_0\mathcal{B}(V,v)||_{0,s} &\leq c_s\varepsilon^4(||v||_{0,s}+||V||_{0,s}||v||_{0,s_0}).
\end{aligned}
$$

2.7.4 First splitting of the space (operator π_0)

We are interested in the inversion of the operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}$ in a certain subspace. The first difficulty comes from the infinite dimension of the system, despite of the use of a projection Π_N suppressing the Fourier modes $e^{i\mathbf{k}\cdot\mathbf{x}}$ such that $N_{\mathbf{k}}>N$. So, we use now the property described in (6.10) for the spectrum of the operator $\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0$ which is selfadjoint in $\mathcal{K}_{0,s}$:

$$
\begin{aligned}
\lambda_0-\lambda_0(|\mathbf{k}|^2) &\geq 0,\\
\lambda_0-\lambda_j(|\mathbf{k}|^2) &> \delta_0>0,\quad j=1,2,...\\
\lambda_0(|\mathbf{k}|^2) &\to 0 \text{ as } |\mathbf{k}|\to 0 \text{ or } \infty.
\end{aligned}
$$

Let us consider $\delta_1>0$, defined at Lemma 6.4. Then for $\mathbf{k}\in\Gamma$, the inequality $||\mathbf{k}|-k_c|>\delta_1$ implies $\lambda_0-\lambda_0(|\mathbf{k}|^2)>\delta_0'$ $(=\mathcal{O}(\delta_1^2)$ (recall that $\lambda_0(|\mathbf{k}|^2)$ is analytic in $|\mathbf{k}|^2$ with a maximum λ_0 in k_c) and choose δ_1 small enough for having $\delta_0'<\delta_0/2$. We now define the projection π_0, orthogonal in $\mathbf{Q}_0\mathcal{K}_{0,s}$, for any $s\geq 0$, which consists in eliminating the Fourier modes $\mathbf{k}\in\Gamma$ such that $||\mathbf{k}|-k_c|>\delta_1$. We give at Figure 7.1 a sketch of the spectrum of the selfadjoint operator $\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0$. We notice that the selfadjoint operator

$$(\mathbb{I}-\pi_0)\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0(\mathbb{I}-\pi_0)$$

has an inverse bounded by $1/\delta_0'$, since its eigenvalues (dense in the spectrum) are in absolute value larger than δ_0'.

Then, for $|\widetilde{\mu}|\leq\varepsilon$ and $||V||_{0,s_0}\leq 1$, $s_0>d/2$, the operator

$$(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)$$

is a perturbation of order ε of $(\mathbb{I}-\pi_0)\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0(\mathbb{I}-\pi_0)$ (see (7.11)): for ε_0 small enough, we have for $s\in[0,s_0]$,

$$||\widetilde{\mu}+\mu_\varepsilon-2\mathbf{Q}_0\mathcal{B}(u_\varepsilon,\cdot)-2\varepsilon^4\mathbf{Q}_0\mathcal{B}(V,\cdot)||_{0,s}\leq c\varepsilon, \tag{7.12}$$

hence, for ε_0 small enough, and $\delta_0'>2c\varepsilon$, the operator $(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)$ has an inverse bounded by $2/\delta_0'$ in $(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s_0}$. Notice that a true estimate of the inverse in $\mathbf{Q}_0\mathcal{K}_{0,s}$ for $s>s_0$ would need a bound for $||V||_{0,s}$, which we have not, except for $s=s_0$. Let us now show that the inversion of $\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}$ reduces to the inversion of a small perturbation $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ of $\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\pi_0$ in $\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}$ for $d/2<s_0$.

Indeed, let us consider the linear system

$$\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}v=f\in\mathbf{Q}_0\mathcal{K}_{0,s_0}. \tag{7.13}$$

This leads to

$$\begin{aligned}\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(v_0+v_1) &= \pi_0 f,\\ (\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(v_0+v_1) &= (\mathbb{I}-\pi_0)f,\end{aligned}$$

where

$$v_0=\pi_0 v,\quad v_1=(\mathbb{I}-\pi_0)v.$$

Solving first with respect to v_1 gives

$$v_1=\mathfrak{Q}^{(1,1)}(\mathbb{I}-\pi_0)f+\mathfrak{Q}^{(1,0)}v_0, \tag{7.14}$$

with bounded operators $\mathfrak{Q}^{(1,1)}$ and $\mathfrak{Q}^{(1,0)}$ defined by

$$\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V} =: [(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)]^{-1}\in\mathcal{L}((\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s_0}), \tag{7.15}$$

$$\mathfrak{Q}^{(1,0)}_{\varepsilon,\widetilde{\mu},V} =: -\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\in\mathcal{L}(\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0},(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s_0}). \tag{7.16}$$

Then the system satisfied by v_0 becomes

$$\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}v_0=\pi_0 f+\mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)f, \tag{7.17}$$

with

$$\mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}=:-\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}\in\mathcal{L}((\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s_0},\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}) \tag{7.18}$$

$$\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}:=\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}[\mathbb{I}+\mathfrak{Q}^{(1,0)}_{\varepsilon,\widetilde{\mu},V}]\pi_0\in\mathcal{L}(\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}). \tag{7.19}$$

We show in the next subsection, for $V \in \mathbf{Q}_0\mathcal{K}_{0,s}$ such that $||V||_{0,s_0} < 1$ and δ_0' well chosen, that there exists $c(s) > 0$ with the following tame estimates, valid for $d/2 < s_0 \leq s \leq \overline{s}$ and $0 \leq \varepsilon \leq \varepsilon_1(\overline{s})$:

$$\begin{aligned} ||\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leq \frac{c(s)}{\delta_0'}\{||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}\} \ \forall v \in (\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s}, \qquad (7.20)\\ ||\mathfrak{Q}^{(1,0)}_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leq \frac{c(s)}{\delta_0'}\varepsilon\{||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}\} \ \forall v \in \pi_0\mathbf{Q}_0\mathcal{K}_{0,s},\\ ||\mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leq \frac{c(s)}{\delta_0'}\varepsilon\{||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}\} \ \forall v \in (\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s}. \end{aligned}$$

2.7.5 Structure of $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$

We need to study the structure of $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ defined by (7.19). This is summed up in the following

Lemma 7.2 *For s such that $\overline{s} \geq s \geq s_0 > d/2$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_1(\overline{s}) \leq \varepsilon_0, |\widetilde{\mu}| \leq \varepsilon_0$, and $V \in \mathbf{Q}_0\mathcal{K}_{0,s}$, with $||V||_{0,s_0} \leq 1$, we have*

$$\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} = \pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0 + \widetilde{\mu} + \mathfrak{B}_\varepsilon + \varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}} + \mathfrak{R}_{\varepsilon,\widetilde{\mu},V}, \qquad (7.21)$$

with

$$\mathfrak{B}_\varepsilon = -2\pi_0\mathbf{Q}_0\mathcal{B}(u_\varepsilon,\cdot)\mathbf{Q}_0\pi_0 + \mathcal{O}(\varepsilon^2),$$

$\mathfrak{B}_\varepsilon$, $\mathfrak{C}_{\varepsilon,\widetilde{\mu}}$ and $\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}$ depend analytically on their arguments, with $\mathfrak{R}_{\varepsilon,\widetilde{\mu},0} = 0$ and a constant $c(s)$ such that for any $v \in \pi_0\mathbf{Q}_0\mathcal{K}_{0,s}$

$$\begin{aligned} ||\mathfrak{B}_\varepsilon v||_{0,s} &\leq c\varepsilon||v||_{0,s},\\ ||\mathfrak{C}_{\varepsilon,\widetilde{\mu}}v||_{0,s} + ||\partial_{\widetilde{\mu}}\mathfrak{C}_{\varepsilon,\widetilde{\mu}}v||_{0,s} &\leq c||v||_{0,s},\\ ||\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leq c\varepsilon^4\{||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}\}, \qquad (7.22)\\ ||\partial_{\widetilde{\mu}}\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leq c\varepsilon^4\{||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}\},\\ ||\partial_\varepsilon\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leq c\varepsilon^3\{||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}\}. \end{aligned}$$

Proof. We examine first $\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}$ which is the inverse of $(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)$. Thanks to (7.11), we can write

$$(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0) = (\mathbb{I}-\pi_0)\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0(\mathbb{I}-\pi_0) + \widetilde{\mu}\mathbb{I}d + (\mathbb{I}-\pi_0)\mathcal{P}(\varepsilon,\widetilde{\mu},V)(\mathbb{I}-\pi_0), \qquad (7.23)$$

where $\mathbb{I}d$ is the identity in the subspace $(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s}$ and

$$\mathcal{P}(\varepsilon,\widetilde{\mu},V) =: \mu_\varepsilon - 2\mathbf{Q}_0\mathcal{B}(u_\varepsilon - \varepsilon^2\widetilde{h}(\varepsilon,\widetilde{\mu}),\cdot) - 2\varepsilon^4\mathbf{Q}_0\mathcal{B}(V,\cdot).$$

Now, for $V \in \mathbf{Q}_0\mathcal{K}_{0,s}$, the operator $\mathcal{P}(\varepsilon, \widetilde{\mu}, V)$ takes values in $\mathcal{L}((\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s})$ for $d/2 < s_0 \leq s \leq \overline{s}$, and satisfies for $\varepsilon \in [0, \varepsilon_0]$, ε_0 small enough and $||V||_{0,s_0} \leq 1, |\widetilde{\mu}| \leq \varepsilon$,

$$||(\mathbb{I}-\pi_0)\mathcal{P}(\varepsilon, \widetilde{\mu}, V)(\mathbb{I}-\pi_0)v||_{0,s} \leq c\{\varepsilon||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}\}.$$

Let us define the operator

$$\mathfrak{S} =: [(\mathbb{I}-\pi_0)\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0(\mathbb{I}-\pi_0)]^{-1}(\mathbb{I}-\pi_0)\{\widetilde{\mu} + \mathcal{P}(\varepsilon, \widetilde{\mu}, V)(\mathbb{I}-\pi_0)\},$$

then

$$[(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)]^{-1} = (\mathbb{I}+\mathfrak{S})^{-1}[(\mathbb{I}-\pi_0)\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0(\mathbb{I}-\pi_0)]^{-1},$$

then, we need to invert $(\mathbb{I}+\mathfrak{S})$ in checking a tame estimate.

For $\varepsilon < \varepsilon_1(\overline{s}) \leq \varepsilon_0$, $|\widetilde{\mu}| \leq \varepsilon$ and $||V||_{0,s_0} \leq 1$, there exists a constant $c > 0$ such that for any $v \in (\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s}$

$$||\mathfrak{S}v||_{0,s} \leq \frac{c}{\delta_0'}[\varepsilon||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}],$$

and for ε_0 small enough such that $(\varepsilon + \varepsilon^4) \leq 2\varepsilon$, we have for any $p \in \mathbb{N}$

$$||\mathfrak{S}^p v||_{0,s} \leq \frac{c}{\delta_0'}(\frac{2c\varepsilon_0}{\delta_0'})^{p-1}[(|\widetilde{\mu}| + \varepsilon)||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}],$$

hence for any $v \in (\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s}$

$$||(\mathbb{I}+\mathfrak{S})^{-1}v||_{0,s} \leq ||v||_{0,s} + \frac{c}{\delta_0'}(1 - \frac{2c\varepsilon_0}{\delta_0'})^{-1}[\varepsilon||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}].$$

It results that, for $\delta_0' > 4c\varepsilon_0$, $[(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)]^{-1} = \mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}$ is analytic in its arguments and satisfies, for ε_0 small enough, the estimate

$$||\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}v||_{0,s} \leq c'/\delta_0'(||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}) \quad \forall v \in (\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s}$$

which is the first part of (7.20). Coming back to (7.19) with (7.16) and (7.18), we observe that

$$(\mathbb{I}-\pi_0)\mathbf{Q}_0[(\mathcal{A}-\lambda_0) + \mu_\varepsilon + \widetilde{\mu}]\mathbf{Q}_0\pi_0 = \pi_0\mathbf{Q}_0[(\mathcal{A}-\lambda_0) + \mu_\varepsilon + \widetilde{\mu}]\mathbf{Q}_0(\mathbb{I}-\pi_0) = 0$$

because the coefficients of the linear operator are independent of $\mathbf{x}$. Then

$$\begin{aligned}
(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\pi_0 &= -2(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{B}(u_\varepsilon - \varepsilon^2\widetilde{h}, \cdot)\mathbf{Q}_0\pi_0 - 2\varepsilon^4(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{B}(V, \cdot)\pi_0,\\
\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0) &= -2\pi_0\mathbf{Q}_0\mathcal{B}(u_\varepsilon - \varepsilon^2\widetilde{h}, \cdot)\mathbf{Q}_0(\mathbb{I}-\pi_0) - 2\varepsilon^4\pi_0\mathbf{Q}_0\mathcal{B}(V, \cdot)(\mathbb{I}-\pi_0),
\end{aligned}$$

both operators being of order ε (with the tame estimates), and depending analytically on their arguments. It finally results from (7.16) and (7.18) that the rest of estimate (7.20) holds. Finally

$$\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\mathfrak{Q}^{(1,0)}_{\varepsilon,\widetilde{\mu},V} = \mathfrak{C}^{(1)}_{\varepsilon} + \varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}} + \mathfrak{R}'_{\varepsilon,\widetilde{\mu},V} \tag{7.24}$$

with $\mathfrak{C}^{(1)}_{\varepsilon}, \mathfrak{C}_{\varepsilon,\widetilde{\mu}}$ and $\mathfrak{R}'_{\varepsilon,\widetilde{\mu},V}$ analytic in their arguments, taking values in $\mathcal{L}(\pi_0\mathbf{Q}_0\mathcal{K}_{0,s})$ for $s_0 \leq s \leq \overline{s}$, and a careful examination of (7.15), (7.16), (7.19), (7.11) leads $\forall v \in \pi_0\mathbf{Q}_0\mathcal{K}_{0,s}$, to

$$\mathfrak{C}^{(1)}_{\varepsilon} = \mathcal{O}(\varepsilon^2),$$

$$\begin{aligned} ||\mathfrak{R}'_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leq c\varepsilon^5(||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}), \\ ||\partial_{\varepsilon,\widetilde{\mu}}\mathfrak{R}'_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leq c\varepsilon^4(||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}). \end{aligned}$$

Finally, from (7.19) we can write

$$\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} = \pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\pi_0 + \mathfrak{C}^{(1)}_{\varepsilon} + \varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}} + \mathfrak{R}'_{\varepsilon,\widetilde{\mu},V}, \tag{7.25}$$

where $\mathfrak{R}'_{\varepsilon,\widetilde{\mu},V}$ is at least linear in V. This leads to (7.21), (7.22) and to the result of the Lemma. ∎

Remark 7.3 *We may observe that the spectrum of $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ in $\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}$ results from a perturbation of order ε of the spectrum of the selfadjoint operator $\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0$, the spectrum of which is the closure of the set of eigenvalues $\lambda_j(|\mathbf{k}|^2)-\lambda_0$, $j=0,1,..$, $\mathbf{k}\in\Gamma$, with*

$$\begin{aligned} -\delta_0' &\leq \lambda_0(|\mathbf{k}|^2)-\lambda_0 < 0, \ \textit{and} \ \pm\lambda_j(|\mathbf{k}|^2)-\lambda_0 < -\delta_0, \ j=1,2,.. \\ \textit{and} \ -\lambda_0(|\mathbf{k}|^2)-\lambda_0 &< -\delta_0 \quad \textit{where } \mathbf{k}\in\Gamma \textit{ with } 0<||\mathbf{k}|-k_c|\leq\delta_1. \end{aligned}$$

It results that, for ε small enough, the spectrum of $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ in $\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}$ has a gap in its real part, between $-3\delta_0/4$ and $-\delta_0/2$. Hence the eigenvalues which might be close to 0, are those coming from $\lambda_0(|\mathbf{k}|^2)-\lambda_0$ uniquely, and this allows us to come back to a situation analogue to the one in [6], except for the selfadjointness of the operator which is not true here, starting at order ε.

Remark 7.4 *We notice that the restriction on δ_1 leads to a restriction on $\delta_0' = \mathcal{O}(\delta_1^2)$. The restriction on δ_0' made in the proof of Lemma above is independent of ε_0, for ε_0 small enough.*

Remark 7.5 *The operator $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ depends analytically on $(\varepsilon,\widetilde{\mu},V)$, therefore, we can give its expression for $\varepsilon=0$. From Lemma 7.2 we have*

$$\mathfrak{L}'_{0,\widetilde{\mu},V} = \pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0 + \widetilde{\mu}\mathbb{I}. \tag{7.26}$$

Coming back to the linear equation (7.13), we finally have

Lemma 7.6 *For $s_0 > d/2$, $\overline{s} > s_0$, $0 < \varepsilon \leq \varepsilon_1(\overline{s}) \leq \varepsilon_0$, $|\widetilde{\mu}| \leq \varepsilon_0$, $V \in \mathbf{Q}_0\mathcal{K}_{0,s}$ such that $||V||_{0,s_0} \leq 1$, and $s \in [s_0, \overline{s}]$, assume that there exists $C(s) > 0$ such that*

$$||\mathfrak{L}'^{-1}_{\varepsilon,\widetilde{\mu},V} f_0||_{0,s} \leq C(s)[||f_0||_{0,s} + ||V||_{0,s}||f_0||_{0,s_0}], \text{ for any } f_0 = \pi_0 f, \text{ with } f \in \mathbf{Q}_0\mathcal{K}_{0,s}.$$

Then, for $s \in [s_0, \overline{s}]$, ε_0 small enough and $f \in \mathbf{Q}_0\mathcal{K}_{0,s}$

$$||\mathfrak{L}^{-1}_{\varepsilon,\widetilde{\mu},V} f||_{0,s} \leq C'(s)[||f||_{0,s} + ||V||_{0,s}||f||_{0,s_0}], \tag{7.27}$$

where $C'(s) = 3Cs) + c(s)/\delta_0'$.

Proof. We start with (7.17) and the estimate for $\mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}$ in (7.20). We obtain, for ε small enough

$$\begin{aligned} ||v_0||_{0,s} &\leq C(s)[||\pi_0 f + \mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I} - \pi_0)f||_{0,s} + ||V||_{0,s}||\pi_0 f + \mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I} - \pi_0)f||_{0,s_0}] \\ &\leq 2C(s)[||f||_{0,s} + ||V||_{0,s}||f||_{0,s_0}]. \end{aligned}$$

Using now (7.14) with (7.20), we obtain successively

$$\begin{aligned} ||\mathfrak{Q}^{(1,0)}_{\varepsilon,\widetilde{\mu},V} v_0||_{0,s} &\leq 2\varepsilon \frac{c(s)}{\delta_0'} C(s)[||f||_{0,s} + ||V||_{0,s}||f||_{0,s_0}], \\ ||v_1||_{0,s} &\leq \frac{2c(s)}{\delta_0'}[||f||_{0,s} + ||V||_{0,s}||f||_{0,s_0}], \end{aligned}$$

and $v_0 + v_1$ is $\mathfrak{L}^{-1}_{\varepsilon,\widetilde{\mu},V} f$ for which (7.27) holds in the norm $\mathbf{Q}_0\mathcal{K}_{0,s}$. ■

2.7.6 Projection Π_N

We define the projection Π_N as the suppression of Fourier modes with $\mathbf{k} \in \Gamma$ such that $N_{\mathbf{k}} > N$. The range of this projection is then

$$E_N := \Pi_N \pi_0 \mathbf{Q}_0 \mathcal{K}_{0,s},$$

which is in fact independent of s (however its norm depends on s), and where we do not forget that coefficients are functions of $z \in [0, 1]$, here in L^2. A difference with the spaces E_N occuring in [6] and [5] (for example), is that our E_N is *infinite dimensional*. However *the spectrum of the linear operator $\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} \Pi_N$ is discrete* since, for a given V, it is a perturbation of the operator $\Pi_N \pi_0 \mathbf{Q}_0(\mathcal{A} - \lambda_0)\mathbf{Q}_0 \pi_0 \Pi_N$, where the number of Fourier modes $e^{i\mathbf{k}\cdot\mathbf{x}}$ is finite (number $\mathcal{N}$ bounded by bN^d, d being defined in section 2.3 and b independent of N), and that for any fixed $|\mathbf{k}|$, the spectrum of $\Pi_N \pi_0 \mathbf{Q}_0(\mathcal{A} - \lambda_0)\mathbf{Q}_0 \pi_0 \Pi_N$ is discrete, only composed with eigenvalues of finite multiplicities. Notice also that

$$\Pi_N \pi_0 = \pi_0 \Pi_N,$$

and that Lemma 7.6 is still valid, when restricted to E_N.

2.8 Estimates of the inverse of $(\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N)$

2.8.1 Estimate of $(\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N)^{-1}$ in $\Pi_N\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}$ for small N

Lemma 8.1 *Let $s_0 > d/2$, $V\in\mathcal{K}_{0,s_0}$ satisfies $||V||_{0,s_0}\leq 1$, and assume $(\varepsilon,\widetilde{\mu})\in[0,\varepsilon_0]\times[-\varepsilon,\varepsilon]$. Then for $N\leq M_\varepsilon$, where M_ε is defined by (8.3), we have the following estimates*

$$||(\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N)^{-1}v||_{0,s_0} \leq 2c(1+N^2)^{2l_0}||v||_{0,s_0},\ \text{for } v\in\Pi_N\pi_0 Q_0\mathcal{K}_{0,s_0}, \tag{8.1}$$
$$||(\Pi_N\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\Pi_N)^{-1}v||_{0,s_0} \leq 2cc'(1+N^2)^{2l_0}||v||_{0,s_0}\ \text{for } v\in\Pi_N Q_0\mathcal{K}_{0,s_0}. \tag{8.2}$$

Proof. We use Lemma 7.2 and

$$\begin{aligned}\Pi_N\pi_0\mathbf{Q}_0\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\mathbf{Q}_0\pi_0\Pi_N &= \Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N+\widetilde{\mu}+\\ &\quad +\Pi_N\pi_0\mathbf{Q}_0[\mathfrak{B}_\varepsilon+\varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}}+\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}]\mathbf{Q}_0\pi_0\Pi_N,\end{aligned}$$

with the estimates (for $|\widetilde{\mu}|\leq\varepsilon$)

$$||\Pi_N\pi_0\mathbf{Q}_0[\widetilde{\mu}+\mathfrak{B}_\varepsilon+\varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}}+\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}]\mathbf{Q}_0\pi_0\Pi_N||_{0,s_0}\leq c_1\varepsilon.$$

Now, by construction, and from Lemma 6.4, we have

$$||(\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N)^{-1}||_{0,s_0}\leq c(1+N^2)^{2l_0}.$$

Then, if we have

$$cc_1\varepsilon(1+N^2)^{2l_0}\leq 1/2$$

we can use Neumann series, to invert the operator $(\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N)$ in $\Pi_N\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}$, and obtain (8.1) provided that

$$N\leq M_\varepsilon=\left[\frac{c_2}{\varepsilon^{1/4l_0}}\right]\leq\left(\frac{1}{(2cc_1\varepsilon)^{1/2l_0}}-1\right)^{1/2}, \tag{8.3}$$

where the brackets $[\cdot]$ mean the integer part of. The result for $(\Pi_N\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\Pi_N)^{-1}$ comes from Lemma 7.6. ■

2.8.2 Good set of $\widetilde{\mu}$

Let us define for $M>0$, $s_0>d/2$

$$\begin{aligned}\mathcal{U}_M^{(N)}\ :\ &=\{u\in C^2([0,\varepsilon_1[\times[-\varepsilon,\varepsilon],E_N);\ u(0,\widetilde{\mu})=0,\\ &\quad ||u||_{0,s_0}\leq 1,\ ||\partial_{\varepsilon,\widetilde{\mu}}u||_{0,s_0}\leq M,\ ||\partial^2_{\varepsilon,\widetilde{\mu}}u||_{0,s_0}\leq M\}.\end{aligned} \tag{8.4}$$

We do not forget that Lemma 7.2 says that operator $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ is analytic in $(\varepsilon,\widetilde{\mu},V)$.

Now, for $V \in \mathcal{U}_M^{(N)}$ we need to study the *inverse* of $\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N$ when it exists, in function of $\widetilde{\mu}$ for ε fixed. As an operator in $\mathcal{L}(E_N)$ with the norm induced by $\mathcal{L}(\mathcal{K}_{0,s_0})$, its eigenvalues result from a small perturbation of the selfadjoint operator $\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\pi_0\Pi_N\mathbf{Q}_0$ which has a discrete set of eigenvalues (notice that since we do not impose a bound on $||V||_{0,s}$, the perturbation might not be small for $s > s_0$). Since we are only interested into the eigenvalues very close to 0, the eigenvalues which interest us are the ones which perturb the *(negative) eigenvalues* $\lambda_0(|\mathbf{k}|^2)-\lambda_0$ *close to* 0, obtained for $|\mathbf{k}|$ near k_c.

For $s = s_0$, let us introduce the projection Π' commuting with $\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N$, associated with this group of eigenvalues close to 0 (separated from the rest of the spectrum at a distance at least $\delta_0/4$). We then apply the results (such as [48] Theorem 6.17 p.178) on bounded operators with a separation of the spectrum in two bounded parts. We then obtain that the spectrum of the operator

$$(\mathbb{I}-\Pi')\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N(\mathbb{I}-\Pi')$$

lies at a distance at least $3\delta_0/4$ from 0, hence its inverse is bounded by a constant C. We can then proceed exactly as with the projection π_0 at section 2.7.4 and prove the following

Lemma 8.2 *For $s_0 > d/2$, $0 < \varepsilon \le \varepsilon_0$, $|\widetilde{\mu}| \le \varepsilon$, $V \in \mathbf{Q}_0\mathcal{K}_{0,s_0}$ such that $||V||_{0,s_0} \le 1$, there exists $c" > 0$ such that*

$$||(\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N)^{-1}||_{0,s_0} \le c"||(\Pi'\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N\Pi')^{-1}||_{0,s_0} .$$

We are in the same finite-dimensional space as in [6]. The definition of the good set of $\widetilde{\mu}$ is only linked with the finite set of eigenvalues perturbing $\lambda_0(|\mathbf{k}|^2)-\lambda_0$ for $\mathbf{k}\in\Gamma$, $||\mathbf{k}|-k_c| \le \delta_1$, and located in the strip

$$-3\delta_0/4 < \mathrm{Re}(\cdot) < \delta_0/4,$$

for ε small enough. However, we cannot use directly the method of [6], since *the operator* $\Pi'\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N\Pi'$ *is not selfadjoint.*

From Lemma 7.2 we have

$$\begin{aligned}\Pi'\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N\Pi' \quad = \quad & \Pi'\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N\Pi' + \widetilde{\mu}\mathbb{I}d + \\ & +\Pi'\Pi_N\mathfrak{B}_\varepsilon\Pi_N\Pi' + \varepsilon^2\widetilde{\mu}\Pi'\Pi_N\mathfrak{C}_{\varepsilon,\widetilde{\mu}}\Pi_N\Pi' + \Pi'\Pi_N\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}\Pi_N\Pi'\end{aligned}$$

where $\mathbb{I}d$ is the identity in $\Pi'E_N$. The new property is that the negative selfadjoint operator $\Pi'\Pi_N\pi_0\mathbf{Q}_0(A-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N\Pi'$ satisfies

$$||\Pi'\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N\Pi'||_{0,s_0} \le \delta_0' \tag{8.5}$$

which is the size of its spectrum even in absence of Π_N (the norm in $\mathcal{L}(E_N)$ is the norm induced by $\mathcal{L}(\mathcal{K}_{0,s_0})$).

In the sequel of this subsection and the next one, we simplify the notations in defining

$$\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}} =: \Pi'\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N\Pi', \tag{8.6}$$

which is analytic in $(\varepsilon,\widetilde{\mu})$ when $V=0$. Then we define

$$V(\varepsilon,\widetilde{\mu}) = V_0(\varepsilon) + V_1(\varepsilon,\widetilde{\mu}),$$

where V_0, V_1 are C^2 in their arguments, and V_1 satisfies (see properties required in $\mathcal{U}^{(N)}_M$)

$$||V_1(\varepsilon,\widetilde{\mu})||_{0,s_0} \leq M|\widetilde{\mu}|,\ ||\partial_{\widetilde{\mu}}V_1(\varepsilon,\widetilde{\mu})||_{0,s_0} \leq M,\ ||\partial_{\widetilde{\mu}}V_1(\varepsilon,\widetilde{\mu}_2) - \partial_{\widetilde{\mu}}V_1(\varepsilon,\widetilde{\mu}_1)|| \leq M|\widetilde{\mu}_2 - \widetilde{\mu}_1|.$$

Then, we also decompose accordingly $\mathfrak{R}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}$ as

$$\mathfrak{R}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})} = \varepsilon^4\mathfrak{R}^{(0)}_{\varepsilon} + \varepsilon^4\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}},$$

where $\mathfrak{R}^{(0)}_{\varepsilon}$, $\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}}$ are C^2 in their arguments, and $\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}}$ satisfies

$$\begin{aligned} ||\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}}v||_{0,s_0} &\leq M|\widetilde{\mu}|||v||_{0,s_0},\ ||\partial_{\widetilde{\mu}}\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}}v||_{0,s_0} \leq M||v||_{0,s_0} \\ ||(\partial_{\widetilde{\mu}}\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}_2} - \partial_{\widetilde{\mu}}\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}_1})v||_{0,s_0} &\leq M|\widetilde{\mu}_2 - \widetilde{\mu}_1|||v||_{0,s_0}. \end{aligned}$$

Then,

$$\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}} = (\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}'^{(N)}_{\varepsilon} + \widetilde{\mu}\mathbb{I}d + \varepsilon^2\mathfrak{C}'^{(N)}_{\varepsilon,\widetilde{\mu}},$$

with

$$(\widetilde{\mathcal{A}-\lambda_0})_N =: \Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N$$

$$\begin{aligned} \mathfrak{B}'^{(N)}_{\varepsilon} &= \Pi_N(\mathfrak{B}_{\varepsilon} + \varepsilon^4\mathfrak{R}^{(0)}_{\varepsilon})\Pi_N, \\ \mathfrak{C}'^{(N)}_{\varepsilon,\widetilde{\mu}} &= \Pi_N(\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}} + \varepsilon^2\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}})\Pi_N, \end{aligned}$$

Let us now consider the selfadjoint operator $\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}}\mathfrak{L}^{(N,V)*}_{\varepsilon,\widetilde{\mu}}$, which may now be written as

$$\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}}\mathfrak{L}^{(N,V)*}_{\varepsilon,\widetilde{\mu}} = \widetilde{\mu}^2\mathbb{I}d + \widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}} + \widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}, \tag{8.7}$$

where (we simplify in omitting below the writting of Π'):

$$\begin{aligned} \widetilde{\mathfrak{B}}^{(N)}_{\varepsilon} &= (\widetilde{\mathcal{A}-\lambda_0})^2_N + \mathfrak{B}'^{(N)}_{\varepsilon}(\widetilde{\mathcal{A}-\lambda_0})_N + (\widetilde{\mathcal{A}-\lambda_0})_N\mathfrak{B}'^{(N)*}_{\varepsilon} + \mathfrak{B}'^{(N)}_{\varepsilon}\mathfrak{B}'^{(N)*}_{\varepsilon}, \\ \widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}} &= \widetilde{\mu}[2(\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}'^{(N)}_{\varepsilon} + \mathfrak{B}'^{(N)*}_{\varepsilon}] + \varepsilon^2[(\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}'^{(N)}_{\varepsilon} + \widetilde{\mu}]\mathfrak{C}'^{(N)*}_{\varepsilon,\widetilde{\mu}} + \\ &\quad + \varepsilon^2\mathfrak{C}'^{(N)}_{\varepsilon,\widetilde{\mu}}[(\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}'^{(N)*}_{\varepsilon} + \widetilde{\mu}] + \varepsilon^4\mathfrak{C}'^{(N)}_{\varepsilon,\widetilde{\mu}}\mathfrak{C}'^{(N)*}_{\varepsilon,\widetilde{\mu}}, \end{aligned}$$

where the adjoint is taken with the scalar product in E_N induced by the scalar product in $\mathcal{K}_{0,s_0}$. Operators $\widetilde{\mathfrak{B}}_\varepsilon^{(N)}$ and $\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)}$ are C^1 in their arguments. Moreover there exists $c > 0$ such that

$$\begin{aligned} ||\widetilde{\mathfrak{B}}_{\varepsilon_2}^{(N)} - \widetilde{\mathfrak{B}}_{\varepsilon_1}^{(N)}||_{0s_0} &\leq c(\delta_0' + \varepsilon)|\varepsilon_2 - \varepsilon_1|, \quad \widetilde{\mathfrak{C}}_{\varepsilon,0}^{(N)} = 0 \\ ||\widetilde{\mathfrak{C}}_{\varepsilon_2,\widetilde{\mu}_2}^{(N)} - \widetilde{\mathfrak{C}}_{\varepsilon_1,\widetilde{\mu}_1}^{(N)}||_{0,s_0} &\leq c(\delta_0' + \varepsilon)(|\varepsilon_2 - \varepsilon_1| + |\widetilde{\mu}_2 - \widetilde{\mu}_1|), \\ ||\partial_{\widetilde{\mu}}\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}_2}^{(N)} - \partial_{\widetilde{\mu}}\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}_1}^{(N)}||_{0,s_0} &\leq c\varepsilon^2|\widetilde{\mu}_2 - \widetilde{\mu}_1|. \end{aligned} \tag{8.8}$$

Let us now define

Definition 8.3 *For $V \in \mathcal{U}_M^{(N)}$ and τ, $\gamma > 0$ (to be determined later), the "good" set of $\widetilde{\mu}$ is the set*

$$G_{\varepsilon,\gamma}^{(N)}(V) := \left\{ \widetilde{\mu} \in [-\varepsilon, \varepsilon]; ||(\Pi'\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N\Pi')^{-1}v||_{0,s_0} \leq \frac{N^\tau}{\gamma}||v||_{0,s_0}, \quad \textit{for any } v \in \Pi' E_N \right\},$$

where $||\cdot||_{0,s}$ means the norm in $\mathcal{L}(E_N)$ induced by $\mathcal{L}(\mathcal{K}_{0,s})$.

Saying that $\widetilde{\mu}$ is "good", i.e. $\widetilde{\mu} \in G_{\varepsilon,\gamma}^{(N)}(V)$, implies that the positive selfadjoint operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*}$ has all its eigenvalues larger than $(\frac{\gamma}{N^\tau})^2$. It is now possible to give a bound for the measure of the bad set for $\widetilde{\mu}$.

2.8.3 Bad set of $\widetilde{\mu}$

By definition, the bad set of $\widetilde{\mu}$ is the complement of the good set. Hence for $V \in \mathcal{U}_M^{(N)}$,

$$B_{\varepsilon,\gamma}^{(N)}(V) := \left\{ \begin{array}{c} \widetilde{\mu} \in [-\varepsilon, \varepsilon]; \ \exists v \in \Pi' E_N \text{ such that} \\ ||(\Pi'\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N\Pi')^{-1}v||_{0,s_0} > \frac{N^\tau}{\gamma}||v||_{0,s_0}. \end{array} \right\}.$$

Now we prove the following

Lemma 8.4 *Assume that $N > M_\varepsilon$, $d/2 < s_0, \tau > d + 12l_0$, $(\varepsilon, \widetilde{\mu}) \in (0, \varepsilon_1] \times [-\varepsilon, \varepsilon]$, and $V \in \mathcal{U}_M^{(N)}$. Moreover assume that Condition 8.7 holds, then there exists a constant $C > 0$, such that the measure of $B_{\varepsilon,\gamma}^{(N)}(V)$ is bounded by*

$$\frac{C\gamma}{N^{\tau-d}},$$

The following proof only considers eigenvalues close to 0, i.e. we use, without mentioning it, the projection Π' which eliminates the infinite dimensional subspace corresponding to "large" eigenvalues.

Let us prove the following.

Lemma 8.5 *For ε small enough, $\widetilde{\mu} \in [-\varepsilon,\varepsilon], s_0 > d/2$, the eigenvalues of $\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}}\mathfrak{L}^{(N,V)*}_{\varepsilon,\widetilde{\mu}}$ take the form*

$$\sigma_j(\varepsilon,\widetilde{\mu}) = \widetilde{\mu}^2 + f_j(\varepsilon,\widetilde{\mu}), \tag{8.9}$$

where $f_j(\varepsilon,\widetilde{\mu})$ is Lipschitz in $(\varepsilon,\widetilde{\mu})$ with

$$|f_j(\varepsilon_2,\widetilde{\mu}_2) - f_j(\varepsilon_1,\widetilde{\mu}_1)| \leq c(\delta'_0+\varepsilon)(|\varepsilon_2-\varepsilon_1| + |\widetilde{\mu}_2-\widetilde{\mu}_1|. \tag{8.10}$$

Moreover, for ε fixed, $f_j(\varepsilon,\widetilde{\mu})$ is C^2 with respect to $\widetilde{\mu}$.

Proof.

We use the Lidskii theorem (see [48] theorem 6.10 p.126) for comparing the eigenvalues f_j of operators $\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon_2,\widetilde{\mu}_2}+\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon_2}$ and $\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon_1,\widetilde{\mu}_1}+\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon_1}$, and the estimate (8.8), which directly leads to (8.10). Then, it remains to add $\widetilde{\mu}^2$ for obtaing the eigenvalues σ_j of $\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}}\mathfrak{L}^{(N,V)*}_{\varepsilon,\widetilde{\mu}}$. The property that $f_j(\varepsilon,\widetilde{\mu})$ is C^2 with respect to $\widetilde{\mu}$ results from the selfadjointness and from [48] see p.115 and the proof of theorem 6.8 p.122 applied on the reduced operator (using the eigenprojection associated with a group of eigenvalues which split for $\widetilde{\mu}$ close to $\widetilde{\mu}_0$).

■

Remark 8.6 *Let us consider eigenvalues $\widetilde{\mu}g_j(\varepsilon,\widetilde{\mu})$ of the selfadjoint operator $\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}}$ which we write as*

$$\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}} = \widetilde{\mu}\widetilde{\mathfrak{C}}^{(1)}_{\varepsilon} + \widetilde{\mu}\widetilde{\mathfrak{C}}^{(2)}_{\varepsilon,\widetilde{\mu}},$$

where $\widetilde{\mathfrak{C}}^{(2)}_{\varepsilon,\widetilde{\mu}}$ is C^1 in $\widetilde{\mu}$ and

$$\widetilde{\mathfrak{C}}^{(2)}_{\varepsilon,0} = 0,\ \partial_{\widetilde{\mu}}\widetilde{\mathfrak{C}}^{(2)}_{\varepsilon,0} = 0,\ ||\widetilde{\mathfrak{C}}^{(2)}_{\varepsilon,\widetilde{\mu}}||_{0,s_0} \leq c\varepsilon^2|\widetilde{\mu}|. \tag{8.11}$$

By the Lidskii theorem we know that

$$\widetilde{\mu}g_j(\varepsilon,\widetilde{\mu}) = \widetilde{\mu}g_j^{(1)}(\varepsilon) + \widetilde{\mu}g_j^{(2)}(\varepsilon,\widetilde{\mu}),$$

with

$$g_j^{(1)}(\varepsilon) \text{ eigenvalue of } \widetilde{\mathfrak{C}}^{(1)}_{\varepsilon},$$

and $\{g_1^{(2)}(\varepsilon,\widetilde{\mu}),...,g_{\mathcal{N}}^{(2)}(\varepsilon,\widetilde{\mu})\}$ belongs to the convex hull of the vectors obtained from $\{\gamma_1,...\gamma_{\mathcal{N}}\}$ by all possible permutations, where γ_j 's are the eigenvalues of $\widetilde{\mathfrak{C}}^{(2)}_{\varepsilon,\widetilde{\mu}}$ in E_N. Then, because of (8.11), we obtain

$$|g_j^{(2)}(\varepsilon,\widetilde{\mu})| \leq c\varepsilon^2|\widetilde{\mu}|.$$

Applying again the Lidskii theorem, in considering the eigenvalues $f_j(\varepsilon,\widetilde{\mu})$ of the selfadjoint operator $\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}} + \widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}$, this leads to

$$f_j(\varepsilon,\widetilde{\mu}) = s_\varepsilon + \widetilde{\mu}f_j^{(1)}(\varepsilon,\widetilde{\mu})$$

where $\widetilde{\mu} f_j^{(1)}(\varepsilon,\widetilde{\mu})$ *belongs to the convex hull of the vectors obtained from* $\{\widetilde{\mu} g_1(\varepsilon,\widetilde{\mu}),...\widetilde{\mu} g_N(\varepsilon,\widetilde{\mu})\}$ *by all possible permutations, where* $\widetilde{\mu} g_j(\varepsilon,\widetilde{\mu})$*'s are the eigenvalues of* $\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}}$ *in* E_N*, and we cannot decompose* $f_j^{(1)}(\varepsilon,\widetilde{\mu})$ *as* $f_j^{(1)}(\varepsilon,0)+f_j^{(2)}(\varepsilon,\widetilde{\mu})$*, with* $f_j^{(2)}(\varepsilon,\widetilde{\mu})$ *Lipschitz in* $\widetilde{\mu}$*. For being able to claim such a decomposition, we need to control the Lipschitz constant with respect to* $\widetilde{\mu}$ *of the second derivative with respect to* $\widetilde{\mu}$*, in 0 of* $f_j(\varepsilon,\widetilde{\mu})$*. It is shown for example in [48] that such an information uses a bound for the pseudo-inverse of* $\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}-s_\varepsilon$*, which is of the size of the inverse of the distance of* s_ε *from the spectrum of* $\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}$*. This distance is unfortunately very small of order* N^{-4l_0}*.*

Let us now try another way. For a given ε*, let us consider an eigenvalue* s_ε *of* $\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}$*, and define the associated orthogonal eigenprojection* $\mathbf{P}_\varepsilon$*. Then, because* $\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}$ *is selfadjoint, we have*

$$\mathbf{P}_\varepsilon(\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}-s_\varepsilon)=0.$$

The operator $\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}}$ *acts as a perturbation, and let us consider* f_j *which belongs to the* s_ε *- group of eigenvalues, resulting from the perturbation of* s_ε*, and denote by* $\mathbf{P}_{\varepsilon,\widetilde{\mu}}$ *the orthogonal eigenprojection associated with the* s_ε *- group of eigenvalues. Then, by definition there is an eigenvector* $\zeta_j(\varepsilon,\widetilde{\mu})$ *satisfying*

$$\{\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}}+\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}-f_j(\varepsilon,\widetilde{\mu})\}\zeta_j(\varepsilon,\widetilde{\mu})=0,$$

which is equivalent to

$$\mathbf{P}_\varepsilon\{\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}}+s_\varepsilon-f_j(\varepsilon,\widetilde{\mu})\}\zeta_j(\varepsilon,\widetilde{\mu})=0.$$

We have $\mathbf{P}_\varepsilon\zeta_j(\varepsilon,\widetilde{\mu})\in\mathbf{P}_\varepsilon E_N$*, and also, since* $\mathbf{P}_\varepsilon\mathbf{P}_{\varepsilon,\widetilde{\mu}}$ *is one to one from* $\mathbf{P}_{\varepsilon,\widetilde{\mu}}E_N$ *onto* $\mathbf{P}_\varepsilon E_N$*,*

$$\zeta_j(\varepsilon,\widetilde{\mu})=(\mathbf{P}_\varepsilon\mathbf{P}_{\varepsilon,\widetilde{\mu}})^{-1}\mathbf{P}_\varepsilon\zeta_j(\varepsilon,\widetilde{\mu}),$$

which means that $\mathbf{P}_\varepsilon\zeta_j(\varepsilon,\widetilde{\mu})$ *is an eigenvector belonging to the eigenvalue* $f_j(\varepsilon,\widetilde{\mu})-s_\varepsilon$ *for the operator* $\mathbf{P}_\varepsilon\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}}(\mathbf{P}_\varepsilon\mathbf{P}_{\varepsilon,\widetilde{\mu}})^{-1}\mathbf{P}_\varepsilon$ *acting in the subspace* $\mathbf{P}_\varepsilon E_N$*. We just need to decompose into a part which is linear in* $\widetilde{\mu}$ *plus a rest of order* $\widetilde{\mu}^2$*. Then, the problem is that we have no nice bound for the derivative* $\partial_{\widetilde{\mu}}(\mathbf{P}_\varepsilon\mathbf{P}_{\varepsilon,\widetilde{\mu}})$ *because there occurs again (see [48] p.77 formula (2.14)) the pseudo-inverse of* $\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}-s_\varepsilon$*, only bounded by the inverse of the (very small) distance of* s_ε *from the rest of spectrum of* $\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}$*.*

Proof of Lemma 8.4. Assume that $\widetilde{\mu}\in B^{(N)}_{\varepsilon,\gamma}(V)$, then it results that the norm of $(\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}}\mathfrak{L}^{(N,V)*}_{\varepsilon,\widetilde{\mu}})^{-1}$ is $>(\frac{N^\tau}{\gamma})^2$ and that there exists j such that

$$0\le\sigma_j(\varepsilon,\widetilde{\mu})<\eta^2=:(\frac{\gamma}{N^\tau})^2. \tag{8.12}$$

We need to measure the set (depending on ε) of $\widetilde{\mu}$ such that

$$0\le\widetilde{\mu}^2+f_j(\varepsilon,\widetilde{\mu})<\eta^2.$$

Let us consider the function of $\widetilde{\mu}$

$$\phi_\varepsilon(\widetilde{\mu}) =: \widetilde{\mu}^2 + f_j(\varepsilon,\widetilde{\mu}),$$

defined for $|\widetilde{\mu}| \leq \varepsilon$. Thanks to (8.10), we then have

$$\widetilde{\mu}^2 - c(\delta_0' + \varepsilon)|\widetilde{\mu}| + f_j(\varepsilon,0) \leq \phi_\varepsilon(\widetilde{\mu}) \leq \widetilde{\mu}^2 + c(\delta_0' + \varepsilon)|\widetilde{\mu}| + f_j(\varepsilon,0), \tag{8.13}$$

which means that the graph of $\widetilde{\mu} \mapsto \phi_\varepsilon(\widetilde{\mu})$ is situated between two close parabolas. This implies that the roots $\widetilde{\mu}$ of $\phi_\varepsilon(\widetilde{\mu}) = \eta^2$ are bounded, when they exist. The maximal and minimal roots are noted $\widetilde{\mu}^\pm$. So we have

$$\widetilde{\mu}^{+2} + f_j(\varepsilon,\widetilde{\mu}^+) = \eta^2,$$

with the same equation for $\widetilde{\mu}^-$. In the case when these roots do not exist, the bad set is empty for the eigenvalue $\sigma_j(\varepsilon,\widetilde{\mu})$.

In all cases, we have (positive operator)

$$\phi_\varepsilon(\widetilde{\mu}) \geq 0 \text{ for } \widetilde{\mu} \in [\widetilde{\mu}^-,\widetilde{\mu}^+],$$

and the function has at least a minimum in $\widetilde{\mu}_m$ such that

$$\widetilde{\mu}^- < \widetilde{\mu}_m < \widetilde{\mu}^+,\ 0 \leq \phi_\varepsilon(\widetilde{\mu}_m) < \eta^2.$$

Then this leads to

$$\widetilde{\mu}^{+2} - \widetilde{\mu}_m^2 + f_j(\varepsilon,\widetilde{\mu}^+) - f_j(\varepsilon,\widetilde{\mu}_m) < \eta^2,$$

and applying (8.10), we obtain

$$\widetilde{\mu}^{+2} - \widetilde{\mu}_m^2 - c(\delta_0' + \varepsilon)(\widetilde{\mu}^+ - \widetilde{\mu}_m) < \eta^2,$$

hence,

$$(\widetilde{\mu}^+ - \frac{c}{2}(\delta_0' + \varepsilon))^2 - (\widetilde{\mu}_m - \frac{c}{2}(\delta_0' + \varepsilon))^2 < \eta^2.$$

If $\widetilde{\mu}_m - \frac{c}{2}(\delta_0' + \varepsilon)$ and $\widetilde{\mu}^+ - \frac{c}{2}(\delta_0' + \varepsilon)$ have the same sign, we use now the property that $0 < a^2 - b^2 < \eta^2$ leads to $|a - b| < \eta$, when a and b have the same sign. This allows to conclude that, in such a case

$$\widetilde{\mu}^+ - \widetilde{\mu}_m < \eta.$$

In the same way if $\widetilde{\mu}_m + \frac{c}{2}(\delta_0' + \varepsilon)$ and $\widetilde{\mu}^- + \frac{c}{2}(\delta_0' + \varepsilon)$ have the same sign,

$$(\widetilde{\mu}^- + \frac{c}{2}(\delta_0' + \varepsilon))^2 - (\widetilde{\mu}_m + \frac{c}{2}(\delta_0' + \varepsilon))^2 < \eta^2.$$

gives

$$\widetilde{\mu}_m - \widetilde{\mu}^- < \eta,$$

and finally the bad interval would be bounded by 2η.

Since we are unable to prove the suitable property for $f_j(\varepsilon,\widetilde{\mu})$, we need the following

Condition 8.7 *Functions $f_j(\varepsilon,\widetilde{\mu})$ defined in (8.9) have their derivative with respect to $\widetilde{\mu}$ which are Lipschitz: for $\widetilde{\mu}\in[-\varepsilon,\varepsilon]$, there exists $0<k<2$ with*

$$|\partial_{\widetilde{\mu}} f_j(\varepsilon,\widetilde{\mu}_2)-\partial_{\widetilde{\mu}} f_j(\varepsilon,\widetilde{\mu}_1)|\leq k|\widetilde{\mu}_2-\widetilde{\mu}_1|. \tag{8.14}$$

We may observe that this assumption takes into account of a loss of boundedness from the estimate (8.8) for the operator $\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}}$, since the Lipschitz constant for the derivative is $k<2$ in place of $c\varepsilon^2$. However, this is a true assumption, with no proof at this time.

Now, in using Hypothesis (8.14), we claim that the function $\widetilde{\mu}\mapsto\phi_\varepsilon(\widetilde{\mu})$ is convex:

$$\partial_{\widetilde{\mu}}\phi_\varepsilon(\widetilde{\mu})=2\widetilde{\mu}+\partial_{\widetilde{\mu}} f_j(\varepsilon,\widetilde{\mu})$$

is an increasing function of $\widetilde{\mu}$, cancelling in $\widetilde{\mu}=\widetilde{\mu}_m$. This property, combined with the property (8.13), leads to a unique minimum in $\widetilde{\mu}_m$, and to a measure of bad $\widetilde{\mu}$ in the (worse) case given when the graph of ϕ_ε is tangent to the axis. We have

$$\begin{aligned}\phi_\varepsilon(\widetilde{\mu})-\phi_\varepsilon(\widetilde{\mu}_m) &= \int_{\widetilde{\mu}_m}^{\widetilde{\mu}}(2\widetilde{\mu}+\partial_{\widetilde{\mu}} f_j(\varepsilon,\widetilde{\mu}))d\widetilde{\mu}\\ &= \int_{\widetilde{\mu}_m}^{\widetilde{\mu}}(2(\widetilde{\mu}-\widetilde{\mu}_m)+\partial_{\widetilde{\mu}} f_j(\varepsilon,\widetilde{\mu})-\partial_{\widetilde{\mu}} f_j(\varepsilon,\widetilde{\mu}_m))d\widetilde{\mu}\\ &\geq \frac{(2-k)}{2}(\widetilde{\mu}-\widetilde{\mu}_m)^2,\end{aligned}$$

Since $\phi_\varepsilon(\widetilde{\mu}^{\pm})=\eta^2$, we obtain

$$\widetilde{\mu}^+-\widetilde{\mu}^-\leq\frac{2\eta}{\sqrt{(1-k/2)}}$$

Summing up for all eigenvalues, using that the dimension $\mathcal{N}$ of E_N is bounded by bN^d, the measure of the set of bad $\widetilde{\mu}$, is bounded by

$$\frac{2b\gamma}{\sqrt{(1-k/2)}N^{\tau-d}}. \tag{8.15}$$

∎

Remark 8.8 *We give precisions at section 2.10 on the structure of the bad set in the plane $(\varepsilon,\tilde{\mu})$. It is shown that the curves $\widetilde{\mu}^-(\varepsilon),\widetilde{\mu}^+(\varepsilon)$ are Hölder continuous functions of ε with exponent 1/2.*

The estimate of Lemma 8.4 is then proved with $C=\frac{2b}{\sqrt{(1-k/2)}}$. Finally let us observe that this measure is small with respect to the length $2\varepsilon^3$ of the interval for $\widetilde{\mu}=\varepsilon^3\mu'$, provided that

$$\varepsilon^3N^{\tau-d}\geq\varepsilon^3M_\varepsilon^{12l_0}N^{\tau-d-12l_0}\geq c_2'N^{\tau-d-12l_0}$$

is large enough. This is the case, as soon as $\tau > d + 12l_0$.

∎

Then we have the following

Proposition 8.9 *Let $d = 2(l_0 + 1)$ be the dimension of the $\mathbb{Q}$- vector space spanned by the wave vectors $k_j, j = 1, ..., 2q$, and $\tau > d + 2 + 24l_0$. Let N be ≥ 1. Assume moreover that $0 < \gamma \leq \widetilde{\gamma} = \frac{c'}{c2^{2l_0+1}}$, (where c is the constant occuring in (8.1)) and $(\epsilon, \widetilde{\mu}, V) \in [0, \epsilon_1] \times [-\varepsilon, \varepsilon] \times \mathcal{U}_M^{(N)}$ with $\widetilde{\mu} \in G_{\varepsilon,\gamma}^{(N)}(V)$, ϵ_1 small enough. For $s_0 > \frac{d}{2}$, there exists $c' > 0$ independent of N and γ, such that for any $v \in \Pi'\pi_0 E_N$, we have*

$$||(\Pi'\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N\Pi')^{-1}v||_{0,s_0} \leq c'\frac{N^\tau}{\gamma}||v||_{0,s_0}, \tag{8.16}$$

and the same estimate holds for $(\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}$ for $v \in E_N$.

Proof. If $N \geq 1$, then $2c\gamma \leq c'/2^{2l_0} \leq \frac{c'N^\tau}{(1+N^2)^{2l_0}}$, i.e.

$$2c(1 + N^2)^{2l_0} \leq c'\frac{N^\tau}{\gamma}.$$

Then the estimate for $(\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)})^{-1}v$ follows for $N \leq M_\varepsilon$ from (8.1). For $N > M_\varepsilon$ by definition of the good set of $\widetilde{\mu}$, the estimate on $(\Pi'\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N\Pi')^{-1}v$ follows. For $(\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}$ the estimate follows from Lemma 8.2. ∎

Remark 8.10 *The choice to take $\tau > d+2+24l_0$ will be explained later (see Lemma 9.1). With such a choice, we have $\frac{1}{N^{\tau-d-2}} \leq \frac{1}{M_\varepsilon^{24l_0}} \leq c\varepsilon^6$.*

Definition 8.11 *For $V \in \mathcal{U}_M^{(N)}$ and τ, $\gamma > 0$, we define the set of good $\tilde{\mu}$ for all $K \leq N$, as*

$$\mathcal{G}_{\varepsilon,\gamma}^{(N)}(V) = \cap_{K\leq N} G_{\varepsilon,\gamma}^{(K)}(V),$$

where we notice that $G_{\varepsilon,\gamma}^{(K)}(V) = [-\varepsilon, \varepsilon]$ for $K < M_\varepsilon$, thanks to Lemma 8.1.

Our aim is now to obtain an estimate for $(\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}$ in $\mathcal{K}_{0,s}$ for $s > s_0$. We may observe that it is not possible to obtain directly such an estimate in $\mathcal{K}_{0,s}$ for $s > s_0$, because the norm $||V||_{0,s}||$ would appear in the estimates for α_j in the eigenvalues σ_j, and this is far to be controlled.

2.8.4 Separation properties (H1) and (H2)

The eigenvalues close to 0 of the unperturbed operator $\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N$ are the negative numbers $\lambda_0(|\mathbf{k}|^2)-\lambda_0$ where $|\mathbf{k}|\neq k_c$, and $1\leq N_{\mathbf{k}}\leq N$. Let $\rho>0$. We need to have good separation properties of the singular set

$$S_{(N)}=\left\{\mathbf{k}\in\Gamma;\lambda_0-\lambda_0(|\mathbf{k}|^2)<\rho,\ 1\leq N_{\mathbf{k}}\leq N\right\}, \tag{8.17}$$

which contains the $\mathbf{k}$'s corresponding to the small denominators, whereas the regular set is

$$R_{(N)}:=\left\{\mathbf{k}\in\Gamma;\lambda_0-\lambda_0(|\mathbf{k}|^2)\geq\rho,\ 1\leq N_{\mathbf{k}}\leq N\right\}. \tag{8.18}$$

We have a bijection between $S_{(N)}$ and $S(N):=\{x\in\Gamma(N);\ \lambda_0-\lambda_0(|\mathbf{k}(x)|^2)<\rho\}$ where $\mathbf{k}(x)$ is defined in (3.3) and

$$\Gamma(N):=\{x\in\mathbb{Z}^d;\ 0\leq|x|\leq N,\ \mathbf{k}(x)\in\Gamma\}.$$

We use the fact that for $||\mathbf{k}|-k_c|\leq\delta_1$, there exist c_1 and $c_2>0$ such that

$$c_1(|\mathbf{k}|^2-k_c^2)^2\leq\lambda_0-\lambda_0(|\mathbf{k}|^2)\leq c_2(|\mathbf{k}|^2-k_c^2)^2 \tag{8.19}$$

and (3.5) holds. Then as in [6], we use the results of Bourgain in [8], Craig in [10], and [4], so that we obtain

Proposition 8.12 *There exists $\rho_0>0$ independent of N such that if $\rho\in]0,\rho_0]$ then there exists a decomposition of $S(N)=\bigcup_{\alpha\in\mathcal{A}}\Omega_\alpha$ into a union of disjoint clusters Ω_α satisfying :*

- *$(H1)$, for all $\alpha\in\mathcal{A}$, $M_\alpha\leq 2m_\alpha$ where $M_\alpha=\max_{x\in\Omega_\alpha}|x|$ and $m_\alpha=\min_{x\in\Omega_\alpha}|x|$;*
- *$(H2)$, there exists $\delta=\delta(d)\in]0,1[$ independent of N such that if $\alpha,\beta\in\mathcal{A},\alpha\neq\beta$ then*

$$\operatorname{dist}(\Omega_\alpha,\Omega_\beta):=\min_{x\in\Omega_\alpha,y\in\Omega_\beta}|x-y|\geq\frac{(M_\alpha+M_\beta)^\delta}{2}.$$

2.8.5 Estimate of $(\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}$ in $\Pi_N\pi_0\mathbf{Q}_0\mathcal{K}_{0,s}$

We use the proof of [4] (see pages 628 to 636). In fact, we need the selfadjointness in $\Pi_N\pi_0Q_0\mathcal{K}_{0,s}$ (i.e. E_N with the adapted scalar product) of the operator

$$D_N=:\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N,$$

diagonal (see Appendix 6.2.5) with respect to Fourier components in $\Pi_N\pi_0Q_0\mathcal{K}_{0,s}$, for which we know all eigenvalues. Moreover, we have

$$\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N=D_N+\varepsilon T(\varepsilon,\widetilde{\mu},V)$$

where the second part εT is a bounded operator (not diagonal) of order ε having the properties of a multiplication operator, as it is needed in [4] (see Lemma 3.9 in [4]): (see the proof in Appendix 6.2.5)

Lemma 8.13 *Let $A, B \subset S(N) \cup R(N)$, and let $s_0 > d/2$. Then for any $s \geq s_0 > d/2$ there exists $C(s) > 0$ such that the following estimate holds for any $V \in Q_0\mathcal{K}_{0,s}$ such that $||V||_{0,s_0} \leq 1$, and $h \in \Pi_N \pi_0 Q_0 \mathcal{K}_{0,0}$*

$$||T_B^A h||_{0,0} \leq \frac{C(s)\varepsilon(1+\varepsilon^3||V||_{0,s})||h||_{0,0}}{(1+d(A,B))^{s-d/2}},$$

where $d(A, B)$ is the distance in $\mathbb{Z}^d$ between A and B, and T_B^A is the operator T acting in E_N restricted to elements with Fourier spectrum with $\{\mathbf{k}(x); x \in A\}$, the action being projected on elements with Fourier spectrum such that $\{\mathbf{k}(x); x \in B\}$.

This property, with the estimate (8.16) used for any $K \leq N$ (replaces the use of eigenvalues of $\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N$ as it is done in [4]), are the basic ingredients for the proof of the following

Proposition 8.14 *Let $d = 2(l_0 + 1)$ be the dimension of the $\mathbb{Q}$- vector space spanned by the wave vectors $k_j, j = 1, ..., 2q$, and $\tau > d + 2 + 24l_0$ as in Lemma 8.9. Assume moreover that $0 < \gamma \leq \widetilde{\gamma} = \frac{c'}{c2^{2l_0+1}}$, and $(\epsilon, \widetilde{\mu}, V) \in [0, \epsilon_1] \times [-\varepsilon, \varepsilon] \times \mathcal{U}_M^{(N)}$, with $\widetilde{\mu} \in \mathcal{G}_{\epsilon,\gamma}^{(N)}(V)$, ϵ_1 small enough. There exists $s_0(d, \delta, \tau) > \frac{d}{2}$ where δ is the number introduced in separation property (H2), and let $\overline{s} > s_0$. There exists $m(d, \delta, \tau)$ such that for all $s \in [s_0, \overline{s}]$ there exists $K(s) > 0$ such that for any $h \in \Pi_N \pi_0 Q_0 \mathcal{K}_{0,s}$, we have*

$$||(\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}h||_{0,s} \leq K(s)\frac{N^m}{\gamma}(||h||_{0,s} + ||V(\epsilon,\widetilde{\mu})||_{0,s}||h||_{0,s_0}), \tag{8.20}$$

and the same estimate holds for $(\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}$.

2.9 Resolution of the range equation

In this section we use [5] for finding $v = V(\epsilon, \widetilde{\mu})$ in $\mathcal{U}_M^{(N)}$, defined for $(\epsilon, \widetilde{\mu})$ in $[0, \epsilon_1] \times [-\varepsilon, \varepsilon]$, bounded by $O(\epsilon)$, of class C^2 in its arguments, solution of $\mathcal{F}(\epsilon, \widetilde{\mu}, v) = 0$ (see (9.1) below) in a suitably large subset of $(0, \epsilon_1) \times [-\varepsilon, \varepsilon]$.

All operators (linear and non linear) satisfy good tame estimates in the scale of Sobolev spaces $\Pi_N \pi_0 Q_0 \mathcal{K}_{0,s}$ $s > d/2$ and the projection Π_N plays the role of a smoothing operator (see [6]):

$$\begin{aligned}
||\Pi_N u||_{0,s+r} &\leq (1+N^2)^{r/2}||u||_{0,s}, \quad \forall u \in \mathcal{K}_{0,s}, \\
||(\mathbb{I} - \Pi_N)u||_{0,s} &\leq (1+N^2)^{-r/2}||u||_{0,r+s}, \quad \forall u \in \mathcal{K}_{0,s+r}.
\end{aligned}$$

Indeed, we have the good functional setting and the good "tame" properties of the map (see Lemmas 5.7, 7.1, 8.14):

$$
\begin{aligned}
\mathcal{F}(\epsilon,\widetilde{\mu},v) &=: \mathfrak{L}_{\varepsilon,\widetilde{\mu}}v + g(\varepsilon,\widetilde{\mu}) - \varepsilon^4\mathbf{Q}_0\mathcal{B}(v,v) \qquad (9.1)\\
(\epsilon,\widetilde{\mu},v) &\mapsto \mathcal{F}(\epsilon,\widetilde{\mu},v) : [0,\epsilon_1]\times[-\varepsilon,\varepsilon]\times Q_0\mathcal{K}_{0,s} \to Q_0\mathcal{K}_{0,s} \text{ for } s\geq s_0 > d/2,
\end{aligned}
$$

with (see (7.5))

$$
\mathfrak{L}_{\varepsilon,\widetilde{\mu}} = \mathbf{Q}_0(\mathcal{A}-\lambda_0+\widetilde{\mu}+\mu_\varepsilon) - 2\mathbf{Q}_0\mathcal{B}(u_\varepsilon - \varepsilon^2\widetilde{h}(\varepsilon,\widetilde{\mu}),\cdot),
$$

$$
\mathcal{F}(0,0,0) = 0 \text{ (for } \varepsilon = 0, \text{ we have } \widetilde{\mu} = 0).
$$

The mapping $\mathcal{F}$ appears to be C^3 with the following estimates for $v \in Q_0\mathcal{K}_{0,s}$, $s \in [s_0,\overline{s}]$, $s_0 > d/2$, and $||v||_{0,s_0} \leq 1$

$$
\begin{aligned}
||\mathfrak{L}_{\varepsilon,\widetilde{\mu}}v||_{0,s} &\leq C(s)||v||_{0,s},\\
||\varepsilon^4\mathbf{Q}_0\mathcal{B}(v,v')||_{0,s} &\leq \varepsilon^4 C(s)[||v||_{0,s}||v'||_{0,s_0} + ||v'||_{0,s}],\\
||g(\varepsilon,\widetilde{\mu})||_{0,s} &\leq \varepsilon^2 C(s),
\end{aligned}
$$

$$
||\partial_{\varepsilon,\widetilde{\mu}}g(\varepsilon,\widetilde{\mu})||_{0,s} + ||\partial^2_{\varepsilon^2}g(\varepsilon,\widetilde{\mu})||_{0,s} + ||\partial^2_{\varepsilon\widetilde{\mu}}g(\varepsilon,\widetilde{\mu})||_{0,s} + ||\partial^2_{\widetilde{\mu}^2}g(\varepsilon,\widetilde{\mu})||_{0,s} \leq C(s),
$$

$$
||\partial_\varepsilon\mathfrak{L}_{\varepsilon,\widetilde{\mu}}v||_{0,s} \leq C(s)||v||_{0,s}.
$$

We may notice that

$$
\begin{aligned}
D_v\mathcal{F}(\epsilon,\widetilde{\mu},v)[u] &= \mathfrak{L}_{\varepsilon,\widetilde{\mu}}u - 2\varepsilon^4\mathbf{Q}_0\mathcal{B}(v,u),\\
D_v^2\mathcal{F}(\epsilon,\widetilde{\mu},v)[v_1,v_2] &= -2\varepsilon^4\mathbf{Q}_0\mathcal{B}(v_1,v_2),\\
D_v^3\mathcal{F}(\epsilon,\mu',v) &= 0,
\end{aligned}
$$

hence

$$
\begin{aligned}
||\partial_\varepsilon D_v\mathcal{F}(\epsilon,\widetilde{\mu},v)[u]||_{0,s} &\leq C(s)[||u||_{0,s} + \varepsilon^3||v||_{0,s}||u||_{0,s_0}],\\
||\partial_{\widetilde{\mu}} D_v\mathcal{F}(\epsilon,\widetilde{\mu},v)[u]||_{0,s} &\leq C(s)||u||_{0,s},
\end{aligned}
$$

Moreover, Lemma 8.14 says that for any $(\epsilon,\widetilde{\mu},V) \in [0,\epsilon_1]\times[-\varepsilon,\varepsilon]\times U_M^{(N)}, V\in\mathcal{K}_{0,s}$ with $\widetilde{\mu} \in \mathcal{G}^{(N)}_{\varepsilon,\gamma}(V)$

$$
||(\Pi_N D_v\mathcal{F}(\epsilon,\widetilde{\mu},V(\epsilon,\widetilde{\mu}))\Pi_N)^{-1}v||_{0,s} \leq K(s)\frac{N^m}{\gamma}(||v||_{0,s} + ||V(\epsilon,\widetilde{\mu})||_{0,s}||v||_{0,s_0}),
$$

so that assumptions (F1), (F2), (F3), (F4) and on the invertibility of the linearized operator, made in [5] are satisfied. We also satisfy additionnal properties $(F2)^+,(F4)^+$ required in Appendix 6.2.6 on higher order derivatives, useful for getting a solution V which is C^2 in $(\varepsilon,\widetilde{\mu})$. Moreover the required property (L) in [5] needs to be satisfied:

Lemma 9.1 *Choose $N_2 \geq N_1 \geq M_\varepsilon$, and $V_1 \in \mathcal{U}_M^{(N_1)}$, $V_2 \in \mathcal{U}_M^{(N_2)}$. For $\varepsilon \in (0, \varepsilon_1)$, consider the set of $\widetilde{\mu}$ which are "good" for V_1, but "bad" for V_2 :*

$$\widetilde{\mu} \in \left(\mathcal{G}_{\varepsilon,\gamma}^{(N_2)}(V_2)\right)^c \cap \mathcal{G}_{\varepsilon,\gamma}^{(N_1)}(V_1)$$

where the apex c denotes the complementary in $[-\varepsilon, \varepsilon]$. Assume that $||V_2 - V_1||_{0,s_0} \leq N_1^{-\sigma}$, with $\sigma > 2d - 6 + 32l_0$, and $\tau > d + 2 + 24l_0$, then for ε_1 small enough, in particular for $\varepsilon_1 \leq \gamma^{4l_0}$:

$$\text{meas}\left\{\left(\mathcal{G}_{\varepsilon,\gamma}^{(N_2)}(V_2)\right)^c \cap \mathcal{G}_{\varepsilon,\gamma}^{(N_1)}(V_1)\right\} \cap [-\varepsilon, \varepsilon] \leq C_1\gamma\frac{\varepsilon^6}{N_1}.$$

Proof.

$$\begin{aligned}
\left(\mathcal{G}_{\varepsilon,\gamma}^{(N_2)}(V_2)\right)^c \cap \mathcal{G}_{\varepsilon,\gamma}^{(N_1)}(V_1) &= \left(\cup_{M_\varepsilon \leq K \leq N_2} B_{\varepsilon,\gamma}^{(K)}(V_2)\right) \cap \left(\cap_{M_\varepsilon \leq K \leq N_1} G_{\varepsilon,\gamma}^{(K)}(V_1)\right) \\
&\subset \left(\cup_{M_\varepsilon \leq K \leq N_1} B_{\varepsilon,\gamma}^{(K)}(V_2) \cap G_{\varepsilon,\gamma}^{(K)}(V_1)\right) \cup \left(\cup_{N_1 \leq K \leq N_2} B_{\varepsilon,\gamma}^{(K)}(V_2)\right).
\end{aligned}$$

Moreover, according to Lemmas 7.1 and 7.2 and a careful study of the form of operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*}$ in (8.7), we have for $K \leq N_1$

$$||\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_2)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_2)*} - \mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_1)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_1)*}||_{0,s_0} \leq c\varepsilon^4||V_2 - V_1||_{0,s_0} \leq \frac{c\varepsilon^4}{N_1^\sigma}.$$

Let us assume that $\widetilde{\mu} \in B_{\varepsilon,\gamma}^{(K)}(V_2) \cap G_{\varepsilon,\gamma}^{(K)}(V_1)$, then there is at least one eigenvalue (> 0) of $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_2)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_2)*}$ which is $< (\frac{\gamma}{K^\tau})^2$. Then, by Lidskii theorem (see [48] p.126) the selfadjoint operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_1)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_1)*}$ has an eigenvalue $< (\frac{\gamma}{K^\tau})^2 + \frac{c\varepsilon^4}{N_1^\sigma}$. Since $\widetilde{\mu} \in G_{\varepsilon,\gamma}^{(K)}(V_1)$, this eigenvalue is $> (\frac{\gamma}{K^\tau})^2$. Hence, the bad $\widetilde{\mu}$ correspond to an interval

$$[(\frac{\gamma}{K^\tau})^2, (\frac{\gamma}{K^\tau})^2 + \frac{c\varepsilon^4}{N_1^\sigma}],$$

containing the above eigenvalue of $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_1)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_1)*}$. The same proof as the one made for Lemma 8.4, shows that the measure of corresponding bad set of $\widetilde{\mu}$ is bounded by

$$\frac{2}{\sqrt{(1-k/2)}}\sqrt{\frac{c\varepsilon^4}{N_1^\sigma}},$$

Hence,

$$\begin{aligned}
\text{meas}\left(\cup_{M_\varepsilon \leq K \leq N_1} B_{\varepsilon,\gamma}^{(K)}(V_2) \cap G_{\varepsilon,\gamma}^{(K)}(V_1)\right) &\leq \frac{2}{\sqrt{(1-k/2)}}\sqrt{\frac{c\varepsilon^4}{N_1^\sigma}}\sum_{M_\varepsilon \leq K \leq N_1} bK^d \leq \frac{2b\sqrt{c}}{\sqrt{(1-k/2)}}\frac{\varepsilon^2}{N_1^{\sigma/2-d-1}} \\
&\leq \frac{2b\sqrt{c}}{\sqrt{(1-k/2)}}\frac{\varepsilon^2}{N_1}\frac{1}{M_\varepsilon^{\sigma/2-d-2}} \leq \frac{c'\gamma}{N_1}\varepsilon^{2+\frac{\sigma/2-d-3}{4l_0}} \leq \frac{c'\gamma\varepsilon^6}{N_1}.
\end{aligned}$$

Now

$$\begin{aligned}\text{meas}\left(\cup_{N_1\le K\le N_2} B^{(K)}_{\varepsilon,\gamma}(V_2)\right) &\le \sum_{N_1\le K\le N_2}\frac{C\gamma}{K^{\tau-d}} \le \frac{C\gamma(\tau-d-1)}{N_1^{\tau-d-1}}\\ &\le \frac{c''\gamma}{N_1}\varepsilon^{\frac{\tau-d-2}{4l_0}} \le \frac{c''\gamma\varepsilon^6}{N_1}.\end{aligned}$$

Finally

$$\text{meas}\left(\mathcal{G}^{(N_2)}_{\varepsilon,\gamma}(V_2)\right)^c \cap \mathcal{G}^{(N_1)}_{\varepsilon,\gamma}(V_1) \le \frac{(c'+c'')\gamma\varepsilon^6}{N_1},$$

which is the result of the Lemma.

■

We may then apply a simple adaptation of theorem 3 of Berti-Bolle-Procesi [5] to solve equation $\mathcal{F}(\epsilon,\widetilde{\mu},V)=0$, and find the solution V which is C^2 in the parameters $(\varepsilon,\widetilde{\mu})$, and such that $V\in\mathcal{U}^{(N)}_M$. Let γ,m,s_0 be as in Proposition 8.14. Moreover, let $\overline{s}>s_0+4(m+1)+8m=s_0+4+12m$.

From proposition 8.14, it follows that if $(\epsilon,\widetilde{\mu},V)\in[0,\epsilon_1]\times[-\varepsilon,\varepsilon]\times\mathcal{U}^{(N)}_M$, $V\in\mathcal{K}_{0,s}$ and $\widetilde{\mu}\in G^{(N)}_{\epsilon,\gamma}(V)$ then $(\epsilon,\widetilde{\mu},V)\in J^{(N)}_{\gamma,m}$ (as defined in (4) of [5], that is (8.20) holds for $s\in[s_0,\overline{s}]$.

In [5] [theorem 3] one considers $N\ge N_0=N_0(\gamma)$ with $N_0(\gamma)$ sufficiently large and $0<\epsilon\le\epsilon_2(\gamma)$ with $\epsilon_2(\gamma)$ sufficiently small. We may choose $N_0=N_0(\gamma)=M_{\epsilon_3(\gamma)}$ with a suitable $\epsilon_3(\gamma)\le\epsilon_2$ and we consider in the following $0<\epsilon\le\epsilon_3(\gamma)$.

Theorem 9.2 *Let s_0 and $\tilde{\gamma}$ be as in Proposition 8.14. Then for all $0<\gamma<\tilde{\gamma}$ there exist $\epsilon_2(\gamma)\in[0,\epsilon_0]$ and a C^2-map $V:(0,\epsilon_2(\gamma))\times[-\varepsilon,\varepsilon]\to\Pi_N\pi_0Q_0\mathcal{K}_{0,s_0}$, such that $V(0,0)=0$, $||\partial_{\widetilde{\mu}}V||_{0,s_0}\le M$, and if $\epsilon\in(0,\epsilon_2(\gamma))$, $\widetilde{\mu}\in([-\varepsilon,\varepsilon]\setminus C_{\epsilon,\gamma})$, the function $V(\epsilon,\widetilde{\mu})$ is solution of $\mathcal{F}(\epsilon,\widetilde{\mu},V)=0$ (9.1). Here $C_{\epsilon,\gamma}$ is a subset of $[-\varepsilon,\varepsilon]$ which is a Hölder continuous function of ε, and has Lebesgue-measure less than $C\gamma\epsilon^6$ for some constant $C>0$ independent of ϵ and γ.*

The proof is the same as in [6], except for Hölder continuity which is proved at next section. In fact $C_{\epsilon,\gamma}$ is a union of intervals $I^{(N_n)}_\varepsilon$ (see definition 10.1, with $N_n=(N_0(\gamma))^{2^n}$, so that each end of each interval is a fonction of ε which is Hölder continuous in ε with exponent $1/2$.

2.10 Resolution of the bifurcation equation

Let V be the function obtained in Theorem 9.2. It is C^2 in $(\varepsilon,\widetilde{\mu})$. Replacing $V(\varepsilon,\widetilde{\mu})$ in the bifurcation equation (7.4), and replacing $\tilde{\mu}$ by $\varepsilon^3\mu'$, we can solve with respect to μ' and find a function $\tilde{h}(\varepsilon)$ which is C^1 in (ε), such that

$$\mu' = \varepsilon\mu_4 + \varepsilon\tilde{h}(\varepsilon), \quad (H),\ \tilde{h}(\varepsilon) = \mathcal{O}(\varepsilon) \tag{10.1}$$

for $\varepsilon \in (0, \varepsilon_2(\gamma))$ provided that ε_2 is small enough, and $\mu' \in [-1, 1]$.

For obtaining solutions valid for our system, *the condition* $\mu_4 \neq 0$ *is not required* (see (6.24) for μ_4). Indeed, in case $\mu_4 = 0$, the curve (H) in the (ε, μ') plane is just more flat near $\varepsilon = 0$. This coefficient μ_4 has not been computed yet, but it can be computed in principle, depending a priori on q only.

Let us show that in the plane (ε, μ') the bad set is located into "bad strips". Then we shall need a transversality condition to insure that these bad strips intersect transversally the "curve" (H), such that any point of this curve, which does not belong to bad strips, gives indeed an eligible solution of our problem.

2.10.1 Transversality condition for "bad strips"

In the plane (ε, μ'), the bad strips are bounded by the curves given by the solutions $\widetilde{\mu}^{\pm}(\varepsilon)$ (where $\widetilde{\mu} = \varepsilon^3\mu'$) of

$$\sigma_j(\varepsilon, \widetilde{\mu}) = \widetilde{\mu}^2 + f_j(\varepsilon, \widetilde{\mu}) = \eta^2,$$

where $\eta = \gamma/N^\tau$, not forgetting that σ_j depends on N.

Definition 10.1 *For N and V fixed, a set of "bad strips" is defined by*

$$BS_N(V) = \{(\varepsilon, \mu') \in [0, \varepsilon_2(\gamma)] \times [-1, 1]; \varepsilon^3\mu' \in I_\varepsilon^{(N)}\},$$

where $I_\varepsilon^{(N)}$ is one of the intervals $(\tilde{\mu}_j^-(\varepsilon), \tilde{\mu}_j^+(\varepsilon))$, or with one of the bounds replaced by ε^3 (right bound), or by $-\varepsilon^3$ (left bound), as defined at section 2.8.3 .

Let us show that the limiting curves $\tilde{\mu}_j^-(\varepsilon), \tilde{\mu}_j^+(\varepsilon)$ are Hölder continuous with exponent 1/2. We have for $\varepsilon_2 > \varepsilon_1$, along a limiting curve

$$\sigma_j(\varepsilon_2, \widetilde{\mu}_2) - \sigma_j(\varepsilon_1, \widetilde{\mu}_1) = 0,\ \widetilde{\mu}_j = \widetilde{\mu}(\varepsilon_j),$$

$$\sigma_j(\varepsilon_2, \widetilde{\mu}_2) - \sigma_j(\varepsilon_1, \widetilde{\mu}_1) = \widetilde{\mu}_2^2 - \widetilde{\mu}_1^2 + f_j(\varepsilon_2, \widetilde{\mu}_2) - f_j(\varepsilon_1, \widetilde{\mu}_1)$$

and thanks to (8.10)), assuming $\widetilde{\mu_2} > \widetilde{\mu_1}$,

$$\widetilde{\mu}_2^2 - \widetilde{\mu}_1^2 \leq c(\delta_0' + \varepsilon)(|\varepsilon_2 - \varepsilon_1| + \widetilde{\mu}_2 - \widetilde{\mu}_1)$$

hence

$$[\widetilde{\mu}_2 - \frac{c}{2}(\delta_0' + \varepsilon)]^2 - [\widetilde{\mu}_1 - \frac{c}{2}(\delta_0' + \varepsilon)]^2 \leq c'(\delta_0' + \varepsilon)(\varepsilon_2 - \varepsilon_1),$$

and since the two quantities in brackets have the same sign when $|\tilde{\mu}_2-\tilde{\mu}_1|$ is small enough, then if

$$[\widetilde{\mu}_2-\frac{c}{2}(\delta_0'+\varepsilon)]^2-[\widetilde{\mu}_1-\frac{c}{2}(\delta_0'+\varepsilon)]^2>0,$$

we may use the argument that when $0<a^2-b^2<\eta^2$, with $ab\geq 0$, then $|a-b|\leq|\eta|$, which leads to

$$\widetilde{\mu}_2-\widetilde{\mu}_1\leq\sqrt{c'(\delta_0'+\varepsilon)(\varepsilon_2-\varepsilon_1)},$$

which is the Hölder continuity. If, on the contrary

$$[\widetilde{\mu}_2-\frac{c}{2}(\delta_0'+\varepsilon)]^2-[\widetilde{\mu}_1-\frac{c}{2}(\delta_0'+\varepsilon)]^2<0,$$

we need to use Condition 8.7, as in Section 2.8.3. For $|\widetilde{\mu}_2-\widetilde{\mu}_1|$ small enough, we may assume that either $\widetilde{\mu}_m(\varepsilon_1)<\widetilde{\mu}_1<\widetilde{\mu}_2$ (upper limit curve), or $\widetilde{\mu}_1<\widetilde{\mu}_2<\widetilde{\mu}_m(\varepsilon_2)$ (lower limit curve). In the first case, we obtain

$$\sigma_j(\varepsilon_1,\widetilde{\mu}_2)-\sigma_j(\varepsilon_1,\widetilde{\mu}_1)\geq(1-k/2)[(\widetilde{\mu}_2-\widetilde{\mu}_m(\varepsilon_1))^2-(\widetilde{\mu}_1-\widetilde{\mu}_m(\varepsilon_1))^2]\geq(1-k/2)(\widetilde{\mu}_2-\widetilde{\mu}_1)^2.$$

In the second case, we obtain

$$|\sigma_j(\varepsilon_2,\widetilde{\mu}_2)-\sigma_j(\varepsilon_2,\widetilde{\mu}_1)|\geq(1-k/2)[(\widetilde{\mu}_m(\varepsilon_2)-\widetilde{\mu}_1)^2-(\widetilde{\mu}_m(\varepsilon_2)-\widetilde{\mu}_2)^2]\geq(1-k/2)|\widetilde{\mu}_2-\widetilde{\mu}_1|^2.$$

On the other hand, we have

$$\begin{aligned}|\sigma_j(\varepsilon_1,\widetilde{\mu}_2)-\sigma_j(\varepsilon_1,\widetilde{\mu}_1)| &= |\sigma_j(\varepsilon_1,\widetilde{\mu}_2)-\sigma_j(\varepsilon_2,\widetilde{\mu}_2)|\leq c'(\delta_0'+\varepsilon)|\varepsilon_2-\varepsilon_1|\\ |\sigma_j(\varepsilon_2,\widetilde{\mu}_2)-\sigma_j(\varepsilon_2,\widetilde{\mu}_1)| &= |\sigma_j(\varepsilon_2,\widetilde{\mu}_1)-\sigma_j(\varepsilon_1,\widetilde{\mu}_1)|\leq c'(\delta_0'+\varepsilon)|\varepsilon_2-\varepsilon_1|.\end{aligned}$$

Hence, in all cases

$$|\widetilde{\mu}_2-\widetilde{\mu}_1|^2\leq(1-k/2)^{-1}c'(\delta_0'+\varepsilon)|\varepsilon_2-\varepsilon_1|,$$

which is Hölder continuity. ∎

In the case when $\widetilde{\mu}$ *is not exceptional, i.e. if the eigenvalue* σ_j *is not multiple,* the slope of the tangent to the curves $\widetilde{\mu}_j^-(\varepsilon),\widetilde{\mu}_j^+(\varepsilon)$ is

$$t(\varepsilon)=-\frac{\partial_\varepsilon\sigma_j(\varepsilon,\widetilde{\mu}_j^+)}{\partial_{\widetilde{\mu}}\sigma_j(\varepsilon,\widetilde{\mu}_j^+)},\tag{10.2}$$

given here for $\widetilde{\mu}_j^+(\varepsilon)$ (analogous formulae holding for the other curve). Now in a more precise way, for $(\varepsilon,\tilde{\mu})$ not exceptional, and taking into account of the form (8.7), we obtain by standard arguments for simple eigenvalues:

$$\begin{aligned}\partial_{\widetilde{\mu}}\sigma_j(\varepsilon,\widetilde{\mu}^+) &= 2\langle(\mathcal{A}-\lambda_0)\zeta_j(\varepsilon,\widetilde{\mu}^+),\zeta_j(\varepsilon,\tilde{\mu}^+)\rangle+2\tilde{\mu}^++\mathcal{O}(\varepsilon)=\mathcal{O}(\delta_0'+\varepsilon),\\ \partial_\varepsilon\sigma_j(\varepsilon,\widetilde{\mu}^+) &= -4\langle\mathcal{B}(u_1,(\mathcal{A}-\lambda_0)\zeta_j(\varepsilon,\widetilde{\mu}^+),\zeta_j(\varepsilon,\widetilde{\mu}^+)\rangle+\mathcal{O}(\varepsilon)=\mathcal{O}(\delta_0'+\varepsilon),\end{aligned}$$

where $\zeta_j(\varepsilon, \widetilde{\mu}^+)$ is the eigenvector with norm 1 belonging to the eigenvalue $\sigma_j(\varepsilon, \widetilde{\mu}^+)$ of the operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*}$. Even though the operator $(\mathcal{A} - \lambda_0)$ is definite negative in the subspace where ζ_j lives, we may notice that $(\mathcal{A} - \lambda_0)\zeta_j(\varepsilon, \widetilde{\mu}^+)$ may be very small, so the term $\mathcal{O}(\varepsilon)$ above might be the dominant order in $\partial_{\widetilde{\mu}}\sigma_j$ and $\partial_\varepsilon\sigma_j$. It is then difficult to be more precise for any transversallity condition of the strips $BS_N(V)$ with respect to the curve (H) defined by (10.1).

Now, let us consider for (N, ε) fixed, the bad set of $\widetilde{\mu}$ which we know is of measure bounded by $c_3\gamma\varepsilon^6/N$ (see Proposition 8.9 and Lemma 9.1). In case of intersection of a bad strip with (H), we need to measure the corresponding set of "bad ε". The proof of Theorem 9.2 via Nash-Moser process considers a sequence $N_n = (N_0(\gamma))^{2^n}$ and successive approximates V_n of the solution V. For estimating the intersections of the bad strips with the curve (H) we are led to make a transversality conjecture.

Conjecture 10.2 *Let $\widetilde{\mu}^{\pm(N_n)}(\varepsilon)$ be any one of the limiting curves of the bad strips of $BS_{N_n}(V_{n-1})$, $n \in \mathbb{N}$. Then we assume that for any of these curves, there exists $c > 0$ independent of N_n, such that for $h \in \mathbb{R}$ in a neighborhood of 0, the following inequality holds:*

$$|\widetilde{\mu}(\varepsilon + h) - \widetilde{\mu}(\varepsilon)| \geq c\varepsilon^2|h|.$$

Remark 10.3 *This is indeed a* very weak assumption *for the slopes defined by (10.2), since this means that the slopes $t(\varepsilon)$ have a lower bound $|t(\varepsilon)| > c\varepsilon^2$. This insures transversality with the bifurcation curve (H) , the slope of which is $\mathcal{O}(\varepsilon^3)$. However we have no means to check its validity. Moreover, if, unluckily, a curve $\widetilde{\mu}(\varepsilon)$ belonging to one of the bad strips of $BS_{N_n}(V_{n-1})$ intersects (H) at an exceptional point $(\varepsilon, \widetilde{\mu}(\varepsilon)$, where an eigenvalue σ_j is multiple, then we cannot a priori define the "slope" of the corresponding limiting curve of the bad strip. This is why we took the above formulation for the Transversality conjecture even though we might just eliminate the corresponding exceptional values of ε (we have no bound for their measure).*

Remark 10.4 *In taking μ_ε in (7.1) at a higher order than ε^3, we should find $\widetilde{\mu}$ of higher order than ε^4 which flattens the slope of the bifurcation curve (H). Then we could* weaken the transversality condition *and replace ε^2 by a an order in ε larger than 2, which still guarantees the transversality with (H).*

Let us denote by $\delta\widetilde{\mu}$ the measure of the bad $\widetilde{\mu}$, and by $\delta\varepsilon$ the corresponding measure for bad ε. Then we have, (see the right side of Figure 10.1):

$$\delta\varepsilon < \frac{\delta\widetilde{\mu}}{|t|} < \frac{\delta\widetilde{\mu}}{c\varepsilon^2}.$$

Let us define for ε fixed, the set $B_\varepsilon S_N(V)$ which is the section of $BS_N(V)$ for some ε. In summing the measure of the bad set for ε after all iterations, we obtain a measure of the bad set for ε, bounded by the measure of $C_{\varepsilon,\gamma} = \cup_{n\geq 1} B_\varepsilon S_{N_n}(V_{n-1})$ divided by $c\varepsilon^2$, i.e. a bad set bounded by $C\gamma\varepsilon^4$ (see Theorem 9.2). The complementary subset in $(0, \varepsilon_3)$, constitutes the good set of ε, which is of asymptotic full measure since $\frac{\varepsilon - C\gamma\varepsilon^4}{\varepsilon} \to 1$ as $\varepsilon \to 0$.

Remark 10.5 *In the case when we need to weaken the transversality condition 10.2, as indicated in the Remark above, we can also increase the order (here ε^6) for the size of bad $\widetilde{\mu}$ in Theorem 9.2, just in increasing τ in Proposition 8.14, so that we can keep an order of smallness ε^4 for the bad ε's.*

Remark 10.6 *If we consider $\widetilde{\mu}$ in an interval independent of ε, we can look at the situation for $\varepsilon = 0$, as in Remark 7.5. We see that the eigenvalues $\sigma_j(0, \widetilde{\mu})$ have the form:*

$$\sigma_j(0, \widetilde{\mu}) = (\widetilde{\mu} + \lambda_0(|\mathbf{k}|^2) - \lambda_0)^2, \quad N_\mathbf{k} \leq N$$

This leads to bad intervals for $\widetilde{\mu}$ of the form

$$[\lambda_0 - \lambda_0(|\mathbf{k}|^2) - \frac{\gamma}{N^\tau}, \lambda_0 - \lambda_0(|\mathbf{k}|^2) + \frac{\gamma}{N^\tau}], \quad \textit{with } \mathbf{k} \textit{ such that } N_\mathbf{k} \leq N. \tag{10.3}$$

We notice that $\lambda_0 - \lambda_0(|\mathbf{k}|^2) \sim c(|\mathbf{k}|^2 - k_c^2)$ with $c \neq 0$ because of Assumption 6.2. Hence

$$\lambda_0 - \lambda_0(|\mathbf{k}|^2) > \frac{c'}{N^{4l_0}},$$

which gives intervals (10.3) "far" from 0 for τ large enough (which is one of our assumptions in Proposition 8.14).

2.10.2 Final result

If the Transversality Conjecture 10.2 is verified, then there is a good set for ε, with asymptotic full measure as $\varepsilon \to 0$, such that there exists a couple $(\varepsilon, \widetilde{\mu}(\epsilon))$ on the curve (H) which lies in the good set (see Figure 10.1). Then this gives the existence of a solution $(\varepsilon, \mu'(\varepsilon))$ of (10.1), as ε tends towards 0.

Now we observe that we can write $\mu' = \varepsilon\overline{\mu}$, with $\overline{\mu}$ centered in μ_4. This defines the good 1-dimensional set $\overline{\Lambda}_\varepsilon$ of all good $\overline{\mu}_\varepsilon$.

Finally with (7.1), we obtain a solution of (5.17) under the form

$$\begin{aligned} u &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4 + \varepsilon^4 (V(\varepsilon, \varepsilon^4\overline{\mu}_\varepsilon) - h(\varepsilon, \varepsilon\overline{\mu}_\varepsilon)) \\ \lambda &= \lambda_0 - \mu_2\varepsilon^2 - \mu_3\varepsilon^3 - \varepsilon^4\overline{\mu}_\varepsilon. \end{aligned}$$

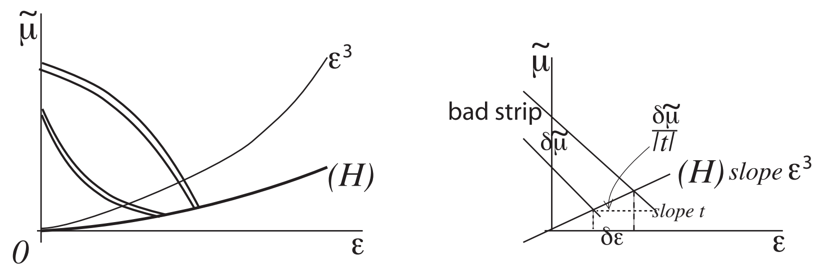

Figure 10.1: Sketch of the bad set in the plane $(\varepsilon, \tilde{\mu})$. (H) is the "curve" given by (10.1) approximated by $\varepsilon^4\mu_4$ ($\mu_4 > 0$ is assumed here). The drawing on the right side explains the bound for the measure of $\delta\varepsilon$.

This ends the proof of Theorem 1.1, with little adaptation of notations. Notice that the solution $(\lambda, u)(\varepsilon)$ is C^1, restricted to the "good" values of ε. (see Figure 1.2).

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Chapter 3

Spatial dynamics

In this Chapter we study the structure of the linearized operator in the spatial dynamics formulation of the steady prpblem, as introduced by K.Kirchgässner [49] and later adapted on Navier-Stokes equations in [41]. More specifically, we give the details on the center spectrum of the operator in main Lemmas 2.2 and 2.3.

3.1 Spatial dynamics

3.1.1 Basic formulation

The starting point of our analysis is a formulation of the steady case of system (2.4),(2.5),(2.6) of Chapter one as a dynamical system in which the evolutionary variable is the horizontal spatial coordinate x. We recall below the system in its steady form,

$$\mu^{-1}\Delta\mathbf{V} + \theta e_z - \mathcal{P}^{-1}(\mathbf{V}\cdot\nabla)\mathbf{V} - \nabla p = 0, \tag{1.1}$$

$$\mu^{-1}\Delta\theta + V_z - (\mathbf{V}\cdot\nabla)\theta = 0 \tag{1.2}$$

$$\nabla\cdot\mathbf{V} = 0, \tag{1.3}$$

$$V_z = \theta = 0 \text{ for } z = 0, 1, \tag{1.4}$$

where

$$\mu = \mathcal{R}^{1/2},$$

and for a rigid boundary in $z = 0$ or 1, we add

$$V_x = V_y = 0, \tag{1.5}$$

while for a free boundary in $z = 0$ or 1, we add

$$\partial_z V_x = \partial_z V_y = 0. \tag{1.6}$$

Set $\mathbf{V} = (V_x, V_\perp)$, where $V_\perp = (V_y, V_z)$, and consider the new variables

$$\mathbf{W} = \mu^{-1}\partial_x \mathbf{V} - p\mathbf{e}_x, \quad \phi = \partial_x \theta, \tag{1.7}$$

in which we write $\mathbf{W} = (W_x, W_\perp)$, and $W_\perp = (W_y, W_z)$. Using the equation (1.3) we obtain the formula for the pressure,

$$p = -\mu^{-1}\nabla_\perp \cdot V_\perp - W_x. \tag{1.8}$$

Then we write the system (1.1,1.2,1.3) in the form

$$\partial_x \mathbf{U} = \mathcal{L}_\mu \mathbf{U} + \mathcal{B}_\mu(\mathbf{U}, \mathbf{U}), \tag{1.9}$$

in which $\mathbf{U}$ is the 8-components vector

$$\mathbf{U} = (V_x, V_\perp, W_x, W_\perp, \theta, \phi),$$

and the operators $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$ are linear and quadratic, respectively, defined by

$$\mathcal{L}_\mu \mathbf{U} = \begin{pmatrix} -\nabla_\perp \cdot V_\perp \\ \mu W_\perp \\ -\mu^{-1}\Delta_\perp V_x \\ -\mu^{-1}\Delta_\perp V_\perp - \theta \mathbf{e}_z - \mu^{-1}\nabla_\perp(\nabla_\perp \cdot V_\perp) - \nabla_\perp W_x \\ \phi \\ -\Delta_\perp \theta - \mu V_z \end{pmatrix},$$

$$\mathcal{B}_\mu(\mathbf{U}, \mathbf{U}) = \begin{pmatrix} 0 \\ 0 \\ \mathcal{P}^{-1}\big((V_\perp \cdot \nabla_\perp)V_x - V_x(\nabla_\perp \cdot V_\perp)\big) \\ \mathcal{P}^{-1}\big((V_\perp \cdot \nabla_\perp)V_\perp + \mu V_x W_\perp\big) \\ 0 \\ \mu\big((V_\perp \cdot \nabla_\perp)\theta + V_x \phi\big) \end{pmatrix}.$$

We look for solutions of (1.9) which are periodic in y and satisfy the boundary conditions (1.4), and (1.5) (for the moment, the case of "rigid-rigid" boundary conditions). For such solutions we have

$$\frac{d}{dx}\int_\Omega V_x \, dy \, dz = -\int_\Omega \nabla_\perp \cdot V_\perp \, dy \, dz = -\int_{\partial\Omega} n \cdot V_\perp \, ds = 0,$$

which implies that the flux

$$\mathcal{F}(x) = \int_\Omega V_x \, dy \, dz$$

is constant. Equivalently, this property implies that the dynamical system (1.9) leaves invariant the subspace orthogonal to the vector $\boldsymbol{\psi}_0 = (1, 0, 0, 0, 0, 0, 0, 0)$. We restrict to this subspace, hence fixing the constant flux to 0. Including this property and the boundary conditions (1.4)-(1.5) in the definition of the phase space $\mathcal{X}$ of the dynamical system (1.9), we take

$$\mathcal{X} = \big\{\mathbf{U} \in (H^1_{per}(\Omega))^3 \times (L^2_{per}(\Omega))^3 \times H^1_{per}(\Omega) \times L^2_{per}(\Omega) \;; \\ V_x = V_\perp = \theta = 0 \text{ on } z = 0, 1, \text{ and } \int_\Omega V_x \, dy \, dz = 0\big\}.$$

where $\Omega = \mathbb{R} \times (0, 1)$ and the subscript *per* means that the functions are $2\pi/k_y$-periodic in y, for some fixed $k_y > 0$ (in order to distinguish between periodicity in x and y, we add the subscript y in the notation of the wavenumber k). The phase space $\mathcal{X}$ is a closed subspace of the Hilbert space

$$\widetilde{\mathcal{X}} = (H^1_{per}(\Omega))^3 \times (L^2_{per}(\Omega))^3 \times H^1_{per}(\Omega) \times L^2_{per}(\Omega),$$

so that it is a Hilbert space endowed with the usual scalar product of $\widetilde{\mathcal{X}}$. Accordingly, we define the domain of definition $\mathcal{Z}$ of the linear operator $\mathcal{L}_\mu$ by

$$\mathcal{Z} = \big\{\mathbf{U} \in \mathcal{X} \cap (H^2_{per}(\Omega))^3 \times (H^1_{per}(\Omega))^3 \times H^2_{per}(\Omega) \times H^1_{per}(\Omega) \;; \\ \nabla_\perp \cdot V_\perp = W_\perp = \phi = 0 \text{ on } z = 0, 1\big\},$$

so that $\mathcal{L}_\mu$ is closed and its domain $\mathcal{Z}$ is dense and compactly embedded in $\mathcal{X}$. In particular, this latter property implies that $\mathcal{L}_\mu$ has purely point spectrum which consists of isolated eigenvalues with finite algebraic multiplicity.

The dynamical system (1.9) inherits the symmetries of the original system (1.1),(1.2),(1.3). As for the two-dimensional convection, horizontal translations $y \to y + a/k_y$ along the y direction give a one-parameter family of linear maps $(\boldsymbol{\tau}_a)_{a \in \mathbb{R}/2\pi\mathbb{Z}}$ defined on $\mathcal{X}$ through

$$\boldsymbol{\tau}_a \mathbf{U}(y, z) = \mathbf{U}(y + a/k_y, z), \tag{1.10}$$

and which commute with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$. The reflection $x \mapsto -x$ now gives a reversibility symmetry

$$\mathbf{S}_1 \mathbf{U}(y, z) = (-V_x, V_\perp, W_x, -W_\perp, \theta, -\phi)(y, z), \tag{1.11}$$

for $\mathbf{U} \in \mathcal{X}$, which anti-commutes with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$, and the reflections $y \mapsto -y$ and $z \mapsto 1 - z$ give the symmetries

$$\mathbf{S}_2 \mathbf{U}(y, z) = (V_x, -V_y, V_z, W_x, -W_y, W_z, \theta, \phi)(-y, z), \tag{1.12} \\ \mathbf{S}_3 \mathbf{U}(y, z) = (V_x, V_y, -V_z, W_x, W_y, -W_z, -\theta, -\phi)(y, 1 - z),$$

for $\mathbf{U} \in \mathcal{X}$, which both commute with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$. Notice that

$$\boldsymbol{\tau}_a \mathbf{S}_2 = \mathbf{S}_2 \boldsymbol{\tau}_{-a}, \quad \boldsymbol{\tau}_0 = \boldsymbol{\tau}_{2\pi} = \mathbb{I},$$

so that the system (1.9) is $O(2)$-equivariant, and that $\mathbf{S}_3$ commutes with $\boldsymbol{\tau}_a$.

In addition to these symmetries inherited from the original system (1.1),(1.2),(1.3), the dynamical system (1.9) has a specific invariance due to the new variable $\mathbf{W} = (W_x, W_\perp)$ in (1.7). While $W_\perp$ satisfies the same boundary conditions as $V_\perp$, included in the domain of definition $\mathcal{Z}$ of the linear operator, there are no such conditions for W_x because the pressure p in the definition of W_x is only defined up to a constant. As a consequence, the dynamical system is invariant upon adding any constant to W_x, i.e., the vector field is invariant under the action of the one-parameter family of maps $(\boldsymbol{T}_b)_{b\in\mathbb{R}}$, defined on $\mathcal{X}$ through

$$\boldsymbol{T}_b \mathbf{U} = \mathbf{U} + b\boldsymbol{\varphi}_0, \quad \boldsymbol{\varphi}_0 = (0,0,0,1,0,0,0,0)^t. \tag{1.13}$$

This invariance introduces the vector $\boldsymbol{\varphi}_0$ in the kernel of $\mathcal{L}_\mu$ (see Lemma 2.1).

3.1.2 Two-dimensional convection

Let us adapt here some results obtained in Chapter one, Section 1.3.4. The simple classical convection problem restricts to velocity fields $\mathbf{V} = (0, V_y, V_z)$ which are two-dimensional and functions which are independent of x and periodic in y. The corresponding function space for the system (1.1)-(1.5) is

$$\mathcal{H} = \{\mathbf{u} \in \{0\} \times (L^2_{per}(\Omega))^3 \; ; \; \nabla \cdot \mathbf{V} = 0, \; V_z = 0 \text{ on } z = 0, 1\},$$

where $\Omega = \mathbb{R} \times (0,1)$ and the subscript *per* means that the functions are $2\pi/k$-periodic in y, for some fixed $k > 0$. The boundary conditions (1.4)-(1.5) (for the rigid-rigid case) are included in the domain $\mathcal{D}$ of the linear operator $\mathbf{L}_\mu$ by taking

$$\mathcal{D} = \{\mathbf{u} \in \{0\} \times (H^2_{per}(\Omega))^3 \; ; \; \nabla \cdot \mathbf{V} = 0, \; V_y = V_z = \theta = 0 \text{ on } z = 0, 1\}.$$

As seen in Chapter 1, the linear operator $\mathbf{L}_\mu$ is selfadjoint, with compact resolvent and the quadratic operator $\mathbf{R}$ is symmetric and bounded from $\mathcal{D}$ to $\mathcal{H}$.

As a consequence of the invariance of the equations (1.1)-(1.5) under horizontal translations and reflections, the system is $O(2)$-equivariant with the one-parameter family of linear maps $(\boldsymbol{\tau}_a)_{a\in\mathbb{R}/2\pi\mathbb{Z}}$ and the discrete symmetry $\mathbf{S}_2$.

Bifurcations are determined by the kernel of $\mathbf{L}_\mu$. As seen in Chapter 1, elements in the kernel of $\mathbf{L}_\mu$ are found by looking for solutions of the form $e^{iky}\widehat{\mathbf{u}}_k(z)$ for the linear equation

$$\mathbf{L}_\mu \mathbf{u} = 0, \tag{1.14}$$

and the boundary conditions $V_y = V_z = \theta = 0$ on $z = 0, 1$. The direct computation made in Chapter 1, section 1.3.3 (see also [14]) gives

$$e^{iky}\widehat{\mathbf{u}}_k(z) = e^{iky}\begin{pmatrix} 0 \\ \frac{i}{k}DV \\ V \\ \theta \end{pmatrix}, \tag{1.15}$$

where $D = d/dz$ denotes the derivative with respect to z, and the functions $V = V(z)$ and $\theta = \theta(z)$ are real-valued solutions of the boundary value problem

$$(D^2 - k^2)^2 V = \mu k^2 \theta, \quad V = DV = 0 \text{ in } z = 0, 1, \tag{1.16}$$

$$(D^2 - k^2)\theta = -\mu V, \quad \theta = 0 \text{ in } z = 0, 1. \tag{1.17}$$

Of particular interest for the classical bifurcation problem, and also in our context, is the global minimum $\mu_c = \mu_0(k_c)$ of $\mu_0(k)$ (see Figure 3.2 in Chapter one).

Going back to the kernel of $\mathbf{L}_\mu$, as expected by the general theory of $O(2)$-equivariant systems, for $\mu = \mu_0(k)$ and any $k > 0$ the kernel of $\mathbf{L}_{\mu_0(k)}$ is two-dimensional and spanned by the vectors

$$\boldsymbol{\xi}_0 = e^{iky}\widehat{\mathbf{u}}_k(z), \quad \overline{\boldsymbol{\xi}_0} = e^{-iky}\overline{\widehat{\mathbf{u}}_k}(z),$$

satisfying

$$\boldsymbol{\tau}_a\boldsymbol{\xi}_0 = e^{ia}\boldsymbol{\xi}_0, \quad \mathbf{S}_2\boldsymbol{\xi}_0 = \overline{\boldsymbol{\xi}_0}, \quad \mathbf{S}_3\boldsymbol{\xi}_0 = -\boldsymbol{\xi}_0.$$

Since the operator has compact resolvent, this shows that 0 is an isolated double semi-simple eigenvalue of $\mathbf{L}_{\mu_0(k)}$, and it turns out that all other eigenvalues are negative. This property is a key ingredient in the proof of existence of rolls, which bifurcate from the trivial solution at $\mu = \mu_0(k)$, for any fixed $k > 0$, in a steady bifurcation with $O(2)$ symmetry.

We made in Chapter one, section 1.3.4, the bifurcation analysis showing the existence of convective rolls (taking $B = C = 0$). The equilibria $\mathbf{U} \in \mathcal{Z}$ of the dynamical system (1.9) can be found as solutions $\mathbf{u} \in \mathcal{D}$ of the two-dimensional problem through the projection

$$\mathbf{u} = \Pi\mathbf{U} = (V_x, V_\perp, \theta). \tag{1.18}$$

In particular, for any $k_y = k > 0$ fixed, the rolls found above give a circle of equilibria $\boldsymbol{\tau}_a(\mathbf{U}^\star_{k,\mu})$, for $a \in \mathbb{R}/2\pi\mathbb{Z}$, which bifurcate for $\mu > \mu_0(k)$ sufficiently close to $\mu_0(k)$, belong to $\mathcal{D}$, and satisfy

$$\mathbf{S}_1\mathbf{U}^\star_{k,\mu} = \mathbf{S}_2\mathbf{U}^\star_{k,\mu} = \mathbf{U}^\star_{k,\mu}, \quad \mathbf{S}_3\mathbf{U}^\star_{k,\mu} = \boldsymbol{\tau}_\pi\mathbf{U}^\star_{k,\mu}. \tag{1.19}$$

Due to the rotation invariance of the three-dimensional problem, horizontally rotated rolls are solutions and relative equilibria of the dynamical system (1.9). For any angle

$\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, we find the rotated rolls $\boldsymbol{\mathcal{R}}_\alpha(\mathbf{U}^\star_{k,\mu})$, where the horizontal rotation $\boldsymbol{\mathcal{R}}_\alpha$ acts on the 4-components vector $\mathbf{u} = \Pi\mathbf{U}$ through

$$\boldsymbol{\mathcal{R}}_\alpha \mathbf{u}(x,y,z) = (\mathcal{R}_\alpha(V_x,V_y),V_z,\theta)(\mathcal{R}_{-\alpha}(x,y),z), \tag{1.20}$$

in which

$$\mathcal{R}_\alpha(x,y) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha).$$

(We do not need here the more complicated representation formula for the 8-components vector $\mathbf{U}$.) These rotated rolls are periodic functions in both x and y with wavenumbers $k\sin\alpha$ and $k\cos\alpha$, respectively. As solutions of the dynamical system (1.9), they belong to the phase space $\mathcal{X}$ provided $k_y = k\cos\alpha$, and in this case they are $2\pi/k\sin\alpha$-periodic solutions in x (see Figure 1.1 for a plot of the possible wavenumbers k_y in y for $\mu > \mu_c$ sufficiently close to μ_c). For the particular angles $\alpha = 0$ and $\alpha = \pi$ the rotated rolls are equilibria in the phase-space $\mathcal{X}$ with $k_y = k$. For the orthogonal angles $\alpha = \pi/2$ and $\alpha = 3\pi/2$, they are solutions $2\pi/k$-periodic in x, for any $k_y > 0$.

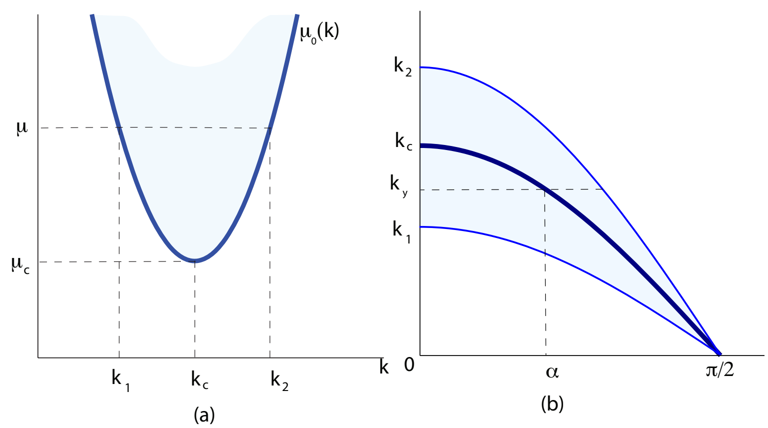

Figure 1.1: **(a)** Graph of $\mu_0(k)$. Two-dimensional rolls bifurcate into the shaded region situated above the curve $\mu_0(k)$. For $\mu > \mu_c$ sufficiently close to μ_c, two-dimensional rolls exist for wavenumbers $k \in (k_1,k_2)$ with $\mu = \mu_0(k_1) = \mu_0(k_2)$.**(b)** Plot of the wavenumbers $k_y = k\cos\alpha$ in y of the rolls rotated by angles $\alpha \in (0,\pi/2)$, for $k = k_1, k_c, k_2$. For $\mu > \mu_c$ sufficiently close to μ_c, rotated rolls exist in the shaded region. In the bifurcation analysis we fix $k_y = k_c\cos\alpha$, for some $\alpha \in (0,\pi/3)$.

The invariance of $\mathbf{U}^*_{k,\mu}$ under the action of the symmetry $\mathbf{S}_2$ implies that rolls rotated by angles α and $\pi+\alpha$ coincide,

$$\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu} = \boldsymbol{\mathcal{R}}_{\pi+\alpha}\mathbf{U}^*_{k,\mu}.$$

Upon rotation, rolls loose their invariance under the horizontal reflections $x \to -x$ and $y \to -y$, the actions of $\mathbf{S}_1$ and $\mathbf{S}_2$ on a roll rotated by an angle $\alpha \notin \{0, \pi\}$ gives the same roll but rotated by the opposite angle,

$$\mathbf{S}_1(\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}(x)) = \boldsymbol{\mathcal{R}}_{-\alpha} \mathbf{U}^*_{k,\mu}(-x), \quad \mathbf{S}_2 \boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu} = \boldsymbol{\mathcal{R}}_{-\alpha} \mathbf{U}^*_{k,\mu}.$$

These equalities imply that rotated rolls keep a reversibility symmetry,

$$\mathbf{S}_1 \mathbf{S}_2(\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}(x)) = \boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}(-x). \tag{1.21}$$

The last equality in (1.19) remains valid for angles $\alpha \notin \{\pi/2, 3\pi/2\}$, whereas for angles $\alpha = \pi/2$ and $\alpha = 3\pi/2$ the rotated rolls are invariant under the action of the entire family of linear maps $(\boldsymbol{\tau}_a)_{a \in \mathbb{R}/2\pi\mathbb{Z}}$.

In the next Chapter, we construct the domain walls as reversible heteroclinic solutions of the dynamical system (1.9) connecting two rotated rolls, $\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}$ at $x = -\infty$ and $\boldsymbol{\mathcal{R}}_{-\alpha} \mathbf{U}^*_{k,\mu}$ at $x = \infty$. In the bifurcation problem, we will suitably fix $k_y \in (0, k_c)$ and take μ, close to μ_c, as bifurcation parameter. The next step of our analysis is to determine the purely imaginary eigenvalues of the linear operator $\mathcal{L}_{\mu_c}$.

3.1.3 Free-free boundary conditions

In the case of two free boundaries, the rigid-rigid boundary conditions $V_x|_{z=0,1} = V_y|_{z=0,1} = 0$ are replaced by the "free-free" boundary conditions

$$\partial_z V_x|_{z=0,1} = \partial_z V_y|_{z=0,1} = 0, \tag{1.22}$$

the horizontal components (V_x, V_y) of the velocity field $\mathbf{V}$ satisfying now Neumann boundary conditions along the vertical axis z, instead of Dirichlet boundary conditions. The equations in the system (1.1)-(1.3), (1.6) are the same, and with these boundary conditions the system has exactly the same symmetries as in the case of rigid-rigid boundary conditions.

In the classical two-dimensional convection, the existence of rolls is shown as in Section 3.1.2. The sequence of parameter values $\mu_0(k) < \mu_1(k) < \mu_2(k) < \dots$ has the same properties as in Chapter one Section 1.3.3, the difference being that in the boundary value problem (1.16)-(1.17) the equality $DV = 0$ is replaced by $D^2V = 0$. Notice that in this case $\mu_0(k)$ is now explicit (see [59] and Section 1.3.3 of chapter 1),

$$\mu_0(k) = \frac{1}{|k|}\left(k^2 + \pi^2\right)^{3/2},$$

from which we easily obtain the numerical values

$$k_c = \frac{\pi}{\sqrt{2}}, \quad \mu_c = \frac{3\sqrt{3}}{2}\pi^2.$$

Furthermore, the solution V of the boundary value problem (1.16)-(1.17) is now explicit,

$$V(z) = \sin(\pi z).$$

In our approach, we replace the spaces $\mathcal{X}$ and $\mathcal{Z}$ in the spatial dynamics formulation (1.9) by

$$\mathcal{X} = \big\{\mathbf{U} \in (H^1_{per}(\Omega))^3 \times (L^2_{per}(\Omega))^3 \times H^1_{per}(\Omega) \times L^2_{per}(\Omega)\ ;\\ V_z = \theta = 0 \text{ on } z = 0,1, \text{ and } \int_\Omega V_x\, dy\, dz = 0\big\},$$

and

$$\mathcal{Z} = \big\{\mathbf{U} \in \mathcal{X} \cap (H^2_{per}(\Omega))^3 \times (H^1_{per}(\Omega))^3 \times H^2_{per}(\Omega) \times H^1_{per}(\Omega)\ ;\\ \partial_z V_x = \partial_z V_y = W_z = \phi = 0 \text{ on } z = 0,1\big\}.$$

The equations in (1.9) and the symmetries $\boldsymbol{\tau}_a$, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\mathbf{T}_b$ in Section 3.1 do not change.

3.1.4 Rigid-free boundary conditions

In the case of one rigid and one free boundaries, the boundary conditions

$$V_x|_{z=1} = V_y|_{z=1} = 0$$

are replaced by the boundary conditions

$$\partial_z V_x|_{z=1} = \partial_z V_y|_{z=1} = 0, \tag{1.23}$$

while the boundary conditions in $z = 0$ are (1.5). As in the previous case, the equations (1.1)-(1.3) remain the same. In contrast to the rigid-rigid and free-free boundary conditions, these rigid-free boundary conditions are asymmetric and the system looses its reflection symmetry in the vertical coordinate z. As an immediate consequence, in the spatial dynamics formulation, the system (1.9) is not equivariant under the action of the symmetry $\mathbf{S}_3$ anymore. While the spectral properties of the linear operator $\mathcal{L}_{\mu_c}$ in next sections and the center manifold reduction in Chapters 4 and 5 remain valid.

3.2 Study of the linearized operator $\mathcal{L}_\mu$

3.2.1 Connection with the classical linear problem

Solutions $\mathbf{U} = (V_x, V_\perp, W_x, W_\perp, \theta, \phi) \in \mathcal{Z}$ of the eigenvalue problem

$$\mathcal{L}_\mu \mathbf{U} = i\omega \mathbf{U}, \tag{2.1}$$

are linear combinations of vectors of the form $\mathbf{U}_{\omega,n}(y,z) = e^{ink_y y}\widehat{\mathbf{U}}_{\omega,n}(z)$, with $n \in \mathbb{Z}$, due to periodicity in y. Projecting with Π given by (1.18), we obtain a solution

$$\mathbf{u}_{\omega,n}(x,y,z) = e^{i(\omega x + nk_y y)}\,\Pi\widehat{\mathbf{U}}_{\omega,n}(z),$$

of the three-dimensional classical problem (1.14) (see $\mathbf{L}_\mu \mathbf{u} = 0$ in Chapter 1, Section 1.3.3), and rotating by a suitable angle α we find a solution $e^{iky}\widehat{\mathbf{u}}_k(z)$ of the linear equation (1.14) with

$$k^2 = \omega^2 + n^2 k_y^2. \tag{2.2}$$

The angle α is determined by the equalities

$$\omega = k\sin\alpha, \quad nk_y = k\cos\alpha, \tag{2.3}$$

and we have the relationship

$$\Pi\widehat{\mathbf{U}}_{\omega,n}(z) = \boldsymbol{\mathcal{R}}_{-\alpha}\widehat{\mathbf{u}}_k(z).$$

Consequently, for a given $k_y > 0$, the eigenvectors $\mathbf{U}_{\omega,n}$ associated with purely imaginary eigenvalues $\nu = i\omega$ of $\mathcal{L}_\mu$ are obtained by rotating with $\boldsymbol{\mathcal{R}}_{-\alpha}$ the elements in the kernel of $\mathbf{L}_\mu$ given by (1.15), through the relationship (2.3) and

$$\Pi\mathbf{U}_{\omega,n}(y,z) = e^{ink_y y}\Pi\widehat{\mathbf{U}}_{\omega,n}(z) = e^{ink_y y}\boldsymbol{\mathcal{R}}_{-\alpha}\widehat{\mathbf{u}}_k(z). \tag{2.4}$$

This holds for all eigenvectors $\mathbf{U}_{\omega,n}$ such that $\Pi\mathbf{U}_{\omega,n} \neq 0$. We obtain in this way all purely imaginary eigenvalues of $\mathcal{L}_\mu$ with associated eigenvectors $\mathbf{U}$ such that $\Pi\mathbf{U} \neq 0$. Using the properties of the kernel of $\mathbf{L}_\mu$ in Section 3.1.2, we obtain the following result, for $\mu = \mu_0(k)$.

Lemma 2.1 *Assume that k_y and k are positive integers. Then the linear operator $\mathcal{L}_{\mu_0(k)}$ has the complex conjugated purely imaginary eigenvalues*

$$\pm\, i\omega_n(k), \quad \omega_n(k) = \sqrt{k^2 - n^2 k_y^2} > 0, \tag{2.5}$$

for any integer $0 \le n < k/k_y$, and the following properties hold.

For $n = 0$, $\omega_0(k) = k$ and the complex conjugated eigenvalues $\pm ik$ are geometrically simple with associated eigenvector of the form

$$\mathbf{U}_{k,0}(y,z) = \widehat{\mathbf{U}}_{k,0}(z),$$

for the eigenvalue ik and the complex conjugated vector for the eigenvalue $-ik$.

For $0 < n < k/k_y$, the complex conjugated eigenvalues $\pm i\omega_n(k)$ are geometrically double with associated eigenvectors of the form

$$\mathbf{U}_{\omega_n(k),\pm n}(y,z) = e^{\pm ink_y y}\widehat{\mathbf{U}}_{\omega_n(k),\pm n}(z),$$

for the eigenvalue $i\omega_n(k)$ and the complex conjugated vectors for the eigenvalue $-i\omega_n(k)$.

The vectors $\widehat{\mathbf{U}}_{k,0}(z)$ and $\widehat{\mathbf{U}}_{\omega_1(k),\pm 1}(z)$ are given by [1]

$$\widehat{\mathbf{U}}_{k,0}(z) = \begin{pmatrix} \frac{i}{k}DV_k \\ 0 \\ V_k \\ -\frac{1}{\mu_0(k)k^2}D^3V_k \\ 0 \\ \frac{ik}{\mu_0(k)}V_k \\ \frac{1}{\mu_0(k)k^2}(D^2-k^2)^2V_k \\ \frac{i}{\mu_0(k)k}(D^2-k^2)^2V_k \end{pmatrix}, \quad \widehat{\mathbf{U}}_{\omega_1(k),\pm 1}(z) = \begin{pmatrix} \frac{i\omega_1(k)}{k^2}DV_k \\ \pm\frac{ik_y}{k^2}DV_k \\ V_k \\ -\frac{1}{\mu_0(k)k^2}(D^2-k_y^2)DV_k \\ \mp\frac{k_y\omega_1(k)}{\mu_0(k)k^2}DV_k \\ \frac{i\omega_1(k)}{\mu_0(k)}V_k \\ \frac{1}{\mu_0(k)k^2}(D^2-k^2)^2V_k \\ \frac{i\omega_1(k)}{\mu_0(k)k^2}(D^2-k^2)^2V_k \end{pmatrix},$$

where the function V_k is a real-valued solution of the boundary value problem

$$(D^2-k^2)^3V_k + \mu_0(k)^2k^2V_k = 0, \quad V_k = DV_k = (D^2-k^2)^2V_k = 0 \text{ in } z = 0, 1. \tag{2.6}$$

Proof. First, notice that for eigenvectors $\mathbf{U}$ with $\Pi\mathbf{U} = 0$, the eigenvalue problem (2.1) is reduced to the system

$$\begin{aligned} \mu W_\perp &= 0 \\ 0 &= i\omega W_x \\ -\nabla_\perp W_x &= 0 \\ \phi &= 0 \end{aligned}$$

for the variables $(W_x, W_\perp, \phi)$. The only nontrivial solution of this system is $(W_x, 0, 0, 0)$, with W_x a constant function, when $\omega = 0$. This implies that 0 is an eigenvalue of $\mathcal{L}_\mu$ with associated eigenvector $\boldsymbol{\varphi}_0$ given by (1.13), and that all other eigenvalues have associated eigenvectors $\mathbf{U}$ with $\Pi\mathbf{U} \neq 0$. In particular, nonzero purely imaginary eigenvalues of $\mathcal{L}_\mu$ and their associated eigenvectors are all determined from the properties of the kernel of the operator $\mathbf{L}_\mu$ in Section 3.1.2 through the equalites (2.2), (2.3), and (2.4).

For $\mu = \mu_0(k)$, we obtain the eigenvalues given by (2.5). The uniqueness, up to a multipicative constant, of the element in the kernel of $\mathbf{L}_{\mu_0(k)}$ given by (1.15), implies that the eigenvalues $\pm ik$, for $n = 0$, are geometrically simple, and since opposite numbers $\pm n$ give the same pair of eigenvalues $\pm i\omega_n(k)$, for $n \neq 0$, these eigenvalues are geometrically double. Finally, the equalities (2.4) and (1.15), allow to compute the projections $\Pi\mathbf{U}_{k,0}$ and $\Pi\mathbf{U}_{\omega_n(k),\pm n}$ of the eigenvectors and the remaining components $(\mathbf{W}, \phi)$ are found from (1.7) and (1.8). We obtain the formulas which complete the proof of the lemma. ■

[1] For our purposes, we do not need the explicit formulas for $n > 1$.

3.2.2 The Center spectrum for $k_c/2 < k_y < k_c$

Lemma 2.1 shows that the linear operator $\mathcal{L}_{\mu_c}$ has the purely imaginary eigenvalues

$$\pm i\sqrt{k_c^2 - n^2 k_y^2},$$

for positive integers n such that $0 \leq n < k_c/k_y$. Upon decreasing k_y, the number of pairs of eigenvalues increases. Counted with geometric multiplicities, for $k_y > k_c$, there is one pair of purely imaginary eigenvalues with $n = 0$, for $k_c \geq k_y > k_c/2$ there are three pairs with $n = 0, \pm 1$, and more generally for $k_c/N \geq k_y > k_c/(N+1)$ there are $2N+1$ pairs with $n = 0, \pm 1, \ldots, \pm N$. For the construction of domain walls we need at least one pair of purely imaginary eigenvalues with opposite Fourier modes $\pm n \neq 0$. We restrict here to the simplest situation when $k_c > k_y > k_c/2$ and $\mathcal{L}_{\mu_c}$ has six purely imaginary eigenvalues with Fourier modes $n = 0, \pm 1$.

For notational convenience, we set

$$k_y = k_c \cos\alpha, \quad k_x = k_c \sin\alpha$$

and take $\alpha \in (0, \pi/3)$. In the following lemma we give a complete description of the purely imaginary spectrum of the linear operator $\mathcal{L}_{\mu_c}$.

Lemma 2.2 *Assume that $k_y = k_c \cos\alpha$ with $\alpha \in (0, \pi/3)$. Then the center spectrum $\sigma_c(\mathcal{L}_{\mu_c})$ of the linear operator $\mathcal{L}_{\mu_c}$ consists of five eigenvalues,*

$$\sigma_c(\mathcal{L}_{\mu_c}) = \{0, \pm ik_c, \pm ik_x\}, \quad k_x = k_c \sin\alpha, \tag{2.7}$$

with the following properties.

(i) The eigenvalue 0 is simple with associated eigenvector $\boldsymbol{\varphi}_0$ given by (1.13), which is invariant under the actions of $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\boldsymbol{\tau}_a$.

(ii) The complex conjugated eigenvalues $\pm ik_c$ are algebraically double and geometrically simple with associated generalized eigenvectors of the form

$$\boldsymbol{\zeta}_0 = \widehat{\mathbf{U}}_0(z), \quad \boldsymbol{\Psi}_0 = \widehat{\boldsymbol{\Psi}}_0(z),$$

for the eigenvalue ik_c and the complex conjugated vectors for the eigenvalue $-ik_c$, such that

$$(\mathcal{L}_{\mu_c} - ik_c)\boldsymbol{\zeta}_0 = \mathbf{0}, \quad (\mathcal{L}_{\mu_c} - ik_c)\boldsymbol{\Psi}_0 = \boldsymbol{\zeta}_0,$$

and

$$\begin{aligned} \mathbf{S}_1\boldsymbol{\zeta}_0 &= \overline{\boldsymbol{\zeta}_0}, \quad \mathbf{S}_2\boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0, \quad \mathbf{S}_3\boldsymbol{\zeta}_0 = -\boldsymbol{\zeta}_0, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0, \\ \mathbf{S}_1\boldsymbol{\Psi}_0 &= -\overline{\boldsymbol{\Psi}_0}, \quad \mathbf{S}_2\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_0, \quad \mathbf{S}_3\boldsymbol{\Psi}_0 = -\boldsymbol{\Psi}_0, \quad \boldsymbol{\tau}_a\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_0. \end{aligned}$$

(iii) *The complex conjugated eigenvalues $\pm ik_x$ are algebraically quadruple and geometrically double with associated generalized eigenvectors of the form*

$$\boldsymbol{\zeta}_\pm = e^{\pm ik_y y}\widehat{\mathbf{U}}_\pm(z), \quad \boldsymbol{\Psi}_\pm = e^{\pm ik_y y}\widehat{\boldsymbol{\Psi}}_\pm(z),$$

for the eigenvalue ik_x and the complex conjugated vectors for the eigenvalue $-ik_x$, such that

$$(\mathcal{L}_{\mu_c} - ik_x)\boldsymbol{\zeta}_\pm = \mathbf{0}, \quad (\mathcal{L}_{\mu_c} - ik_x)\boldsymbol{\Psi}_\pm = \boldsymbol{\zeta}_\pm,$$

and

$$\begin{aligned}
&\mathbf{S}_1\boldsymbol{\zeta}_+ = \overline{\boldsymbol{\zeta}_-}, \quad \mathbf{S}_2\boldsymbol{\zeta}_+ = \boldsymbol{\zeta}_-, \quad \mathbf{S}_3\boldsymbol{\zeta}_+ = -\boldsymbol{\zeta}_+, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_+ = e^{ia}\boldsymbol{\zeta}_+,\\
&\mathbf{S}_1\boldsymbol{\zeta}_- = \overline{\boldsymbol{\zeta}_+}, \quad \mathbf{S}_2\boldsymbol{\zeta}_- = \boldsymbol{\zeta}_+, \quad \mathbf{S}_3\boldsymbol{\zeta}_- = -\boldsymbol{\zeta}_-, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_- = e^{-ia}\boldsymbol{\zeta}_-,\\
&\mathbf{S}_1\boldsymbol{\Psi}_+ = -\overline{\boldsymbol{\Psi}_-}, \quad \mathbf{S}_2\boldsymbol{\Psi}_+ = \boldsymbol{\Psi}_-, \quad \mathbf{S}_3\boldsymbol{\Psi}_+ = -\boldsymbol{\Psi}_+, \quad \boldsymbol{\tau}_a\boldsymbol{\Psi}_+ = e^{ia}\boldsymbol{\Psi}_+,\\
&\mathbf{S}_1\boldsymbol{\Psi}_- = -\overline{\boldsymbol{\Psi}_+}, \quad \mathbf{S}_2\boldsymbol{\Psi}_- = \boldsymbol{\Psi}_+, \quad \mathbf{S}_3\boldsymbol{\Psi}_- = -\boldsymbol{\Psi}_-, \quad \boldsymbol{\tau}_a\boldsymbol{\Psi}_- = e^{-ia}\boldsymbol{\Psi}_-.
\end{aligned}$$

Proof. The result in Lemma 2.1 shows that $\pm ik_c$ and $\pm ik_x$ are purely imaginary eigenvalues of $\mathcal{L}_{\mu_c}$ and the first part of its proof implies that 0 is an eigenvalue of $\mathcal{L}_{\mu_c}$. Since μ_c is the unique global minimum of $\mu_0(k)$, there are no other eigenvalues with zero real part. This proves the property (2.7). Furthermore, the eigenvalue 0 is geometrically simple, with associated eigenvector $\boldsymbol{\varphi}_0$ given by (1.13), and the eigenvalues $\pm ik_c$ and $\pm ik_x$ have geometric multiplicities one and two, respectively. The associated eigenvectors $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_\pm$ are computed from the formulas in Lemma 2.1, by taking $n = 0$ and $n = \pm 1$, respectively, for $k = k_c$ and $k_y = k_c\cos\alpha$. We obtain

$$\boldsymbol{\zeta}_0 = \widehat{\mathbf{U}}_0(z), \quad \boldsymbol{\zeta}_\pm = e^{\pm ik_y y}\widehat{\mathbf{U}}_\pm(z),$$

where

$$\widehat{\mathbf{U}}_0(z) = \begin{pmatrix} \frac{i}{k_c}DV \\ 0 \\ V \\ -\frac{1}{\mu_c k_c^2}D^3V \\ 0 \\ \frac{ik_c}{\mu_c}V \\ \frac{1}{\mu_c k_c^2}(D^2-k_c^2)^2V \\ \frac{i}{\mu_c k_c}(D^2-k_c^2)^2V \end{pmatrix}, \quad \widehat{\mathbf{U}}_\pm(z) = \begin{pmatrix} \frac{i\sin\alpha}{k_c}DV \\ \pm\frac{i\cos\alpha}{k_c}DV \\ V \\ -\frac{1}{k_c^2\mu_c}(D^2-k_c^2\cos^2\alpha)DV \\ \mp\frac{\sin\alpha\cos\alpha}{\mu_c}DV \\ \frac{ik_c\sin\alpha}{\mu_c}V \\ \frac{1}{\mu_c k_c^2}(D^2-k_c^2)^2V \\ \frac{i\sin\alpha}{\mu_c k_c}(D^2-k_c^2)^2V \end{pmatrix},$$

and the function V is a real-valued solution of the boundary value problem

$$(D^2 - k_c^2)^3 V + \mu_c^2 k_c^2 V = 0, \quad V = DV = (D^2 - k_c^2)^2 V = 0 \text{ in } z = 0, 1. \tag{2.8}$$

This boundary value problem being equivalent to (1.16)-(1.17) for $\mu = \mu_c$, the function V is positive and symmetric with respect to $z = 1/2$. The latter property and the explicit formulas above imply the symmetry properties of $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_\pm$.

Next, the algebraic multiplicity of the eigenvalue 0 is directly determined by solving the equation

$$\mathcal{L}_{\mu_c} \boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_0.$$

Up to an element in the kernel of $\mathcal{L}_{\mu_c}$, we find

$$\boldsymbol{\varphi}_1 = \left(\frac{\mu_c}{2} z(1-z), 0, 0, 0, 0, 0, 0, 0 \right)^t.$$

Since $\boldsymbol{\varphi}_1 \notin \mathcal{X}$, this proves that the eigenvalue 0 is algebraically simple. The invariance of $\boldsymbol{\varphi}_0$ under the actions of $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\boldsymbol{\tau}_a$ is easily checked, which completes the proof of part (i) of the Lemma.

For the algebraic multiplicities of the nonzero eigenvalues $\pm ik_c$ and $\pm ik_x$, we use their continuation as eigenvalues of $\mathcal{L}_{\mu_0(k)}$, for k close to k_c. The latter eigenvalues are the geometrically simple eigenvalues $\pm ik$ and the geometrically double eigenvalues $\pm i\omega_1(k)$ in Lemma 2.1. In Appendix 6.3.2 we prove that their algebraic multiplicities are equal to their geometric multiplicities. Then a standard continuation argument implies that the eigenvalues $\pm ik_c$ and $\pm ik_x$ of $\mathcal{L}_{\mu_c}$ are algebraically double and quadruple, respectively.

Finally, we compute the generalized eigenvectors $\boldsymbol{\Psi}_0$ and $\boldsymbol{\Psi}_\pm$ associated with the eigenvalues ik_c and ik_x, respectively, from the eigenvectors associated with the eigenvalues ik and $i\omega_1(k)$ of $\mathcal{L}_{\mu_0(k)}$ given in Lemma 2.1. Differentiating the eigenvalue problems

$$\mathcal{L}_{\mu_0(k)} \mathbf{U}_{k,0} = ik \mathbf{U}_{k,0}, \quad \mathcal{L}_{\mu_0(k)} \mathbf{U}_{\omega_1(k),\pm 1} = i\omega_1(k) \mathbf{U}_{\omega_1(k),\pm 1},$$

with respect to k at $k = k_c$, and using the properties

$$\mu_0'(k_c) = 0, \quad \omega_1'(k_c) = \frac{k_c}{\sqrt{k_c^2 - k_y^2}} = \frac{1}{\sin \alpha},$$

we obtain the equalities

$$(\mathcal{L}_{\mu_c} - ik_c) \left(\frac{d}{dk} \mathbf{U}_{k,0}\big|_{k=k_c} \right) = i\boldsymbol{\zeta}_0,$$

$$(\mathcal{L}_{\mu_c} - ik_x) \left(\frac{d}{dk} \mathbf{U}_{\omega_1(k),\pm 1}\big|_{k=k_c} \right) = \frac{i}{\sin \alpha} \boldsymbol{\zeta}_\pm.$$

Consequently, the generalized eigenvectors are given by

$$\boldsymbol{\Psi}_0 = -i\left(\frac{d}{dk}\mathbf{U}_{k,0}\big|_{k=k_c}\right), \quad \boldsymbol{\Psi}_\pm = -i\sin\alpha\left(\frac{d}{dk}\mathbf{U}_{\omega_1(k),\pm1}\right)\big|_{k=k_c}. \tag{2.9}$$

In particular, they have the same form

$$\boldsymbol{\Psi}_0 = \widehat{\boldsymbol{\Psi}}_0(z), \quad \boldsymbol{\Psi}_\pm = e^{\pm ik_y y}\widehat{\boldsymbol{\Psi}}_\pm(z),$$

as the eigenvectors $\mathbf{U}_{k,0}$ and $\mathbf{U}_{\omega_1(k),\pm1}$ given in Lemma 2.1. Furthermore, since the function V_k in the expressions of $\widehat{\mathbf{U}}_{k,0}(z)$ and $\widehat{\mathbf{U}}_{\omega_1(k),\pm1}(z)$ is symmetric with respect to $z = 1/2$, just as the function V in (2.8), the eigenvectors $\mathbf{U}_{k,0}$ and $\mathbf{U}_{\omega_1(k),\pm1}$ have the same symmetry properties as the eigenvectors $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_\pm$, respectively. Together with the formulas (2.9), this implies that $\boldsymbol{\Psi}_0$ and $\boldsymbol{\Psi}_\pm$ have the symmetry properties given in (*ii*) and (*iii*), and completes the proof of the lemma. ■

3.2.3 The Center spectrum for $k_y = k_c$

We consider the parameter regime with (k,μ) close to (k_c,μ_c), where $\mu_c = \mu_0(k_c)$. We set

$$\mu = \mu_c + \tilde{\mu}, \quad k = k_c(1+\tilde{k}),$$

in which $\tilde{\mu}$ and $\tilde{k}$ are small parameters. We also eliminate the dependence on k of the phase space $\mathcal{X}$ of the dynamical system (1.9) by normalizing to $2\pi/k_c$ the period in y of the solutions. The resulting system is of the form (1.9) in which now $\Delta_\perp = (1+\tilde{k})^2\partial_{yy} + \partial_{zz}$, $\nabla_\perp = ((1+\tilde{k})\partial_y, \partial_z)$, and its phase space is $\mathcal{X}$ with $k = k_c$. We write this system in the form

$$\partial_x\mathbf{U} = \mathcal{L}_c\mathbf{U} + \mathcal{R}(\mathbf{U},\tilde{\mu},\tilde{k}), \tag{2.10}$$

where

$$\mathcal{L}_c = \mathcal{L}_{\mu_c}\big|_{\tilde{k}=0}, \quad \mathcal{R}(\mathbf{U},\tilde{\mu},\tilde{k}) = (\mathcal{L}_\mu - \mathcal{L}_{\mu_c}\big|_{\tilde{k}=0})\mathbf{U} + \mathcal{B}_\mu(\mathbf{U},\mathbf{U}), \tag{2.11}$$

and $\mathcal{R}$ is a smooth map from $\mathcal{Z}\times(-\mu_c,\infty)\times\mathbb{R}$ into $\mathcal{X}$ satisfying

$$\mathcal{R}(0,\tilde{\mu},\tilde{k}) = 0, \quad D_{\mathbf{U}}\mathcal{R}(0,0,0) = 0. \tag{2.12}$$

For the spectrum of $\mathcal{L}_c$ the arguments are the same as in previous section, except for the purely imaginary eigenvalues which are different. The following result is obtained by taking the limit $\alpha = 0$ in Lemma 2.1.

Lemma 2.3 *The center spectrum of the linear operator $\mathcal{L}_c$ consists of the three eigenvalues $0, \pm ik_c$ with the following properties.*

(i) *The eigenvalue* 0 *has algebraic multiplicity* 9 *and geometric multiplicity* 3*, and the complex conjugated eigenvalues* $\pm ik_c$ *are algebraically double and geometrically simple.*

(ii) *For the eigenvalue* 0*, there are three linearly independent eigenvectors:* $\boldsymbol{\varphi}_0$ *given by* (1.13)*,* $\boldsymbol{\zeta}_0$ *of the form* $\boldsymbol{\zeta}_0(y,z) = \widehat{U}_{k_c}(z)e^{ik_cy}$*,*[2] *and the complex conjugated vector* $\overline{\boldsymbol{\zeta}}_0$*, and two chains of generalized eigenvectors:* $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3$ *associated to* $\boldsymbol{\zeta}_0$*,*

$$\mathcal{L}_c\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_0, \quad \mathcal{L}_c\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_1, \quad \mathcal{L}_c\boldsymbol{\zeta}_3 = \boldsymbol{\zeta}_2,$$

and the conjugated vectors $\overline{\boldsymbol{\zeta}_1}, \overline{\boldsymbol{\zeta}_2}, \overline{\boldsymbol{\zeta}_3}$ *associated to* $\overline{\boldsymbol{\zeta}_0}$*. The eigenvector* $\boldsymbol{\varphi}_0$ *is invariant under the actions of* $\boldsymbol{S}_1$*,* $\boldsymbol{S}_2$*, and* $\boldsymbol{\tau}_a$*, and the other generalized eigenvectors satisfy:*

$$\begin{aligned}
&\boldsymbol{S}_1\boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_0 = \overline{\boldsymbol{\zeta}_0}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_0 = e^{ia}\boldsymbol{\zeta}_0,\\
&\boldsymbol{S}_1\boldsymbol{\zeta}_1 = -\boldsymbol{\zeta}_1, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_1 = \overline{\boldsymbol{\zeta}_1}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_1 = e^{ia}\boldsymbol{\zeta}_1,\\
&\boldsymbol{S}_1\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_2, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_2 = \overline{\boldsymbol{\zeta}_2}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_2 = e^{ia}\boldsymbol{\zeta}_2,\\
&\boldsymbol{S}_1\boldsymbol{\zeta}_3 = -\boldsymbol{\zeta}_3, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_3 = \overline{\boldsymbol{\zeta}_3}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_3 = e^{ia}\boldsymbol{\zeta}_3.
\end{aligned}$$

(iii) *For the eigenvalue* ik_c*, there is one eigenvector* $\boldsymbol{\xi}_0$ *of the form* $\boldsymbol{\xi}_0(y,z) = \widehat{\mathbf{U}}_0(z)$*, and an associated generalized eigenvector* $\boldsymbol{\xi}_1$ *with the properties*

$$(\mathcal{L}_c - ik_c)\boldsymbol{\xi}_1 = \boldsymbol{\xi}_0,$$

and

$$\begin{aligned}
&\boldsymbol{S}_1\boldsymbol{\xi}_0 = \overline{\boldsymbol{\xi}_0}, \quad \boldsymbol{S}_2\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0, \quad \boldsymbol{\tau}_a\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0,\\
&\boldsymbol{S}_1\boldsymbol{\xi}_1 = -\overline{\boldsymbol{\xi}_1}, \quad \boldsymbol{S}_2\boldsymbol{\xi}_1 = \boldsymbol{\xi}_1, \quad \boldsymbol{\tau}_a\boldsymbol{\xi}_1 = \boldsymbol{\xi}_1.
\end{aligned}$$

The complex conjugated vectors $\overline{\boldsymbol{\xi}_0}$ *and* $\overline{\boldsymbol{\xi}_1}$ *are eigenvector and generalized eigenvector, respectively, for the eigenvalue* $-ik_c$*.*

[2] For our purposes, we do not need the explicit formulas for eigenvectors and generalized eigenvectors.

Chapter 4

Symmetric domain walls

In this Chapter we study the bifurcation of symmetric domain walls. This is based on the papers [29] and [30], and uses the results on the linearized operator obtained in Lemma 2.2 of Chapter three.

4.1 Introduction

At least locally, the most frequently observed patterns are convective rolls aligned along a certain direction (see Figure 1.1 (a) and (c)). However, such a pattern is only observed in a part of the apparatus, while the rolls take another direction in another part of the apparatus. The connection between the two regimes is quite sharp, occurring along a plane, and the two regimes of rolls make a definite angle between them (see Figure 1.1(b) and [36, 55, 22, 2] for experimental evidences not all on pure Bénard-Rayleigh convection). These line defects are referred to as domain walls or grain boundaries. In this Chapter, we consider the case where two systems of rolls connect symmetrically with respect to a plane, even though such a perfectly symmetric pattern is not yet observed experimentally.

The aim of this Chapter is to prove mathematically that such domain walls are indeed solutions of the steady Navier-Stokes-Boussinesq equations. Many works gave tentative justifications of the existence of such patterns using formally derived amplitude equations (see [58, 57, 24] and the references therein). Beyond amplitude equations, the only mathematical results existing before [29] have been obtained for the Swift-Hohenberg equation, a toy model which exhibits many of the properties of the Bénard-Rayleigh convection problem [32, 65] (see also [56]). The domain walls constructed in [32] are symmetric, connecting rolls rotated by opposite angles $\pm\alpha$, for $\alpha \in (0, \pi/3)$. This result has been extended to arbitrary angles $\alpha \in (0, \pi/2)$ in [65]. We point out that next Chapter gives results for orthogonal domain walls (not symmetric in such a case).

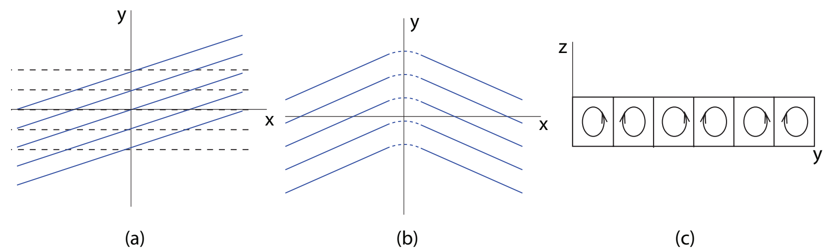

Figure 1.1: **(a)** Schematic plots of two-dimensional rotated rolls, rotated by an angle α (solid lines). **(b)** symmetric domain walls constructed as heteroclinic connections between rolls rotated by opposite angles $\pm\alpha$. **(c)** In the vertical (y, z)-plane, streamlines of two-dimensional rolls (cross-section through the dashed lines in **(a)**)

The main results of this Chapter are Theorems 3.1 and 4.1.

4.2 Reduction of the nonlinear problem

We start with the spatial dynamics system (1.9) in Chapter 3

$$\partial_x \mathbf{U} = \mathcal{L}_\mu \mathbf{U} + \mathcal{B}_\mu(\mathbf{U}, \mathbf{U}), \tag{2.1}$$

for which we gave the structure of the center spectrum of the linear operator $\mathcal{L}_\mu$ at Lemma 2.2 of Chapter 3.

The next step is the center manifold reduction. Using the symmetries of the system (2.1), we identify a twelve-dimensional invariant submanifold of the center manifold, which contains the heteroclinic orbits of (2.1) corresponding to domain walls. In the case when symmetry $\mathbf{S}_3$ applies (in rigid-rigid or free-free cases) we reduce the search to a 8-dimensional invariant submanifold, while when symmetry $\mathbf{S}_3$ does not apply (rigid-free case) we need to study a 12-dimensional system.

4.2.1 Center manifold reduction

We set $\varepsilon = \mu - \mu_c$ [1] and write the dynamical system (2.1) in the form

$$\partial_x \mathbf{U} = \mathcal{L}_{\mu_c} \mathbf{U} + \mathcal{R}(\mathbf{U}, \varepsilon), \tag{2.2}$$

where

$$\mathcal{R}(\mathbf{U}, \varepsilon) = (\mathcal{L}_\mu - \mathcal{L}_{\mu_c})\mathbf{U} + \mathcal{B}_\mu(\mathbf{U}, \mathbf{U}),$$

[1] For simplification, in this Chapter ε is identical to $\tilde{\mu}$ defined in Chapter 1

is a smooth map from $\mathcal{Z} \times (-\mu_c, \infty)$ into $\mathcal{X}$, and

$$\mathcal{R}(0, \varepsilon) = 0, \quad D_{\mathbf{U}}\mathcal{R}(0, 0) = 0.$$

In particular, $\mathcal{R}$ satisfies the hypotheses of the center manifold theorem (see [28] [Section 2.3.1]). We also have to check two hypotheses on the linear operator $\mathcal{L}_{\mu_c}$. The first one requires that the center spectrum of $\mathcal{L}_{\mu_c}$ consists of finitely many purely imaginary eigenvalues with finite algebraic multiplicity and the result in Lemma 2.2 of Chapter 3 shows that this hypothesis holds. The second one is the estimate on the norm of the resolvent of $\mathcal{L}_{\mu_c}$ obtained by taking $\mu = \mu_c$ in the lemma below.

Lemma 2.1 *For any $\mu > 0$, there exist positive constants C_μ and ω_μ such that*

$$\|(\mathcal{L}_\mu - i\omega)^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq \frac{C_\mu}{|\omega|}, \tag{2.3}$$

for any real number ω, with $|\omega| > \omega_\mu$.

Proof. We write $\mathcal{L}_\mu = \mathcal{A}_\mu + \mathcal{B}_\mu$, where

$$\mathcal{A}_\mu\mathbf{U} = \begin{pmatrix} -\nabla_\perp \cdot V_\perp \\ \mu W_\perp \\ -\mu^{-1}\Delta_\perp V_x \\ -\mu^{-1}\Delta_\perp V_\perp - \mu^{-1}\nabla_\perp(\nabla_\perp \cdot V_\perp) - \nabla_\perp W_x \\ \phi \\ -\Delta_\perp \theta \end{pmatrix}, \quad \mathcal{B}_\mu\mathbf{U} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\theta\mathbf{e}_z \\ 0 \\ -\mu V_z \end{pmatrix}.$$

Since the operator $\mathcal{B}_\mu$ is bounded in $\mathcal{X}$, the resolvent equality

$$(\mathcal{L}_\mu - i\omega)^{-1} = (\mathbb{I} + (\mathcal{A}_\mu - i\omega)^{-1}\mathcal{B}_\mu)(\mathcal{A}_\mu - i\omega)^{-1},$$

implies that it is enough to prove the result for $\mathcal{A}_\mu$. The action of $\mathcal{A}_\mu$ on the components $(\mathbf{V}, \mathbf{W})$ and (θ, ϕ) of $\mathbf{U}$ being decoupled, the operator is diagonal, $\mathcal{A}_\mu = \mathrm{diag}(\mathcal{A}_\mu^{\mathrm{St}}, \mathcal{A}_\mu^{\mathrm{so}})$, where $\mathcal{A}_\mu^{\mathrm{St}}$ acting on $(\mathbf{V}, \mathbf{W})$ is a Stokes operator and $\mathcal{A}_\mu^{\mathrm{so}}$ acting on (θ, ϕ) is a Laplace operator. The estimate (2.3) has been proved for the Stokes operator $\mathcal{A}_\mu^{\mathrm{St}}$ in [41, Appendix 2], and it is easily obtained for the Laplace operator $\mathcal{A}_\mu^{\mathrm{so}}$. This implies the result for $\mathcal{A}_\mu$ and completes the proof of the lemma. ∎

Denote by $\mathcal{X}_c$ the spectral subspace associated with the center spectrum of $\mathcal{L}_{\mu_c}$, by $\mathcal{P}_c$ the corresponding spectral projection, and set $\mathcal{Z}_h = (\mathbb{I} - \mathcal{P}_c)\mathcal{Z}$. Applying the center manifold theorem [28], for any arbitrary, but fixed, $k \geq 3$, there exists a map $\mathbf{\Phi} \in \mathcal{C}^k(\mathcal{X}_c \times \mathbb{R}, \mathcal{Z}_h)$, with

$$\mathbf{\Phi}(0, \varepsilon) = 0, \quad D_{\mathbf{U}}\mathbf{\Phi}(0, 0) = 0, \tag{2.4}$$

and a neighborhood $\mathcal{U}_1 \times \mathcal{U}_2$ of $(0,0)$ in $\mathcal{Z} \times \mathbb{R}$ such that for any $\varepsilon \in \mathcal{U}_2$, the manifold

$$\mathcal{M}_c(\varepsilon) = \{\mathbf{U}_c + \mathbf{\Phi}(\mathbf{U}_c, \varepsilon)\ ;\ \mathbf{U}_c \in \mathcal{X}_c\}, \tag{2.5}$$

has the following properties:

(i) $\mathcal{M}_c(\varepsilon)$ is locally invariant, i.e., if $\mathbf{U}$ is a solution of (2.2) satisfying $\mathbf{U}(0) \in \mathcal{M}_c(\varepsilon) \cap \mathcal{U}_1$ and $\mathbf{U}(x) \in \mathcal{U}_1$ for all $x \in [0, L]$, then $\mathbf{U}(x) \in \mathcal{M}_c(\varepsilon)$ for all $x \in [0, L]$;

(ii) $\mathcal{M}_c(\varepsilon)$ contains the set of bounded solutions of (2.2) staying in $\mathcal{U}_1$ for all $x \in \mathbb{R}$, i.e., if $\mathbf{U}$ is a solution of (2.2) satisfying $\mathbf{U}(x) \in \mathcal{U}_1$ for all $x \in \mathbb{R}$, then $\mathbf{U}(0) \in \mathcal{M}_c(\varepsilon)$;

(iii) the invariant dynamics on the center manifold is determined by the reduced system

$$\frac{d\mathbf{U}_c}{dx} = \mathcal{L}_{\mu_c}\big|_{\mathcal{X}_c}\mathbf{U}_c + \mathcal{P}_c\mathcal{R}(\mathbf{U}_c + \mathbf{\Phi}(\mathbf{U}_c, \varepsilon), \varepsilon) \stackrel{def}{=} f(\mathbf{U}_c, \varepsilon), \tag{2.6}$$

where

$$f(0, \varepsilon) = 0, \quad D_{\mathbf{U}_c} f(0,0) = \mathcal{L}_{\mu_c}\big|_{\mathcal{X}_c};$$

(iv) the reduced system (2.6) inherits the symmetries of (2.2), i.e., the reduced vector field $f(\cdot, \varepsilon)$ anti-commutes with $\mathbf{S}_1$, commutes with $\mathbf{S}_2$, $\mathbf{S}_3$ (when this symmetry applies), and $\boldsymbol{\tau}_a$, and is invariant under the action of $\boldsymbol{T}_b$.

An immediate consequence of these properties is that the heteroclinic solutions of (2.2) representing domain walls belong to the center manifold $\mathcal{M}_c(\varepsilon)$, for sufficiently small ε, and can be constructed as solutions of the reduced system (2.6).

4.2.2 Reduced system

According to Lemma 2.2 of Chapter 3, the center space $\mathcal{X}_c$ has dimension 13 and we can write

$$\begin{aligned} \mathbf{U}_c \;=\; & w\boldsymbol{\varphi}_0 + A_0\boldsymbol{\zeta}_0 + B_0\mathbf{\Psi}_0 + A_+\boldsymbol{\zeta}_+ + B_+\mathbf{\Psi}_+ + A_-\boldsymbol{\zeta}_- + B_-\mathbf{\Psi}_- \\ & + \overline{A_0\boldsymbol{\zeta}_0} + \overline{B_0\mathbf{\Psi}_0} + \overline{A_+\boldsymbol{\zeta}_+} + \overline{B_+\mathbf{\Psi}_+} + \overline{A_-\boldsymbol{\zeta}_-} + \overline{B_-\mathbf{\Psi}_-}, \end{aligned} \tag{2.7}$$

where $w \in \mathbb{R}$ and $X = (A_0, B_0, A_+, B_+, A_-, B_-) \in \mathbb{C}^6$. Then the reduced system (2.6) takes the form

$$\frac{dw}{dx} = h(w, X, \overline{X}, \varepsilon), \tag{2.8}$$

$$\frac{dX}{dx} = F(w, X, \overline{X}, \varepsilon), \tag{2.9}$$

in which h is real-valued and $F = (f_0, g_0, f_+, g_+, f_-, g_-)$ has six complex-valued components. This system is completed by the complex conjugated equation of (2.9) for $\overline{X}$. Notice that the symmetries of the reduced system act on these variables through

$$\begin{aligned}
\mathbf{S}_1(w, A_0, B_0, A_+, B_+, A_-, B_-) &= (w, \overline{A_0}, -\overline{B_0}, \overline{A_-}, -\overline{B_-}, \overline{A_+}, -\overline{B_+}),\\
\mathbf{S}_2(w, A_0, B_0, A_+, B_+, A_-, B_-) &= (w, A_0, B_0, A_-, B_-, A_+, B_+),\\
\boldsymbol{\tau}_a(w, A_0, B_0, A_+, B_+, A_-, B_-) &= (w, A_0, B_0, e^{ia}A_+, e^{ia}B_+, e^{-ia}A_-, e^{-ia}B_-),\\
\boldsymbol{T}_b(w, A_0, B_0, A_+, B_+, A_-, B_-) &= (w + b, A_0, B_0, A_+, B_+, A_-, B_-),
\end{aligned}$$

and, when the symmetry $\mathbf{S}_3$ applies,

$$\mathbf{S}_3(w, A_0, B_0, A_+, B_+, A_-, B_-) = (w, -A_0, -B_0, -A_+, -B_+, -A_-, -B_-).$$

Using the symmetries above, we obtain the following result.

Lemma 2.2 *For any ε sufficiently small, the reduced system (2.8)-(2.9) has the following properties:*

(i) the reduced vector field (h, F) does not depend on w;

(ii) the components (f_0, g_0) of F are even functions while the components (f_+, g_+, f_-, g_-) of F are odd functions in the variables $(A_+, B_+, \overline{A_+}, \overline{B_+}, A_-, B_-, \overline{A_-}, \overline{B_-})$.

(iii) In addition, when the symmetry $\mathbf{S}_3$ applies, components (f_0, g_0) are odd functions in the variables $(A_0, B_0, \overline{A_0}, \overline{B_0})$, while (f_+, g_+, f_-, g_-) are even functions in the variables $(A_0, B_0, \overline{A_0}, \overline{B_0})$.

Proof. Due to the invariance of the reduced system (2.8)- (2.9) under the action of $\boldsymbol{T}_b$, the vector field (h, F) satisfies

$$(h, F)(w + b, X, \overline{X}, \varepsilon) = (h, F)(w, X, \overline{X}, \varepsilon),$$

for any real number b. This implies that (h, F) does not depend on w and proves (i).

Next, the vector field F, which only depends on X and $\overline{X}$, commutes with the symmetries $\boldsymbol{\tau}_\pi$ acting on these components through

$$\boldsymbol{\tau}_\pi(A_0, B_0, A_+, B_+, A_-, B_-) = (A_0, B_0, -A_+, -B_+, -A_-, -B_-),$$

The first equality implies the parity properties (ii) of the components $(f_0, g_0, f_+, g_+, f_-, g_-)$ of F in the variables $(A_+, B_+, \overline{A_+}, \overline{B_+}, A_-, B_-, \overline{A_-}, \overline{B_-})$.

When $\mathbf{S}_3$ applies the action of $\mathbf{S}_3\boldsymbol{\tau}_\pi$ gives the commutation with

$$\mathbf{S}_3\boldsymbol{\tau}_\pi(A_0, B_0, A_+, B_+, A_-, B_-) = (-A_0, -B_0, A_+, B_+, A_-, B_-),$$

which implies the parity properties (iii) in the variables $(A_0, B_0, \overline{A_0}, \overline{B_0})$. ■

An immediate consequence of the first property in the lemma above being that the two equations (2.8) and (2.9) are decoupled, we can first solve (2.9) for X, and then integrate (2.8) to determine w. We therefore restrict our existence analysis to the equation

$$\frac{dX}{dx} = F(X, \overline{X}, \varepsilon), \tag{2.10}$$

which together with the complex conjugate equation for $\overline{X}$ form a 12-dimensional system. For this system, the parity properties (ii) of the vector field F in Lemma 2.2, imply that there exist a first invariant subspaces:

$$E_0 = \left\{(X, \overline{X}),\ X \in \mathbb{C}^6\ ;\ (A_+, B_+, A_-, B_-) = 0\right\},$$

which is 4-dimensional, and in the case when $\mathbf{S}_3$ applies a second invariant subspace is

$$E_\pm = \left\{(X, \overline{X}),\ X \in \mathbb{C}^6\ ;\ (A_0, B_0) = 0\right\},$$

which is 8-dimensional. Each of these subspaces give an invariant submanifold of the center manifold. Solutions in the submanifold associated with E_0 are invariant under the action of $(\boldsymbol{\tau}_a)_{a\in\mathbb{R}/2\pi\mathbb{Z}}$ and therefore correspond to solutions of the full dynamical system (2.1) which do not depend on y. This submanifold contains the rolls rotated by an angle $\pi/2$.

Solutions in the submanifold associated with $E_\pm$ (when $\mathbf{S}_3$ applies) are invariant under the action of $\mathbf{S}_3\boldsymbol{\tau}_\pi$ and correspond to truly three-dimensional solutions of the full dynamical system (2.1). This submanifold contains the rotated rolls $\boldsymbol{\mathcal{R}}_\beta\mathbf{U}^*_{k,\mu}$ for k close to k_c and rotation angle β chosen such that they are $2\pi/k_y-$ periodic in y, i.e. such that $k_y = k\cos\beta$. For the construction of domain walls in the case when the symmetry $\mathbf{S}_3$ applies, we restrict our study to the subspace $E_\pm$.

4.3 Bifurcation of symmetric domain walls when symmetry $\mathbf{S}_3$ applies

We summarize the main result of this Section in the next theorem.

Theorem 3.1 *Consider the steady Navier-Stokes-Boussinesq system (2.1) with either "'rigid-rigid" boundary conditions:*

$$\mathbf{V}|_{z=0,1} = 0, \quad \theta|_{z=0,1} = 0, \tag{3.1}$$

or "free-free" boundary conditions:

$$V_z|_{z=0,1} = \partial_z V_x|_{z=0,1} = \partial_z V_y|_{z=0,1} = 0, \quad \theta|_{z=0,1} = 0. \tag{3.2}$$

Denote by $\mathcal{R}_c$ the critical Rayleigh number at which convective rolls with wavenumbers k_c bifurcate from the conduction state. Then for any Prandtl number $\mathcal{P}$, there exists a positive number $\alpha_*(\mathcal{P}) \leq \pi/3$ such that for angles $\alpha \in (0, \alpha_*(\mathcal{P}))$, a symmetric domain wall bifurcates for Rayleigh numbers $\mathcal{R} = \mathcal{R}_c + \epsilon$, with $\epsilon > 0$ sufficiently small. The domain wall connects two rotated rolls which are the rotations by opposite angles $\pm(\alpha + O(\epsilon))$ of a roll with wavenumber $k_c + O(\epsilon)$, continuously linked to the amplitude which is of order $O(\epsilon^{1/2})$.

We determine the leading order dynamics of the restriction to $E_\pm$ of the reduced system (2.10) with the help of a normal form transformation to cubic order, followed by suitable scalings of variables. For the resulting systems, we identify particular solutions which correspond to rotated rolls.

4.3.1 Cubic normal form of the reduced system

We write the reduced system (2.10) restricted to the invariant 8-dimensional subspace $E_\pm$ in the from

$$\frac{dY}{dx} = G(Y, \overline{Y}, \varepsilon), \tag{3.3}$$

in which $Y = (A_+, B_+, A_-, B_-) \in \mathbb{C}^4$. Taking into account the properties of the reduced system (2.6), the formula (2.4), and the choice for the generalized eigenvectors in Lemma 2.2 of Chapter 3, we find

$$G(0,0,\varepsilon) = 0, \quad D_Y G(0,0,0) = L_0, \quad D_{\overline{Y}} G(0,0,0) = 0,$$

where L_0 is a Jordan matrix acting on Y through

$$L_0 = \begin{pmatrix} ik_x & 1 & 0 & 0 \\ 0 & ik_x & 0 & 0 \\ 0 & 0 & ik_x & 1 \\ 0 & 0 & 0 & ik_x \end{pmatrix}. \tag{3.4}$$

Using a general normal forms theorem for parameter-dependent vector fields in the presence of symmetries (e.g., see [28, Chapter 3]), we determine a normal form of the system (3.3) up to cubic order.

Lemma 3.2 *For any $k \geq 3$, there exist neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 in $\mathbb{C}^4$ and $\mathbb{R}$, respectively, such that for any $\varepsilon \in \mathcal{V}_2$, there is a polynomial $\boldsymbol{P}_\varepsilon : \mathbb{C}^4 \times \overline{\mathbb{C}^4} \to \mathbb{C}^4$ of degree 3 in the variables $(Z, \overline{Z})$, such that for $Z \in \mathcal{V}_1$, the polynomial change of variable*

$$Y = Z + \boldsymbol{P}_\varepsilon(Z, \overline{Z}), \tag{3.5}$$

transforms the equation (3.3) into the normal form

$$\frac{dZ}{dx} = L_0 Z + N(Z, \overline{Z}, \varepsilon) + \rho(Z, \overline{Z}, \varepsilon), \tag{3.6}$$

with the following properties:

(i) the map ρ belongs to $\mathcal{C}^k(\mathcal{V}_1 \times \overline{\mathcal{V}_1} \times \mathcal{V}_2, \mathbb{C}^4)$, and

$$\rho(Z, \overline{Z}, \varepsilon) = O(|\varepsilon|^2 \|Z\| + \varepsilon \|Z\|^3 + \|Z\|^5);$$

(ii) both $N(\cdot, \cdot, \varepsilon)$ and $\rho(\cdot, \cdot, \varepsilon)$ anti-commute with $\mathbf{S}_1$ and commute with $\mathbf{S}_2$, $\mathbf{S}_3$, and $\boldsymbol{\tau}_a$, for any $\varepsilon \in \mathcal{V}_2$;

(iii) the four components (N_+, M_+, N_-, M_-) of N are of the form

$$\begin{aligned}
N_+ &= iA_+P_+ + A_-R_+ \\
M_+ &= iB_+P_+ + B_-R_+ + A_+Q_+ + iA_-S_+ \\
N_- &= iA_-P_- - A_+\overline{R_+} \\
M_- &= iB_-P_- - B_+\overline{R_+} + A_-Q_- - iA_+\overline{S_+}
\end{aligned}$$

in which

$$\begin{aligned}
P_+ &= \beta_0\varepsilon + \beta_1 A_+\overline{A_+} + i\beta_2(A_+\overline{B_+} - \overline{A_+}B_+) + \beta_3 A_-\overline{A_-} + i\beta_4(A_-\overline{B_-} - \overline{A_-}B_-) \\
P_- &= \beta_0\varepsilon + \beta_3 A_+\overline{A_+} + i\beta_4(A_+\overline{B_+} - \overline{A_+}B_+) + \beta_1 A_-\overline{A_-} + i\beta_2(A_-\overline{B_-} - \overline{A_-}B_-) \\
Q_+ &= b_0\varepsilon + b_1 A_+\overline{A_+} + ib_2(A_+\overline{B_+} - \overline{A_+}B_+) + b_3 A_-\overline{A_-} + ib_4(A_-\overline{B_-} - \overline{A_-}B_-) \\
Q_- &= b_0\varepsilon + b_3 A_+\overline{A_+} + ib_4(A_+\overline{B_+} - \overline{A_+}B_+) + b_1 A_-\overline{A_-} + ib_2(A_-\overline{B_-} - \overline{A_-}B_-) \\
R_+ &= \gamma_5(A_+\overline{B_-} - \overline{A_-}B_+), \quad S_+ = c_5(A_+\overline{B_-} - \overline{A_-}B_+),
\end{aligned}$$

where (A_+, B_+, A_-, B_-) are the four components of Z and the coefficients β_j, b_j, γ_5 and c_5 are all real.

The proof of this lemma can be found in Appendix 6.4.1. We point out that the result is valid for any system of the form (3.3) which has a linear part as in (2.6) and the symmetries $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\boldsymbol{\tau}_a$ given in Section 4.2.2.

4.3.2 Rotated rolls as periodic solutions

The normal form (3.6) truncated at cubic order has the property to leave invariant the two 4-dimensional subspaces

$$E_+ = \left\{(Z,\overline{Z}),\ Z\in\mathbb{C}^4\ ;\ (A_-,B_-)=0\right\},\quad E_- = \left\{(Z,\overline{Z}),\ Z\in\mathbb{C}^4\ ;\ (A_+,B_+)=0\right\},$$

which is not the case for the full system (3.6). The systems obtained by restricting the normal form truncated at cubic order to E_+ and E_- being similar, we consider the one restricted to E_+,

$$\frac{dA_+}{dx} = ik_xA_+ + B_+ + iA_+P_+ \tag{3.7}$$

$$\frac{dB_+}{dx} = ik_xB_+ + iB_+P_+ + A_+Q_+ \tag{3.8}$$

with

$$\begin{aligned}P_+ &= \beta_0\varepsilon + \beta_1A_+\overline{A_+} + i\beta_2(A_+\overline{B_+} - \overline{A_+}B_+),\\ Q_+ &= b_0\varepsilon + b_1A_+\overline{A_+} + ib_2(A_+\overline{B_+} - \overline{A_+}B_+).\end{aligned}$$

Notice that (3.7)-(3.8) is the system found at cubic order in the case of the classical reversible 1 : 1 resonance bifurcation, or reversible Hopf bifurcation. In our case, the reversibility symmetry is given by $\mathbf{S}_1\mathbf{S}_2$. This system is integrable and we refer to [28, Section 4.3.3] for a detailed discussion of its bounded solutions.

We consider here the periodic solutions of (3.7)-(3.8) with wavenumbers $k_x+\theta$ close to k_x, for small ε. According to [28, Section 4.3.3], these periodic solutions are determined, up to the action of $(\boldsymbol{\tau}_a)_{a\in\ \mathbb{R}/2\pi\mathbb{Z}}$ and to translations in x, by the reversible periodic solutions

$$A_+ = r_0e^{i(k_x+\theta)x},\quad B_+ = iq_0e^{i(k_x+\theta)x}, \tag{3.9}$$

with real numbers $r_0>0$ and q_0 satisfying the equalities

$$\begin{aligned}\theta &= \frac{q_0}{r_0} + \beta_0\varepsilon + \beta_1r_0^2 + 2\beta_2r_0q_0,\\ 0 &= q_0^2 + r_0^2\left(b_0\varepsilon + b_1r_0^2 + 2b_2r_0q_0\right),\end{aligned}$$

obtained by replacing (3.9) into the system (3.7)-(3.8). Solving for q_0 and r_0, we find

$$q_0 = \frac{r_0\left(\theta - \beta_0\varepsilon - \beta_1r_0^2\right)}{1+2\beta_2r_0^2},\quad r_0^2 = -\frac{b_0}{b_1}\varepsilon - \frac{1}{b_1}\theta^2 + O(|\varepsilon\theta| + |\varepsilon|^2 + |\theta|^3), \tag{3.10}$$

as $(\varepsilon,\theta)\to(0,0)$. For ε such that $b_0\varepsilon/b_1<0$, the right hand side in the formula for r_0^2 is positive for small ε and θ small enough, and we have a solution (A_+,B_+) given by (3.9) for the system (3.7)-(3.8). Notice that θ must be $O(|\varepsilon|^{1/2})$-small when $b_1 > 0$, which, as we shall see later in this section, is the case here.

For the 8-dimensional normal form (3.6) truncated at cubic order we obtain the solutions $(A_+, B_+, 0, 0)$ which belong to the invariant subspace E_+. The persistence of these solutions for the full normal form (3.6) can be proved via the implicit function theorem, for instance, by adapting the method used in the case of reversible 1 : 1 resonance bifurcations in [42, Section III.1] and Appendix 6.5.4 of Chapter 5. For small ε such that $b_0\varepsilon/b_1 < 0$ and θ small enough, we obtain a family of reversible periodic solutions $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ of the normal form (3.6), which are uniquely determined by their leading order part

$$(r_0 e^{i(k_x+\theta)x}, 0, 0, 0), \quad r_0^2 = -\frac{b_0}{b_1}\varepsilon - \frac{1}{b_1}\theta^2, \quad r_0 > 0. \tag{3.11}$$

This leading order part belongs to E_+, which is not the case for $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$, and it is the same as the one of the solutions (3.9) of the truncated system. As it follows from the implicit function theorem, the periodic solutions $\boldsymbol{\tau}_a(\widetilde{\mathbf{Z}}_{\varepsilon,\theta})$, $a \in \mathbb{R}/2\pi\mathbb{Z}$, are, up to translations in x, the only periodic solutions of the system (3.6) with leading order part of the form (3.11) in E_+ and wavenumbers $k_x + \theta$ sufficiently close to k_x, for sufficiently small ε. Notice that there are precisely two *reversible* solutions, $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ with $r_0 > 0$ and $\tau_\pi \widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ with $r_0 < 0$. We show below that the solutions $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ correspond to solutions of dynamical system (2.1) which are rotated rolls $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$, with k and μ sufficiently close to k_c and μ_c, respectively. We use this correspondence to compute the coefficients b_0 and b_1 of the normal form.

Consider the rotated roll $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$, for $\mu > \mu_c$ close to μ_c, wavenumber k close to k_c such that

$$k \in (k_1, k_2), \quad \mu_0(k_1) = \mu_0(k_2) = \mu,$$

(see Figure 1.1 of Chapter 3), and rotation angle $\beta \in (0, \pi/2)$ chosen such that the rotated roll is a solution of the dynamical system (2.1), i.e., such that

$$k \cos\beta = k_y = k_c \cos\alpha. \tag{3.12}$$

The rotation angle $\beta \in (0, \pi/2)$ is uniquely determined through this formula, and from the Taylor expansion of $\mu_0(k)$,

$$\mu_0(k) = \mu_c + \frac{1}{2}\mu_0''(k_c)(k-k_c)^2 + O(|k-k_c|^3), \tag{3.13}$$

for k close to k_c, we find that the unique values k_1 and k_2 above are $O(|\mu-\mu_c|^{1/2})$-close to k_c. The rotated roll $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ is periodic in x with wavenumber

$$k_x' = k\sin\beta = \sqrt{k^2 - k_c^2\cos^2\alpha} = k_c\sin\alpha + \frac{1}{\sin\alpha}(k-k_c) + O(|k-k_c|^2), \tag{3.14}$$

where we used (3.12) to obtain the second equality, and has the reversibility symmetry mentioned in Chapter 3

$$\mathbf{S}_1\mathbf{S}_2(\mathcal{R}_\beta\mathbf{U}^*_{k,\mu}(x)) = \mathcal{R}_\beta\mathbf{U}^*_{k,\mu}(-x)) \tag{3.15}$$

According to the formulas obtained In Appendix 6.1.2 of Chapter 1, and denoting by δ the amplitude of rolls, we have that

$$\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{\Pi}\mathbf{U}^*_{k,\mu}(x,y,z) = \delta e^{i(k'_x x+k_y y)}\boldsymbol{\mathcal{R}}_{-\beta}\widehat{\mathbf{u}}_k(z) + \delta e^{-i(k'_x x+k_y y)}\boldsymbol{\mathcal{R}}_{-\beta}\overline{\widehat{\mathbf{u}}_k(z)} + O(\delta^2), \tag{3.16}$$

where $\delta > 0$ is the small parameter and $\widehat{\mathbf{u}}_k(z)$ is given by (1.15) in Chapter 3. Furthermore, from (2.4) of Chapter 3 we obtain

$$e^{ik_y y}\boldsymbol{\mathcal{R}}_{-\beta}\widehat{\mathbf{u}}_k(z) = \mathbf{\Pi}\mathbf{U}_{\omega_1(k),1}(y,z) = \mathbf{\Pi}\boldsymbol{\zeta}_+(y,z) + O(|k-k_c|), \tag{3.17}$$

where $\mathbf{U}_{\omega_1(k),1}$ and $\boldsymbol{\zeta}_+$ are the eigenvectors in Lemmas 2.1 and 2.2 of Chapter 3, respectively.

For $\mu = \mu_c + \varepsilon$, the rotated roll $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ is a solution of the dynamical system (2.2), which is the same as (2.1). From formulas of Appendix 6.1.2 of Chapter 1, and (3.13) we obtain the relationship

$$\varepsilon = (\mu - \mu_0(k)) + (\mu_0(k) - \mu_c) = \mu_2\delta^2 + \frac{1}{2}\mu_0''(k_c)(k-k_c)^2 + O(|\delta|^3 + |k-k_c|^3), \tag{3.18}$$

implying that $\delta = O(\varepsilon^{1/2})$ and $|k-k_c| = O(\varepsilon^{1/2})$, since the values μ_2 and $\mu_0''(k_c)$ given by Appendix 6.1.4 of Chapter 1 and (1.2) of Chapter 1, respectively, are positive. In particular, the rotated roll $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ has small amplitude of order $O(\varepsilon^{1/2})$ and therefore belongs to the center manifold (2.5) of (2.2), provided ε is sufficiently small. Furthermore, we saw in Section 3.1.2 of chapter 3 that for rotation angles $\beta \in (0, \pi/2)$, the rolls $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ are invariant under the action of $\mathbf{S}_3\boldsymbol{\tau}_\pi$. This implies that $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ belongs to the center submanifold associated to $E_\pm$ found in Section 4.2.2. Consequently, it provides a periodic solution of the reduced system (3.3), from which we obtain a periodic solution for the normal form system (3.6) through the change of variables (3.5). These periodic solutions inherit the reversibility symmetry (3.15) of the rotated rolls.

We set

$$\theta = k'_x - k_x = k'_x - k_c\sin\alpha = \frac{1}{\sin\alpha}(k-k_c) + O(|k-k_c|^2), \tag{3.19}$$

where k'_x is the wavenumber given by (3.14), and denote by $\mathbf{Z}_{\varepsilon,\theta}$ the periodic solution of the normal form (3.6) corresponding to $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$. The parameters (ε,θ) are related to (k,μ) through the equalities $\varepsilon = \mu - \mu_c$ and (3.19), which define a one-to-one map $(k,\mu) \to (\varepsilon,\theta)$, for k in a neighborhood of k_c and any μ. Comparing the expressions of $\mathbf{\Pi}\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ given by (3.16) and by the formulas (2.5) and (2.4)for the solutions on the center manifold, using the equalities (3.17) and (3.19), we obtain the expansion

$$\mathbf{Z}_{\varepsilon,\theta}(x) = \left(\delta e^{i(k_x+\theta)x}, 0, 0, 0\right) + O(|\delta||\theta| + |\delta|^2), \tag{3.20}$$

with $\delta > 0$ determined through (3.18) and (3.19),

$$\delta^2 = \frac{1}{\mu_2}\varepsilon - \frac{\mu_0''(k_c)\sin^2\alpha}{2\mu_2}\,\theta^2 + O(|\varepsilon|^{3/2} + |\varepsilon|^{1/2}|\theta|^2 + |\theta|^3). \tag{3.21}$$

The existence and the above properties of the periodic solutions $\mathbf{Z}_{\varepsilon,\theta}$ of the normal form system (3.6) are directly obtained from the existence and properties of the rotated rolls $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$, without using the solutions $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ found from the periodic solutions (3.9) of the truncated system. With $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$, the solutions $\mathbf{Z}_{\varepsilon,\theta}$ share the property of being reversible periodic solutions of the system (3.6) with leading order parts in E_+ and wavenumbers $k_x + \theta$ sufficiently close to k_x, for sufficiently small ε. The solutions $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ and $\tau_\pi\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ being the only ones with these properties, taking into account that δ in (3.20) and r_0 in (3.11) are both positive, we deduce that $\mathbf{Z}_{\varepsilon,\theta}$ and $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ are the same solutions of the system (3.6), for sufficiently small ε and θ. In particular, their leading order parts are the same. Identifying the leading order part of δ^2 in (3.21) with r_0^2 in (3.11), we can compute the coefficients

$$b_0 = -\frac{2}{\mu_0''(k_c)\sin^2\alpha} < 0, \quad b_1 = \frac{2\mu_2}{\mu_0''(k_c)\sin^2\alpha} > 0. \tag{3.22}$$

The signs of these two coefficients are needed in the subsequent arguments.

Remark 3.3 *As usual in this type of approach, the coefficient b_0 can be determined from the property that the eigenvalues of the matrix obtained by linearizing the normal form (3.6) at $Z = 0$ are equal to the continuation of the eigenvalues $\pm ik_x$ of $\mathcal{L}_{\mu_c}$ as eigenvalues of $\mathcal{L}_\mu$ for $\mu = \mu_c + \varepsilon$ and sufficiently small ε. In the proof of Lemma 2.2 of Chapter 3 we saw that the latter eigenvalues are the purely imaginary eigenvalues $\pm i\omega_1(k_1)$ and $\pm i\omega_1(k_2)$ given by (2.5) (in Chapter3), with $k_1 < k_c < k_2$ such that $\mu = \mu_0(k_1) = \mu_0(k_2)$. Computing the eigenvalues of the normal form (3.6) we obtain*

$$i\omega_1(k_1) = i\left(k_x - \sqrt{-b_0\varepsilon} + O(\varepsilon)\right),$$

whereas from (2.5) (in Chapter3) we find

$$i\omega_1(k_1) = i\sqrt{k_1^2 - k_c^2\cos^2\alpha} = i\left(k_c\sin\alpha + \frac{1}{\sin\alpha}(k_1 - k_c) + O(|k_1 - k_c|^2)\right).$$

These two equalities and the Taylor expansion (3.13) of $\mu_0(k)$, taken at $k = k_1$, give the value of b_0 in (3.22). Furthermore, by replacing the expansions (3.20) and (3.21) with $\theta = 0$ into the equation for B_+ of the normal form (3.6) and identifying the coefficients of the terms of order $O(\varepsilon^{3/2})$, we easily obtain that $b_1 = -\mu_2 b_0$. These arguments give an alternative way for the computation of b_0 and b_1, without using the solutions $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$.

4.3.3 Leading order system when S_3 applies

From now on we restrict to $\varepsilon > 0$, which corresponds to values $\mu > \mu_c$ for which rolls exist. We further transform the normal form (3.6) by introducing new variables

$$\widehat{x} = |b_0\varepsilon|^{1/2}x, \quad A_\pm(x) = \left|\frac{b_0\varepsilon}{b_1}\right|^{1/2} e^{ik_x x}C_\pm(\widehat{x}), \quad B_\pm(x) = \frac{|b_0\varepsilon|}{|b_1|^{1/2}}e^{ik_x x}D_\pm(\widehat{x}). \tag{3.23}$$

Taking into account the signs of b_0 and b_1 in (3.22), we obtain the first order system,

$$\begin{aligned}
C_+' &= D_+ + \widehat{f}_+(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}), && (3.24)\\
D_+' &= \left(-1 + |C_+|^2 + g|C_-|^2\right) C_+ + \widehat{g}_+(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}),\\
C_-' &= D_- + \widehat{f}_-(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}), && (3.25)\\
D_-' &= (-1 + g|C_+|^2 + |C_-|^2)C_- + \widehat{g}_-(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}),
\end{aligned}$$

in which g is the quotient

$$g = \frac{b_3}{b_1}, \tag{3.26}$$

and $\widehat{f}_\pm$, $\widehat{g}_\pm$ are C^k-functions in their arguments of the form

$$\begin{aligned}
&\widehat{f}_\pm = \widehat{f}_{\pm,0} + \widehat{f}_{\pm,1}, \quad \widehat{g}_\pm = \widehat{g}_{\pm,0} + \widehat{g}_{\pm,1},\\
&\widehat{f}_{\pm,0} = \widehat{f}_{\pm,0}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, \varepsilon^{1/2}) = O(\varepsilon^{1/2}(|C_\pm| + |D_\pm|)),\\
&\widehat{f}_{\pm,1} = \widehat{f}_{\pm,1}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}) = O(\varepsilon^{3/2}(|C_\pm| + |D_\pm|)),\\
&\widehat{g}_{\pm,0} = \widehat{g}_{\pm,0}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, \varepsilon^{1/2}) = O(\varepsilon^{1/2}(|C_\pm| + |D_\pm|)),\\
&\widehat{g}_{\pm,1} = \widehat{g}_{\pm,1}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}) = O(\varepsilon(|C_\pm| + |D_\pm|)).
\end{aligned}$$

Solving the equations (3.24) and (3.25) for D_+ and D_-, respectively, we rewrite the first order system (3.24)-(3.25) as a second order system,

$$C_+'' = \left(-1 + |C_+|^2 + g|C_-|^2\right) C_+ + h_+(C_\pm, C_\pm', \overline{C_\pm}, \overline{C_\pm'}, e^{\pm ik_x x/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}), \tag{3.27}$$

$$C_-'' = \left(-1 + g|C_+|^2 + |C_-|^2\right) C_- + h_-(C_\pm, C_\pm', \overline{C_\pm}, \overline{C_\pm'}, e^{\pm ik_x x/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}), \tag{3.28}$$

where we replaced $\widehat{x}$ by x, for notational convenience, and $h_\pm$ are C^k-functions in their arguments of the form

$$\begin{aligned}
&h_\pm = h_{\pm,0} + h_{\pm,1},\\
&h_{\pm,0} = h_{\pm,0}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, \varepsilon^{1/2}) = O(\varepsilon^{1/2}(|C_\pm| + |D_\pm|)),\\
&h_{\pm,1} = h_{\pm,1}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x x/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}) = O(\varepsilon(|C_\pm| + |D_\pm|)).
\end{aligned}$$

Notice that both systems above inherit the symmetries of the normal form (3.6).

Through the change of variables (3.23), after rescaling θ, from the periodic solutions $\mathbf{Z}_{\varepsilon,\theta}$ of the normal form (3.6) we obtain a family of solutions $\mathbf{P}_{\varepsilon,\theta}$ of the second order system (3.27)-(3.28). The properties below are easily obtained from the ones found for $\mathbf{Z}_{\varepsilon,\theta}$ in Section 4.3.2.

Lemma 3.4 *For any $\varepsilon > 0$ and θ sufficiently small, the system (3.27)-(3.28) possesses a two-parameter family of solutions $\mathbf{P}_{\varepsilon,\theta}$ with the following properties:*

(i) $e^{-i\theta x}\mathbf{P}_{\varepsilon,\theta}$ *is periodic in x with wavenumber $\theta + k_x/|b_0\varepsilon|^{1/2}$;*

(ii) $\mathbf{S}_1\mathbf{S}_2(\mathbf{P}_{\varepsilon,\theta}(x)) = \mathbf{P}_{\varepsilon,\theta}(-x)$*, for all $x \in \mathbb{R}$;*

(iii) $\mathbf{P}_{\varepsilon,\theta}(x) = \big((1-\theta^2)^{1/2}e^{i\theta x}, 0\big) + O(\varepsilon^{1/2})$*, as $(\varepsilon,\theta) \to (0,0)$;*

(iv) $\mathbf{P}_{\varepsilon,\theta}$ *corresponds to a solution of the system (2.1) which is a rotated roll $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ with*

$$\cos\beta = k_y/k, \quad \mu = \mu_c + \varepsilon, \quad k = k_c + |b_0\varepsilon|^{1/2}\theta\sin\alpha + O(\varepsilon\theta^2). \tag{3.29}$$

Notice that $\mathbf{P}_{\varepsilon,\theta}$ is periodic in x when $\theta = 0$, whereas for $\theta \neq 0$ it is a quasiperiodic function. This comes from the change of variables (3.23) where in the expressions of $A_\pm$ and $B_\pm$ we only factored out the exponential $e^{ik_x x}$, instead of the exponential $e^{i(k_x+\theta)x}$ which would have preserved periodicity. This lack of periodicity does not pose any problem for the remaining arguments, in which we only use the properties *(ii)*-*(iv)* above.

The second property in Lemma 3.4 shows that the solutions $\mathbf{P}_{\varepsilon,\theta}$ are reversible, the reversibility symmetry being $\mathbf{S}_1\mathbf{S}_2$. Using the reversibility symmetry $\mathbf{S}_1$, we obtain a second family of solutions of the system (3.27)-(3.28),

$$\mathbf{Q}_{\varepsilon,\theta}(x) = \mathbf{S}_1(\mathbf{P}_{\varepsilon,\theta}(-x)) = \Big(0, (1-\theta^2)^{1/2}e^{i\theta x}\Big) + O(\varepsilon^{1/2}). \tag{3.30}$$

These solutions have the properties *(i)* and *(ii)* in Lemma 3.4 and correspond to the rotated rolls $\boldsymbol{\mathcal{R}}_\beta\mathbf{U}^*_{k,\mu}$ satisfying (3.29). In addition, the family of maps $(\boldsymbol{\tau}_a)_{a\in\,\mathbb{R}/2\pi\mathbb{Z}}$ provides the circles of solutions $\boldsymbol{\tau}_a(\mathbf{P}_{\varepsilon,\theta})$ and $\boldsymbol{\tau}_a(\mathbf{Q}_{\varepsilon,\theta})$, $a \in \mathbb{R}/2\pi\mathbb{Z}$.

The existence proof in the next section requires that the quotient g in (3.26) takes values in the interval $(1, 4+\sqrt{13})$. The lemma below shows that this property holds at least for small angles α.

Lemma 3.5 *For any Prandtl number $\mathcal{P}$, there exists an angle $\alpha_*(\mathcal{P}) \in (0, \pi/3]$ such that $1 < g < 4+\sqrt{13}$, for any $\alpha \in (0, \alpha_*(\mathcal{P}))$.*

Proof. We compute the coefficient g in Appendix 6.4.2 in the rigid-rigid and free-free cases. The result in formula (4.12) of this Appendix shows that the limit as α tends to 0 of g is equal to 2, which proves the result. ■

A symbolic computation, using the package Maple, of g shows in the rigid-rigid and free-free cases, that the inequality $g > 1$ holds for any Prandtl number $\mathcal{P} > 0$ and any angle $\alpha \in (0, \pi/3)$, and that the inequality $g < 4 + \sqrt{13}$ holds in the shaded region of the $(\alpha, \mathcal{P})$-plane indicated in Figures 3.1 and 3.2.

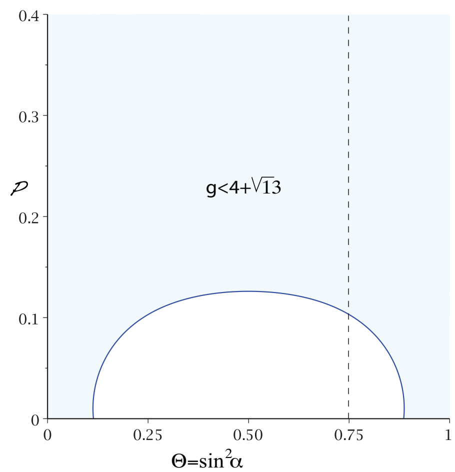

Figure 3.1: “Rigid-rigid” case. In the $(\Theta, \mathcal{P})$-plane, with $\Theta = \sin^2 \alpha$, Maple plot of the curve along which $g = 4 + \sqrt{13}$, for $\Theta \in (0, 1)$. The inequality $g < 4 + \sqrt{13}$ holds in the shaded regions, whereas the inequality $g > 1$ holds everywhere. Domain walls are constructed in the shaded region situated to the left of the vertical line $\Theta = \sin^2(\pi/3) = 0.75$.

4.3.4 Existence of domain walls when $\mathbf{S}_3$ applies

We construct domain walls as reversible heteroclinic solutions of (3.27)-(3.28) connecting the solutions $\mathbf{Q}_{\varepsilon,\theta}$ as $x \to -\infty$ with $\mathbf{P}_{\varepsilon,\theta}$ as $x \to \infty$, for a suitable $\theta = \theta(\varepsilon^{1/2})$ and $\varepsilon > 0$ sufficiently small. While the asymptotic solutions $\mathbf{P}_{\varepsilon,\theta}$ and $\mathbf{Q}_{\varepsilon,\theta}$ have the reversibility symmetry $\mathbf{S}_1\mathbf{S}_2$, the heteroclinic solutions will have the reversibility symmetry $\mathbf{S}_1$.

Following the approach developed in [32], we start by constructing a heteroclinic solution for the leading order system obtained at $\varepsilon = 0$ and then using the implicit function theorem we show that it persists for the full system. In contrast to the reduced system in [32] which was 12-dimensional, we have here an 8-dimensional system, only. This simplifies a part of the proof of Lemma 3.8 below. On the other hand, the quotient g takes here different values

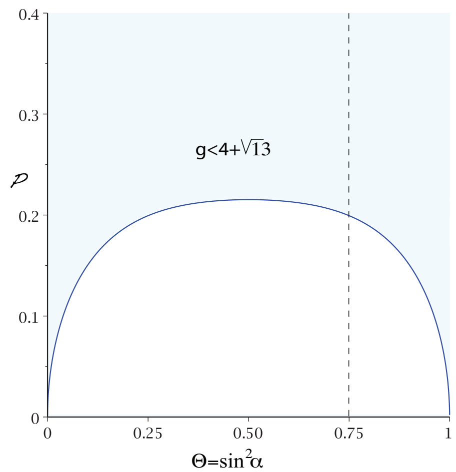

Figure 3.2: "Free-free" case. In the $(\Theta, \mathcal{P})$-plane, with $\Theta = \sin^2 \alpha \in (0,1)$, Maple plot of the curve along which $g = 4 + \sqrt{13}$, in the case of "free-free" boundary conditions. The inequality $g < 4 + \sqrt{13}$ holds in the shaded regions, whereas the inequality $g > 1$ holds everywhere. Domain walls are constructed in the shaded region situated to the left of the vertical line $\Theta = \sin^2(\pi/3) = 0.75$.

depending on the Prandtl number $\mathcal{P}$ and the angle α (see Figures 3.1 and 3.2), whereas $g = 2$ in [32]. We therefore need to extend the arguments from [32] to more general values g. We obtain a persistence result for $g \in (1, 4 + \sqrt{13})$.

Leading order heteroclinic

Consider the leading order system

$$C_+'' = \left(-1 + |C_+|^2 + g|C_-|^2\right) C_+, \tag{3.31}$$

$$C_-'' = \left(-1 + g|C_+|^2 + |C_-|^2\right) C_-, \tag{3.32}$$

obtained by setting $\varepsilon = 0$ in (3.27)-(3.28). According to Lemma 3.4, this system has the solutions

$$\mathbf{P}_{0,\theta}(x) = \left((1-\theta^2)^{1/2} e^{i\theta x}, 0\right), \quad \mathbf{Q}_{0,\theta}(x) = \left(0, (1-\theta^2)^{1/2} e^{i\theta x}\right),$$

with θ sufficiently small. The leading order heteroclinic is constructed for $\theta = 0$, as a real-valued solution of (3.31)-(3.32) connecting the equilibrium $\mathbf{Q}_{0,0} = (0,1)$ as $x \to -\infty$ with the equilibrium $\mathbf{P}_{0,0} = (1,0)$ as $x \to \infty$.

Under the assumption that $g > 1$ [2], the existence of such a heteroclinic solution has been proved in [68]. According to [68, Theorem 5], for any $g > 1$, the system (3.31)-(3.32) possesses a heteroclinic solution (C_+^*, C_-^*), where $C_\pm^*$ are smooth real-valued functions defined on $\mathbb{R}$ and have the following properties:

(i) $\lim_{x\to-\infty}(C_+^*(x), C_-^*(x)) = (0,1)$ and $\lim_{x\to\infty}(C_+^*(x), C_-^*(x)) = (1,0)$;

(ii) $C_+^*(x) = C_-^*(-x), \ \forall\, x \in \mathbb{R}$;

(iii) $C_+^*(x)^2 + C_-^*(x)^2 \leq 1$ and $C_+^*(x) + C_-^*(x) \geq \min(1, 2/\sqrt{g+1}), \ \forall\, x \in \mathbb{R}$;

(iv) $(C_+^{*\prime}(x))^2 + (C_-^{*\prime}(x))^2 = \dfrac{1}{2}\left(C_+^*(x)^2 + C_-^*(x)^2 - 1\right)^2 + (g-1)C_+^*(x)^2 C_-^*(x)^2, \ \forall\, x \in \mathbb{R}$.

The second property above shows that (C_+^*, C_-^*) is reversible, with reversibility symmetry $\mathbf{S}_1$. The last property is a consequence of the Hamiltonian structure of the system (3.31)-(3.32), which was one of the key ingredients in the existence proof in [68]. Notice that the equilibria $(1,0)$ and $(0,1)$ of the system (3.31)-(3.32) are both saddles having a two-dimensional stable manifold and a two-dimensional unstable manifold. The heteroclinic connection (C_+^*, C_-^*) belongs to the intersection of the two-dimensional stable manifold of $(1,0)$ with the two-dimensional unstable manifold of $(0,1)$.

In addition to these properties, in the proof of Lemma 3.8 below we need the two results in the following lemma.

Lemma 3.6 *Consider the heteroclinic solution (C_+^*, C_-^*) of the system (3.31)-(3.32).*

(i) For any $g > 1$, the functions C_+^* and C_-^* have the asymptotic behavior

$$C_+^*(x) = \alpha_* e^{\sqrt{g-1}\,x} + O(e^{(\sqrt{g-1}+\delta_*)x}), \quad C_-^*(x) = 1 - \beta_* e^{d_*\,x} + O(e^{(d_*+\delta_*)x}), \tag{3.33}$$

as $x \to -\infty$, for some positive constants α_*, d_*, δ_* and $\beta_* \geq 0$.

(ii) For any $g \in (1, 4+\sqrt{13})$, the functions C_+^* and C_-^* satisfy the inequality

$$3C_+^{*2}(x) + gC_-^{*2}(x) > 1, \quad \forall\, x \in \mathbb{R}. \tag{3.34}$$

Proof. (*i*) The heteroclinic connection (C_+^*, C_-^*) being included in the unstable manifold of the equilibrium $(0,1)$, the functions C_+^* and $1 - C_-^*$ decay exponentially to 0, as $x \to -\infty$. This implies the behavior of C_-^* and by taking into account the behavior of the different terms in the equation (3.31), we obtain the result for C_+^*.

[2]It turns out that this condition is necessary and sufficient.

(ii) For $g \in (3/2, 4+\sqrt{13})$ the property (3.34) is an immediate consequence of the inequality

$$C_+^*(x) + C_-^*(x) \geq \min(1, 2/\sqrt{g+1}), \quad \forall\, x \in \mathbb{R},$$

given above. We set

$$f_g(x) = 3C_+^{*2}(x) + gC_-^{*2}(x) - 1,$$

so that f_g is a smooth function defined on $\mathbb{R}$ and f_g is positive for any $g \in (3/2, 4+\sqrt{13})$. Assuming that there exists $g \in (1, 3/2]$ such that (3.34) does not hold, since f_g has positive limits at $x = \pm\infty$,

$$\lim_{x\to-\infty} f_g(x) = g - 1 > 0, \quad \lim_{x\to\infty} f_g(x) = 2,$$

and since the property holds for any $g \in (3/2, 4+\sqrt{13})$, there exists $g \in (1, 3/2]$ and $x_* \in \mathbb{R}$ such that

$$f_g(x_*) = 0, \quad f_g'(x_*) = 0, \quad f_g''(x_*) \geq 0, \tag{3.35}$$

i.e., f_g vanishes at a local minimum x_*.

For notational simplicity, we set

$$U = C_+^{*2}(x_*), \quad V = C_-^{*2}(x_*), \quad X = (C_+'(x_*))^2, \quad Y = (C_-'(x_*))^2.$$

Then the two equalities in (3.35) imply,

$$3U + gV = 1, \quad 9UX = g^2VY,$$

and from the property (iv) above we find that

$$X + Y = \frac{1}{2}(U+V-1)^2 + (g-1)UV.$$

Consequently, we can write V, X, Y as functions of U,

$$\begin{aligned}
V &= \frac{1}{g}(1-3U),\\
X &= \frac{1}{2}\frac{(1-3U)((5g^2-9)U^2 + 6(1-g)U - (g-1)^2)}{g(3(g-3)U-g)},\\
Y &= \frac{9}{2}\frac{U((5g^2-9)U^2 + 6(1-g)U - (g-1)^2)}{g^2(3(g-3)U-g)},
\end{aligned}$$

and then compute

$$\begin{aligned}
f_g''(x_*) &= 2(3X + gY + 3U(-1+U+gV) + gV(-1+gU+V)\\
&= \big(18(g-1)(g^2-9)U^3 + (12g(9-g^2) - 27(3+g^2))U^2\\
&\quad +2g(g^2+6g-9)U + (g-1)(g-3)\big)/(g(g-3(g-3)U)).
\end{aligned}$$

For $g \in (1, 3/2)$ and $U \in (0,1)$ we find that $f_g''(x_*) < 0$, which proves the result. ■

Remark 3.7 *(i) As pointed out in [68], the system (3.31)-(3.32) is integrable in the case $g = 3$, and the heteroclinic solution (C_+^*, C_-^*) can be explicitly computed in this case. We find that*

$$C_\pm^*(x) = \frac{1}{2}\left(1 \pm \tanh\left(\frac{x}{\sqrt{2}}\right)\right).$$

These formulas allow to easily check the properties in Lemma 3.6, and also the ones in Lemma 3.8 below, in this particular case.

(ii) The heteroclinic connection (C_+^*, C_-^*) being real-valued, it is in fact a solution of the 4-dimensional system obtained by restricting (3.31)-(3.32) to the invariant subspace of real-valued solutions. As a solution of the (complex) 8-dimensional system, it belongs to the circle of heteroclinic solutions $\boldsymbol{\tau}_a(C_+^*, C_-^*)$, for $a \in \mathbb{R}/2\pi\mathbb{Z}$, and all these heteroclinic solutions are reversible. Notice that such a property does not hold for the circle of solutions $\boldsymbol{\tau}_a(\mathbf{P}_{\varepsilon,\theta})$ found in Section 4.3.3, the reason being that the reversibility symmetries are different, $\mathbf{S}_1$ for (C_+^*, C_-^*) and $\mathbf{S}_1\mathbf{S}_2$ for $\mathbf{P}_{\varepsilon,\theta}$.

Persistence of the heteroclinic

The heteroclinic solution (C_+^*, C_-^*) is a particular reversible solution of the system (3.27)-(3.28) for $\varepsilon = 0$. Its persistence for small $\varepsilon > 0$ is proved by applying the implicit function theorem in a space of reversible exponentially decaying functions,

$$\mathcal{X}_\eta^r = \{(C_+, C_-, \overline{C_+}, \overline{C_-}) \in \mathcal{X}_\eta \ ; \ C_+(x) = \overline{C_-}(-x),\ x \in \mathbb{R}\}, \tag{3.36}$$

where, for $\eta > 0$,

$$\mathcal{X}_\eta = \left\{(C_+, C_-, \overline{C_+}, \overline{C_-}) \in (L_\eta^2)^4\right\}, \quad L_\eta^2 = \left\{f : \mathbb{R} \to \mathbb{C} \ ; \ \int_{\mathbb{R}} e^{2\eta|x|}|f(x)|^2 < \infty\right\}.$$

A key step of the proof is the analysis of the operator obtained by linearizing the leading order system (3.31)-(3.32), together with the complex conjugated equations, at (C_+^*, C_-^*), i.e., the linear operator $\mathcal{L}_*$ acting on (C_+, C_-) through

$$\mathcal{L}_*\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} C_+'' - \left(-1 + 2C_+^{*2} + gC_-^{*2}\right)C_+ - C_+^{*2}\overline{C_+} - gC_+^*C_-^*(C_- + \overline{C_-}) \\ C_-'' - \left(-1 + gC_+^{*2} + 2C_-^{*2}\right)C_- - C_-^{*2}\overline{C_-} - gC_+^*C_-^*(C_+ + \overline{C_+}) \end{pmatrix}.$$

In the space of exponentially decaying functions $\mathcal{X}_\eta$, the operator $\mathcal{L}_*$ is closed with dense domain

$$\mathcal{Y}_\eta = \left\{(C_+, C_-, \overline{C_+}, \overline{C_-}) \in (H_\eta^2)^4\right\}, \quad H_\eta^2 = \left\{f : \mathbb{R} \to \mathbb{C} \ ; \ f, f', f'' \in L_\eta^2\right\}, \tag{3.37}$$

and the subspace $\mathcal{X}_\eta^r$ of reversible functions is invariant under the action of $\mathcal{L}_*$, due to the reversibility of both the system (3.27)-(3.28) and the heteroclinic (C_+^*, C_-^*). The following lemma extends the result in [32, Lemma 4.1] to values $g \in (1, 4 + \sqrt{13})$.

Lemma 3.8 *Assume that $g \in (1, 4+\sqrt{13})$. For any $\eta > 0$ sufficiently small, the operator $\mathcal{L}_*$ acting in $\mathcal{X}^r_\eta$ is Fredholm with index -1. The kernel of $\mathcal{L}_*$ is trivial, and the one-dimensional kernel of its L^2-adjoint is spanned by $(\mathrm{i}C^*_+, -\mathrm{i}C^*_-, -\mathrm{i}C^*_+, \mathrm{i}C^*_-)$.*

Proof. Taking as new variables the real and imaginary parts of $C_\pm$,

$$U_\pm = \frac{1}{2}(C_\pm + \overline{C_\pm}), \quad V_\pm = \frac{1}{2i}(C_\pm - \overline{C_\pm}),$$

we obtain the matrix operator

$$\mathcal{M}_* = \begin{pmatrix} \mathcal{M}_r & 0 \\ 0 & \mathcal{M}_i \end{pmatrix},$$

with

$$\mathcal{M}_r \begin{pmatrix} U_+ \\ U_- \end{pmatrix} = \begin{pmatrix} U''_+ - \left(-1 + 3C^{*2}_+ + gC^{*2}_-\right) U_+ - 2gC^*_+C^*_-U_- \\ U''_- - \left(-1 + gC^{*2}_+ + 3C^{*2}_-\right) U_- - 2gC^*_+C^*_-U_+ \end{pmatrix},$$

$$\mathcal{M}_i \begin{pmatrix} V_+ \\ V_- \end{pmatrix} = \begin{pmatrix} V''_+ - \left(-1 + C^{*2}_+ + gC^{*2}_-\right) V_+ \\ V''_- - \left(-1 + gC^{*2}_+ + C^{*2}_-\right) V_- \end{pmatrix},$$

acting in, respectively,

$$X^r_\eta = \left\{(U_+, U_-) \in (L^2_\eta)^2 \; ; \; U_+(x) = U_-(-x), \; x \in \mathbb{R}\right\},$$
$$X^i_\eta = \left\{(V_+, V_-) \in (L^2_\eta)^2 \; ; \; V_+(x) = -V_-(-x), \; x \in \mathbb{R}\right\}.$$

The properties of $\mathcal{L}_*$ are found from the ones of $\mathcal{M}_r$ and $\mathcal{M}_i$. In the case $g = 2$, the operator $\mathcal{M}_r$ has been studied in [31, Lemma 4.6] and the operator $\mathcal{M}_i$ in [32, Lemma 4.1]. Using the same arguments, it is straightforward to show that, for any $g > 1$, the operator $\mathcal{M}_r$ is Fredholm with index 0, whereas the operator $\mathcal{M}_i$ is Fredholm with index -1, has a trivial kernel, and the one-dimensional kernel of its L^2-adjoint is spanned by $(C^*_+, -C^*_-)$. To complete the proof it remains to show that the kernel of $\mathcal{M}_r$ is trivial. In this part of the proof, we use the two properties given in Lemma 3.6, the second one leading to the restriction $g \in (1, 4+\sqrt{13})$.

Elements in the kernel of $\mathcal{M}_r$ are couples of functions $(U_+, U_-) \in X^r_\eta$, solving the linear system

$$U''_+ = \left(-1 + 3C^{*2}_+ + gC^{*2}_-\right) U_+ + 2gC^*_+C^*_-U_-, \tag{3.38}$$

$$U''_- = \left(-1 + gC^{*2}_+ + 3C^{*2}_-\right) U_- + 2gC^*_+C^*_-U_+. \tag{3.39}$$

Due to the translation invariance of the leading order system (3.31)-(3.32), the derivative $(C^{*\prime}_+, C^{*\prime}_-)$ is a solution of this linear system, but it does not satisfy the reversibility condition

$U_+(x) = U_-(-x)$, and therefore it does not belong to the kernel of $\mathcal{M}_r$. We show below that the space of bounded solutions of this linear system is one-dimensional, hence spanned by the derivative $(C_+^{*\prime}, C_-^{*\prime})$ of the heteroclinic solution. This implies that the kernel of $\mathcal{M}_r$ is trivial and proves the result.

In the limit $x = -\infty$, the system (3.38)-(3.39) is autonomous, and the equations are decoupled,

$$U_+'' = (g-1)U_+, \quad U_-'' = 2U_-.$$

Consequently, the set of solutions of (3.38)-(3.39) which are bounded as $x \to -\infty$ is a two-dimensional vector space consisting of pairs (U_+, U_-) of exponentially decaying functions. Taking into account the exponential decay of solutions of the autonomous system and the asymptotic behavior of the heteroclinic solution in (3.33) we obtain that

$$U_+(x) = \alpha_+ e^{\sqrt{g-1}\,x} + O(e^{(\sqrt{g-1}+\delta_*)x}), \tag{3.40}$$

as $x \to -\infty$, for some $\alpha_+ \in \mathbb{R}$ and $\delta_* > 0$. We show below that $\alpha_+ \neq 0$, which implies that the space of bounded solutions of this linear system is one-dimensional. Indeed, assuming that there are two linearly independent solutions of (3.38)-(3.39), then a suitable linear combination of these solutions gives a solution with $\alpha_+ = 0$, which contradicts the property $\alpha_+ \neq 0$.

Assume that $\alpha_+ = 0$. Then the exponential decay of U_+ is given to leading order by the coupling term $2gC_+^*C_-^*U_-$ in (3.38). The product $2gC_+^*C_-^*$ being positive, this implies that U_+ and U_- have the same sign as $x \to -\infty$. Since both functions decay exponentially as $x \to -\infty$, they have constant signs on an interval $(-\infty, m)$, for some real number m. Assume, for instance, that they are both positive for x in $(-\infty, m)$, and take the first local maximum x_* of U_-, hence satisfying

$$U_-(x_*) > 0, \quad U_-'(x_*) = 0, \quad U_-''(x_*) \le 0, \quad U_-(x) > 0, \ \forall\ x < x_*.$$

From the equation (3.39) we find

$$2gC_+^*(x_*)C_-^*(x_*)U_+(x_*) \le -\left(-1 + gC_+^{*2}(x_*) + 3C_-^{*2}(x_*)\right)U_-(x_*),$$

which together with the property (3.34) in Lemma 3.6 and the positivity of $U_-(x_*)$, C_+^*, and C_-^*, implies that $U_+(x_*) < 0$. We claim that $U_+(x) < 0$, for all $x \le x_*$. Indeed, assuming that U_+ is not negative, there exists a local maximum at some point $\widetilde{x}_* < x_*$ such that

$$U_+(\widetilde{x}_*) \ge 0, \quad U_+'(\widetilde{x}_*) = 0, \quad U_+''(\widetilde{x}_*) \le 0.$$

Using now the equation (3.38), and arguing as above we obtain that $U_-(\widetilde{x}_*) \leq 0$, which contradicts the positivity of U_- for $x < x_*$. This implies that U_+ and U_- cannot have the same signs as $x \to -\infty$, which contradicts the assumption $\alpha_+ = 0$, and completes the proof. ■

The remaining part of the persistence proof consists in applying the implicit function theorem to show the existence of a heteroclinic solution for the full system (3.27)-(3.28), connecting $\mathbf{Q}_{\varepsilon,\theta}$, as $x \to -\infty$, to $\mathbf{P}_{\varepsilon,\theta}$, as $x \to \infty$. The operator $\mathcal{L}_*$ being Fredholm with index -1, the presence of the parameter θ is essential in these last arguments. In the proof, θ plays the role of an additional unknown which is determined as a function of ε when applying the implicit function theorem.

Theorem 3.9 *Assume that $g \in (1, 4 + \sqrt{13})$. For any $\varepsilon > 0$ sufficiently small, there exists $\theta = O(\varepsilon^{1/2})$, continuously depending on $\varepsilon^{1/2}$, such that the system (3.27)-(3.28) possesses a reversible heteroclinic solution $\mathbf{C}_\varepsilon = (C_{+,\varepsilon}, C_{-,\varepsilon})$ connecting the solutions $\mathbf{Q}_{\varepsilon,\theta}$, as $x \to -\infty$, to $\mathbf{P}_{\varepsilon,\theta}$, as $x \to \infty$.*

Proof. We follow the proofs in [32, Theorem 2] and [65, Theorem 2].

The system (3.27)-(3.28) together with the complex conjugated equations is of the form

$$\mathcal{F}(\mathbf{C}, \overline{\mathbf{C}}, \varepsilon^{1/2}) = 0, \quad \mathbf{C} = (C_+, C_-), \tag{3.41}$$

and it has the particular solutions $\mathbf{P}_{\varepsilon,\theta}$ and $\mathbf{Q}_{\varepsilon,\theta}$ found in Section 4.3.3, for sufficiently small θ and $\varepsilon > 0$, and the heteroclinic solution $\mathbf{C}^* = (C_+^*, C_-^*)$ from Section 4.3.4, for $\varepsilon = 0$. We set

$$\widetilde{\mathbf{P}}_{\varepsilon,\theta} = \mathbf{P}_{\varepsilon,\theta} - (1,0)\, e^{i\theta x}, \quad \widetilde{\mathbf{Q}}_{\varepsilon,\theta} = \mathbf{Q}_{\varepsilon,\theta} - (0,1)\, e^{i\theta x},$$

and take a smooth function $\chi : \mathbb{R} \to [0,1]$ such that

$$\chi(x) = 1, \text{ if } x \geq M, \quad \chi(x) = 0, \text{ if } x \leq m,$$

for some positive constants $m < M$. We look for solutions of (3.41) of the form

$$\mathbf{C}(x) = e^{i\theta x}\mathbf{C}^*(x) + \chi(x)\widetilde{\mathbf{P}}_{\varepsilon,\theta}(x) + \chi(-x)\widetilde{\mathbf{Q}}_{\varepsilon,\theta}(x) + \mathbf{V}(x), \tag{3.42}$$

with $(\mathbf{V}, \overline{\mathbf{V}}) \in \mathcal{Y}_\eta^r = \mathcal{Y}_\eta \cap \mathcal{X}_\eta^r$, where $\mathcal{X}_\eta^r$ and $\mathcal{Y}_\eta$ are defined in (3.36) and (3.37), respectively. Notice that the difference $\mathbf{C} - \mathbf{P}_{\epsilon,\theta}$ (resp. $\mathbf{C} - \mathbf{Q}_{\epsilon,\theta}$) decays exponentially to 0, as $x \to \infty$ (resp. $x \to -\infty$), with the same decay rate as $\mathbf{V}$, and that $\mathbf{C}$ and $\mathbf{V}$ have the same reversibility symmetry $\mathbf{S}_1$.

Substituting (3.42) into (3.41) we obtain an equation of the form

$$\mathcal{T}(\mathbf{V}, \overline{\mathbf{V}}, \theta, \varepsilon^{1/2}) = 0.$$

As shown in [32, Theorem 2], $\mathcal{T}(\mathbf{V},\overline{\mathbf{V}},\theta,\varepsilon^{1/2}) \in \mathcal{X}_\eta^r$, for any $(\mathbf{V},\overline{\mathbf{V}}) \in \mathcal{Y}_\eta^r$ and $(\theta,\varepsilon^{1/2})$ sufficiently small, and from the properties of $h_\pm$ in (3.27)-(3.28) we find that

$$\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_1, \quad \mathcal{T}_1 = O(\varepsilon), \tag{3.43}$$

with $\mathcal{T}_0$ continuously differentiable and $\mathcal{T}_1$ continuous and continuously differentiable with respect to $(\mathbf{V},\overline{\mathbf{V}},\theta)$. Furthermore,

$$\mathcal{T}(0,0,0,0) = \mathcal{F}(\mathbf{C}^*,\overline{\mathbf{C}^*},0) = 0,$$

and a direct calculation shows that

$$D_{\mathbf{V}}\mathcal{T}(0,0,0,0) = \mathcal{L}_*, \quad D_\theta\mathcal{T}(0,0,0,0) = \mathcal{L}_*\begin{pmatrix} ix\mathbf{C}^* \\ -ix\mathbf{C}^* \end{pmatrix} = \begin{pmatrix} 2i\mathbf{C}^{*\prime} \\ -2i\mathbf{C}^{*\prime} \end{pmatrix}.$$

According to Lemma 3.8, the operator $\mathcal{L}_*$ is Fredholm with index -1, injective, and its range is L^2-orthogonal to $(iC_+^*,-iC_-^*,-iC_+^*,iC_-^*)$. The L^2-scalar product of this vector with the differential $D_\theta\mathcal{T}(0,0,0,0)$ is given by

$$2\int_{\mathbb{R}}\left(2C_+^{*\prime}(x)C_+^*(x) - 2C_-^{*\prime}(x)C_-^*(x)\right)\mathrm{d}x = 2\int_{\mathbb{R}}\left(C_+^{*2}(x) - C_-^{*2}(x)\right)'\mathrm{d}x = 4 \neq 0, \tag{3.44}$$

which implies that $D_\theta\mathcal{T}(0,0,0,0)$ does not belong to the range of $\mathcal{L}^*$. Consequently, the differential $D_{(\mathbf{V},\theta)}\mathcal{T}(0,0,0,0)$ is bijective, and the result in the lemma follows from the implicit function theorem [23, Theorems 10.1.1 and 10.1.2] and (3.43). ■

Going back to the Bénard-Rayleigh problem, the result in this theorem, together with Lemma 3.4, implies the existence of a symmetric domain wall connecting two rotated rolls, $\boldsymbol{\mathcal{R}}_\beta\mathbf{U}_{k,\mu}^*$, as $x\to-\infty$, to $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}_{k,\mu}^*$, as $x\to\infty$, with $k = k_c + O(\varepsilon)$ and $\beta = \alpha + O(\varepsilon)$, for positive $\varepsilon = \mu - \mu_c$ sufficiently small. The family of maps $(\boldsymbol{\tau}_a)_{a\in\,\mathbb{R}/2\pi\mathbb{Z}}$ provides the circle of reversible heteroclinic solutions $\boldsymbol{\tau}_a(C_{+,\varepsilon},C_{-,\varepsilon})$, for $a \in \mathbb{R}/2\pi\mathbb{Z}$, which corresponds to translations in y of the symmetric domain wall. This proves Theorem 3.1. Notice that $\epsilon = \mathcal{R} - \mathcal{R}_c$ in Theorem 3.1 is linked to $\varepsilon = \mu - \mu_c$ in Theorem 3.9 through $\mathcal{R}^{1/2} = \mu$ and $\mathcal{R}_c^{1/2} = \mu_c$.

4.4 Bifurcation of symmetric domain walls without symmetry $\mathbf{S}_3$

This is the case with rigid-free boundary conditions, i.e.

$$\begin{aligned} &V_x = V_y = V_z = \theta = 0 \text{ for } z = 0, \\ &\partial_z V_x = \partial_z V_y = V_z = \theta = 0 \text{ for } z = 1. \end{aligned} \tag{4.1}$$

In such a case the symmetry $\mathbf{S}_3$ does not apply. The main result of this Section is summarized in the following theorem.

Theorem 4.1 *Consider the steady Navier-Stokes-Boussinesq system (2.1) with "rigid-free" boundary conditions (4.1). Denote by $\mathcal{R}_c$ the critical Rayleigh number at which convective rolls with wavenumbers k_c bifurcate from the conduction state. Then for any angle $\alpha \in (0, \pi/3)$, $\alpha \neq \pi/6$, there exists $\mathcal{P}_*(\alpha) \geq 0$ such that, up to a finite set, for any Prandtl number $\mathcal{P} > \mathcal{P}_*(\alpha)$, a symmetric domain wall bifurcates for Rayleigh numbers $\mathcal{R} = \mathcal{R}_c + \epsilon$, with $\epsilon > 0$ sufficiently small. The domain wall connects two rotated rolls which are the rotations by opposite angles $\pm(\alpha + O(\epsilon))$ of a roll with wavenumber $k_c + O(\epsilon)$, continuously linked to the amplitude which is of order $O(\epsilon^{1/2})$.*

4.4.1 A cubic normal form for 12-dimensional vector fields

In this section we prove a normal form theorem for reversible and $O(2)$-equivariant 12-dimensional vector fields having the same linear part and symmetries action as the one in (2.10), i.e.

$$\boldsymbol{S}_1(A_0, B_0, A_+, B_+, A_-, B_-) = (\overline{A_0}, -\overline{B_0}, \overline{A_-}, -\overline{B_-}, \overline{A_+}, -\overline{B_+}), \tag{4.2}$$

$$\boldsymbol{S}_2(A_0, B_0, A_+, B_+, A_-, B_-) = (A_0, B_0, A_-, B_-, A_+, B_+), \tag{4.3}$$

$$\boldsymbol{\tau}_a(A_0, B_0, A_+, B_+, A_-, B_-) = (A_0, B_0, e^{ia}A_+, e^{ia}B_+, e^{-ia}A_-, e^{-ia}B_-). \tag{4.4}$$

Theorem 4.2 *Consider a system of ordinary differential equations*

$$\frac{dX}{dx} = G(X, \overline{X}, \varepsilon), \tag{4.5}$$

in which $X = (A_0, B_0, A_+, B_+, A_-, B_-) \in \mathbb{C}^6$ and the vector field G is of class C^k, for some $k \geq 5$, in a neighborhood $\mathcal{U}_1 \times \overline{\mathcal{U}_1} \times \mathcal{U}_2 \subset \overline{\mathbb{C}^6} \times \mathbb{C}^6 \times \mathbb{R}$ of the origin. Assume that

$$G(0, 0, \varepsilon) = 0, \quad D_X G(0, 0, 0) = L_0, \quad D_{\overline{X}} G(0, 0, 0) = 0,$$

where L_0 is a Jordan matrix acting on X through

$$L_0 = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} ik_c & 1 \\ 0 & ik_c \end{pmatrix}, \quad B_2 = \begin{pmatrix} ik_x & 1 \\ 0 & ik_x \end{pmatrix}, \tag{4.6}$$

with $1 < k_c/k_x \neq 2$, and that $G(\cdot, \cdot, \varepsilon)$ anti-commutes with $\boldsymbol{S}_1$ given by (4.2) and commutes with $\boldsymbol{S}_2$ and $\boldsymbol{\tau}_a$ given by (4.3) and (4.4), respectively.

There exist neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 in $\mathbb{C}^6$ and $\mathbb{R}$, respectively, such that for any $\varepsilon \in \mathcal{V}_2$, there is a polynomial $P(\cdot,\cdot,\varepsilon) : \mathbb{C}^6 \times \overline{\mathbb{C}^6} \to \mathbb{C}^6$ of degree 3 in the variables $(Z, \overline{Z})$, such that for $Z \in \mathcal{V}_1$, the change of variable

$$X = Z + P(Z, \overline{Z}, \varepsilon), \tag{4.7}$$

transforms the equation (4.5) into the normal form

$$\frac{dZ}{dx} = L_0 Z + N(Z, \overline{Z}, \varepsilon) + \rho(Z, \overline{Z}, \varepsilon), \tag{4.8}$$

with the following properties:

(i) the map ρ belongs to $\mathcal{C}^k(\mathcal{V}_1 \times \overline{\mathcal{V}_1} \times \mathcal{V}_2, \mathbb{C}^6)$, and

$$\rho(Z, \overline{Z}, \varepsilon) = O(|\varepsilon|^2 \|Z\| + \varepsilon \|Z\|^3 + \|Z\|^4);$$

(ii) both $N(\cdot,\cdot,\varepsilon)$ and $\rho(\cdot,\cdot,\varepsilon)$ anti-commute with $\boldsymbol{S}_1$ and commute with $\boldsymbol{S}_2$ and $\boldsymbol{\tau}_a$, for any $\varepsilon \in \mathcal{V}_2$;

(iii) the six components $(N_0, M_0, N_+, M_+, N_-, M_-)$ of N are of the form

$$\begin{aligned}
N_0 &= iA_0P_0 + \alpha_5(A_+u_7 + A_-u_8),\\
M_0 &= iB_0P_0 + A_0Q_0 + \alpha_5(B_+u_7 + B_-u_8) + ia_5(A_+u_7 + A_-u_8),\\
N_+ &= iA_+P_+ + \beta_7 A_0\overline{u_7} + \beta_8 A_-u_9,\\
M_+ &= iB_+P_+ + A_+Q_+ + \beta_7 B_0\overline{u_7} + ib_7A_0\overline{u_7} + \beta_8 B_-u_9 + ib_8A_-u_9,\\
N_- &= iA_-P_- + \beta_7 A_0\overline{u_8} - \beta_8 A_+\overline{u_9},\\
M_- &= iB_-P_- + A_-Q_- + \beta_7 B_0\overline{u_8} + ib_7A_0\overline{u_8} - \beta_8 B_+\overline{u_9} - ib_8A_+\overline{u_9},
\end{aligned}$$

with

$$\begin{aligned}
P_0 &= \alpha_0\varepsilon + \alpha_1u_1 + \alpha_2u_2 + \alpha_3(u_3 + u_5) + \alpha_4(u_4 + u_6),\\
Q_0 &= a_0\varepsilon + a_1u_1 + a_2u_2 + a_3(u_3 + u_5) + a_4(u_4 + u_6),\\
P_+ &= \beta_0\varepsilon + \beta_1u_1 + \beta_2u_2 + \beta_3u_3 + \beta_4u_4 + \beta_5u_5 + \beta_6u_6,\\
Q_+ &= b_0\varepsilon + b_1u_1 + b_2u_2 + b_3u_3 + b_4u_4 + b_5u_5 + b_6u_6,\\
P_- &= \beta_0\varepsilon + \beta_1u_1 + \beta_2u_2 + \beta_5u_3 + \beta_6u_4 + \beta_3u_5 + \beta_4u_6,\\
Q_- &= b_0\varepsilon + b_1u_1 + b_2u_2 + b_5u_3 + b_6u_4 + b_3u_5 + b_4u_6,
\end{aligned}$$

where $(A_0, B_0, A_+, B_+, A_-, B_-)$ *are the six components of* Z*, the coefficients* α_j*,* a_j*,* β_j*,* b_j *are all real, and*

$$u_1 = A_0\overline{A_0}, \quad u_2 = i(A_0\overline{B_0} - \overline{A_0}B_0),$$
$$u_3 = A_+\overline{A_+}, \quad u_4 = i(A_+\overline{B_+} - \overline{A_+}B_+), \quad u_5 = A_-\overline{A_-}, \quad u_6 = i(A_-\overline{B_-} - \overline{A_-}B_-),$$
$$u_7 = (A_0\overline{B_+} - \overline{A_+}B_0), \quad u_8 = (A_0\overline{B_-} - \overline{A_-}B_0), \quad u_9 = (A_+\overline{B_-} - \overline{A_-}B_+).$$

Proof. From general normal form theorems (e.g., see [28, Sections 3.2.1, 3.3.1, and 3.3.2]), we obtain the existence of two polynomials $P(\cdot,\cdot,\varepsilon)$ and $N(\cdot,\cdot,\varepsilon)$ of degree 3 in the variables $(Z,\overline{Z})$ such that the properties (i) and (ii) hold, the polynomial N is of the form

$$N(Z,\overline{Z},\varepsilon) = \widetilde{N}_1(Z,\overline{Z})\varepsilon + N_2(Z,\overline{Z}) + \widetilde{N}_2(Z,\overline{Z})\varepsilon + N_3(Z,\overline{Z}), \tag{4.9}$$

with N_p and $\widetilde{N}_p$ homogeneous polynomials of degree p in $(Z,\overline{Z})$, and satisfies the identity

$$D_Z N(Z,\overline{Z},\varepsilon)L_0^* Z + D_{\overline{Z}} N(Z,\overline{Z},\varepsilon)\overline{L_0^* Z} = L_0^* N(Z,\overline{Z},\varepsilon), \quad \forall\ (Z,\varepsilon) \in \mathbb{C}^6 \times \mathcal{V}_2, \tag{4.10}$$

in which L_0^* is the adjoint of L_0. Due to the equivariance of the normal form under the action of the symmetry $\boldsymbol{S}_2$, it is enough to determine the first four components (N_0, M_0, N_+, M_+) of N, the result for (N_-, M_-) being obtained by exchanging the indices $+$ and $-$ in the expressions of (N_+, M_+).

Monomials in N_0 are M_0 are of the form

$$A_0^{p_0}\overline{A_0}^{q_0} B_0^{r_0}\overline{B_0}^{s_0} A_+^{p_+}\overline{A_+}^{q_+} B_+^{r_+}\overline{B_+}^{s_+} A_-^{p_-}\overline{A_-}^{q_-} B_-^{r_-}\overline{B_-}^{s_-}, \tag{4.11}$$

with nonnegative exponents such that

$$p_0 + q_0 + r_0 + s_0 + p_+ + q_+ + r_+ + s_+ + p_- + q_- + r_- + s_- = m, \quad m \in \{1,2,3\}. \tag{4.12}$$

From the commutativity of N and $\boldsymbol{\tau}_a$, we obtain that their exponents also satisfy the equality

$$(p_+ - q_+ + r_+ - s_+) - (p_- - q_- + r_- - s_-) = 0, \tag{4.13}$$

and we claim that we also have the equalities

$$p_0 - q_0 + r_0 - s_0 = 1, \quad (p_+ - q_+ + r_+ - s_+) + (p_- - q_- + r_- - s_-) = 0, \tag{4.14}$$

when $k_c/k_x \neq 2$.

Indeed, the identity (4.10) implies that N_0 and M_0 satisfy the equalities

$$(\mathcal{D}^* + ik_c)N_0 = 0, \quad (\mathcal{D}^* + ik_c)M_0 = N_0,$$

in which

$$\begin{aligned}\mathcal{D}^* \;=\; & -ik_cA_0\frac{\partial}{\partial A_0}+(A_0-ik_cB_0)\frac{\partial}{\partial B_0}+ik_c\overline{A_0}\frac{\partial}{\partial \overline{A_0}}+(\overline{A_0}+ik_c\overline{B_0})\frac{\partial}{\partial \overline{B_0}}\\ & -ik_xA_+\frac{\partial}{\partial A_+}+(A_+-ik_xB_+)\frac{\partial}{\partial B_+}+ik_x\overline{A_+}\frac{\partial}{\partial \overline{A_+}}+(\overline{A_+}+ik_x\overline{B_+})\frac{\partial}{\partial \overline{B_+}}\\ & -ik_xA_-\frac{\partial}{\partial A_-}+(A_--ik_xB_-)\frac{\partial}{\partial B_-}+ik_x\overline{A_-}\frac{\partial}{\partial \overline{A_-}}+(\overline{A_-}+ik_x\overline{B_-})\frac{\partial}{\partial \overline{B_-}},\end{aligned}$$

is a linear map which preserves the degree of homogeneous polynomials. For a fixed degree m, taking a basis of monomials ordered by decreasing exponents p_0, q_0, r_0, s_0, p_+, q_+, r_+, s_+, p_-, q_-, r_-, and s_-, the action of $\mathcal{D}^*$ is represented by a lower triangular matrix with equal elements on the diagonal given by

$$-ik_c\left(p_0-q_0+r_0-s_0\right)-ik_x\left(p_+-q_++r_+-s_+\right)-ik_x\left(p_--q_-+r_--s_-\right).$$

Consequently, the polynomials N_0 and M_0, which belong to the kernel and generalized kernel of $\mathcal{D}_*+ik_c$, respectively, belong to the subspace spanned by monomials for which the quantity above is equal to $-ik_c$. Taking into account that $k_c/k_x>1$ and the properties (4.12)-(4.13), we conclude that (4.14) holds when $k_c/k_x\neq 2$. This proves the claim.

Next, taking $m=1$ in (4.12) and using (4.12)-(4.14) it is straightforward to check that the first two components of $\widetilde{N}_1$ in (4.9) have the form given in (iii). For even integers m, and in particular for $m=2$, from the equalities (4.12) and (4.13) we obtain that $p_0-q_0+r_0-s_0$ must be an even integer. This contradicts the first equality in (4.14). Consequently, there are no mononomials of even degree in the first two components of N. It remains to consider the cubic monomials, $m=3$. Collecting all cubic monomials satisfying (4.12)-(4.14), we directly compute the action of $(\mathcal{D}^*+ik_c)$ on all these monomials. Then we identify a basis for the kernel of $(\mathcal{D}^*+ik_c)$ which gives the result for N_0, and a basis for the generalized kernel of $(\mathcal{D}^*+ik_c)$ which gives the result for M_0.

For the components N_+ and M_+ of N the result is obtained in the same way. We only point out that for these polynomials the exponents of the monomials (4.11) satisfy (4.12), the equality

$$(p_+-q_++r_+-s_+)-(p_--q_-+r_--s_-)=1, \tag{4.15}$$

replacing (4.13), and

$$p_0-q_0+r_0-s_0=0,\quad (p_+-q_++r_+-s_+)+(p_--q_-+r_--s_-)=1, \tag{4.16}$$

instead of (4.14).

Finally, taking into account the action of the reversibility $\boldsymbol{S}_1$, it is straightforward to check that the coefficients α_j, a_j, β_j and b_j are real. This completes the proof of the theorem. ■

Remark 4.3 *In the resonant case $k_c/k_x = 2$, the two polynomials N_0 and M_0 contain the additional quadratic terms $\alpha_6 A_+ A_-$ and $i\alpha_6(B_+A_- + A_+B_-) + a_6 A_+A_-$, respectively, with real coefficients α_6 and c_6.*

Applying the result in Theorem 4.2 to our reduced system (2.10) we obtain its normal form (4.8) for $k_c/k_x \neq 2$, i.e., for any angle $\alpha \in (0, \pi/3)$, $\alpha \neq \pi/6$.

4.4.2 Existence of domain walls

In this section, we prove the existence of a heteroclinic connection for the normal form system (4.8). Following the analysis from Section 4.3, we focus on the main differences and refer to Section 4.3 for the technical details which remain the same.

We restrict to $\varepsilon > 0$, which corresponds to values $\mu > \mu_c$ for which rolls exist and exclude the resonant angle $\alpha = \pi/6$ which requires a different analysis.

Rotated rolls and coefficients of the normal form

The rotated rolls $\boldsymbol{\mathcal{R}}_{-\beta}\boldsymbol{U}^*_{k,\mu}$ with rotation angle $\beta \in (0, \pi/2)$ are $2\pi/k\sin\beta$-periodic solutions of the dynamical system (2.1) and belong to the phase space $\mathcal{X}$ when their wavenumber in y is equal to k_y,

$$k\cos\beta = k_y = k_c\cos\alpha.$$

For (k, μ) close enough to (k_c, μ_c) they are small bounded solutions which belong to the center manifold (2.5) of (2.2). Projected on the center space $\mathcal{X}_c$ they become $2\pi/k\sin\beta$-periodic solutions of the reduced system (4.5) and also of the normal form system (4.8). As in Section 4.3 we obtain a family $\boldsymbol{Z}_{\varepsilon,\theta}$ of $2\pi/k\sin\beta$-periodic solutions of the normal form system, where the parameters (ε, θ) are related to (k, μ) through the equalities

$$\varepsilon = \mu - \mu_c, \quad \theta = k\sin\beta - k_x = k\sin\beta - k_c\sin\alpha = \frac{1}{\sin\alpha}(k - k_c) + O(|k - k_c|^2).$$

These solutions have the expansion

$$\boldsymbol{Z}_{\varepsilon,\theta}(x) = \Big(0, 0, \delta e^{i(k_x+\theta)x}, 0, 0, 0\Big) + O(|\delta||\theta| + |\delta|^2), \tag{4.17}$$

with $\delta > 0$ as in (3.21). As shown in Section 4.3, we can use this family of solutions to determine two coefficients of the normal form system

$$b_0 = -\frac{2}{\mu_0''(k_c)\sin^2\alpha} < 0, \quad b_3 = \frac{2\mu_2}{\mu_0''(k_c)\sin^2\alpha} > 0, \tag{4.18}$$

where $\mu_0''(k) > 0$ is the second order derivative of $\mu_0(k)$ with respect to k and $\mu_2 > 0$. The symmetry properties of rotated rolls, seen in Chapter 3:

$$\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}(x) = \boldsymbol{\mathcal{R}}_{\pi+\alpha}\mathbf{U}^*_{k,\mu}(x)$$

$$\mathbf{S}_1(\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}(x)) = \boldsymbol{\mathcal{R}}_{-\alpha}\mathbf{U}^*_{k,\mu}(-x), \quad \mathbf{S}_2\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu} = \boldsymbol{\mathcal{R}}_{-\alpha}\mathbf{U}^*_{k,\mu}, \tag{4.19}$$

$$\mathbf{S}_1\mathbf{S}_2(\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}(x)) = \boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}(-x).$$

are preserved so that $\boldsymbol{Z}_{\varepsilon,\theta}$ is $\boldsymbol{S}_1\boldsymbol{S}_2$-reversible and the rolls $\boldsymbol{\mathcal{R}}_\beta \boldsymbol{U}^*_{k,\mu}$ rotated by the opposite angle β give the periodic solutions $\boldsymbol{S}_2\boldsymbol{Z}_{\varepsilon,\theta}$. We refer to Section 4.3 for more details.

Similarly, by taking the rotation angle $\beta = \pi/2$, from the rotated rolls $\boldsymbol{\mathcal{R}}_{\pi/2}\boldsymbol{U}^*_{k,\mu}$, which are constant in y and therefore $2\pi/k$-periodic solutions of the dynamical system (2.1), we obtain a second family of periodic solutions for the normal form system. With the help of these solutions we can compute two other coefficients of the normal form system,

$$a_0 = -\frac{2}{\mu_0''(k_c)} < 0, \quad a_1 = \frac{2\mu_2}{\mu_0''(k_c)} > 0. \tag{4.20}$$

Notice that we have the following relationship between coefficients:

$$a_0 = b_0 \sin^2\alpha < 0, \quad a_1 = b_3 \sin^2\alpha > 0. \tag{4.21}$$

Leading order dynamics

We consider the new variables

$$\widehat{x} = |b_0\varepsilon|^{1/2}x, \quad A_0(x) = \left|\tfrac{b_0\varepsilon}{b_3}\right|^{1/2} e^{ik_cx}C_0(\widehat{x}), \quad B_0(x) = \tfrac{|b_0\varepsilon|}{|b_3|^{1/2}}e^{ik_cx}D_0(\widehat{x}), \tag{4.22}$$

$$A_\pm(x) = \left|\tfrac{b_0\varepsilon}{b_3}\right|^{1/2} e^{ik_xx}C_\pm(\widehat{x}), \quad B_\pm(x) = \tfrac{|b_0\varepsilon|}{|b_3|^{1/2}}e^{ik_xx}D_\pm(\widehat{x}). \tag{4.23}$$

Replacing these variables into the normal form system (4.8) and taking into account the signs of b_0 and b_3 in (4.18) and the relationship (4.21), we obtain the new system

$$C_0' = D_0 + O(\varepsilon^{1/2}), \tag{4.24}$$

$$D_0' = \sin^2\alpha\left(-1 + |C_0|^2 + g_1(|C_+|^2 + |C_-|^2)\right)C_0 + O(\varepsilon^{1/2}), \tag{4.25}$$

$$C_+' = D_+ + O(\varepsilon^{1/2}), \tag{4.26}$$

$$D_+' = \left(-1 + g_2|C_0|^2 + |C_+|^2 + g_3|C_-|^2\right)C_+ + O(\varepsilon^{1/2}), \tag{4.27}$$

$$C_-' = D_- + O(\varepsilon^{1/2}), \tag{4.28}$$

$$D_-' = \left(-1 + g_2|C_0|^2 + g_3|C_+|^2 + |C_-|^2\right)C_- + O(\varepsilon^{1/2}), \tag{4.29}$$

in which

$$g_1 = \frac{a_3}{a_1}, \quad g_2 = \frac{b_1}{b_3}, \quad g_3 = \frac{b_5}{b_3}.$$

Solving the equations (4.24), (4.26) and (4.28) for D_0, D_+ and D_-, respectively, we rewrite the first order system (4.26)-(4.29) as a second order system,

$$C_0'' = \sin^2\alpha \left(-1 + |C_0|^2 + g_1(|C_+|^2 + |C_-|^2)\right) C_0 + O(\varepsilon^{1/2}), \tag{4.30}$$

$$C_+'' = \left(-1 + g_2|C_0|^2 + |C_+|^2 + g_3|C_-|^2\right) C_+ + O(\varepsilon^{1/2}), \tag{4.31}$$

$$C_-'' = \left(-1 + g_2|C_0|^2 + g_3|C_+|^2 + |C_-|^2\right) C_- + O(\varepsilon^{1/2}), \tag{4.32}$$

in which the $O(\varepsilon^{1/2})$-terms are continuous in $(C_0, C_\pm, \epsilon^{1/2})$ and continuously differentiable in $(C_0, C_\pm)$.[3]

From the periodic solutions $\boldsymbol{Z}_{\varepsilon,\theta}$ and $\boldsymbol{S}_2\boldsymbol{Z}_{\varepsilon,\theta}$ of the normal form system we obtain the $\boldsymbol{S}_1\boldsymbol{S}_2$-reversible solutions $\boldsymbol{P}_{\varepsilon,\theta}$ and $\boldsymbol{Q}_{\varepsilon,\theta} = \boldsymbol{S}_2\boldsymbol{P}_{\varepsilon,\theta}$, respectively, for the system (4.30)-(4.32) with expansions

$$\boldsymbol{P}_{\varepsilon,\theta}(x) = \Big(0, (1-\theta^2)^{1/2}e^{i\theta x}, 0\Big) + O(\varepsilon^{1/2}), \quad \boldsymbol{Q}_{\varepsilon,\theta}(x) = \Big(0, 0, (1-\theta^2)^{1/2}e^{i\theta x}\Big) + O(\varepsilon^{1/2}) \tag{4.33}$$

(see also Lemma 3.4). Notice that these periodic solutions represent the rotated rolls $\boldsymbol{\mathcal{R}}_\beta\boldsymbol{U}^*_{k,\mu}$ and $\boldsymbol{\mathcal{R}}_{-\beta}\boldsymbol{U}^*_{k,\mu}$. While they are periodic at leading order, the terms $O(\varepsilon^{1/2})$ are quasiperiodic in x due to the presence of the exponentials $e^{ik_c x}$ and $e^{ik_x x}$ in the change of variables (4.22)-(4.23).

An important property of the leading order system obtained by setting $\varepsilon = 0$ in (4.30)-(4.32) is that it leaves the four-dimensional subspace $\{(C_0, C_+, C_-)\ ;\ C_0 = 0\}$ invariant. Restricting to this subspace we recover the leading order system of Section 4.3:

$$C_+'' = \left(-1 + |C_+|^2 + g_3|C_-|^2\right) C_+, \tag{4.34}$$

$$C_-'' = \left(-1 + g_3|C_+|^2 + |C_-|^2\right) C_-. \tag{4.35}$$

The existence of a heteroclinic solution for this system has been proved in [68]. According to [68, Theorem 5], for any $g_3 > 1$, the system (4.34)-(4.35) possesses a heteroclinic solution (C_+^*, C_-^*), with $C_\pm^*$ smooth real-valued functions defined on $\mathbb{R}$, which is $\boldsymbol{S}_1$-reversible and connects the equilibrium $(0,1)$ as $x \to -\infty$ with the equilibrium $(1,0)$ as $x \to \infty$. This heteroclinic solution represents the leading order part of the domain walls constructed in Section 4.3.

The invariance of the subspace $\{(C_0, C_+, C_-)\ ;\ C_0 = 0\}$ implies that the leading order system from (4.30)-(4.32) possesses the $\boldsymbol{S}_1$-reversible heteroclinic solution $(0, C_+^*, C_-^*)$, with $C_\pm^*$ as above, which connects the equilibrium $(0,0,1)$ as $x \to -\infty$ with the equilibrium $(0,1,0)$ as $x \to \infty$. Notice that the equilibria $(0,0,1)$ and $(0,1,0)$ are equal to $\boldsymbol{Q}_{0,0}$

[3] This property of the $O(\varepsilon^{1/2})$-terms is less strong than the one from Section 4.3 but it is enough for our purposes. It is needed when applying the implicit function theorem in the proof of Theorem 3.9.

and $\boldsymbol{P}_{0,0}$ and therefore represent the rotated rolls with wavenumber k_c, $\boldsymbol{\mathcal{R}}_\alpha \boldsymbol{U}^*_{k_c,\mu}$ and $\boldsymbol{\mathcal{R}}_{-\alpha} \boldsymbol{U}^*_{k_c,\mu}$, respectively.

4.4.3 Existence of heteroclinic solutions

The existence of domain walls is obtained by proving that the $\boldsymbol{S}_1$-reversible heteroclinic solution $(0, C_+^*, C_-^*)$ found for $\varepsilon = 0$ persists for $\varepsilon > 0$. More precisely, we have the following result.

Theorem 4.4 *Assume that $g_1 > 1$ and $g_3 \in (1, 4+\sqrt{13})$. Except for at most a finite number of values g_1, for any $\varepsilon > 0$ sufficiently small, there exists $\theta = O(\varepsilon^{1/2})$, continuously depending on $\varepsilon^{1/2}$, such that the system (4.31)-(4.32) possesses a reversible heteroclinic solution $\boldsymbol{C}_\varepsilon = (C_{0,\varepsilon}, C_{+,\varepsilon}, C_{-,\varepsilon})$ connecting the solutions $\boldsymbol{Q}_{\varepsilon,\theta}$, as $x \to -\infty$, to $\boldsymbol{P}_{\varepsilon,\theta}$, as $x \to \infty$.*

The quotients g_1 and g_3 depend on the angle α and the Prandtl number $\mathcal{P}$ through the complicated analytical formulas (4.15) and (4.16) computed in Appendix 6.4.3. Taking $\alpha = 0$ in (4.16) of Appendix 6.4.3 we obtain that the limit as $\alpha \to 0$ of g_3 is equal to 2, just as in Section 4.3. Consequently, the condition $g_3 \in (1, 4+\sqrt{13})$ holds at least for small angles $\alpha \in (0, \alpha_*(\mathcal{P}))$, for some positive $\alpha_*(\mathcal{P})$. For g_1, we have the same property when $\alpha = \pi/2$, but these angles are excluded from our analysis and it seems difficult to check analytically the inequality $g_1 > 1$ for angles $\alpha \in (0, \pi/3)$. Instead, we compute g_3 symbolically, using Maple, and obtain that the inequality $g_3 > 1$ is always satisfied, for any positive Prandtl number $\mathcal{P}$ and any angle $\alpha \in (0, \pi/2)$. By comparing the formulas (4.15) and (4.16) in Appendix 6.4.3 we then obtain that the inequality $g_1 > 1$ always holds, as well. The same Maple computation also allows to determine the values $(\alpha, \mathcal{P})$ for which the second condition on g_3 is satisfied, i.e., $g_3 < 4 + \sqrt{13}$. We summarize these properties in Figure 4.1.

The solutions $\boldsymbol{Q}_{\varepsilon,\theta}$ and $\boldsymbol{P}_{\varepsilon,\theta}$ in Theorem 3.9 representing the rotated rolls $\boldsymbol{\mathcal{R}}_\beta \boldsymbol{U}^*_{k,\mu}$ and $\boldsymbol{\mathcal{R}}_{-\beta} \boldsymbol{U}^*_{k,\mu}$, respectively, the result in Theorem 4.1 is an immediate consequence of Theorem 4.4 and the properties of g_1 and g_3 in Figure 4.1.

The proof of Theorem 4.4 is based on the implicit function theorem applied in the space of $\boldsymbol{S}_1$-reversible exponentially decaying functions,

$$\mathcal{X}_\eta^r = \{(C_0, C_+, C_-, \overline{C_0}, \overline{C_+}, \overline{C_-}) \in \mathcal{X}_\eta \ ; \ C_0(x) = \overline{C_0}(-x), \ C_+(x) = \overline{C_-}(-x), \ x \in \mathbb{R}\}, \tag{4.36}$$

where, for $\eta > 0$,

$$\mathcal{X}_\eta = \left\{(C_0, C_+, C_-, \overline{C_0}, \overline{C_+}, \overline{C_-}) \in (L^2_\eta)^4\right\}, \ L^2_\eta = \left\{ f : \mathbb{R} \to \mathbb{C} \ ; \ \int_{\mathbb{R}} e^{2\eta|x|} |f(x)|^2 < \infty \right\}.$$

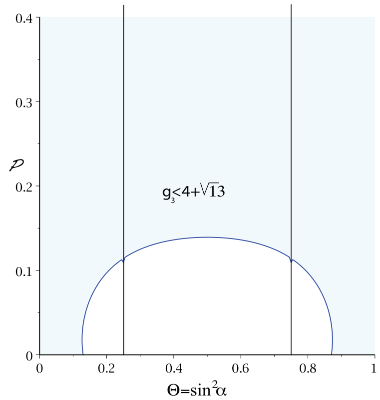

Figure 4.1: In the $(\Theta, \mathcal{P})$-plane, with $\Theta = \sin^2\alpha$, Maple plot of the curve along which $g_3 = 4+\sqrt{13}$, for $\Theta \in (0,1)$. The inequality $g_3 < 4+\sqrt{13}$ holds in the shaded region, whereas the inequalities $g_3 > 1$ and $g_1 > 1$ hold everywhere. Domain walls are constructed in the shaded region situated to the left of the vertical line $\Theta = \sin^2(\pi/3) = 0.75$, except for the values on the vertical line $\Theta = \sin^2(\pi/6) = 0.25$ which correspond to the resonant case $k_c/k_x = 2$, and perhaps a finite number of curves corresponding to the finite number of (unknown) values g_1 in Theorem 4.4.

It turns out that the linear operator needed in the implicit function theorem has exactly the same properties as the one in Lemma 3.8. Therefore, the implicit function theorem can be applied in the same way as done in Theorem 3.9 and we omit these details of proof here. We focus on the properties of this linear operator, which is obtained by linearizing the system (4.30)-(4.32) with $\varepsilon = 0$, together with the complex conjugated equations, at $(0, C_+^*, C_-^*)$, i.e., the linear operator $\mathcal{L}_*$ acting on (C_0, C_+, C_-) through

$$\mathcal{L}_* \begin{pmatrix} C_0 \\ C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} C_0'' - \sin^2\alpha\left(-1 + g_1(C_+^{*2} + C_-^{*2})\right) C_0 \\ C_+'' - \left(-1 + 2C_+^{*2} + g_3 C_-^{*2}\right) C_+ - C_+^{*2}\overline{C_+} - g_3 C_+^* C_-^* (C_- + \overline{C_-}) \\ C_-'' - \left(-1 + g_3 C_+^{*2} + 2C_-^{*2}\right) C_- - C_-^{*2}\overline{C_-} - g_3 C_+^* C_-^* (C_+ + \overline{C_+}) \end{pmatrix}.$$

We prove that this operator has the same properties as in Lemma 3.8, which completes the proof of Theorem 4.4.

Lemma 4.5 *Assume that $g_1 > 1$ and $g_3 \in (1, 4+\sqrt{13})$. Except for at most a finite number of values g_1, for any $\eta > 0$ sufficiently small, the operator $\mathcal{L}_*$ acting in $\mathcal{X}_\eta^r$ is Fredholm with index -1. The kernel of $\mathcal{L}_*$ is trivial and the one-dimensional kernel of its L^2-adjoint is spanned by $(0, \mathrm{i}C_+^*, -\mathrm{i}C_-^*, 0, -\mathrm{i}C_+^*, \mathrm{i}C_-^*)$.*

Proof. The action of the operator $\mathcal{L}_*$ on the C_0-component being decoupled from the action on the (C_+, C_-)-components, we may write

$$\mathcal{L}_* = \begin{pmatrix} \mathcal{L}_0 & 0 \\ 0 & \mathcal{L}_\pm \end{pmatrix},$$

with

$$\mathcal{L}_0 C_0 = C_0'' - \sin^2\alpha \left(-1 + g_1(C_+^{*2} + C_-^{*2})\right) C_0,$$

$$\mathcal{L}_\pm \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} C_+'' - \left(-1 + 2C_+^{*2} + g_3 C_-^{*2}\right) C_+ - C_+^{*2}\overline{C_+} - g_3 C_+^* C_-^* (C_- + \overline{C_-}) \\ C_-'' - \left(-1 + g_3 C_+^{*2} + 2C_-^{*2}\right) C_- - C_-^{*2}\overline{C_-} - g_3 C_+^* C_-^* (C_+ + \overline{C_+}) \end{pmatrix},$$

and we have the complex conjugated actions on the components $\overline{C_0}$ and $(\overline{C_+}, \overline{C_-})$. The operator $\mathcal{L}_\pm$ is precisely the one from Lemma 3.8. For any $g_3 > 1$, it is a Fredholm operator with index -1, has a trivial kernel, and the one-dimensional kernel of its L^2-adjoint is spanned by $(\mathrm{i}C_+^*, -\mathrm{i}C_-^*, -\mathrm{i}C_+^*, \mathrm{i}C_-^*)$. To complete the proof it remains to show that the operator $\mathcal{L}_0$ is invertible.

Taking as new variables

$$y = (\sin\alpha)x, \quad U_0 = \frac{1}{2}(C_0 + \overline{C_0}), \quad V_0 = \frac{1}{2i}(C_0 - \overline{C_0}), \quad \widetilde{C}_\pm^*(y) = C_\pm^*(x),$$

we obtain the matrix operator

$$\mathcal{M}_0 \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} U_0'' + U_0 - g_1(\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2})U_0 \\ V_0'' + V_0 - g_1(\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2})V_0 \end{pmatrix},$$

acting in

$$X_\eta^0 = \left\{(U_0, V_0) \in (L_\eta^2)^2 \; ; \; U_0(y) = U_0(-y), \; V_0(y) = -V_0(-y), \; y \in \mathbb{R}\right\}.$$

The invertibility of $\mathcal{L}_0$ is equivalent to the invertibility of the matrix operator $\mathcal{M}_0$. The action of $\mathcal{M}_0$ on the two components U_0 and V_0 being the same its invertibility in X_η^0 is equivalent to that of the scalar operator

$$L_0[g_1] = \partial_{yy} + 1 - g_1(\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2})$$

acting in L_η^2. We show the invertibility of this operator in L^2, except for at most a finite number of values $g_1 > 1$, which by a standard perturbation argument implies its invertibility L_η^2 for sufficiently small η.

The function $\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2}$ converging towards 1 as $x \to \pm\infty$, the operator $L_0[g_1]$ is a relatively compact perturbation of the asymptotic selfadjoint operator $\partial_{yy} + 1 - g_1$. Consequently, they have the same essential spectrum,

$$\sigma_{ess}(L_0[g_1]) = \sigma_{ess}(\partial_{yy} + 1 - g_1) = \sigma(\partial_{yy} + 1 - g_1) = (-\infty, 1 - g_1].$$

This implies that for any $g_1 > 1$ the operator $L_0[g_1]$ is Fredholm with index 0, so that its invertibility is equivalent to the property that its kernel is trivial.

We claim that if $L_0[g_1^*]$ has a nontrivial kernel for some $g_1^* > 1$, then $L_0[g_1]$ is invertible for any $g_1 = g_1^* + \gamma \neq g_1^*$ with sufficiently small γ. Indeed, consider an orthogonal basis $\{\xi_1^*, \dots, \xi_n^*\}$ of the finite dimensional kernel of $L_0[g_1^*]$, which is the spectral subspace associated with the eigenvalue 0, because 0 is an isolated eigenvalue and the operator is selfadjoint. For sufficiently small γ, the operator $L_0[g_1]$ has at most n eigenvalues close to 0, which are the continuation of the eigenvalue 0 of $L_0[g_1^*]$, and the spectral subspace associated to these eigenvalues has a basis $\{\xi_1(\gamma), \dots, \xi_n(\gamma)\}$ which is the smooth continuation of the basis above. These eigenvalues of $L_0[g_1]$ are the eigenvalues of the $n \times n$-matrix $M[g_1]$ representing the action of $L_0[g_1]$ on the basis $\{\xi_1(\gamma), \dots, \xi_n(\gamma)\}$. A direct computation of this matrix shows that

$$M[\gamma] = M_1\gamma + O(\gamma^2), \quad M_1 = \big(\langle B\xi_i^*, \xi_j^*\rangle\big)_{1\leq i,j\leq n},$$

where

$$B = \frac{d}{dg_1} L_0[g_1]\big|_{g_1 = g_1^*} = -(\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2}).$$

The function $\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2}$ being continuous and positive with limits equal to 1 at $x = \pm\infty$, it is bounded from below by a positive constant c_*. This implies that B is a negative selfadjoint operator, so that the eigenvalues of M_1 are all negative. Consequently, 0 is not an eigenvalue of $M[\gamma]$ for $\gamma \neq 0$ sufficiently small, which implies that $L[g_1]$ is invertible, for g_1 close enough to g_1^* and proves the claim. As a consequence of this property, the set of values $g_1 > 1$ for which the operator $L_0[g_1]$ is not invertible is countable and has no accumulation point in $\mathbb{R}$. In addition, the function $\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2}$ being bounded from below by $c_* > 0$, it is straightforward to check that the operator $L_0[g_1]$ is negative, and in particular invertible, for any $g_1 > 1/c_*$. We conclude that the set of values $g_1 > 1$ for which $L_0[g_1]$ is not invertible is at most a finite set, which completes the proof of the lemma. ∎

Chapter 5

Orthogonal domain walls

This Chapter is based on papers [12], [39], [40], related to orthogonal domain walls. We use the results of Lemma 2.3 in Chapter 3.

5.1 Introduction

In this Chapter, we consider the case where two systems of rolls connect orthogonally. We refer to the works [2, 57, 58], and the references therein, for experimental and previous analytical results, and detailed descriptions of these patterns and defects.

Mathematically, the governing equations are the Navier-Stokes-Boussinesq (N-S-B) equations, completed by boundary conditions at the two plates. Observed patterns are then found as particular steady solutions of these equations. In Chapter 4, as in [29] and [33] we handled the full governing (N-S-B) equations and proved, for various boundary conditions, the existence of symmetric domain walls in convection.

The existence of orthogonal domain walls (effectively observed experimentally) has been studied formally by Manneville and Pomeau in [58]. In [3] and [35], (this is named "planar 90^0 grain boundary separating two stripe domains of mutually perpendicular orientations"), this is completed by the study of the dynamics of these defects, function of the waves numbers of each set of rolls, however only on a Swift-Hohenberg type of model ODE so that these previous works do not start with the Navier-Stokes-Boussinesq system of equations, and just give interesting asymptotic non rigorous results in the mathematical sense.

In this Chapter, as initiated by Buffoni et al [12], we handle the full governing equations, showing that the study leads to a small perturbation of the reduced system of amplitude equations in $\mathbb{R}^6$, the same system as the one predicted in [58]:

$$\begin{aligned} A^{(4)} &= A(1-A^2-gB^2) \\ B'' &= \varepsilon^2 B(-1+gA^2+B^2), \end{aligned} \tag{1.1}$$

where ε^2 is the amplitude of rolls at infinities, and g a number, function of the Prandtl number of the flow. By a variational argument Boris Buffoni et al [12] prove the existence of an heteroclinic orbit, for any $g > 1$, and ε small enough, such that

$$\begin{aligned} A_*(x) &> 0,\ 0 < B_*(x) < 1 \\ (A_*(x), B_*(x)) &\to \begin{cases} M_- = (1,0) \text{ as } x \to -\infty \\ M_+ = (0,1) \text{ as } x \to +\infty \end{cases}. \end{aligned}$$

This orbit is expected to represent the connection between a set of convecting rolls parallel to the x direction, with a set of orthogonal rolls. This type of elegant proof does not allow to prove the persistence of such heteroclinic curve under reversible perturbations of the vector field, such that the one resulting from the full N-S-B system. Our purpose here is to use the analytic results, as done in [39] and [40] for proving the persistence of the above heteroclinic, hence applied to orthogonal domain walls in Bénard-Rayleigh convection. It should be noticed that even though the present analysis looks similar to the one made in Chapter 4, it really needs serious adaptation since, here we loose the symmetry of the wall defect, which plays an important role in Chapter 4. Moreover, contrary to the symmetric case considered in Chapter 4, the size of the perturbation which depends on ε, appears in the rescaled heteroclinic of system (1.1). This introduces lot of computations for controlling higher order terms (see section 5.4). For obtaining steady solutions of N-S-B system, we are led to consider the connection between rolls of different wave numbers; we give the link between them and a modulated "shift" of the system of rolls parallel to the wall, leading to a one parameter set of solutions, for a fixed Rayleigh number slightly above criticality, and a fixed Prandtl number. Contrary to the symmetric case, the wave numbers of rolls at infinities need not be the same.

The first result is Theorem 1.1 with its 2 corollaries, giving the heteroclinic curve of system (1.1) with some useful properties:

Theorem 1.1 *Let us choose $1/3 \leq \delta \leq 1$, where δ is defined by $g = 1 + \delta^2$, then for ε small enough, the 3-dim unstable manifold of M_- intersects transversally the 3-dim stable manifold of M_+, except maybe for a finite set of values of δ. The connecting curve obtained by this intersection, is the unique heteroclinic connecting M_- towards M_+. Moreover its dependency in parameters (ε, δ) is analytic. In addition we have $B(x) > 0$ and $B'(x) > 0$*

on $(-\infty, +\infty)$. *For* $x \to -\infty$ *we have* $(A-1, A', A'', A''', B, B') \to 0$ *at least as* $e^{\varepsilon\delta x}$, *while for* $x \to +\infty$, $(A, A', A'', A''') \to 0$ *at least as* $e^{-\sqrt{\frac{\delta}{2}}x}$, *and* $(B-1, B') \to 0$ *at least as* $e^{-\sqrt{2}\varepsilon x}$.

Corollary 1.2 *For* $x \in (-\infty, 0]$ *and choosing* $\delta^* < \delta$, *there exists* $c > 0$ *independent of* ε *small enough, such that the heteroclinic curve satisfies*

$$\begin{aligned} |A(x) - \sqrt{1-(1+\delta^2)B(x)}| &\leq c\varepsilon^{2/5}B(x)e^{\varepsilon\delta^* x} \\ |A^{(m)}(x)| &\leq c\varepsilon^{3/5}B(x)e^{\varepsilon\delta^* x},\ m = 1, 2, 3. \end{aligned}$$

Corollary 1.3 *For* $x \in [0, +\infty)$ *and* $\delta_* = \frac{1}{10}\delta^{2/5}$, *there exists* $c > 0$ *independent of* ε *small enough, such that the heteroclinic curve satisfies*

$$|A^{(m)}(x)| \leq c\varepsilon^{2/5}e^{-\delta_*\varepsilon^{1/5}x},\ m = 0, 1, 2, 3.$$

We give a sketch of the proof of Theorem 1.1 in Appendix 6.5.1.

The second main result is Theorem 1.4 proving the existence of orthogonal domain walls (see Figure 1.1):

Theorem 1.4 *Except for a finite number of values of* $g = 1 + \delta^2$ *and for* ε *small enough, such that Theorem 1.1 applies, the heteroclinic solution connecting an equilibrium at* $-\infty$ *(representing convective rolls parallel to* x *- axis and symmetric in coordinate* y*) and a periodic solution at* $+\infty$ *(representing convective rolls orthogonal to the previous ones, parallel to the wall), exists as a family of orthogonal domain walls. Denoting by* ε^2 *the amplitude of rolls at infinities, the wave number of rolls orthogonal to the wall (resp. parallel to the wall) being* $k_c(1+\varepsilon^2 k_-)$ *(resp.* $k_c(1+\varepsilon^2 k_+)$*), where* k_c *is the critical wave number, the result is the following:* k_+ *and* k_- *are functions of* ε *and of a parameter* φ, *such that*

$$\begin{aligned} |k_+(\varepsilon, \varphi)| &\leq c\varepsilon^2, \\ k_-(\varepsilon, \varphi) &= \mp\gamma_1\varepsilon^{1+2/5}\exp(-\varphi) + \mathcal{O}(\varepsilon^{1+3/5}),\ \text{with}\ \exp|\varphi| \leq \varepsilon^{-2/5}. \end{aligned}$$

The parameter φ *is linked to the "shift"* z *of rolls parallel to the wall in such a way that*

$$z = \gamma_2\varepsilon^{1+1/5}(\exp\varphi \mp \exp(-\varphi)) + \mathcal{O}(\varepsilon^{1+2/5}),$$

where the numbers γ_1, γ_2 *of order 1, the choice of* $\pm$ *in* z, *and* k_- *and the possibility to obtain* $k_- = k_+$ *only depend on* g *and on the cubic coefficient* $(d_2 - d_4)$ *in the normal form (see (5.7) in Appendix 6.5.2), all being functions of the Prandtl number.*

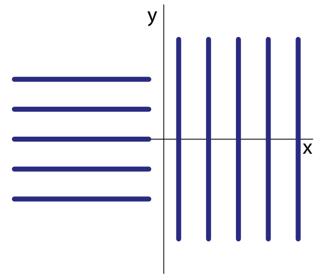

Figure 1.1: Orthogonal domain wall

Remark 1.5 *Numbers* γ_1 *and* γ_2 *are given by*

$$\gamma_1 = 2\sqrt{\frac{2|a_5|}{3}}, \ \gamma_2 = \frac{\varepsilon^{1/5}}{2a_1}\sqrt{\frac{3|a_5|}{2}},$$

where a_1 *and* a_5 *are defined by the following integrals, where* (A_*, B_*) *is the heteroclinic solution given by Theorem 1.1*

$$\begin{aligned} a_1 &= \int_{\mathbb{R}} A_*''^2 dx = \mathcal{O}(\varepsilon^{1/5}), \quad \mp = -sgn(a_5), \\ a_5\varepsilon^{4/5} &= (d_2 - d_4)\int_{\mathbb{R}} A_* A_*'^3 dx. \end{aligned}$$

Moreover, the equality $k_+ = k_- = \mathcal{O}(\varepsilon^2)$ *is possible (for a suitable choice of* φ*) if*

$$d_2 - d_4 < 0.$$

Remark 1.6 *Our method may be used for other physical problem displaying analogue patterns, such as, for example at a fluid-ferro-fluid interface, as studied in the symmetric case ("corner defect") by J.Horn in [34]. More generally, any physical problem leading to a normal form such as (5.2) (see Appendix 6.5.2) introduces the 4 important coefficients* (g, d_2, d_4, c_9) *of the cubic normal form, and should, after validation of the reduction, lead to a Theorem such as Theorem 1.4.*

Remark 1.7 *Values of* δ *such that* $0.476 \leq \delta$ *include values obtained for* δ *in the Bénard-Rayleigh convection problem where* $g = 1 + \delta^2$ *is function of the Prandtl number* $\mathcal{P}$ *(as computed in [29]) and Chapter 4. With rigid-rigid, rigid-free, or free-free boundaries the minimum values of* g *are respectively* $(g_{\min} = 1.227,\ 1.332,\ 1.423)$ *corresponding to* $\delta_{\min} =$ $0.476, 0.576, 0.650$. *The restriction in Theorems 1.1 and 1.4 corresponds to* $1 < g \leq 2$. *Then, the eligible values for the Prandtl number are respectively* $\mathcal{P} > 0.5308, > 0.6222, > 0.8078$.

5.2 Center manifold reduction for orthogonal domain walls

5.2.1 System on the center manifold

We consider the parameter regime with (k, μ) close to (k_c, μ_c), where $\mu_c = \mu_0(k_c)$. We set

$$\mu = \mu_c + \tilde{\mu}, \quad k = k_c(1 + \tilde{k}),$$

in which $\tilde{\mu}$ and $\tilde{k}$ are small parameters. We also eliminate the dependence on k of the phase space $\mathcal{X}$ of the dynamical system by normalizing to $2\pi/k_c$ the period in y of the solutions. The resulting system is of the same form as (1.9) in Chapter 3, in which now $\Delta_\perp = (1 + \tilde{k})^2 \partial_{yy} + \partial_{zz}$, $\nabla_\perp = ((1 + \tilde{k})\partial_y, \partial_z)$, and its phase space is $\mathcal{X}$ with $k = k_c$. We write this system in the form

$$\partial_x \mathbf{U} = \mathcal{L}_c \mathbf{U} + \mathcal{R}(\mathbf{U}, \tilde{\mu}, \tilde{k}), \tag{2.1}$$

where

$$\mathcal{L}_c = \mathcal{L}_{\mu_c}\big|_{\tilde{k}=0}, \quad \mathcal{R}(\mathbf{U}, \tilde{\mu}, \tilde{k}) = (\mathcal{L}_\mu - \mathcal{L}_{\mu_c}\big|_{\tilde{k}=0})\mathbf{U} + \mathcal{B}_\mu(\mathbf{U}, \mathbf{U}), \tag{2.2}$$

and $\mathcal{R}$ is a smooth map from $\mathcal{Z} \times (-\mu_c, \infty) \times \mathbb{R}$ into $\mathcal{X}$ satisfying

$$\mathcal{R}(0, \tilde{\mu}, \tilde{k}) = 0, \quad D_{\mathbf{U}}\mathcal{R}(0, 0, 0) = 0. \tag{2.3}$$

We apply a center manifold theorem to obtain a reduced system of ordinary differential equations which describes the dynamics of (2.1) in a neighborhood of the equilibrium $\mathbf{U} = 0$ for small $(\tilde{\mu}, \tilde{k})$. The arguments are the same as the ones in Chapter 4, except for the purely imaginary eigenvalues of the linear operator $\mathcal{L}_c$ which are different. The following result is obtained in Lemma 2.3 of Chapter 3.

Lemma 2.1 *The center spectrum of the linear operator $\mathcal{L}_c$ consists of the three eigenvalues $0, \pm ik_c$ with the following properties.*

(i) The eigenvalue 0 has algebraic multiplicity 9 and geometric multiplicity 3, and the complex conjugated eigenvalues $\pm ik_c$ are algebraically double and geometrically simple.

(ii) For the eigenvalue 0, there are three linearly independent eigenvectors: $\boldsymbol{\varphi}_0$ given by (13) in Chapter 3, $\boldsymbol{\zeta}_0$ of the form $\boldsymbol{\zeta}_0(y,z) = \widehat{U}_{k_c}(z)e^{ik_c y}$, with $\widehat{U}_{k_c}(z) \in \mathbb{C}^8$, and the complex conjugated vector $\overline{\boldsymbol{\zeta}}_0$, and two chains of generalized eigenvectors: $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3$ associated to $\boldsymbol{\zeta}_0$,

$$\mathcal{L}_c\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_0, \quad \mathcal{L}_c\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_1, \quad \mathcal{L}_c\boldsymbol{\zeta}_3 = \boldsymbol{\zeta}_2,$$

and the conjugated vectors $\overline{\boldsymbol{\zeta}_1}, \overline{\boldsymbol{\zeta}_2}, \overline{\boldsymbol{\zeta}_3}$ *associated to* $\overline{\boldsymbol{\zeta}_0}$. *The eigenvector* $\boldsymbol{\varphi}_0$ *is invariant under the actions of* $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, *and* $\boldsymbol{\tau}_a$, *and the other generalized eigenvectors satisfy:*

$$\begin{aligned}
&\boldsymbol{S}_1\boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_0 = \overline{\boldsymbol{\zeta}_0}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_0 = e^{ia}\boldsymbol{\zeta}_0,\\
&\boldsymbol{S}_1\boldsymbol{\zeta}_1 = -\boldsymbol{\zeta}_1, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_1 = \overline{\boldsymbol{\zeta}_1}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_1 = e^{ia}\boldsymbol{\zeta}_1,\\
&\boldsymbol{S}_1\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_2, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_2 = \overline{\boldsymbol{\zeta}_2}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_2 = e^{ia}\boldsymbol{\zeta}_2,\\
&\boldsymbol{S}_1\boldsymbol{\zeta}_3 = -\boldsymbol{\zeta}_3, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_3 = \overline{\boldsymbol{\zeta}_3}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_3 = e^{ia}\boldsymbol{\zeta}_3.
\end{aligned}$$

(iii) For the eigenvalue ik_c, *there is one eigenvector* $\boldsymbol{\xi}_0$ *of the form* $\boldsymbol{\xi}_0(y,z) = \widehat{\mathbf{U}}_0(z) \in \mathbb{C}^8$, *and an associated generalized eigenvector* $\boldsymbol{\xi}_1$ *with the properties*

$$(\mathcal{L}_c - ik_c)\boldsymbol{\xi}_1 = \boldsymbol{\xi}_0,$$

and

$$\begin{aligned}
&\boldsymbol{S}_1\boldsymbol{\xi}_0 = \overline{\boldsymbol{\xi}_0}, \quad \boldsymbol{S}_2\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0, \quad \boldsymbol{\tau}_a\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0,\\
&\boldsymbol{S}_1\boldsymbol{\xi}_1 = -\overline{\boldsymbol{\xi}_1}, \quad \boldsymbol{S}_2\boldsymbol{\xi}_1 = \boldsymbol{\xi}_1, \quad \boldsymbol{\tau}_a\boldsymbol{\xi}_1 = \boldsymbol{\xi}_1.
\end{aligned}$$

The complex conjugated vectors $\overline{\boldsymbol{\xi}_0}$ *and* $\overline{\boldsymbol{\xi}_1}$ *are eigenvector and generalized eigenvector, respectively, for the eigenvalue* $-ik_c$.

As a result of the center manifold theorem, we obtain that the small bounded solutions of the infinite-dimensional dynamical system (2.1) belong to a 13-dimensional center manifold, for any sufficiently small $\tilde{\mu}$ and $\tilde{k}$, and are of the form

$$\begin{aligned}
\boldsymbol{U} \;=\; & w\boldsymbol{\phi}_0 + A_0\boldsymbol{\zeta}_0 + A_1\boldsymbol{\zeta}_1 + A_2\boldsymbol{\zeta}_2 + A_3\boldsymbol{\zeta}_3 + B_0\boldsymbol{\xi}_0 + B_1\boldsymbol{\xi}_1\\
& + \overline{A_0\boldsymbol{\zeta}_0} + \overline{A_1\boldsymbol{\zeta}_1} + \overline{A_2\boldsymbol{\zeta}_2} + \overline{A_3\boldsymbol{\zeta}_3} + \overline{B_0\boldsymbol{\xi}_0} + \overline{B_1\boldsymbol{\xi}_1} + \boldsymbol{\Phi}(X, \overline{X}, \widetilde{\mu}, \widetilde{k}),
\end{aligned}$$

in which w and $X = (A_0, A_1, A_2, A_3, B_0, B_1)$ are x-dependent functions with values in $\mathbb{R}$ and $\mathbb{C}^6$, respectively, and Φ is of class C^m in its arguments, for any fixed $m \geq 1$. The eigenvectors and generalized eigenvalues being complex-valued, it is convenient to use here complex variables $(X, \overline{X})$, instead of 12 real variables, hence by identifying $\mathbb{R}^{12}$ with the space $\mathbb{C}^6 \times \overline{\mathbb{C}^6} = \{(Z, \overline{Z})\ ;\ Z \in \mathbb{C}^6\}$.

The reduced 13-dimensional system for w, X, and $\overline{X}$ inherits the properties of the infinite-dimensional dynamical system (2.1). In particular, as in Chapter 4, the invariance of (2.1) under the action of $\boldsymbol{T}_b$, implies that the reduced vector field is invariant under the action of the induced transformation $w \mapsto w + b$, for any $b \in \mathbb{R}$, and therefore does not depend on w. Consequently, the equations for w and $(X, \overline{X})$ in the reduced system are decoupled,

$$\frac{dw}{dx} = h(X, \overline{X}, \widetilde{\mu}, \widetilde{k}),$$

and

$$\frac{dX}{dx} = F(X, \overline{X}, \widetilde{\mu}, \widetilde{k}), \quad \frac{d\overline{X}}{dx} = \overline{F(X, \overline{X}, \widetilde{\mu}, \widetilde{k})}. \tag{2.4}$$

Taking into account the properties (2.2)-(2.3) and the result in Lemma 2.1 we obtain that

$$F(0, 0, \tilde{\mu}, \tilde{k}) = 0, \quad D_X F(0, 0, 0, 0) = L_c, \quad D_{\overline{X}} F(0, 0, 0, 0) = 0, \tag{2.5}$$

where L_c is the 6×6 Jordan matrix

$$L_c = \begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} ik_c & 1 \\ 0 & ik_c \end{pmatrix}. \tag{2.6}$$

In addition, from the symmetry properties of the eigenvectors and generalized eigenvectors in Lemma 2.1, we deduce their actions on the variable X,

$$\boldsymbol{S}_1(A_0, A_1, A_2, A_3, B_0, B_1) = (A_0, -A_1, A_2, -A_3, \overline{B_0}, -\overline{B_1}), \tag{2.7}$$

$$\boldsymbol{S}_2(A_0, A_1, A_2, A_3, B_0, B_1) = (\overline{A_0}, \overline{A_1}, \overline{A_2}, \overline{A_3}, B_0, B_1), \tag{2.8}$$

$$\boldsymbol{\tau}_a(A_0, A_1, A_2, A_3, B_0, B_1) = (e^{ia}A_0, e^{ia}A_1, e^{ia}A_2, e^{ia}A_3, B_0, B_1). \tag{2.9}$$

Then, the vector field in the reduced system (2.4) anti-commutes with $\boldsymbol{S}_1$ and commutes with $\boldsymbol{S}_2$ and $\boldsymbol{\tau}_a$. Notice that the equivariance under the action of $\boldsymbol{S}_2$ implies that the reduced system leaves invariant the 8-dimensional subspace $\{(X, \overline{X})\ ;\ A_j = \overline{A_j},\ j = 0, 1, 2, 3\}$. Solutions in this subspace correspond to solutions of (2.1) which are even in y. There is a second invariant subspace $\{(X, \overline{X})\ ;\ A_j = 0,\ j = 0, 1, 2, 3\}$, which corresponds to solutions of (2.1) which do not depend on y.

After some calculations and rescaling (see (5.6) in Appendix 6.5.2) the perturbed system becomes

$$\begin{aligned} A_0^{(4)} &= k_- A_0'' + A_0(1 - \frac{k_-^2}{4} - A_0^2 - g|B_0|^2) + \widehat{f}, \\ B_0'' &= \varepsilon^2 B_0(-1 + gA_0^2 + |B_0|^2) + \widehat{g}. \end{aligned} \tag{2.10}$$

Parameters are defined as (see Appendix 6.5.2)

$$\begin{aligned} \varepsilon^4 \quad \sim \quad & \tilde{\mu} = \mathcal{R}^{1/2} - \mathcal{R}_c^{1/2},\ \mathcal{R} \text{ Rayleigh number}, \\ & k_c(1 + \varepsilon^2 k_-) \text{ wave number in } y \text{ direction}, \end{aligned}$$

Remark 2.2 *Notice that the system (2.10) becomes just system (1.1) for* $k_- = \widehat{f} = \widehat{g} = 0$, *and* B_0 *real.*

In (2.10) we have

$$\begin{aligned}\widehat{f}(k_-,\varepsilon,\exp(\pm i\frac{x}{2\varepsilon}),X,Y,\overline{Y}) &= \widehat{f_0}+\widehat{f_1}\\ \widehat{g}(k_-,\varepsilon,\exp(\pm i\frac{x}{2\varepsilon}),X,Y,\overline{Y}) &= \widehat{g_0}+\widehat{g_1},\end{aligned}$$

where we rename X, Y as

$$\begin{aligned}X &= (A_0,A_0',A_0'',A_0''')^t\in\mathbb{R}^4,\\ Y &= (B_0,B_0')^t\in\mathbb{C}^2.\end{aligned}$$

The dependency in $\exp(\pm i\frac{x}{2\varepsilon})$ of $\widehat{f}$ and $\widehat{g}$ comes from terms not in normal form, of degree at least 5 in (X, Y) and the rescaling of the original amplitude B of the rolls parallel to the wall. In fact (see Appendix 6.5.2) B is rescaled as $\varepsilon^2 B_0 e^{ix/2\varepsilon}$, where x is the rescaled coordinate. "Cubic" terms $\widehat{f_0},\widehat{g_0}$, are autonomous, and given by (5.7) and (5.8) of this Appendix where oefficients c_j, d_j are real (due to symmetries as seen in Appendix 6.5.2). Higher order terms, not in normal form are non autonomous and such that

$$\begin{aligned}\widehat{f_1} &= \varepsilon^4\mathcal{O}[|X|(|X|^2+|Y|^2+\varepsilon^4)^2],\\ \widehat{g_1} &= \varepsilon^6\mathcal{O}[(|X|^2+|Y|)(|X|^2+|Y|^2+\varepsilon^4)^2].\end{aligned}$$

Moreover the system (2.10) commutes with the reversibility symmetry S_1 :

$$(x,A_0,A_0',A_0'',A_0''',B_0,B_0')\mapsto(-x,A_0,-A_0',A_0'',-A_0''',\overline{B_0},-\overline{B_0'}),$$

and we have the additional symmetry property resulting from the equivariance of the original system under the shift by half of a wave length in the y direction (fixing the symmetry $y\mapsto -y$):

$$\begin{aligned}&\text{r.h.s. of } A_0^{(4)} \text{ is odd in } X,\\ &\text{r.h.s. of } B_0'' \text{ is even in } X.\end{aligned}$$

The estimates for non normal form terms $\widehat{f_1}$ and $\widehat{g_1}$, result from the property that they start at order 5, since the normal form does not contain terms of degree 4 in (X, Y), and from the inequality

$$(a+b)^4\le 4(a^2+b^2)^2 \text{ for } a,b\in\mathbb{R}.$$

Remark 2.3 *Notice that the above reduction is valid for the three classical boundary conditions for the Bénard-Rayleigh convection problem: rigid-rigid, free-free, free-rigid. However in the case of rigid-rigid or free-free boundary conditions, $Y = 0$ is an invariant subspace (see Remark 5.5 in Appendix 6.5.2), which simplifies the estimate for $\widehat{g_1}$.*

Remark 2.4 *Notice also that the high order terms $\widehat{f_1}$ and $\widehat{g_1}$, of size $\mathcal{O}(\varepsilon^4)$ for $A_0^{(4)}$ and $\mathcal{O}(\varepsilon^6)$ for B_0'' are functions of $e^{\pm i\frac{x}{2\varepsilon}}$. This is due to the fact that $\varepsilon^2 B_0 e^{i\frac{x}{2\varepsilon}}$ is the original amplitude of the Y mode (see (5.4) in Appendix 6.5.2).*

5.3 Setting of the perturbed system

5.3.1 Solutions at infinities

Since we leave now some freedom to the wave numbers, as well in the y direction, as in the x direction, the "end points" of the expected heteroclinic are no longer $(1,0)$ at $-\infty$, and the circle $(0,e^{i\phi})$ at $+\infty$. In fact the classical study of steady convective rolls, shows that these should be respectively $(A_0^{(-\infty)}(k_-), B_0^{(-\infty)}(k_-))$ and $(0, B_0^{(+\infty)}(\omega,x))$. From Appendix 6.5.3 for the equilibrium at $-\infty$, we have

$$\begin{aligned}
(A_0^{(-\infty)})^2 &= 1-\frac{k_-^2}{4}+\sigma_0\varepsilon^2k_- + \mathcal{O}(\varepsilon^2|k_-|^3+\varepsilon^4),\\
1-(A_0^{(-\infty)}) \stackrel{def}{=} -\frac{\widetilde{\omega}_-^2}{2},\ \text{with } \widetilde{\omega}_-^2 &= \frac{k_-^2}{4}-\sigma_0\varepsilon^2k_- + \mathcal{O}[k_-^4+\varepsilon^2|k_-|^3+\varepsilon^4],\\
B_0^{(-\infty)} &= \mathcal{O}(\varepsilon^6).
\end{aligned}$$

From Appendix 6.5.4 for the periodic solutions at $+\infty$, we have

$$\begin{aligned}
e^{i\frac{x}{2\varepsilon}}B_0^{(+\infty)}(\omega,x) &= r_0e^{i\omega x}+\mathcal{O}(\varepsilon^6),\ A_0^{(+\infty)}=0,\\
\omega \stackrel{def}{=} \frac{1}{2\varepsilon}+\varepsilon\widetilde{\omega}_+ &= \frac{1+\varepsilon^2k_+}{2\varepsilon}+\mathcal{O}(\varepsilon^7),
\end{aligned}$$

$$B_0^{(+\infty)}e^{-i\varepsilon\widetilde{\omega}_+x} = C_0^{(+\infty)}+iD_0^{(+\infty)}$$

$$\begin{aligned}
r_0^2 &= 1-\frac{k_+^2}{4}+\mathcal{O}(\varepsilon^2|k_+|+\varepsilon^4) = 1-\mathcal{O}[(|\widetilde{\omega}_+|+\varepsilon^2)^2],\\
C_0^{(+\infty)} &= r_0+\mathcal{O}(\varepsilon^6),\ \text{oscil. part}(C_0^{(+\infty)})=\mathcal{O}(\varepsilon^6),\\
D_0^{(+\infty)} &= \mathcal{O}(\varepsilon^6).
\end{aligned}$$

Remark 3.1 *The coefficient σ_0 introduced in the expression of $(A_0^{(-\infty)})^2$ depends on the Prandtl number.*

Remark 3.2 *We may notice that in case the system has the symmetry S_0 representing $z\mapsto 1-z$ (OK for rigid-rigid, or free-free boundary conditions), then $B_0^{(-\infty)}=0$, which simplifies computations (see Appendix 6.5.3).*

5.3.2 First change of variable

Let us set

$$B_0e^{-i\varepsilon\widetilde{\omega}_+x} = C_0+iD_0,$$

then (2.10) becomes

$$A_0^{(4)} = k_- A_0'' + A_0[1 - \frac{k_-^2}{4} - A_0^2 - g(C_0^2 + D_0^2)] + f \tag{3.1}$$

$$\begin{aligned} C_0'' &= 2\varepsilon\widetilde{\omega}_+ D_0' + \varepsilon^2 C_0(-1 + \widetilde{\omega}_+^2 + gA_0^2 + C_0^2 + D_0^2) + g_r \\ D_0'' &= -2\varepsilon\widetilde{\omega}_+ C_0' + \varepsilon^2 D_0(-1 + \widetilde{\omega}_+^2 + gA_0^2 + C_0^2 + D_0^2) + g_i \end{aligned} \tag{3.2}$$

with

$$f = \widehat{f}, \quad g_r + ig_i = \widehat{g}e^{-i\varepsilon\widetilde{\omega}_+ x},$$

and where the exponential factor disappears in the cubic part when we replace B_0 by $(C_0 + iD_0)e^{i\varepsilon\widetilde{\omega}_+ x}$. Let us define

$$\begin{aligned} f &= f_0(\varepsilon, k_-, X, Y, \overline{Y}) + f_1(\omega x, \varepsilon, k_-, X, Y, \overline{Y}) \\ g_r &= g_{r0}(\varepsilon, X, Y, \overline{Y}) + g_{r1}(\omega x, \varepsilon, k_-, X, Y, \overline{Y}) \\ g_i &= g_{i0}(\varepsilon, X, Y, \overline{Y}) + g_{i1}(\omega x, \varepsilon, k_-, X, Y, \overline{Y}), \end{aligned}$$

where f_0, g_{r0}, g_{i0} come only from cubic terms of the normal form in (2.10), and where f_1, g_{r1}, g_{i1} are $2\pi-$periodic in ωx, smooth in their arguments, and satisfy estimates

$$\begin{aligned} |f_1(\omega x, \varepsilon, k_-, X, Y, \overline{Y})| &\leq c\varepsilon^4 |X|(|X|^2 + |Y|^2)^2 \\ |g_{r1}(\omega x, \varepsilon, k_-, X, Y, \overline{Y})| + |g_{i1}(\omega x, \varepsilon, k_-, X, Y, \overline{Y})| &\leq c\varepsilon^6(|X|^2 + |Y|)(|X|^2 + |Y|^2)^2, \end{aligned}$$

with

$$\begin{aligned} X &= (A_0, A_0', A_0'', A_0''') \\ Y &= (C_0 + iD_0, C_0' + iD_0'). \end{aligned}$$

Then we have from (5.7), (5.8) of Appendix 6.5.2 :

$$\begin{aligned} f_0 = {} & d_1\varepsilon A_0(C_0D_0' - D_0C_0') + \sigma_0\varepsilon^2 k_- A_0^3 + d_2\varepsilon^2 A_0 A_0'^2 + d_3\varepsilon^2 A_0'' \\ & + d_4\varepsilon^2 A_0^2 A_0'' + d_5\varepsilon^2 A_0''(C_0^2 + D_0^2) + d_6\varepsilon^2 A_0(C_0'^2 + D_0'^2) + \\ & + d_7\varepsilon^2 A_0'(C_0C_0' + D_0D_0') + d_8\varepsilon^3 A_0''(C_0D_0' - D_0C_0') + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{3.3}$$

$$\begin{aligned} g_{r0} + ig_{i0} = {} & i\varepsilon^3(C_0' + iD_0')[c_0 + c_1A_0^2 + c_2(C_0^2 + D_0^2)] \\ & + \varepsilon^3 c_3(C_0 + iD_0)(C_0D_0' - D_0C_0') + i\varepsilon^3 c_9(C_0 + iD_0)A_0A_0' \\ & + \varepsilon^4 c_4(C_0' + iD_0')(C_0D_0' - D_0C_0') + c_5\varepsilon^4 A_0A_0''(C_0 + iD_0) \\ & + \varepsilon^4[c_6A_0'^2(C_0 + iD_0) + c_7A_0A_0'(C_0' + iD_0')] \\ & + i\varepsilon^5(C_0' + iD_0')(c_7A_0A_0'' + c_{10}A_0'^2) \\ & + i\varepsilon^5(C_0 + iD_0)(c_8A_0A_0''' + c_{11}A_0'A_0'') + \mathcal{O}(\varepsilon^6). \end{aligned} \tag{3.4}$$

Now, let us set a first change of variables

$$
\begin{aligned}
A_0 &= A_* + \widetilde{A_0} \\
C_0 &= B_* + \widetilde{C_0} \\
D_0 &= \widetilde{D_0}
\end{aligned}
\tag{3.5}
$$

where (A_*, B_*) is the heteroclinic solution found at Theorem 1.1, and where we observe that we expect

$$
\begin{aligned}
\widetilde{A_0} \underset{x=-\infty}{\rightarrow} A_0^{(-\infty)} - 1 &= -\frac{\widetilde{\omega}_-^2}{2}, \\
C_0 + iD_0 \underset{x=-\infty}{\rightarrow} C_0^{(-\infty)} &= B_0^{(-\infty)} = \mathcal{O}(\varepsilon^6), \\
\widetilde{C_0} + i\widetilde{D_0} \underset{x=+\infty}{\rightarrow} C_0^{(+\infty)} + iD_0^{(+\infty)} - 1 &\sim -\frac{(\widetilde{\omega}_+ + \mathcal{O}(\varepsilon^2))^2}{2}.
\end{aligned}
$$

Then (3.1,3.2) becomes the "perturbed system"

$$
\mathcal{M}_g(\widetilde{A_0}, \widetilde{C_0}) = \begin{pmatrix} -k_-(A_*'' + \widetilde{A_0}'') + \frac{k_-^2}{4}(A_* + \widetilde{A_0}) + \widetilde{\phi_0} \\ \frac{2\widetilde{\omega}_+}{\varepsilon}\widetilde{D_0}' + \widetilde{\omega}_+^2(B_* + \widetilde{C_0}) + \widetilde{\psi_{0r}} \end{pmatrix}, \tag{3.6}
$$

$$
\mathcal{L}_g\widetilde{D_0} = -\frac{2\widetilde{\omega}_+}{\varepsilon}(B_*' + \widetilde{C_0}') + \widetilde{\omega}_+^2\widetilde{D_0} + \widetilde{\psi_{0i}}, \tag{3.7}
$$

where linear operators $\mathcal{M}_g$ and $\mathcal{L}_g$ are defined as

$$
\mathcal{M}_g\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} + (1 - 3A_*^2 - gB_*^2)A - 2gA_*B_*C \\ \frac{1}{\varepsilon^2}C'' + (1 - gA_*^2 - 3B_*^2)C - 2gA_*B_*A \end{pmatrix}, \tag{3.8}
$$

$$
\mathcal{L}_gD = \frac{1}{\varepsilon^2}D'' + (1 - gA_*^2 - B_*^2)D, \tag{3.9}
$$

and where $\widetilde{\phi_0}, \widetilde{\psi_{0r}}, \widetilde{\psi_{0i}}$ are smooth functions of $(\omega x, \varepsilon, k_-, \widetilde{\omega}_+, \widetilde{X}, \widetilde{Y})$ where

$$
\begin{aligned}
\widetilde{X} &= (\widetilde{A_0}, \widetilde{A_0}', \widetilde{A_0}'', \widetilde{A_0}''') \\
\widetilde{Y} &= (\widetilde{C_0}, \widetilde{D_0}, \widetilde{C_0}', \widetilde{D_0}')
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\phi_0} &= \widetilde{\phi_{00}}(\varepsilon, k_-, \widetilde{X}, \widetilde{Y}) + \widetilde{\phi_{01}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y}) \\
\widetilde{\psi_{0r}} &= \widetilde{\psi_{0r0}}(\varepsilon, k_-, \widetilde{X}, \widetilde{Y}) + \widetilde{\psi_{0r1}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y}) \\
\widetilde{\psi_{0i}} &= \widetilde{\psi_{0i0}}(\varepsilon, k_-, \widetilde{X}, \widetilde{Y}) + \widetilde{\psi_{0i1}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y})
\end{aligned}
$$

$$
\begin{aligned}
|\widetilde{\phi_{01}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y})| &\leq c\varepsilon^4 \\
|\widetilde{\psi_{0r1}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y})| + |\widetilde{\psi_{0i1}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y})| &\leq c\varepsilon^4.
\end{aligned}
\tag{3.10}
$$

More precisely, we have

$$\widetilde{\phi_{00}}(\varepsilon,k_-,\widetilde{X},\widetilde{Y}) = 3A_*\widetilde{A_0}^2+\widetilde{A_0}^3+2gB_*\widetilde{A_0}\widetilde{C_0} + g(A_*+\widetilde{A_0})(\widetilde{C_0}^2+\widetilde{D_0}^2)+f_{00}, \tag{3.11}$$

$$\widetilde{\psi_{0r0}}(\varepsilon,k_-,\widetilde{X},\widetilde{Y}) = 2gA_*\widetilde{A_0}\widetilde{C_0}+gB_*\widetilde{A_0}^2+2B_*\widetilde{C_0}^2+g\widetilde{A_0}^2\widetilde{C_0} + (B_*+\widetilde{C_0})(\widetilde{C_0}^2+\widetilde{D_0}^2)+g_{00r}, \tag{3.12}$$

$$\widetilde{\psi_{0i0}}(\varepsilon,k_-,\widetilde{X},\widetilde{Y}) = 2gA_*\widetilde{A_0}\widetilde{D_0}+2B_*\widetilde{C_0}\widetilde{D_0}+g\widetilde{A_0}^2\widetilde{D_0} + \widetilde{D_0}(\widetilde{C_0}^2+\widetilde{D_0}^2)+g_{00i}, \tag{3.13}$$

and in using Theorem 1.1, Corollaries 1.2 and 1.3, and assuming

$$|\widetilde{X}|\leq 1, |\widetilde{Y}|\leq 1, |\widetilde{D_0}|\leq\varepsilon, \tag{3.14}$$

$$\begin{aligned}
f_{00} &= \sigma_0\varepsilon^2k_-A_*^3+\mathcal{O}[\varepsilon^{2+3/5}e^{\varepsilon\delta_*x}\chi_{(-\infty,0)}+\varepsilon^{2+2/5}e^{-\varepsilon^{1/5}\delta_*x}\chi_{(0,\infty)}+\varepsilon^2(|\widetilde{X}|+|\widetilde{Y}|)+\varepsilon|\widetilde{D_0}'|],\\
g_{00r} &= \mathcal{O}[\varepsilon^{2+3/5}e^{\varepsilon\delta_*x}\chi_{(-\infty,0)}+\varepsilon^{2+4/5}e^{-\varepsilon^{1/5}\delta_*x}\chi_{(0,\infty)}+\varepsilon^2(|\widetilde{X}|+|\widetilde{Y}|)+\varepsilon|\widetilde{D_0}'|],\\
g_{00i} &= \mathcal{O}[\varepsilon^{1+3/5}e^{\varepsilon\delta_*x}\chi_{(-\infty,0)}+\varepsilon^{1+4/5}e^{-\varepsilon^{1/5}\delta_*x}\chi_{(0,\infty)}+\varepsilon(|\widetilde{X}|+|\widetilde{Y}|)+\varepsilon(|\widetilde{C_0}'|],
\end{aligned}$$

where f_{00} and $g_{00r}+ig_{00i}$ are smooth functions which come from the rest of the cubic normal form written in (3.3,3.4)) and $\chi_{(-\infty,0)}$ and $\chi_{(0,\infty)}$ are the characteristic functions on the corresponding intervals.

Remark 3.3 *We notice that the estimates for the main terms independent of $\widetilde{X},\widetilde{Y}$ come from*

$$\begin{aligned}
\textit{for } f_{00} &: \quad \sigma_0\varepsilon^2k_-A_*^3+d_2\varepsilon^2A_*A_*'^2+d_3\varepsilon^2A_*''+d_4\varepsilon^2A_*^2A_*''+d_5\varepsilon^2A_*''B_*^2,\\
\textit{for } g_{00r} &: \quad c_5\varepsilon^2A_*A_*''B_*+c_6\varepsilon^2A_*'^2B_*+c_7\varepsilon^2A_*A_*'B_*'\\
\textit{for } g_{00i} &: \quad \varepsilon B_*'(c_0+c_1A_*^2+c_2B_*^2)+\varepsilon c_9B_*A_*A_*'.
\end{aligned}$$

Moreover, notice that, below, we need to compute $\int f_{00}A_*'dx$, $\int g_{00r}B_*'dx$, $\int g_{00i}B_*dx$, which, for terms independent of $\widetilde{X},\widetilde{Y}$ leads to

$$\begin{aligned}
\textit{for } \int f_{00}A_*'dx &= -\frac{\sigma_0\varepsilon^2k_-}{4}+\varepsilon^2\int(d_2A_*A_*'^3+d_4A_*^2A_*'A_*'')dx+\mathcal{O}(\varepsilon^3)\\
&= -\frac{\sigma_0\varepsilon^2k_-}{4}+\mathcal{O}(\varepsilon^{2+4/5}),\\
\textit{for } \int g_{00r}B_*'dx &\sim \varepsilon^2\int_{\mathbb{R}}c_5A_*A_*''B_*B_*'dx+\varepsilon^2\int_{\mathbb{R}}c_6A_*'^2B_*B_*'dx=\mathcal{O}(\varepsilon^{3+1/5}),\\
\textit{for } \int g_{00i}B_*dx &= \varepsilon(\frac{c_0}{2}+\frac{c_2}{4})+\varepsilon(c_1-c_9)\int_{\mathbb{R}}A_*^2B_*B_*'=\mathcal{O}(\varepsilon),
\end{aligned}$$

where we notice

$$\begin{aligned}\int A_*''A_*'dx &= 0,\ \varepsilon^2\int A_*''B_*^2A_*'dx = -\varepsilon^2\int A_*'^2B_*B_*'dx = \mathcal{O}(\varepsilon^3),\\ \int(d_2A_*A_*'^3+d_4A_*^2A_*'A_*'')dx &= (d_2-d_4)\int A_*A_*'^3dx = \mathcal{O}(\varepsilon^{1/2}),\\ \int_{\mathbb{R}}A_*A_*''B_*B_*'dx &= -\int_{\mathbb{R}}[A_*'^2B_*B_*'+A_*A_*'(B_*B_*')']dx,\end{aligned}$$

taking care of the convergence in $e^{\varepsilon\delta_*x}$ *(resp* $e^{-\varepsilon^{1/5}\delta_*x}$*) at* $-\infty$ *(resp at* $+\infty$*), which implies a division by* ε *in the integral on* $(-\infty,0)$ *(resp. by* $\varepsilon^{1/5}$ *in the integral on* $(0,+\infty)$*).*

5.3.3 Second change of variables

Before solving the system we need to change variables so that the variables and the right hand side of (3.6,3.7) tend towards 0 at infinity. Let us denote

$$\begin{aligned}\widetilde{X}^{(-\infty)} &= (A_0^{(-\infty)}-1,0,0,0) = (\mathcal{O}(\widetilde{\omega}_-^2),0,0,0)\\ \widetilde{Y}^{(-\infty)} &= (C_0^{(-\infty)},0,0,0) = (\mathcal{O}(\varepsilon^6),0,0,0),\\ \widetilde{X}^{(+\infty)} &= 0\\ \widetilde{Y}^{(+\infty)} &= (C_0^{(+\infty)}-1,D_0^{(+\infty)},C_0^{(+\infty)\prime},D_0^{(+\infty)\prime}) = [\mathcal{O}((\widetilde{\omega}_++\varepsilon^2)^2),\mathcal{O}(\varepsilon^6),\mathcal{O}(\varepsilon^5),\mathcal{O}(\varepsilon^5)],\end{aligned}$$

then, taking care in (3.1,3.2), of the forms of f, g_r, g_i, we notice that the limit terms in the right hand side of (3.6,3.7) as $x\to-\infty$ are

$$\begin{aligned}&\frac{k_-^2}{4}A_0^{(-\infty)}+\widetilde{\phi_0}(\omega x,\varepsilon,k_-,\widetilde{X}^{(-\infty)},\widetilde{Y}^{(-\infty)}) \text{ exp limit as } e^{\varepsilon\delta_*x} \text{ (see } f_{00}),\\ &\widetilde{\omega}_+^2C_0^{(-\infty)}+\widetilde{\psi_{0r}}(\omega x,\varepsilon,k_-,\widetilde{X}^{(-\infty)},\widetilde{Y}^{(-\infty)}) \text{ exp limit as } e^{\varepsilon\delta_*x} \text{ (see } g_{00r})\\ &\widetilde{\psi_{0i}}(\omega x,\varepsilon,k_-,\widetilde{X}^{(-\infty)},\widetilde{Y}^{(-\infty)}) \text{ exp limit as } e^{\varepsilon\delta_*x} \text{ (as } B_*' \text{ and see } g_{00i}).\end{aligned}$$

The limit terms of the right hand side of (3.6,3.7) as $x\to+\infty$ is

$$\begin{aligned}&0 \text{ exp limit as } e^{-\varepsilon^{1/5}\delta_*x} \text{ (as } A_*)\\ &\frac{2\widetilde{\omega}_+}{\varepsilon}(D_0^{(+\infty)})'+\widetilde{\omega}_+^2C_0^{(+\infty)}+\widetilde{\psi_{0r}}(\omega x,\varepsilon,k_-,0,\widetilde{Y}^{(+\infty)}) \text{ exp limit as } e^{-\varepsilon^{1/5}\delta_*x} \text{ (see } g_{00r}),\\ &-\frac{2\widetilde{\omega}_+}{\varepsilon}(C_0^{(+\infty)})'+\widetilde{\omega}_+^2D_0^{(+\infty)}+\widetilde{\psi_{0i}}(\omega x,\varepsilon,k_-,0,\widetilde{Y}^{(+\infty)}) \text{ exp limit as } e^{-\varepsilon\sqrt{2}x} \text{ (see } g_{00i}).\end{aligned}$$

Let us make a second change of variables as

$$\begin{aligned}\widetilde{A_0} &= \alpha_-\chi_-+\widehat{A_0}\\ \widetilde{C_0} &= \beta_-\chi_-+\beta_+\chi_++\widehat{C_0},\\ \widetilde{D_0} &= \gamma_+\chi_++\widehat{D_0},\end{aligned}\tag{3.15}$$

with (in using Appendix 6.5.3 and (5.12) in Appendix 6.5.4)

$$\begin{aligned} \alpha_- &= (A_0^{(-\infty)} - 1) = -\widetilde{\omega}_-^2/2,\ \beta_- = B_0^{(-\infty)}, \\ \beta_+ &= (C_0^{(+\infty)}(\omega x) - 1),\ \gamma_+ = D_0^{(+\infty)}(\omega x), \end{aligned} \tag{3.16}$$

$$\text{const part of } \beta_+ \overset{def}{=} \beta_+^{(c)} = -\frac{\widetilde{\omega}_+^2}{2} + \frac{\sigma_1\varepsilon^2\widetilde{\omega}_+}{2} + \frac{\sigma_2\varepsilon^4}{2} + \mathcal{O}[(|\widetilde{\omega}_+| + \varepsilon^2)^4], \tag{3.17}$$

and where χ_- and χ_+ are smooth functions, such that

$$\begin{aligned} \chi_- &= 1 \text{ for } x \in (-\infty, -1), \\ &= 0 \text{ for } x > 0 \\ 0 &< \chi_- < 1 \text{ for } x \in (-1, 0), \end{aligned}$$

$$\begin{aligned} \chi_+ &= 1 \text{ for } x \in (1, \infty), \\ &= 0 \text{ for } x < 0 \\ 0 &< \chi_+ < 1 \text{ for } x \in (0, 1), \end{aligned}$$

such that

$$(\widehat{A_0}, \widehat{C_0}, \widehat{D_0}) \to 0 \text{ as } |x| \to \infty.$$

5.3.4 Properties of linear operators $\mathcal{M}_g$ and $\mathcal{L}_g$ (defined in (3.8,3.9))

We now give a precise definition of the function spaces where we will solve the problem with respect to $(\widehat{A_0}, \widehat{C_0}, \widehat{D_0})$. Indeed, let us define the Hilbert spaces

$$L_\eta^2 = \{u; u(x)e^{\eta|x|} \in L^2(\mathbb{R})\},$$

$$\begin{aligned} \mathcal{D}_0 &= \{(A, C) \in H_\eta^4 \times H_\eta^2; A \in H_\eta^4, C \in \mathcal{D}_1\} \\ \mathcal{D}_1 &= \{C \in H_\eta^2; \varepsilon^{-2}||C''||_{L_\eta^2} + \varepsilon^{-1}||C'||_{L_\eta^2} + ||C||_{L_\eta^2} \overset{def}{=} ||C||_{\mathcal{D}_1} < \infty\} \end{aligned}$$

equiped with natural scalar products. Then we have the following result proved in Appendix 6.5.5:

Lemma 3.4 *Except maybe for a set of isolated values of g, the kernel of $\mathcal{M}_g$ in L_η^2 is one dimensional, spanned by (A'_*, B'_*), and its range has codimension 1, L^2- orthogonal to (A'_*, B'_*). $\mathcal{M}_g$ has a pseudo-inverse acting from L_η^2 to $\mathcal{D}_0$ for any $\eta > 0$ small enough, with bound independent of ε.*

The operator $\mathcal{L}_g$ has a trivial kernel, and its range which has codimension 1, is L^2-orthogonal to B_* ($B_* \notin L^2$). $\mathcal{L}_g$ has a pseudo-inverse acting respectively from L_η^2 to $\mathcal{D}_1$ for $\eta > 0$ small enough, with bound independent of ε.

Remark 3.5 *We might expect a two-dimensional kernel since we have a "circle" of heteroclinics $\{(A_*, B_*e^{i\theta}); \theta \in \mathbb{R}\}$. The one-dimensional kernel of $\mathcal{M}_g$ is the usual one, while we also have $\mathcal{L}_g B_* = 0$. However $B_* \notin L^2_\eta$ so that the kernel of $\mathcal{L}_g$ is $\{0\}$, and we pay this by a codimension one range for $\mathcal{L}_g$. This is explicitely computed in Appendix 6.5.5.*

5.4 Estimates for the right hand sides of $\mathcal{M}_g(\widehat{A_0},\widehat{C_0})$ and $\mathcal{L}_g\widehat{D_0}$

After the second change of variables (3.15) the remaining terms in the right hand side of $\mathcal{M}_g(\widehat{A_0},\widehat{C_0})$ and $\mathcal{L}_g\widehat{D_0}$ coming from

$$\widetilde{\phi_{01}}(\omega x,\varepsilon,k_-,\widetilde{X},\widetilde{Y}),\ \widetilde{\psi_{0r1}}(\omega x,\varepsilon,k_-,\widetilde{X},\widetilde{Y}),\ \widetilde{\psi_{0i1}}(\omega x,\varepsilon,k_-,\widetilde{X},\widetilde{Y})$$

now cancel for $(\widehat{X},\widehat{Y},\widehat{\overline{Y}})=0$, they are then estimated in L^2_η by

$$\mathcal{O}(\varepsilon^4(||(\widehat{A_0},\widehat{C_0})||_{\mathcal{D}_0}+||\widehat{D_0}||_{\mathcal{D}_1}), \tag{4.1}$$

provided that the following condition

$$|\widehat{A_0}(x)|+|\widehat{A_0'}(x)|+|\widehat{A_0''}(x)|+|\widehat{A_0'''}(x)|+|\widehat{C_0}(x)|+|\widehat{C_0'}(x)|+|\widehat{D_0}(x)|+|\widehat{D_0'}(x)|<<1 \tag{4.2}$$

holds. We need to check this condition at the end of subsection 5.5.3. The unknowns in the problem are now

$$(\widehat{A_0},\widehat{C_0})\in\mathcal{D}_0,\ \widehat{D_0}\in\mathcal{D}_1,\ (k_-,\widetilde{\omega}_+)\in\mathbb{R}^2,$$

and ε is supposed to be small enough. In the following we use extensively the estimates (see (3.16,3.17))

$$\begin{aligned}
\alpha_- &= \mathcal{O}(|k_-|+\varepsilon^2)^2,\ \ \beta_+=\mathcal{O}(|\widetilde{\omega}_+|+\varepsilon^2)^2,\\
\beta_- &= \mathcal{O}(\varepsilon^6),\ \text{oscil part }(\beta_+)=\mathcal{O}(\varepsilon^6),\ \gamma_+=\mathcal{O}(\varepsilon^6),\\
\beta_+' &= \mathcal{O}(\varepsilon^5),\ \gamma_+'=\mathcal{O}(\varepsilon^5).
\end{aligned}$$

5.4.1 First component of $\mathcal{M}_g(\widehat{A_0},\widehat{C_0})$

The first component is now the sum of small terms linear in $(\widehat{A_0},\widehat{C_0})$ plus quadratic terms and terms independent of $(\widehat{A_0},\widehat{C_0})$ which tend exponentially to 0 as $e^{\varepsilon\delta_* x}$ for $x\to-\infty$ and $e^{-\sqrt{2}\varepsilon x}$ for $x\to+\infty$:

$$\mathcal{M}_g(\widehat{A_0},\widehat{C_0})|_1 = -k_-\widehat{A_0}''+\frac{k_-^2}{4}\widehat{A_0}+\widehat{\phi_0}+\varphi_1(k_-) \tag{4.3}$$

with

$$\begin{aligned}
\varphi_1(k_-) &= -k_-(A_*''+\alpha_-\chi_-'')+\frac{k_-^2}{4}(A_*-\chi_-)+\alpha_-\chi_-^{(4)} \\
&\quad -3(1-A_*^2)\alpha_-\chi_-+gB_*^2\alpha_-\chi_-+2gA_*B_*(\beta_-\chi_-+\beta_+\chi_+), \\
\widehat{\phi_0} &= \widetilde{\phi_0}(\omega x,\varepsilon,k_-,\widetilde{X},\widetilde{Y})-\chi_-\widetilde{\phi_0}(\omega x,\varepsilon,k_-,\widetilde{X}^{(-\infty)},\widetilde{Y}^{(-\infty)}).
\end{aligned} \tag{4.4}$$

More precisely we have, from (3.11), and taking into account (4.1)

$$\begin{aligned}
\widehat{\phi_0} &= 3[\alpha_-^2(A_*\chi_-^2-\chi_-)+2\alpha_-A_*\chi_-\widehat{A_0}+A_*\widehat{A_0}^2]+\alpha_-^3(\chi_-^3-\chi_-) \\
&\quad +3\alpha_-^2\chi_-^2\widehat{A_0}+3\alpha_-\chi_-\widehat{A_0}^2+\widehat{A_0}^3+2gB_*[\alpha_-\chi_-\widehat{C_0}+(\beta_-\chi_-+\beta_+\chi_+)\widehat{A_0}+\widehat{A_0}\widehat{C_0}] \\
&\quad +g(A_*+\alpha_-\chi_-+\widehat{A_0})[(\beta_-\chi_-+\beta_+\chi_++\widehat{C_0})^2+(\gamma_+\chi_++\widehat{D_0})^2] \\
&\quad -\chi_-g(1+\alpha_-)\beta_-^2+\widehat{f_{00}}, \\
\widehat{f_{00}} &= \sigma_0\varepsilon^2k_-(A_*^3-\chi_-)+\mathcal{O}[\varepsilon^{2+2/5}(e^{\varepsilon\delta_*x}\chi_{(-\infty,0)}+e^{-\varepsilon^{1/5}\delta_*x}\chi_{(0,\infty)})+\varepsilon^2(|\widehat{X}|+|\widehat{Y}|)+\varepsilon|\widehat{D_0}'|].
\end{aligned} \tag{4.5}$$

We notice that for $\eta=\varepsilon\delta_*/2$ ($\eta<\varepsilon\delta$ is necessary), and due to Corollary 1.3,

$$\begin{aligned}
\frac{1}{\varepsilon^2}\beta_+' &= \mathcal{O}(\varepsilon^3),\ \frac{1}{\varepsilon^2}\gamma_+'=\mathcal{O}(\varepsilon^3), \\
\|A_*'\|_{L^2_\eta} &= \mathcal{O}(\varepsilon^{1/10}),\ \|B_*'\|_{L^2_\eta}=\mathcal{O}(\varepsilon^{1/2}), \\
\|A_*'^2\|_{L^2_\eta} &= \mathcal{O}(\varepsilon^{7/10}),\ \|B_*'^2\|_{L^2_\eta}=\mathcal{O}(\varepsilon^{3/2}), \\
\|A_*''\|_{L^2_\eta} &= \mathcal{O}(\varepsilon^{1/10}),\ \|B_*''\|_{L^2_\eta}=\mathcal{O}(\varepsilon^{3/2}).
\end{aligned}$$

Then, in using extensively $2|ab|\leq a^2+b^2$) and, for example

$$\frac{k_-^2}{4}\|A_*-\chi_-\|_{L^2_\eta}=\mathcal{O}(\frac{k_-^2}{\sqrt{\varepsilon}}),$$

we obtain the estimates (here and in the following c is a generic constant, independent of ε)

$$\begin{aligned}
\|\varphi_1(k_-)\|_{L^2_\eta} &\leq c\left(\varepsilon^{1/10}|k_-|+\frac{k_-^2+\varepsilon^4}{\sqrt{\varepsilon}}+\widetilde{\omega}_+^2+\varepsilon^2|\widetilde{\omega}_+|)\right), \\
\int_{\mathbb{R}}\varphi_1(k_-)A_*'dx &= \mathcal{O}[(|k_-|+|\widetilde{\omega}_+|+\varepsilon^2)^2],
\end{aligned} \tag{4.6}$$

using integration by parts and

$$\begin{aligned}
\int_{\mathbb{R}}A_*'A_*''dx &= 0, \\
\int_{\mathbb{R}}(A_*-\chi_-)A_*'dx &= \mathcal{O}(1) \\
\int_{\mathbb{R}}(1-A_*^2)A_*'\chi_-dx &= \mathcal{O}(1).
\end{aligned}$$

In next estimates, we use the following little Lemma (adapted from a simple Sobolev inequality) where we notice that we loose one ε, due to the weak exponential decay at ∞ (η is of order ε):

Lemma 4.1 *For any $u \in H^1_\eta$ and ε sufficiently small, we have*

$$|u(x)| \le c(||u||_{L^2_\eta} + \frac{1}{\varepsilon}||u'||_{L^2_\eta})$$

where c is independent of ε.

Then we may use

$$\begin{array}{rcl} |\widehat{A_0}^{(m)}(x)| & \le & \frac{c}{\varepsilon}||\widehat{A_0}||_{H^4_\eta},\ m=0,1,2,3 \\ |\widehat{C_0}^{(m)}(x)| & \le & c\varepsilon^m||\widehat{C_0}||_{\mathcal{D}_1},\ m=0,1, \\ |\widehat{D_0}^{(m)}(x)| & \le & c\varepsilon^m||\widehat{D_0}||_{\mathcal{D}_1},\ m=0,1. \end{array}$$

Now, from f_{00} in (4.5), we have (see Remark 3.3)

$$||d_3\varepsilon^2 A''_* + d_4\varepsilon^2 A_*^2 A''_* + d_5\varepsilon^2 A''_* B_*^2||_{L^2_\eta} = \mathcal{O}(\varepsilon^{2+1/10}),$$

and for example, from Lemma 4.1

$$2g||B_*\widehat{A_0}\widehat{C_0}||_{L^2_\eta} \le c||\widehat{A_0}||_{H^4_\eta}||\widehat{C_0}||_{\mathcal{D}_1} \le c||(\widehat{A_0},\widehat{C_0})||^2_{\mathcal{D}_0},$$

$$||\widehat{A_0}^2||_{L^2_\eta} \le \frac{c}{\varepsilon}||\widehat{A_0}||^2_{\mathcal{D}_0}, \quad ||\widehat{A_0}^3||_{L^2_\eta} \le \frac{c}{\varepsilon^2}||\widehat{A_0}||^3_{\mathcal{D}_0}.$$

We then obtain, for sufficiently small $\varepsilon, |k_-|, |\widetilde{\omega}_+|, \widehat{A_0}, \widehat{C_0}, \widehat{D_0}$ in $\mathbb{R}^3_+ \times \mathcal{D}_0 \times \mathcal{D}_1$

$$||\widehat{\phi_0}||_{L^2_\eta} \le c\left(\varepsilon^{2+1/10} + \varepsilon^{3/2}|k_-| + \frac{k_-^4}{\sqrt{\varepsilon}} + \widetilde{\omega}_+^4 + \frac{1}{\varepsilon}||\widehat{A_0}||^2_{H^4_\eta} + \frac{1}{\varepsilon^2}||\widehat{A_0}||^3_{H^4_\eta} + ||\widehat{C_0}||^2_{\mathcal{D}_1} + ||\widehat{D_0}||^2_{\mathcal{D}_1}\right). \tag{4.7}$$

5.4.2 Second component of $\mathcal{M}_g(\widehat{A_0},\widehat{C_0})$

For the second component we have

$$\mathcal{M}_g(\widehat{A_0},\widehat{C_0})|_2 = \frac{2\widetilde{\omega}_+}{\varepsilon}\widehat{D_0}' + \widetilde{\omega}_+^2\widehat{C_0} + \widehat{\psi_{0r}} + \varphi_2(k_-), \tag{4.8}$$

with

$$\begin{array}{rcl} \varphi_2(k_-) & = & \widetilde{\omega}_+^2(B_* - \chi_+) - \frac{1}{\varepsilon^2}\beta_-\chi''_- - \frac{2}{\varepsilon^2}\beta'_+\chi'_+ - \frac{1}{\varepsilon^2}\beta_+\chi''_+ + \frac{2\widetilde{\omega}_+}{\varepsilon}\gamma_+\chi'_+ \qquad (4.9) \\ & & -(3 - gA_*^2 - 3B_*^2)\beta_+\chi_+ + [1 - \chi_- - g(A_*^2 - \chi_-)]\beta_-\chi_- + 2gA_*B_*\alpha_-\chi_-, \\ \widehat{\psi_{0r}} & = & \widetilde{\psi_{0r}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y}) - \chi_+\widetilde{\psi_{0r}}(\omega x, \varepsilon, k_-, 0, \widetilde{Y}^{(+\infty)}) \\ & & -\chi_-\widetilde{\psi_{0r}}(\omega x, \varepsilon, k_-, \widetilde{X}^{(-\infty)}, \widetilde{Y}^{(-\infty)}), \end{array}$$

where $\gamma_+ = D_0^{(+\infty)}$. For $\widehat{\psi_{0r}}$ we have

$$\begin{aligned}
\widehat{\psi_{0r}} &= 2gA_*(\alpha_-\chi_-\widehat{C_0} + (\beta_-\chi_- + \beta_+\chi_+)\widehat{A_0} + \widehat{A_0}\widehat{C_0}) \\
&+g(B_* + \beta_+\chi_+ + \widehat{C_0})(\alpha_-^2\chi_-^2 + 2\alpha_-\chi_-\widehat{A_0} + \widehat{A_0}^2) \\
&+g\beta_-\chi_-[(\alpha_-^2(\chi_-^2 - 1) + 2\alpha_-\chi_-\widehat{A_0} + \widehat{A_0}^2] \\
&+[B_*(\beta_-\chi_- + \beta_+\chi_+)^2 - \chi_+\beta_+^2] + [B_*(\gamma_+\chi_+)^2 - \chi_+\gamma_+^2] \\
&+\beta_+\chi_+(\chi_+^2 - 1)(\beta_+^2 + \gamma_+^2) + \beta_-^3\chi_-(\chi_-^2 - 1) \\
&+\widehat{C_0}[(\beta_-\chi_- + \beta_+\chi_+ + \widehat{C_0})^2 + (\gamma_+\chi_+ + \widehat{D_0})^2] \\
&+2(B_* + \beta_+\chi_+)(\beta_-\chi_- + \beta_+\chi_+\widehat{C_0} + \gamma_+\chi_+\widehat{D_0}) \\
&+(B_* + \beta_-\chi_- + \beta_+\chi_+)(\widehat{C_0}^2 + \widehat{D_0}^2) + \widehat{g_{00r}},
\end{aligned} \tag{4.10}$$

$$\widehat{g_{00r}} = \mathcal{O}(\varepsilon^{2+3/5}e^{\varepsilon\delta_* x}\chi_{(-\infty,0)} + \varepsilon^{2+4/5}e^{-\varepsilon^{1/5}\delta_* x}\chi_{(0,\infty)} + \varepsilon^2(|\widehat{X}| + |\widehat{Y}|) + \varepsilon|\widehat{D_0}'|).$$

Now we use

$$||c_5\varepsilon^2 A_* A_*'' B_*||_{L^2_\eta} \leq c\varepsilon^2,$$

and, as above

$$2g||A_*\widehat{A_0}\widehat{C_0}||_{L^2_\eta} \leq \frac{c}{\varepsilon}||(\widehat{A_0}, \widehat{C_0})||^2_{\mathcal{D}_0},$$

so that we obtain for sufficiently small $\varepsilon, k_-, \widetilde{\omega}_+, \widehat{A_0}, \widehat{C_0}, \widehat{D_0}$ in $\mathbb{R}^3 \times \mathcal{D}_0 \times \mathcal{D}_1$ (taking into account of (4.1))

$$\begin{aligned}
||\widehat{\psi_{0r}}||_{L^2_\eta} &\leq c\left(\varepsilon^{2+1/10} + \frac{k_-^4 + \widetilde{\omega}_+^4}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon}||\widehat{A_0}||^2_{\mathcal{D}_0} + ||\widehat{C_0}||^2_{\mathcal{D}_1} + ||\widehat{D_0}||^2_{\mathcal{D}_1}\right) \\
&+c\left((k_-^2 + \widetilde{\omega}_+^2)||(\widehat{A_0}, \widehat{C_0})||_{\mathcal{D}_0}\right),
\end{aligned} \tag{4.11}$$

In using, for example

$$||2gA_*B_*\alpha_-\chi_-||_{L^2_\eta} \leq c\frac{\widetilde{\omega}_-^2}{\sqrt{\varepsilon}},$$

we obtain easily

$$\begin{aligned}
||\varphi_2(k_-)||_{L^2_\eta} &\leq c(\frac{\widetilde{\omega}_-^2}{\sqrt{\varepsilon}} + \frac{(|\widetilde{\omega}_+| + \varepsilon^2)^2}{\varepsilon^2}), \\
\int_{\mathbb{R}} \varphi_2(k_-)B_*' dx &= \mathcal{O}[(k_-^2 + \widetilde{\omega}_+^2 + \varepsilon^4)],
\end{aligned} \tag{4.12}$$

where the last estimates use

$$\begin{aligned}
\frac{1}{\varepsilon^2}\int_0^1 \beta_+'\chi_+' B_*' dx &= \mathcal{O}(\varepsilon^4) \\
\frac{1}{\varepsilon^2}\int_0^1 \beta_+\chi_+'' B_*' dx &= \mathcal{O}(|\widetilde{\omega}_+| + \varepsilon^2)^2
\end{aligned}$$

obtained, for the first integral in integrating by parts, and for the second one in separating the oscillating part of order ε^6 from the constant part $\beta_+^{(c)}$ of β_+, for which we make an integration by parts, in using $B_*''=\mathcal{O}(\varepsilon^2 B_*)$. More precisely we have

$$\begin{aligned}\int_{\mathbb{R}}\varphi_1(k_-)A_*'dx+\int_{\mathbb{R}}\varphi_2(k_-)B_*'dx &= a_2\frac{k_-^2}{4}+a_3\sigma_0\varepsilon^2k_- \\ &\quad +\mathcal{O}(|k_-^3|+\varepsilon^2k_-^2+\widetilde{\omega}_+^2+\varepsilon^4),\end{aligned} \tag{4.13}$$

with

$$\begin{aligned}a_2 &= \int_{\mathbb{R}}(A_*-\chi_-)A_*'dx-a_3,\\ 2a_3 &= \int_{-1}^0\chi_-^{(4)}A_*'-3\int_{\mathbb{R}}(1-A_*^2)A_*'\chi_-dx+g\int_{\mathbb{R}}(A_*B_*^2)'\chi_-dx,\end{aligned}$$

We observe that (see Corollary 1.2)

$$\begin{aligned}\int_{\mathbb{R}}(A_*-\chi_-)A_*'dx &= \frac{1}{2}+\mathcal{O}(\varepsilon^{3/5})\\ \int_{-1}^0\chi_-^{(4)}A_*'dx &= \mathcal{O}(\varepsilon^{3/5})\\ g\int_{-\infty}^0(A_*B_*^2)'\chi_-dx &= -g\int_{-1}^0(A_*B_*^2)\chi_-'dx=\mathcal{O}(\varepsilon^{2/5})\end{aligned}$$

$$-3\int_{-\infty}^0(1-A_*^2)\chi_-A_*'dx=3\int_{-1}^0(A_*-\frac{A_*^3}{3}-\frac{2}{3})\chi_-'dx=2+\mathcal{O}(\varepsilon^{2/5}),$$

so that

$$a_2 = -3/2+\mathcal{O}(\varepsilon^{2/5}), \tag{4.14}$$

$$a_3 = 4+\mathcal{O}(\varepsilon^{2/5}). \tag{4.15}$$

5.4.3 Component $\mathcal{L}_g\widehat{D_0}$

For the third component we obtain

$$\mathcal{L}_g\widehat{D_0}=-\frac{2\widetilde{\omega}_+}{\varepsilon}\widehat{C_0}'+\widetilde{\omega}_+^2\widehat{D_0}+\widehat{\psi_{0i}}+\varphi_3(k_-), \tag{4.16}$$

$$\begin{aligned}\varphi_3(\widetilde{\omega},k_-,\omega x) &= -\frac{2\widetilde{\omega}_+}{\varepsilon}[B_*'+\beta_-\chi_-'+\beta_+\chi_+']-\frac{2}{\varepsilon^2}\gamma_+'\chi_+'\\ &\quad -\frac{1}{\varepsilon^2}\gamma_+\chi_+''-(1-gA_*^2-B_*^2)\gamma_+\chi_+,\end{aligned}$$

and

$$\begin{aligned}\widehat{\psi_{0i}} &= \widetilde{\psi_{0i}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y}) - \chi_+ \widetilde{\psi_{0i}}(\omega x, \varepsilon, k_-, 0, \widetilde{Y}^{(+\infty)}) \\ & -\chi_- \widetilde{\psi_{0i}}(\omega x, \varepsilon, k_-, \widetilde{X}^{(-\infty)}, \widetilde{Y}^{(-\infty)}).\end{aligned}$$

For sufficiently small $\varepsilon, k_-, \widetilde{\omega}_+, \widehat{A_0}, \widehat{C_0}, \widehat{D_0}$ in $\mathbb{R}^3 \times \mathcal{D}_0 \times \mathcal{D}_1$, we obtain the estimates

$$||\varphi_3||_{L^2_\eta} \leq c(\varepsilon^3 + \frac{|\widetilde{\omega}_+|}{\sqrt{\varepsilon}} + \frac{|\widetilde{\omega}_+^3|}{\varepsilon}), \tag{4.17}$$

and taking into account of (4.1),

$$\begin{aligned}||\widehat{\psi_{0i}}||_{L^2_\eta} &\leq c\{\varepsilon^{1+1/10} + (k_-^2 + \widetilde{\omega}_+^2)||\widehat{D_0}||_{\mathcal{D}_1} + ||\widehat{A_0}\widehat{D_0}||_{L^2_\eta} \\ & +||(\widehat{C_0}\widehat{D_0})||_{L^2_\eta} + ||\widehat{D_0}||^2_{\mathcal{D}_1}\},\end{aligned} \tag{4.18}$$

where the term of order $\varepsilon^{1+1/10}$ comes from

$$\varepsilon c_9 ||B_* A_* A'_*||_{L^2_\eta} = \mathcal{O}(\varepsilon^{1+1/10}).$$

5.5 Bifurcation equation

Let us use an adapted Lyapunov-Schmidt method. Since

$$\mathcal{M}_g(A'_*, B'_*) = 0,$$

we now decompose $(\widehat{A_0}, \widehat{C_0}, \widehat{D_0})$ as

$$\begin{aligned}\widehat{A_0} &= zA'_* + u, \\ \widehat{C_0} &= zB'_* + v, \\ \widehat{D_0} &= w.\end{aligned} \tag{5.1}$$

For ε small enough, the unknowns are now

$$(u, v) \in \mathcal{D}_0,\ w \in \mathcal{D}_1,\ (z, k_-, \widetilde{\omega}_+) \in \mathbb{R}^3.$$

Remark 5.1 *It might be interesting to give a physical interpretation of z. By construction of the basic heteroclinic, it corresponds to a shift in x of the heteroclinic. However, z occurs in the component w which modifies the phase of B_0 controlling the rolls parallel to the wall, themselves affected by the slight change of wave length (due to k_+). This "shift" has no effect on the equilibrium at $-\infty$. We interpret this in saying that the system of rolls parallel to the wall (in $x = 0$), adapts itself to fit with the rolls on the other side, orthogonal to the wall. Notice that z corresponds to a "shift" of size of order z/ε for the original phase of the amplitude B of rolls parallel to the wall.*

Then, equations (4.3,4.8) give (Q_0 is the projection in L^2 on the range of $\mathcal{M}_g$)

$$\mathcal{M}_g(u,v) = Q_0\begin{pmatrix} -k_-(zA'_*+u)'' + \frac{k_-^2}{4}(zA'_*+u) + \widehat{\phi_0} + \varphi_1(k_-) \\ \frac{2\widetilde{\omega}_+}{\varepsilon}w' + \widetilde{\omega}_+^2(zB'_*+v) + \widehat{\psi_{0r}} + \varphi_2(k_-) \end{pmatrix}. \tag{5.2}$$

5.5.1 Resolution with respect to $\widetilde{\omega}_+$ and w

We observe that (u,v) and w appear non symmetrically, so we choose to first solve equation (4.16), where the kernel of $\mathcal{L}_g$ is empty, and its range of codimension 1 (see Lemma 3.4). This has the advantage to give w and $\widetilde{\omega}_+$ in function of (u,v,z,k_-,ε). So, let us start by solving the compatibility condition.

Since

$$\int_0^1 \frac{1}{\varepsilon^2}\gamma'_+\chi'_+B_*dx = -\int_0^1 \frac{1}{\varepsilon^2}\gamma_+(\chi'_+B_*)'dx = \mathcal{O}(\varepsilon^4),$$

and using Remark 3.3, we obtain the estimates

$$\begin{aligned}
\int_{\mathbb{R}} \varphi_3 B_* dx &= -\frac{\widetilde{\omega}_+}{\varepsilon}[1+\mathcal{O}(|\widetilde{\omega}_+|+\varepsilon^2)^2] + \mathcal{O}(\varepsilon^4)\ , \\
\int_{\mathbb{R}} \widehat{\psi_{0i}} B_* dx &= \mathcal{O}[\varepsilon^{1+1/10} + (k_-^2+\widetilde{\omega}_+^2)||\widehat{D_0}||_{\mathcal{D}_1} + ||\widehat{D_0}||^2_{\mathcal{D}_1} + ||\widehat{A_0}\widehat{D_0}||_{L^2_\eta} + ||\widehat{C_0}\widehat{D_0}||_{L^2_\eta}] \\
&= \mathcal{O}[\varepsilon^{1+1/10} + \varepsilon^{3/5}|z|||w||_{\mathcal{D}_1} + ||(u,v)||^2_{\mathcal{D}_0} + ||w||^2_{\mathcal{D}_1} + (k_-^2+\widetilde{\omega}_+^2)||w||_{\mathcal{D}_1})].
\end{aligned}$$

Then the compatibility condition for equation (4.16) leads to

$$\frac{2\widetilde{\omega}_+}{\varepsilon}\int_{\mathbb{R}} B'_*B_*dx = \int_{\mathbb{R}}\left[-\frac{2\widetilde{\omega}_+}{\varepsilon}(zB''_*+v') + \widetilde{\omega}_+^2 w + \widehat{\psi_{0i}} + \varphi_3\right]B_*dx,$$

which gives

$$\begin{aligned}
\widetilde{\omega}_+ &= \int_{\mathbb{R}}\left[-2\widetilde{\omega}_+(zB''_*+v') + \varepsilon\widetilde{\omega}_+^2 w\right]B_*dx \\
&\quad +\mathcal{O}[\varepsilon^2 + |\widetilde{\omega}_+|(|\widetilde{\omega}_+|+\varepsilon^2)^2 + \varepsilon^{1+2/5}|z|||w||_{\mathcal{D}_1}] \\
&\quad +\varepsilon\mathcal{O}(||(u,v)||^2_{\mathcal{D}_0} + ||w||^2_{\mathcal{D}_1} + (\widetilde{\omega}_-^2+\widetilde{\omega}_+^2)||w||_{\mathcal{D}_1}).
\end{aligned}$$

The right hand side is a smooth function of its arguments, and may be solved with respect to $\widetilde{\omega}_+$ (or equivalently with respect to k_+ since $\widetilde{\omega}_+ \sim \frac{k_+}{2}$) by implicit function theorem in the neighborhood of 0 for

$$(u,v)\in\mathcal{D}_0,\ w\in\mathcal{D}_1,\ (\varepsilon,\widetilde{\omega}_-,z)\in\mathbb{R}^3,$$

with

$$\widetilde{\omega}_+ = \mathfrak{k}_+(\varepsilon,\widetilde{\omega}_-,z,(u,v),w)\in C^1(\mathbb{R}^3\times\mathcal{D}_0\times\mathcal{D}_1).$$

Moreover, we have the estimate

$$|\mathfrak{k}_+| \leq c[\varepsilon^2 + \varepsilon^{1+2/5}|z|||w||_{\mathcal{D}_1} + \varepsilon\widetilde{\omega}_-^2||w||_{\mathcal{D}_1} + \varepsilon(||(u,v)||^2_{\mathcal{D}_0} + ||w||^2_{\mathcal{D}_1})]. \tag{5.3}$$

For solving equation (4.16) we now have

$$w = \mathcal{L}_g^{-1}[-\frac{2\mathfrak{k}_+}{\varepsilon}(zB_*'' + v') + \mathfrak{k}_+^2 w + \varphi_3 + \widehat{\psi_{0i}}]$$

which may be solved with respect to w in $\mathcal{D}_1$, in the neighborhood of 0, by implicit function theorem, for

$$(\varepsilon, k_-, z, (u,v)) \in \mathbb{R}^3 \times \mathcal{D}_0 \text{ in a neighborhood of } 0.$$

Using (4.17), (4.18), (5.3) and

$$||\frac{B_*''}{\varepsilon}||_{L^2_\eta} = \mathcal{O}(\varepsilon^{1/2}), \quad ||\frac{v'}{\varepsilon}||_{L^2_\eta} \leq ||v||_{\mathcal{D}_1},$$

we obtain

$$w = \mathfrak{w}(\varepsilon, \widetilde{\omega}_-, z, u, v)$$

with

$$||\mathfrak{w}||_{\mathcal{D}_1} \leq c(\varepsilon^{1+1/10} + \varepsilon^{1/2}||(u,v)||^2_{\mathcal{D}_0}), \tag{5.4}$$

and we deduce

$$|\mathfrak{k}_+| \leq c(\varepsilon^2 + \varepsilon||(u,v)||^2_{\mathcal{D}_0}). \tag{5.5}$$

Remark 5.2 *The term of order $\varepsilon^{1+1/10}$ in $\mathfrak{w}$ is $\varepsilon w_1 + \mathcal{O}(\varepsilon^{3/2})$ with w_1 coming from $\widehat{\psi_{0i}}$ and given by (see Appendix 6.5.5 for an explicit formula of the pseudo-inverse of $\mathcal{L}_g$)*

$$w_1 = c_9\mathcal{L}_g^{-1}[B_*A_*A_*' - 2B_*'\int_{\mathbb{R}} B_*^2 A_* A_*' dx], \;\; ||w_1||_{\mathcal{D}_1} = \mathcal{O}(\varepsilon^{1/10}), \tag{5.6}$$

and the compatibility condition (orthogonality to B_*) is satisfied with

$$||2B_*'\int_{\mathbb{R}} B_*^2 A_* A_*' dx||_{L^2_\eta} = \mathcal{O}(\varepsilon^{1/10}).$$

5.5.2 Resolution with respect to (u, v)

Now, we replace w and $\widetilde{\omega}_+$ by their expressions $\mathfrak{w}$ and $\mathfrak{k}_+$, and consider (5.2) which may be solved by implicit function theorem (by Lemma 3.4 the pseudo-inverse of $\mathcal{M}_g$ is bounded from L^2_η to $\mathcal{D}_0$) with respect to (u, v) in a neighborhood of 0 in $\mathcal{D}_0$ for (ε, k_-, z) close to 0 in $\mathbb{R}^3$. Indeed, the right hand side of (5.2) is smooth in its arguments and assuming

$$|k_-| < \varepsilon, \tag{5.7}$$

$$|z| < \varepsilon^{1/5}, \tag{5.8}$$

$$||u||_{\mathcal{D}_0} < \varepsilon^{1+1/20}, \tag{5.9}$$

using (5.1) and collecting results of (4.3,4.6,4.7) for the first component, and (4.8,4.12,4.11) for the second component, estimates in L^2_η of the right hand side are as follows

$$\begin{aligned} \text{1st comp.} &= \mathcal{O}\Bigg(\frac{k_-^2}{\sqrt{\varepsilon}} + \varepsilon^{1/10}|k_-| + \varepsilon^{2+1/10} + \varepsilon^{7/10}z^2 + |k_-||z|||u||_{\mathcal{D}_0} \\ &\quad + |z|\varepsilon^{2/5}||(u,v)||_{\mathcal{D}_0} + \frac{1}{\varepsilon}||u||^2_{\mathcal{D}_0} + ||v||^2_{\mathcal{D}_1} + (1/\varepsilon^2)||u||^3_{\mathcal{D}_0}\Bigg), \\ \text{2nd comp.} &= \mathcal{O}\Bigg(\varepsilon^2 + \frac{k_-^2}{\sqrt{\varepsilon}} + \varepsilon^{3/2}|k_-| + \varepsilon^{7/10}z^2 + \frac{1}{\varepsilon}||u||^2_{\mathcal{D}_0} + ||v||^2_{\mathcal{D}_1} \\ &\quad + (k_-^2 + \varepsilon^{2/5}|z|)||(u,v)||_{\mathcal{D}_0}\Bigg). \end{aligned}$$

where we notice that, for example

$$\begin{aligned} ||\widehat{A_0}^2||_{L^2_\eta} &\leq c(\varepsilon^{7/10}z^2 + |z|\varepsilon^{2/5}||u||_{\mathcal{D}_0} + \frac{1}{\varepsilon}||u||^2_{\mathcal{D}_0}), \\ ||\widehat{C_0}^2||_{L^2_\eta} &\leq c(\varepsilon z^2 + |z|\varepsilon||v||_{\mathcal{D}_1} + ||v||^2_{\mathcal{D}_1}). \end{aligned}$$

Applying implicit function theorem for (ε, k_-, z) satisfying (5.7,5.8) in $\mathbb{R}^3$, leads to

$$(u,v) = (\mathfrak{u},\mathfrak{v})(\varepsilon, k_-, z) \in \mathcal{D}_0$$

with

$$||(\mathfrak{u},\mathfrak{v})||_{\mathcal{D}_0} \leq c(\varepsilon^2 + \frac{k_-^2}{\sqrt{\varepsilon}} + \varepsilon^{1/10}|k_-| + \varepsilon^{7/10}z^2), \tag{5.10}$$

which satisfies the a priori estimate (5.9). Now using (5.4), (5.5), (5.7), (5.8) and (5.10) we obtain

$$\begin{aligned} ||\mathfrak{w}||_{\mathcal{D}_1} &\leq c\varepsilon^{1+1/10}, &(5.11)\\ |\mathfrak{k}_+| &\leq c\varepsilon^2, &(5.12) \end{aligned}$$

where (5.7), (5.8), (4.2) and (3.14) need to be checked at the end. In fact we have the following

Lemma 5.3 *Assuming that (5.7) and (5.8) hold, then (4.2) and (3.14) are satisfied.*

Proof. Condition (4.2) results immediately from the definition (5.1), Lemma 4.1 and estimates (5.10) and (5.11). Then (3.14) results from (3.15), from the same estimates as above, and from (5.11). ■

5.5.3 Final bifurcation equation

It remains to satisfy the orthogonality in L^2 of the right hand side of $\mathcal{M}_g(\widehat{A_0}, \widehat{C_0})$ with (A'_*, B'_*) (see Lemma 3.4). This provides one relationship, expressed as the cancelling of a function of (z, k_-, ε), from which we extract the family of bifurcating solutions. It gives

$$\begin{aligned} 0 &= \int_{\mathbb{R}} [-k_-(zA'''_* + u'') + \frac{k_-^2}{4}(zA'_* + u)]A'_* dx + \int_{\mathbb{R}} (\widehat{\phi_0} + \varphi_1)A'_* dx \\ &\quad + \int_{\mathbb{R}} [\frac{2\widetilde{\omega}_+}{\varepsilon} w' + \widetilde{\omega}_+^2 (zB'_* + v)]B'_* dx + \int_{\mathbb{R}} (\widehat{\psi_{0r}} + \varphi_2)B'_* dx. \end{aligned} \tag{5.13}$$

Let us define

$$a_1 = -\int_{\mathbb{R}} A'''_* A'_* dx = \int_{\mathbb{R}} A''^2_* dx > 0, \;\; a_1 = \mathcal{O}(\varepsilon^{1/5}) \tag{5.14}$$

so that, using Corollaries 1.2, 1.3 and (5.10), (5.11), (5.12), we obtain

$$\begin{aligned} &\int_{\mathbb{R}} [-k_-(zA'''_* + u'') + \frac{k_-^2}{4}(zA'_* + u)]A'_* dx \\ &= a_1 k_- z + \mathcal{O}(\varepsilon^{2+1/10}|k_-| + \varepsilon^{1/5}k_-^2 + \frac{|k_-^3|}{\varepsilon^{2/5}} + \varepsilon^{4/5}|k_-|z^2), \end{aligned} \tag{5.15}$$

$$\int_{\mathbb{R}} [\frac{2\widetilde{\omega}_+}{\varepsilon} w' + \widetilde{\omega}_+^2 (zB'_* + v)]B'_* dx = \mathcal{O}(\varepsilon^{3+1/10}). \tag{5.16}$$

From (4.13) we also have

$$\int_{\mathbb{R}} \varphi_1(k_-)A'_* dx + \int_{\mathbb{R}} \varphi_2(k_-)B'_* dx = a_2 \frac{k_-^2}{4} + a_3\sigma_0\varepsilon^2 k_- + \mathcal{O}(|k_-^3| + \varepsilon^2 k_-^2 + \varepsilon^4). \tag{5.17}$$

We have, from (4.5), (4.10), (5.9), (5.11), (5.12), (5.7), (5.8) and Remark 3.3

$$\int_{\mathbb{R}} \widehat{\phi_0} A'_* dx = z^2[a'_0 + \mathcal{O}(\varepsilon^{8/5})] + \sigma'_0 \varepsilon^2 k_- + \mathcal{O}[\varepsilon^{2+4/5} + \varepsilon^{3/2}k_-^2 + \varepsilon^{1/5}|z|(\varepsilon^2 + k_-^2)], \tag{5.18}$$

with

$$\begin{aligned} a'_0 &= \int_{\mathbb{R}} (3A_* A'^3_* + 2gB_* B'_* A'^2_* + gA_* A'_* B'^2_*)dx + \mathcal{O}(\varepsilon^{8/5}) = \mathcal{O}(\varepsilon^{4/5}), \\ \sigma'_0 &= \sigma_0 \int_{\mathbb{R}} A'_*(A_*^3 - \chi_-)dx + \mathcal{O}(\varepsilon^{2+1/10}) = \sigma_0[\frac{3}{4} + \mathcal{O}(\varepsilon^{2/5})], \end{aligned} \tag{5.19}$$

where (for example) the estimated term in $\varepsilon^{2+4/5}$ comes from

$$\varepsilon^2(d_2 - d_4)\int_{\mathbb{R}} A_* A'^3_* dx \leq c\varepsilon^{2+4/5}, \tag{5.20}$$

occuring (see Remark 3.3) in $\int_{\mathbb{R}} \widehat{f_{00}} A_* dx$.

We also obtain

$$\int_{\mathbb{R}} \widehat{\psi_{0r}} B'_* dx = z^2 a''_0 + \mathcal{O}(\varepsilon^{3+1/5} + \varepsilon^{2+3/5}|z| + k_-^4 + \varepsilon^{1+1/5} k_-^2 + \varepsilon^{11/20}|z|k_-^2 + \varepsilon^{1+3/20}|k_-||z|), \tag{5.21}$$

with

$$a''_0 = \int_{\mathbb{R}} (gB_* B'_* A'^2_* + 2gA_* A'_* B'^2_* + B_* B'^3_*) dx + \mathcal{O}(\varepsilon^{1+3/4}) = \mathcal{O}(\varepsilon^{1+1/5}).$$

Hence collecting (5.15), (5.16), (5.17), (5.18), (5.21), and using a priori estimates (5.7), (5.8), we obtain the bifurcation equation, in identifying main orders of independent coefficients,

$$a_0 z^2 + a'_1 k_- z + a'_2 \frac{k_-^2}{4} + a'_3 \varepsilon^2 k_- + a_4 \varepsilon^{2+1/5} z + a_5 \varepsilon^{2+4/5} = 0, \tag{5.22}$$

where we define

$$a_0 = a'_0 + a''_0 + \mathcal{O}(\varepsilon^{1+4/5}) = \mathcal{O}(\varepsilon^{4/5}). \tag{5.23}$$

Using Corollaries 1.2 and 1.3, we notice that the main contribution of this coefficient is precisely

$$a_0 \sim \int_{-\infty}^{0} 3A_* A'^3_* dx = \mathcal{O}(\varepsilon^{4/5}).$$

From (5.14), (4.13), (5.23) and (5.20) we obtain

$$\begin{aligned}
a_0 &= \varepsilon^{4/5}\overline{a_0} = \mathcal{O}(\varepsilon^{4/5}) \\
a'_1 &= \int_{\mathbb{R}} A''^2_* dx + \mathcal{O}(\varepsilon^{1/2}) = \mathcal{O}(\varepsilon^{1/5}) \\
a'_2 &= a_2 + \mathcal{O}(\varepsilon^{3/2}) = -3/2 + \mathcal{O}(\varepsilon^{2/5}) \\
a'_3 &= a_3\sigma_0 + \sigma'_0 + \mathcal{O}(\varepsilon^{1/10}) = \frac{19}{4}\sigma_0 + \mathcal{O}(\varepsilon^{1/10}), \\
a_5\varepsilon^{4/5} &\sim (d_2 - d_4)\int_{\mathbb{R}} A_* A'^3_* dx \sim \frac{(d_2 - d_4)}{3} a_0 = \mathcal{O}(\varepsilon^{4/5}).
\end{aligned} \tag{5.24}$$

The discriminant of the principal part of the quadratic form in (z, k_-) of the left hand side of (5.22) is

$$\Delta = a'^2_1 - a_0 a'_2 = a'^2_1 + \mathcal{O}(\varepsilon^{4/5}) = \mathcal{O}(\varepsilon^{2/5}) \tag{5.25}$$

which *it is positive*. The bifurcation equation (5.22) may then be written as

$$\left(\frac{a'_2 k_-}{2} + a'_1 z + a'_3 \varepsilon^2\right)^2 - \Delta\left(z + \frac{a''_3 \varepsilon^2}{\Delta}\right)^2 = -a'_2 a_5 \varepsilon^{2+4/5} + \mathcal{O}(\varepsilon^{3+3/5})$$

where

$$a''_3 = a'_1 a'_3 - \frac{a_4 a'_2}{2}\varepsilon^{1/5} = \mathcal{O}(\varepsilon^{1/5}).$$

Using the implicit function theorem, we obtain a family of solutions such that z and k_- are given by (notice that $a_1' = \mathcal{O}(\varepsilon^{1/5})$)

i) if $a_5 < 0$

$$\begin{aligned} z &= \sqrt{\frac{-3a_5}{2}}\frac{\varepsilon^{1+2/5}}{a_1'}\cosh\phi + \mathcal{O}(\varepsilon^{1+2/5}). \\ k_- &= 2\sqrt{\frac{-2a_5}{3}}\varepsilon^{1+2/5}\exp(-\phi) + \mathcal{O}(\varepsilon^{1+3/5}). \\ \phi &\in \mathbb{R}; \end{aligned} \tag{5.26}$$

ii) if $a_5 > 0$

$$\begin{aligned} z &= \frac{1}{a_1'}\sqrt{\frac{3a_5}{2}}\varepsilon^{1+2/5}\sinh\phi + \mathcal{O}(\varepsilon^{1+2/5}) \\ k_- &= -2\sqrt{\frac{2a_5}{3}}\varepsilon^{1+2/5}\exp(-\phi) + \mathcal{O}(\varepsilon^{1+3/5}) \\ \phi &\in \mathbb{R}. \end{aligned} \tag{5.27}$$

For ε small enough, we notice that the principal part of the solution only depends on g and on coefficient $(d_2 - d_4)$ of the cubic normal form (5.7). The above estimates on u, v, w, z, k_- and Lemma 4.1 imply that the conditions (5.7), (5.8), are satisfied for $\exp|\phi| \leq \varepsilon^{-2/5}$. So, Lemma 5.3 applies and Theorem 1.4 is then proved.

Remark 5.4 *It should be noted that the one parameter family of solutions which are obtained for a fixed ε, correspond to convective rolls at $-\infty$ with wave numbers*

$$k_c(1 + \varepsilon^2 k_-)$$

connected to convective rolls at $+\infty$ with wave numbers

$$k_c(1 + 2\varepsilon^2\widetilde{\omega}_+).$$

The calculations made above, show that we obtain $\widetilde{\omega}_+$ and k_- as functions of ε, ϕ where $\phi \in \mathbb{R}$ such that $\exp|\phi| \leq \varepsilon^{-2/5}$. This is a one parameter family of relationships between wave numbers at each infinity, depending on the amplitude ε^2 of rolls.

Remark 5.5 *We might examine the limit size of k_-. For example, is it possible to obtain the case $k_- = k_+ = 2\widetilde{\omega}_+ = \mathcal{O}(\varepsilon^2)$? Then, looking at the bifurcation equation we need to solve at main orders $(\overline{a_0}z^2 + a_5\varepsilon^2)\varepsilon^{4/5} = \mathcal{O}(\varepsilon^{3+1/5})$.*

Since $a_5 \sim \frac{(d_2-d_4)}{3}\overline{a_0}$, this is only possible with $z \sim \varepsilon\sqrt{\frac{d_4-d_2}{3}}$ provided that

$$d_4 - d_2 > 0,$$

which coefficient of the cubic normal form (5.7) of Appendix 6.5.2 is a function of the Prandtl number.

Chapter 6

Appendix

6.1 Appendix of Chapter 1

6.1.1 Relations between eigenvectors and neutral stability curve

The eigenvalue (double in the bidimensional case) which crosses 0 for $\mu = \mu_0(k)$ is precisely of the form

$$\lambda(\widetilde{\mu}, k) = \lambda_{10}\widetilde{\mu} + \lambda_{02}(k - k_c)^2 + \lambda_{11}\widetilde{\mu}(k - k_c) + \lambda_{20}^2\widetilde{\mu} + \mathcal{O}(|\widetilde{\mu}| + |k - k_c|)^3, \tag{1.1}$$

where

$$\widetilde{\mu} = \mu - \mu_c.$$

Indeed, we know by construction, that

$$\lambda(\widetilde{\mu}, k) = 0 \text{ for } \widetilde{\mu} = \mu_0(k) - \mu_c = \frac{\mu_0''}{2}(k - k_c)^2 + \mathcal{O}(|k - k_c|^3),$$

which implies that there is no term of order $(k - k_c)$ in the Taylor expansion of $\lambda(\widetilde{\mu}, k)$. We deduce that

$$\mu_0'' = -2\frac{\lambda_{02}}{\lambda_{10}}. \tag{1.2}$$

Below we give explicit expressions for λ_{10} and λ_{02} in terms of eigenvectors found in (3.10,3.11). After factorization of e^{ikx}, we have at main order (here we have functions of $z \in (0, 1)$ and upper index means (order in $\widetilde{\mu}$, order in $k - k_c$)

$$\begin{aligned}
\mathcal{P}(D^2 - k_c^2)V_x^{00} + ik_c q^{00} &= 0, \\
\mathcal{P}(D^2 - k_c^2)V_z^{00} + \mathcal{P}\mu_c\theta^{00} + Dq^{00} &= 0, \\
(D^2 - k_c^2)\theta^{00} + \mu_c V_z^{00} &= 0, \\
ik_c V_x^{00} + DV_z^{00} &= 0,
\end{aligned} \tag{1.3}$$

where

$$\begin{aligned} V_z^{00} &= \theta^{00} \text{ at } z=0,1 \\ V_x^{00} &= DV_z^{00}=0 \text{ OR } DV_x^{00}=D^2V_z^{00}=0 \text{ at } z=0,1 \end{aligned}$$

depending on the boundary conditions with $V_\perp=(V_x,0)$. Now at order $\widetilde{\mu}$ we obtain

$$\begin{aligned} \mathcal{P}(D^2-k_c^2)V_x^{10}+ik_cq^{10} &= \lambda_{10}V_x^{00}, \\ \mathcal{P}(D^2-k_c^2)V_z^{10}+\mathcal{P}\mu_c\theta^{10}+Dq^{10}+\mathcal{P}\theta^{00} &= \lambda_{10}V_z^{00}, \\ (D^2-k_c^2)\theta^{10}+\mu_cV_z^{10}+V_z^{00} &= \lambda_{10}\theta^{00}, \\ ik_cV_x^{10}+DV_z^{10} &= 0, \end{aligned}$$

with the same boundary conditions as above for $(V_x^{10},V_z^{10},\theta^{10})$. Making the scalar product in $\{L^2(0,1)\}^3$of the 3 first equations with $(V_x^{00},V_z^{00},\mathcal{P}\theta^{00})$ we find, in using (1.3), and noticing that $\overline{V_x^{00}}=-V_x^{00}$,

$$2\mathcal{P}\int_0^1 V_z^{00}\theta^{00}dz=\lambda_{10}\int_0^1[|V_x^{00}|^2+(V_z^{00})^2+\mathcal{P}(\theta^{00})^2]dz,$$

which gives λ_{10}. Notice that, using the expression (1.4) below, we have, as expected

$$\lambda_{10}>0.$$

At order $k-k_c$ we get

$$\begin{aligned} \mathcal{P}(D^2-k_c^2)V_x^{01}+ik_cq^{01}+iq^{00}-2k_c\mathcal{P}V_x^{00} &= \lambda_{01}V_x^{00}, \\ \mathcal{P}(D^2-k_c^2)V_z^{01}+\mathcal{P}\mu_c\theta^{01}+Dq^{01}-2k_c\mathcal{P}V_z^{00} &= \lambda_{01}V_z^{00}, \\ (D^2-k_c^2)\theta^{01}+\mu_cV_z^{01}-2k_c\theta^{00} &= \lambda_{01}\theta^{00}, \\ ik_cV_x^{01}+DV_z^{01}+iV_x^{00} &= 0, \end{aligned}$$

with the same boundary conditions as above for $(V_x^{01},V_z^{01},\theta^{01})$. Making the scalar product in $\{L^2(0,1)\}^3$of the 3 first equations with $(V_x^{00},V_z^{00},\mathcal{P}\theta^{00})$ we find, in using (1.3), the fourth equation above, and the fact that we have $\lambda_{01}=0$ (expressing that $k=k_c$ gives the minimum of the neutral stability curve)

$$\frac{-1}{k_c}\int_0^1 V_z^{00}Dq^{00}dz=2k_c\mathcal{P}\int_0^1[|V_x^{00}|^2+(V_z^{00})^2+(\theta^{00})^2]dz.$$

We deduce a property of the eigenvector

$$\mu_c\int_0^1 V_z^{00}\theta^{00}dz=\int_0^1[3\left(DV_z^{00}\right)^2+3k_c^2(V_z^{00})^2+2k_c^2(\theta^{00})^2]dz. \qquad (1.4)$$

At order $(k-k_c)^2$ we have

$$\begin{aligned}
\mathcal{P}(D^2-k_c^2)V_x^{02}+ik_cq^{02}+iq^{01}-2k_c\mathcal{P}V_x^{01}-\mathcal{P}V_x^{00} &= \lambda_{02}V_x^{00},\\
\mathcal{P}(D^2-k_c^2)V_z^{02}+\mathcal{P}\mu_c\theta^{02}+Dq^{02}-2k_c\mathcal{P}V_z^{01}-\mathcal{P}V_z^{00} &= \lambda_{02}V_z^{00},\\
(D^2-k_c^2)\theta^{02}+\mu_cV_z^{02}-2k_c\theta^{01}-\theta^{00} &= \lambda_{02}\theta^{00},\\
ik_cV_x^{02}+DV_z^{02}+iV_x^{01} &= 0,
\end{aligned}$$

with the same boundary conditions as above for $(V_x^{02},V_z^{02},\theta^{02})$. Making the scalar product in $\{L^2(0,1)\}^3$of the 3 first equations with $(V_x^{00},V_z^{00},\mathcal{P}\theta^{00})$ we find, in using (1.3) and the fourth equation above,

$$\begin{aligned}
&\int_0^1(iV_x^{01}q^{00}+iV_x^{00}q^{01}+2k_c\mathcal{P}(-V_x^{01}V_x^{00}+V_z^{01}V_z^{00}+\theta^{01}\theta^{00})dz\\
&+\mathcal{P}\int_0^1[|V_x^{00}|^2+(V_z^{00})^2+(\theta^{00})^2]dz\\
=\ &-\lambda_{02}\int_0^1[|V_x^{00}|^2+(V_z^{00})^2+\mathcal{P}(\theta^{00})^2]dz, \qquad (1.5)
\end{aligned}$$

which gives λ_{02} in terms of $(V_x^{00},V_z^{00},\theta^{00},q^{00})$ and $(V_x^{01},V_z^{01},\theta^{01},q^{01})$. By construction, we have (1.2) hence we should have

$$\lambda_{02}<0.$$

6.1.2 Relationship between amplitude of rolls and $\lambda(\widetilde{\mu},k)$

By definition, the eigenvector $\mathbf{U}_1(\widetilde{\mu},k)$ is of the form

$$\mathbf{U}_1(\mu,k)=\widehat{U}_1(z)e^{ikx}$$

and belongs to λ; we have

$$\mathbf{L}_\mu\mathbf{U}_1(\widetilde{\mu},k)=\lambda(\widetilde{\mu},k)\mathbf{U}_1(\widetilde{\mu},k).$$

Let us define

$$\begin{aligned}
\widetilde{\mu} &= \mu-\mu_0(k),\\
\mathbf{L}_\mu &= \mathbf{L}_{\mu_0}+\widetilde{\mu}\mathbf{L}_1,\\
\mathbf{U}_1(\widetilde{\mu},k) &= \mathbf{U}_1^0+\widetilde{\mu}\mathbf{U}_1^1+\mathcal{O}(\widetilde{\mu}^2),
\end{aligned}$$

then, by construction

$$\mathbf{L}_{\mu_0}\mathbf{U}_1^0=0,\ \lambda(\widetilde{\mu},k)=\lambda_{10}\widetilde{\mu}+\mathcal{O}(\widetilde{\mu}^2+|\widetilde{\mu}||k-k_c|), \qquad (1.6)$$

and

$$\begin{aligned}\widetilde{\mu}\mathbf{L}_1\mathbf{U}_1^0+\widetilde{\mu}\mathbf{L}_{\mu_0}\mathbf{U}_1^1 &= \lambda(\widetilde{\mu},k)\mathbf{U}_1^0+\mathcal{O}(\widetilde{\mu}^2)\\ &= \lambda_{10}\widetilde{\mu}\mathbf{U}_1^0+\mathcal{O}(\widetilde{\mu}^2+|\widetilde{\mu}||k-k_c|).\end{aligned}$$

Now the steady convective rolls parallel to x_2 axis, and bifurcating for $\mu>\mu_0(k)$ are solution of

$$(\mathbf{L}_{\mu_0}+\widetilde{\mu}\mathbf{L}_1)\mathbf{U}+(\mu_0+\widetilde{\mu})\mathbf{R}(\mathbf{U},\mathbf{U})=0, \tag{1.7}$$

which is solved below by Lyapunov-Schmidt method. We set

$$\begin{aligned}\mathbf{U} &= \delta(u_1^0+\widetilde{u}_2),\ \widetilde{u}_2\in\{u_1^0\}^\perp,\\ u_1^0 &= \mathbf{U}_1^0+\overline{\mathbf{U}_1^0}\ (=\zeta_1+\overline{\zeta_1}\text{ when }k=k_c),\end{aligned}$$

then

$$\begin{aligned}0 &= \widetilde{\mu}\mathbf{L}_1u_1^0+\mathbf{L}_{\mu_0}\widetilde{u}_2+\widetilde{\mu}\mathbf{L}_1\widetilde{u}_2+\delta(\mu_0+\widetilde{\mu})\mathbf{R}(u_1^0,u_1^0)\\ &\quad +2\delta(\mu_0+\widetilde{\mu})\mathbf{R}(u_1^0,\widetilde{u}_2)+\delta(\mu_0+\widetilde{\mu})\mathbf{R}(\widetilde{u}_2,\widetilde{u}_2).\end{aligned}$$

Defining the orthogonal projection $\mathbf{Q}_0$ on the range of $\mathbf{L}_{\mu_0}$ (self adjoint opeator), we obtain the range equation (we use below the fact that $\mathbf{R}(u_1^0,u_1^0)\in$ range of $\mathbf{L}_{\mu_0}$, as shown below in (1.18))

$$\begin{aligned}0 &= \mathbf{L}_{\mu_0}\widetilde{u}_2+\delta(\mu_0+\widetilde{\mu})\mathbf{R}(u_1^0,u_1^0)+\widetilde{\mu}\mathbf{Q}_0\mathbf{L}_1u_1^0+\\ &\quad +\mathbf{Q}_0[\widetilde{\mu}\mathbf{L}_1\widetilde{u}_2+2\delta(\mu_0+\widetilde{\mu})\mathbf{R}(u_1^0,\widetilde{u}_2)+\delta(\mu_0+\widetilde{\mu})\mathbf{R}(\widetilde{u}_2,\widetilde{u}_2)],\end{aligned}$$

which may be solved by the implicit function theorem (its analytic version):

$$\widetilde{u}_2(\delta,\widetilde{\mu})=-\widetilde{\mathbf{L}_{\mu_0}}^{-1}[\delta\mu_0\mathbf{R}(u_1^0,u_1^0)+\widetilde{\mu}\mathbf{Q}_0\mathbf{L}_1u_1^0]+\mathcal{O}(|\widetilde{\mu}|+|\delta|)^2, \tag{1.8}$$

where $\widetilde{\mathbf{L}_{\mu_0}}^{-1}$ is the pseudo-inverse operator of $\mathbf{L}_{\mu_0}$. Then the projection on the kernel of $\mathbf{L}_{\mu_0}$, obtained via the scalar product with u_1^0 leads to

$$0=\widetilde{\mu}\langle\mathbf{L}_1u_1^0,u_1^0\rangle+2\delta(\mu_0+\widetilde{\mu})\langle\mathbf{R}(u_1^0,\widetilde{u}_2),u_1^0\rangle+\langle\widetilde{\mu}\mathbf{L}_1\widetilde{u}_2+\delta(\mu_0+\widetilde{\mu})\mathbf{R}(\widetilde{u}_2,\widetilde{u}_2),u_1^0\rangle,$$

and after replacing $\widetilde{u}_2$ by its expression (1.8), and using the definition of coefficients a and b computed in Appendix 6.1.4

$$a\widetilde{\mu}+b\delta^2+\mathcal{O}(|\delta\widetilde{\mu}|+\widetilde{\mu}^2+|\delta|^3)=0, \tag{1.9}$$

which is a detailed form of (3.14). Other forms of the result are given by

$$\widetilde{\mu}=\frac{-b}{a}\delta^2+\mathcal{O}(\delta^3),$$

and, using (1.6)

$$\delta^2 = -\frac{a}{b\lambda_{10}}\lambda(\widetilde{\mu}, k)\{1 + \mathcal{O}(|\lambda(\widetilde{\mu}, k)|^{1/2})\}, \tag{1.10}$$

where $a > 0, b < 0, \lambda_{10} > 0$. This detailed expression is used in the problem of orthogonal domain walls (rolls intersecting themselves orthogonaly along a line) at Chapter 5.

6.1.3 Form of the system for amplitudes

We wish to find the general form of smooth vector fields in $\mathbb{R}^6$ (in fact in its complexified space $\mathbb{C} \times \mathbb{C}$) with commute with the following symmetry actions

$$\begin{aligned} \boldsymbol{\tau}_a(A, B, C) &= (Ae^{ik_1\cdot a}, Be^{ik_2\cdot a}, Ce^{ik_3\cdot a}) \text{ for all } a \in \mathbb{R}^2/\Gamma, \\ \mathbf{S}(A, B, C) &= (\overline{A}, \overline{C}, \overline{B}), \quad \mathbf{R}_{2\pi/3}(A, B, C) = (C, A, B), \\ \mathbf{R}_\pi(A, B, C) &= (\overline{A}, \overline{B}, \overline{C}). \end{aligned}$$

For the component along ζ_1 we need to satisfy

$$e^{ik_1\cdot a}P(A, B, C, c.c.) = P(Ae^{ik_1\cdot a}, Be^{ik_2\cdot a}, Ce^{ik_3\cdot a}, c.c.) \ \forall a \in \mathbb{R}^2/\Gamma, \tag{1.11}$$

$$\overline{P(A, B, C, \overline{A}, \overline{B}, \overline{C})} = P(\overline{A}, \overline{B}, \overline{C}, A, B, C) \tag{1.12}$$

Let us first consider the coefficients of the Taylor series of P. Studying monomials $A^p\overline{A}^{p'}B^q\overline{B}^{q'}C^r\overline{C}^{r'}$, we see by (1.11) that the only possible monomials are such that

$$(p - p' - 1)k_1 + (q - q')k_2 + (r - r')k_3 = 0$$

then, taking into account of

$$k_1 + k_2 + k_3 = 0,$$

this implies

$$p - p' - 1 = q - q' = r - r'.$$

Hence the only possible monomials of P are

$$\begin{aligned} A|A|^{2p'}|B|^{2q'}|C|^{2r'}(ABC)^l, \ l &\geq 0, \\ \overline{BC}(\overline{ABC})^{l'-1}|A|^{2p}|B|^{2q}|C|^{2r}, \ l' &\geq 1. \end{aligned}$$

The symmetries under (1.12) and under S imply that coefficients are real and symmetric in (B, C). Then the first component, factor of ζ_1, takes the form

$$AP_1[|A|^2, |B|^2, |C|^2, (ABC)] + \overline{BC}P_2[|A|^2, |B|^2, |C|^2, (\overline{ABC})],$$

with in addition the symmetry in (B, C) where P_1 and P_2 are smooth functions.Components on ζ_2 and ζ_3 are obtained in $\mathbf{R}_{2\pi/3}$, by circular permutation on (A, B, C). Hence the component on ζ_2 becomes

$$BP_1[|B|^2, |C|^2, |A|^2, (ABC)] + \overline{CA}P_2[|B|^2, |C|^2, |A|^2, (\overline{ABC})],$$

while on ζ_3 it becomes

$$CP_1[|C|^2, |A|^2, |B|^2, (ABC)] + \overline{AB}P_2[|C|^2, |A|^2, |B|^2, (\overline{ABC})].$$

Then, if we truncate the system at cubic order, this gives the system (3.13). Notice that higher order terms start at order 4, with

$$eA^2BC + f|A|^2\overline{BC} + g\overline{BC}(|B|^2 + |C|^2) \tag{1.13}$$

for the component along ζ_1.

Remark 1.1 *In the case of free-free boundary conditions the additional symmetry S_z acts as*

$$S_z(A, B, C) = -(A, B, C).$$

This symmetry implies that function P_1 is even in (ABC) while P_2 is odd in (ABC) (see the proof in [38]). It then results that the terms of higher orders start now at order 5, with

$$e\overline{A}\left(\overline{BC}\right)^2 + fA|A|^4 + gA|A|^2(|B|^2 + |C|^2) + hA(|B|^2 + |C|^2)^2 + jA|B|^2|C|^2. \tag{1.14}$$

for the component along ζ_1.

6.1.4 Calculation of coefficients

It is shown in [28] that the dynamics in a neighborhood of μ_c is reduced to a center manifold of dimension 6, of the form

$$\mathbf{U} = \mathbf{U}_0 + \Phi(\widetilde{\mu}, \mathbf{U}_0) \tag{1.15}$$

with $\widetilde{\mu} = \mu - \mu_c$, $\mathbf{U}_0 \in \mathcal{E}_0$ given by (3.12) and Φ of class C^k, k fixed as large as we wish. Its Taylor expansion is written as

$$\Phi(\widetilde{\mu}, \mathbf{U}_0) = \sum_{s+p+p'+q+q'+r+r' \geq 2} \widetilde{\mu}^s A^p \overline{A}^{p'} B^q \overline{B}^{q'} C^r \overline{C}^{r'} \Phi^{(s)}_{pp'qq'rr'},$$

$$\begin{aligned} \Phi^{(s)}_{pp'qq'rr'} &= e^{i[(p-p')k_1 + (q-q')k_2 + (r-r')k_3]\cdot X} \widehat{\Phi}^{(s)}_{pp'qq'rr'}(z), \\ \widehat{\Phi}^{(s)}_{p'pq'qr'r} &= \overline{\widehat{\Phi}^{(s)}_{pp'qq'rr'}}. \end{aligned}$$

We then have the identity

$$(1 + D_{U_0}\Phi)\frac{d\mathbf{U}_0}{dt} = \mathbf{L}_\mu \Phi(\widetilde{\mu}, \mathbf{U}_0) + \mu \mathbf{R}(\mathbf{U}_0 + \Phi(\widetilde{\mu}, \mathbf{U}_0), \mathbf{U}_0 + \Phi(\widetilde{\mu}, \mathbf{U}_0)). \tag{1.16}$$

Then we replace $\frac{d\mathbf{U}_0}{dt}$ by its form given a priori by the amplitude equations (3.13) and it remain to identify monomials $\widetilde{\mu}^s A^p \overline{A}^{p'} B^q \overline{B}^{q'} C^r \overline{C}^{r'}$.

Calculation of a

First we define the perturbation operator $\mathbf{L}_1$

$$\mathbf{L}_\mu = \mathbf{L}_{\mu_c} + \mu \mathbf{L}_1$$

with

$$\mathbf{L}_1 \mathbf{U} = (0, V_z).$$

Then coefficient a is obtained by the coefficient of $\widetilde{\mu} A$

$$a\zeta_1 = \mathbf{L}_{\mu_c} \Phi^{(1)}_{100000} + \mathbf{L}_1 \zeta_1,$$

hence taking the scalar product with ζ_1, since $\mathbf{L}_{\mu_c}$ is selfadjoint,

$$a\langle \zeta_1, \zeta_1 \rangle = \langle \mathbf{L}_1 \zeta_1, \zeta_1 \rangle,$$

and we notice that this is the same as what was given by (1.9) for $k = k_c$ since $u_1^0 = \zeta_1 + \overline{\zeta_1}$:

$$a\langle u_1^0, u_1^0 \rangle = \langle \mathbf{L}_1 u_1^0, u_1^0 \rangle.$$

We can show that $a > 0$. Indeed

$$\langle \mathbf{L}_1 \zeta_1, \zeta_1 \rangle = \mathcal{P} \int_\Omega V_z \theta dx_1 dx_2 dz$$

and from (3.10) this gives (using the boundary condition on θ)

$$\begin{aligned} a\langle \zeta_1, \zeta_1 \rangle &= -\frac{\mathcal{P}}{\mu_c} \int_\Omega \theta (D^2 - k^2)\theta dx_1 dx_2 dz \\ &= \frac{\mathcal{P}}{\mu_c} \int_\Omega [(D\theta)^2 + k^2 \theta^2] dx_1 dx_2 dz > 0. \end{aligned}$$

Calculation of c

The coefficient c is obtained by identifying the coefficient of $\overline{BC}$

$$c\zeta_1 = \mathbf{L}_{\mu_c} \overline{\Phi_{000101}} + 2\mu_c \mathbf{R}(\overline{\zeta_2}, \overline{\zeta_3}),$$

hence

$$c\langle \zeta_1, \zeta_1 \rangle = \langle 2\mu_c \mathbf{R}(\overline{\zeta_2}, \overline{\zeta_3}), \zeta_1 \rangle.$$

Let us show that this leads to

$$c = 0.$$

Indeed, coefficient c, which is known to be real, may also be written as

$$6c = \mu_c \langle \mathbf{R}(U, U), U \rangle \tag{1.17}$$

with

$$U = \zeta_1 + \zeta_2 + \zeta_3 + \overline{\zeta_1} + \overline{\zeta_2} + \overline{\zeta_3}$$

since the only contribution in the scalar product comes from monomials where the dependency in X comes from

$$e^{\pm i(k_1+k_2+k_3)\cdot X} = 1, \text{ since } k_1 + k_2 + k_3 = 0,$$

while all other monomials give 0 because of periodicity. Now, a remarkable property of the quadratic operator $\boldsymbol{R}$ in (3.4) is that

$$\langle \mathbf{R}(\mathbf{U}, \mathbf{U}), \mathbf{U} \rangle = 0, \ \forall \mathbf{U} \text{ real in } \mathcal{Z}. \tag{1.18}$$

Proof of (1.18) From the expressions of $\mathbf{R}$ and of the scalar product in $\mathcal{X}$, we obtain

$$\begin{aligned} \langle \mathbf{R}(\mathbf{U}, \mathbf{U}), \mathbf{U} \rangle &= -\langle \Pi_0(\mathbf{V} \cdot \nabla \mathbf{V}), \mathbf{V} \rangle - \mathcal{P} \langle (\mathbf{V} \cdot \nabla \theta)\theta \rangle \\ &= -\frac{1}{2} \langle \mathbf{V} \cdot \nabla \mathbf{V}^2 \rangle - \frac{\mathcal{P}}{2} \langle \mathbf{V} \cdot \nabla \theta^2 \rangle. \end{aligned}$$

By Stokes formula, these integrals reduce to integrals on the boundary of the periodic cell. This is $= 0$ since $\mathbf{V} \cdot n = 0$ for $z = 0, 1$ and by periodicity on the other boundaries of the periodic cell. QED

Finally, $c = 0$.

Calculation of b

Coefficient b is obtained from identification of monomials in $A^2\overline{A}$:

$$\begin{aligned} \mathbf{L}_{\mu_c} \Phi_{200000} + \mu_c \mathbf{R}(\zeta_1, \zeta_1) &= 0 \\ \mathbf{L}_{\mu_c} \Phi_{110000} + 2\mu_c \mathbf{R}(\zeta_1, \overline{\zeta_1}) &= 0 \end{aligned}$$

$$b\zeta_1 = \mathbf{L}_{\mu_c} \Phi_{210000} + 2\mu_c \mathbf{R}(\zeta_1, \Phi_{110000}) + 2\mu_c \mathbf{R}(\overline{\zeta_1}, \Phi_{200000}),$$

$$b\langle\zeta_1,\zeta_1\rangle = \mu_c\langle 2\mathbf{R}(\zeta_1,\Phi_{110000}) + 2\mathbf{R}(\overline{\zeta_1},\Phi_{200000}),\zeta_1\rangle.$$

We notice that

$$\mathbf{R}(\overline{\zeta_1},\Phi_{020000}) \text{ and } \mathbf{R}(\zeta_1,\Phi_{200000})$$

are orthogonal to $\zeta_1+\overline{\zeta_1}$ since $e^{\pm 3ik_cx_1}$ is orthogonal to $e^{\pm ik_cx_1}$. Hence

$$b\langle\zeta_1+\overline{\zeta_1},\zeta_1+\overline{\zeta_1}\rangle = \mu_c\langle 2\mathbf{R}(\zeta_1+\overline{\zeta_1},\Phi_2),\zeta_1+\overline{\zeta_1}\rangle$$

where Φ_2 is defined by

$$\mathbf{L}_{\mu_c}\Phi_2 + \mu_c\mathbf{R}(\zeta_1+\overline{\zeta_1},\zeta_1+\overline{\zeta_1}) = 0.$$

We notice that it is the same formula as what is given by (1.9) for $k=k_c$. Replacing U by U_1+tU_2 in property (1.18) of the quadratic operator $\boldsymbol{R}$ and identifying the coefficient of t implies that

$$2\langle\boldsymbol{R}(\mathbf{U}_1,\mathbf{U}_2),\mathbf{U}_1\rangle + \langle\boldsymbol{R}(\mathbf{U}_1,\mathbf{U}_1),\mathbf{U}_2\rangle = 0\ \forall\mathbf{U}_1,\mathbf{U}_2 \text{ real in } \mathcal{Z}.$$

We then obtain

$$\begin{aligned} b\langle\zeta_1+\overline{\zeta_1},\zeta_1+\overline{\zeta_1}\rangle &= -\mu_c\langle\mathbf{R}(\zeta_1+\overline{\zeta_1},\zeta_1+\overline{\zeta_1}),\Phi_2\rangle \\ &= \langle\mathbf{L}_{\mu_c}\Phi_2,\Phi_2\rangle \end{aligned}$$

which is <0 since Φ_2 is orthogonal to the kernel of $\mathbf{L}_{\mu_c}$ which is self adjoint with all nonzero eigenvalues which are <0. This property was noticed by V.Yudovich [75].

6.2 Appendix of Chapter 2

6.2.1 Inverse of L

In this appendix we compute and estimate the inverse of the operator L. By construction, solving the equation

$$LU = G = (F,g) \in \mathcal{K}_{0,s},\ \text{with } U \in D_s(L), \tag{2.1}$$

means that we have

$$\begin{aligned} \Delta V - \nabla q &= F,\ \nabla\cdot V = 0, \\ \Delta\theta &= g, \end{aligned}$$

i.e.

$$(D^2-k^2)v_{\mathbf{k}}^{(z)} - Dq_{\mathbf{k}} = f_{\mathbf{k}}^{(z)}, \tag{2.2}$$
$$(D^2-k^2)V_{\mathbf{k}}^{(H)} - i\mathbf{k}q_{\mathbf{k}} = F_{\mathbf{k}}^{(H)}, \tag{2.3}$$
$$(D^2-k^2)\theta_{\mathbf{k}} = g_{\mathbf{k}}, \tag{2.4}$$
$$Dv_{\mathbf{k}}^{(z)} + i\mathbf{k}\cdot V_{\mathbf{k}}^{(H)} = 0, \tag{2.5}$$

where $k = |k|$, and with the boundary conditions :

$$\begin{aligned} \theta &= v_{\mathbf{k}}^{(z)} = 0 \text{ in } z = 0,1, \\ V_{\mathbf{k}}^{(H)}|_{z=0,1} &= 0, \text{ or } V_{\mathbf{k}}^{(H)}|_{z=0} = DV_{\mathbf{k}}^{(H)}|_{z=1} = 0, \text{ or } V_{\mathbf{k}}^{(H)}|_{z=1} = DV_{\mathbf{k}}^{(H)}|_{z=0} = 0. \end{aligned}$$

The above system is a classical one (Stokes operator and Laplace operator), already obtained in the periodic case. The only thing to check concerns the estimates with respect to $k \in \Gamma$. The scalar product of (2.2) with $v_{\mathbf{k}}^{(z)}$ plus the scalar product of (2.3) with $V_{\mathbf{k}}^{(H)}$ and integration by parts, taking into account of $Dv_{\mathbf{k}}^{(z)} + ik \cdot V_{\mathbf{k}}^{(H)} = 0$ and of the boundary values, leads to

$$||DV_{\mathbf{k}}||_0^2 + k^2 ||V_{\mathbf{k}}||_0^2 = -\int_0^1 F_{\mathbf{k}} \cdot \overline{V}_{\mathbf{k}} dz \leq ||F_{\mathbf{k}}||_0 ||V_{\mathbf{k}}||_0 \leq \frac{k^2}{2} ||V_{\mathbf{k}}||_0^2 + \frac{1}{2k^2} ||F_{\mathbf{k}}||_0^2. \quad (2.6)$$

Moreover, thanks to the boundary conditions, we have also the Poincaré estimate (see (5.9))

$$||V_{\mathbf{k}}||_0 \leq \frac{1}{\sqrt{2}} ||DV_{\mathbf{k}}||_0.$$

It results that there exists $c > 0$ such that for any $k \in \Gamma$ we have

$$(1+k^2)||DV_{\mathbf{k}}||_0^2 + (1+k^2)^2 ||V_{\mathbf{k}}||_0^2 \leq c||F_{\mathbf{k}}||_0^2. \quad (2.7)$$

The same (simpler) is valid for θ :

$$(1+k^2)||D\theta_{\mathbf{k}}||_0^2 + (1+k^2)^2 ||\theta_{\mathbf{k}}||_0^2 \leq c||g_{\mathbf{k}}||_0^2. \quad (2.8)$$

Now (2.4) leads to

$$||D^2\theta_{\mathbf{k}}||_0 \leq k^2 ||\theta_{\mathbf{k}}||_0 + ||g_{\mathbf{k}}||_0 \leq c' ||g_{\mathbf{k}}||_0,$$

hence, using (2.8),

$$||\theta_{\mathbf{k}}||_2 \leq c_1 ||g_{\mathbf{k}}||_0. \quad (2.9)$$

Let us show that the same type of estimate holds for $V_{\mathbf{k}} = (V_{\mathbf{k}}^{(H)}, v_{\mathbf{k}}^{(z)})$. We observe that the divergence free condition on F leads to

$$Df_{\mathbf{k}}^{(z)} + i\mathbf{k} \cdot F_{\mathbf{k}}^{(H)} = 0$$

which implies

$$\begin{aligned} (D^2 - k^2) q_{\mathbf{k}} &= 0, & (2.10) \\ (D^2 - k^2)^2 v_{\mathbf{k}}^{(z)} &= (D^2 - k^2) f_{\mathbf{k}}^{(z)}, & (2.11) \end{aligned}$$

with boundary conditions on $v_{\mathbf{k}}^{(z)}$ as $v_{\mathbf{k}}^{(z)}|_{z=0,1} = 0$, $Dv_{\mathbf{k}}^{(z)}|_{z=0,1} = 0$ or $Dv_{\mathbf{k}}^{(z)}|_{z=0} = 0, D^2 v_{\mathbf{k}}^{(z)}|_{z=1} = 0$, or $Dv_{\mathbf{k}}^{(z)}|_{z=1} = 0, D^2 v_{\mathbf{k}}^{(z)}|_{z=0} = 0$. Now taking the scalar product of (2.11) with $v_{\mathbf{k}}^{(z)}$ in $L^2(0,1)$, and integrations by parts, lead to

$$\begin{aligned} ||D^2 v_{\mathbf{k}}^{(z)}||^2 + 2k^2 ||Dv_{\mathbf{k}}^{(z)}||^2 + k^4 ||v_{\mathbf{k}}^{(z)}||^2 &= \int_0^1 i\mathbf{k} \cdot F_{\mathbf{k}}^{(H)} D\overline{v}_{\mathbf{k}}^{(z)} dz - k^2 \int_0^1 f_{\mathbf{k}}^{(z)} \overline{v}_{\mathbf{k}}^{(z)} dz \\ &\leq k ||Dv_{\mathbf{k}}^{(z)}||_0 ||F_{\mathbf{k}}^{(H)}||_0 + k^2 ||v_{\mathbf{k}}^{(z)}||_0 ||f_{\mathbf{k}}^{(z)}||_0. \end{aligned}$$

Taking into account of (2.7), we obtain immediately

$$||v_{\mathbf{k}}^{(z)}||_2 \leq c_2 ||F_{\mathbf{k}}||_0. \tag{2.12}$$

Now, in using (2.2) we can say that

$$||Dq_{\mathbf{k}}|| \leq c_3 ||F_{\mathbf{k}}||_0, \tag{2.13}$$

where c_3 is independent of $k \in \Gamma$. Now (2.10) gives

$$q_{\mathbf{k}} = \alpha_{\mathbf{k}} e^{kz} + \beta_{\mathbf{k}} e^{-kz},$$

and

$$Dq_{\mathbf{k}} = k\alpha_{\mathbf{k}} e^{kz} - k\beta_{\mathbf{k}} e^{-kz}$$

should satisfy (2.13). It is easy to check that this implies that

$$k\alpha_{\mathbf{k}}^2 e^{2k} + k\beta_{\mathbf{k}}^2 - 4k^2 \alpha_{\mathbf{k}} \beta_{\mathbf{k}}$$

is bounded by $2c_3^2 ||F_{\mathbf{k}}||_0^2$ for large k. Now since

$$|4k^2 \alpha_{\mathbf{k}} \beta_{\mathbf{k}}| \leq 8k^3 \alpha_{\mathbf{k}}^2 + \frac{k}{2} \beta_{\mathbf{k}}^2,$$

and since, for large k, $8k^3 << ke^{2k}$, we conclude that for large k the quantity $k\alpha_{\mathbf{k}}^2 e^{2k} + k\beta_{\mathbf{k}}^2$ is bounded by $c_4 ||F_{\mathbf{k}}||_0^2$. Now computing $||kq_{\mathbf{k}}||^2$, we see the same behavior in $k\alpha_{\mathbf{k}}^2 e^{2k} + k\beta_{\mathbf{k}}^2$ for large k. It follows that we have

$$||kq_{\mathbf{k}}|| \leq c_5 ||F_{\mathbf{k}}||_0,$$

and (2.3) allows to conclude that

$$||V_{\mathbf{k}}^{(H)}||_2 \leq c_6 ||F_{\mathbf{k}}||_0.$$

Collecting all the above estimates gives for a certain constant $c > 0$

$$||U_{\mathbf{k}}||_2 \leq c ||G_{\mathbf{k}}||_0,$$

which is the desired estimate for L^{-1} now bounded from $\mathcal{K}_{0,s}$ to $D_s(L) \subset \mathcal{K}_{2,s}$.

Extension of the inverse of L

Let us consider now the same equation (2.1) but with a less regular right hand side. Now we take $G \in (\mathcal{D}_{1/2,s})^*$ which is the dual of $\mathcal{D}_{1/2,s}$ defined in (5.5). This means that for any $V \in \mathcal{D}_{1/2,s}$, we have the following bound for the semi-linear form $\langle G, V\rangle_{0,s}$:

$$|\langle G, V\rangle_{0,s}| \leq ||G||_{(\mathcal{D}_{1/2,s})^*}||V||_{\widetilde{1,s}}.$$

We are now looking for $U \in \mathcal{D}_{1/2,s}$ defined by a variational formulation (also classical for the Stokes linear operator, as well as for the Laplace operator (see [76]), both written in Fourier components)

$$\langle U, V\rangle_{\widetilde{1,s}} = -\langle G, V\rangle_{0,s} \text{ for any } V \in \mathcal{D}_{1/2,s},$$

where the definition of $\langle U, V\rangle_{\widetilde{1,s}}$ comes from (5.10). For the type of discussion which follows, we may also refer to [54] p.223-224, adapted to each Fourier component here.

It is easy to check, in looking at the first equality in (2.6) and its analogue for $\theta_{\mathbf{k}}$, that

$$\langle LU, V\rangle_{0,s} = \langle G, V\rangle_{0,s} \text{ for any } V \in \mathcal{D}_{1/2,s},$$

holds, where the brackets are dual products. This proves that the unique solution $U \in \mathcal{D}_{1/2,s}$, hence by definition $(-L)^{1/2}U \in \mathcal{K}_{0,s}$ and

$$||U||_{\widetilde{1,s}} \leq ||G||_{(\mathcal{D}_{1/2,s})^*}, \tag{2.14}$$

which means that the operator L which is bounded from $\mathcal{D}_{1/2,s}$ to $(\mathcal{D}_{1/2,s})^*$, has its inverse bounded from $(\mathcal{D}_{1/2,s})^*$ to $\mathcal{D}_{1/2,s}$.

6.2.2 Proof of Lemmas 4.8

Let u be a scalar function in $\mathcal{H}^{(1)}_{1,s}$, which means that

$$u(\mathbf{x}, z) = \sum_{\mathbf{k}\in\Gamma} u_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}},$$

with

$$\sum_{\mathbf{k}\in\Gamma}(1 + N_{\mathbf{k}}^2)^s||u_{\mathbf{k}}||_1^2 < \infty, \quad ||u_{\mathbf{k}}||_1^2 = \int_0^1 \left(|Du_{\mathbf{k}}|^2 + (1 + |\mathbf{k}|^2)|u_{\mathbf{k}}|^2\right) dz.$$

Assume now that u and v are scalar functions in $\mathcal{H}^{(1)}_{1,s}$, then

$$||uv||^2_{\mathcal{H}_{1,s}} = \int_0^1 \sum_{\mathbf{k}\in\Gamma}(1 + N_{\mathbf{k}}^2)^s \left(|D(uv)_{\mathbf{k}}|^2 + (1 + |\mathbf{k}|^2)|(uv)_{\mathbf{k}}|^2\right) dz$$

and using $(a+b)^2 \leq 2a^2+2b^2$

$$\leq \int_0^1 \sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}^2)^s \left(2|(vDu)_{\mathbf{k}}|^2 + 2|(uDv)_{\mathbf{k}}|^2 + (1+|\mathbf{k}|^2)|(uv)_{\mathbf{k}}|^2\right).$$

From Lemma 4.2 we have

$$\begin{aligned}\sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}^2)^s|(uDv)_{\mathbf{k}}(z)|^2 \quad \leq \quad & 2C(s,s_0)^2(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^s|u_{\mathbf{l}}(z)|^2)(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^{s_0}|Dv_{\mathbf{m}}(z)|^2)+\\ & +2C(s,s_0)^2(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^{s_0}|u_{\mathbf{l}}(z)|^2)(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^s|Dv_{\mathbf{m}}(z)|^2),\end{aligned}$$

and the analogue holds for vDu.

Now, introduce u' and $\widetilde{u}$ defined by

$$\widetilde{u}_{\mathbf{k}} = |u_{\mathbf{k}}| \quad u'_{\mathbf{k}} = \sqrt{1+\mathbf{k}^2}\,\widetilde{u}_{\mathbf{k}},$$

then, in using $(1+|\mathbf{l}+\mathbf{m}|^2) \leq 2((1+|\mathbf{l}|^2)+2(1+|\mathbf{m}|^2)$

$$\begin{aligned}\sqrt{1+\mathbf{k}^2}|(uv)_{\mathbf{k}}| \quad & \leq \quad \sqrt{1+\mathbf{k}^2}\sum_{\mathbf{k}=\mathbf{l}+\mathbf{m}}\frac{u'_{\mathbf{l}}v'_{\mathbf{m}}}{\sqrt{1+\mathbf{l}^2}\sqrt{1+\mathbf{m}^2}}\\ & \leq \quad \sqrt{2}\sum_{\mathbf{k}=\mathbf{l}+\mathbf{m}}\widetilde{u}_{\mathbf{l}}v'_{\mathbf{m}} + u'_{\mathbf{l}}\widetilde{v}_{\mathbf{m}} = \sqrt{2}[(\widetilde{u}v')_{\mathbf{k}} + (u'\widetilde{v})_{\mathbf{k}}].\end{aligned}$$

Hence

$$(1+\mathbf{k}^2)|(uv)_{\mathbf{k}}|^2 \leq 4(|(\widetilde{u}v')_{\mathbf{k}}|^2 + |(u'\widetilde{v})_{\mathbf{k}}|^2),$$

and using again Lemma 4.2 we obtain

$$\begin{aligned}\sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}^2)^s(1+|\mathbf{k}|^2)|(uv)_{\mathbf{k}}|^2 \quad \leq \quad & 8C(s,s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^s(1+|\mathbf{l}|^2)|u_{\mathbf{l}}|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^{s_0}|v_{\mathbf{m}}|^2\right)+\\ & +8C(s,s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^s|u_{\mathbf{l}}|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^{s_0}(1+|\mathbf{m}|^2)|v_{\mathbf{m}}|^2\right)+\\ & +8C(s,s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^{s_0}(1+|\mathbf{l}|^2)|u_{\mathbf{l}}|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^s|v_{\mathbf{m}}|^2\right)+\\ & +8C(s,s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^{s_0}|u_{\mathbf{l}}|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^s(1+|\mathbf{m}|^2)|v_{\mathbf{m}}|^2\right).\end{aligned}$$

Now we can use

$$\begin{aligned}\int_0^1 |Du_{\mathbf{l}}|^2|v_{\mathbf{m}}|^2dz \quad & \leq \quad c||u_{\mathbf{l}}||_{H^1}^2||v_{\mathbf{m}}||_{H_1}^2\\ \int_0^1(1+|\mathbf{l}|^2)|u_{\mathbf{l}}|^2|v_{\mathbf{m}}|^2dz \quad & \leq \quad c(1+|\mathbf{l}|^2)||u_{\mathbf{l}}||_{L^2}^2||v_{\mathbf{m}}||_{H_1}^2\end{aligned}$$

and the similar symmetric estimates to show that there is a constant $c^2(s,s_0) = 10cC^2(s,s_0)$ such that finally

$$||uv||_{1,s}^2 \le c^2(s,s_0)(||u||_{1,s}^2||v||_{1,s_0}^2 + ||u||_{1,s_0}^2||v||_{1,s}^2),$$

Lemma 4.8 is proved.

Assume now that u and v are scalar functions, respectively in $\mathcal{H}_{1,s}^{(1)}$ and $\mathcal{H}_{0,s}^{(1)}$ with $s \ge s_0 > d/2$. Then

$$||uv||_{0,s}^2 = \int_0^1 \sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}^2)^s|(uv)_{\mathbf{k}}|^2 dz$$

which gives, by Lemma 4.2

$$\begin{aligned}\sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}^2)^s|(uv)_{\mathbf{k}}|^2 \quad &\le \quad 2C(s,s_0)^2(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^s|u_{\mathbf{l}}|^2)(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^{s_0}|v_{\mathbf{m}}|^2)+\\ &\quad +2C(s,s_0)^2(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^{s_0}|u_{\mathbf{l}}|^2)(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^{s}|v_{\mathbf{m}}|^2).\end{aligned}$$

Now we use

$$\int_0^1 |u_{\mathbf{l}}|^2|v_{\mathbf{m}}|^2 dz \le c||u_{\mathbf{l}}||_{H^1}^2||v_{\mathbf{m}}||_{L^2}^2,$$

which leads to

$$||uv||_{0,s}^2 \le 2cC(s,s_0)^2(||u||_{1,s}^2||v||_{0,s_0}^2 + ||u||_{1,s_0}^2||v||_{0,s}^2)$$

which gives Lemma 4.9.

Now by Lemma 4.1 we have for all $z\in(0,1)$ the two inequalities

$$\sum_{\mathbf{k}\in\Gamma}|(uv)_{\mathbf{k}}|^2 \le 2c_s(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^s|u_{\mathbf{l}}|^2)(\sum_{\mathbf{m}\in\Gamma}|v_{\mathbf{m}}|^2), \tag{2.15}$$

and

$$\sum_{\mathbf{k}\in\Gamma}|(uv)_{\mathbf{k}}|^2 \le 2c_s(\sum_{\mathbf{l}\in\Gamma}|u_{\mathbf{l}}|^2)(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^s|v_{\mathbf{m}}|^2). \tag{2.16}$$

We also have for some $c>0$:

$$\int_0^1 |u_{\mathbf{l}}|^2|v_{\mathbf{m}}|^2 dz \le c\min\{||u_{\mathbf{l}}||_{H^1}^2||v_{\mathbf{m}}||_{L^2}^2, ||u_{\mathbf{l}}||_{L^2}^2||v_{\mathbf{m}}||_{H^1}^2\}.$$

Then summing (2.15) on $(0,1)$ and using the last inequality leads to Lemma 4.10, while summing (2.16) and using the last inequality leads to Lemma 4.11.

6.2.3 Proofs of the bounds for the quadratic term

For $U \in \mathcal{K}_{2,s}$, we have all components of ∇V and $\nabla\theta$ which are in $\mathcal{H}_{1,s}^{(1)}$. Moreover, for $U \in \mathcal{K}_{2,s}$ and $U' \in \mathcal{K}_{2,s}$ Lemma 4.8 says that the components of

$$V \cdot \nabla V', \quad V \cdot \nabla\theta', \quad V' \cdot \nabla V, \quad V' \cdot \nabla\theta$$

satisfy estimates given by this Lemma in $\mathcal{H}_{1,s}$. The projection $\mathfrak{P}$ does not change the estimates, hence

$$||B(U,U')||_{1,s} \leq c(s,s_0)(||U||_{2,s}||U'||_{2,s_0} + ||U||_{2,s_0}||U'||_{2,s}),$$

which is (5.2).

For proving (5.3) we have $U \in \mathcal{K}_{1,s}$, hence components of ∇V and $\nabla\theta \in \mathcal{H}_{0,s}^{(1)}$ and Lemma 4.9 shows that the components of $V \cdot \nabla V'$, and $V \cdot \nabla\theta'$ lie in $\mathcal{H}_{0,s}^{(1)}$. To obtain $B(U,U')$ we just need to apply the projection $\mathfrak{P}$ to $V \cdot \nabla V'$ and to $V' \cdot \nabla V$. Then estimate (5.3) results immediately from estimate of Lemma 4.9.

For proving (5.15) we need to prove that for $(U,V) \in \mathcal{K}_{1,s} \times \mathcal{K}_{1,0}$ then

$$||B(U,V)||_{0,0} \leq c'||U||_{1,s}||V||_{1,0}.$$

Indeed, components of ∇U and ∇V belong respectively to $\mathcal{H}_{0,s}$ and $\mathcal{H}_{0,0}$ and we need to consider products of functions of the forms $\mathcal{H}_{0,s} \times \mathcal{H}_{1,0}$ and $\mathcal{H}_{1,s} \times \mathcal{H}_{0,0}$. Then Lemmas 4.10 and 4.11 and projecting by $\mathfrak{P}$ (as above) allow to prove that $B(U,V) \in \mathcal{K}_{0,0}$ with the required estimate (5.15).

6.2.4 Study of the nondegeneracy condition leading to (6.6)

Let us come back to the homogeneous system associated with (6.3), which gives for every fixed $\mathbf{k} \in \Gamma$ the discrete set of eigenvalues $\lambda_j(|\mathbf{k}|), j = 0,1,2,...$ (below, for the sake of simplicity, we omit to consider λ_0 as a function of $|k|^2$). Below, we only consider $\mathbf{k}$ in $\mathbb{R}^+$ since we know that only its modulus matters. We are interested in the concavity of the graph of $\lambda_0(k)$ in the neighborhood of $k = k_c > 0$, where $\frac{d\lambda_0}{dk}(k_c) = 0$.

By construction, we have

$$\begin{aligned}
\lambda_0(D^2 - k^2)v_{\mathbf{k}}^{(z)} + \theta_{\mathbf{k}} - Dq_{\mathbf{k}} &= 0, \\
\lambda_0(D^2 - k^2)V_{\mathbf{k}}^{(H)} - ik\mathbf{e}_1 q_{\mathbf{k}} &= 0, \\
\lambda_0(D^2 - k^2)\theta_{\mathbf{k}} + v_{\mathbf{k}}^{(z)} &= 0, \\
Dv_{\mathbf{k}}^{(z)} + ik\mathbf{e}_1 \cdot V_{\mathbf{k}}^{(H)} &= 0,
\end{aligned} \tag{2.17}$$

where $D = d/dz$, $\mathbf{e}_1$ is the unit vector along the x axis, and where

$$v_{\mathbf{k}}^{(z)}|_{z=0,1} = \theta_{\mathbf{k}}|_{z=0,1} = 0,$$

and either

$$V_{\mathbf{k}}^{(H)}|_{z=0,1} = 0, \quad \text{or } V_{\mathbf{k}}^{(H)}|_{z=0} = DV_{\mathbf{k}}^{(H)}|_{z=1} = 0, \text{ or } V_{\mathbf{k}}^{(H)}|_{z=1} = DV_{\mathbf{k}}^{(H)}|_{z=0} = 0.$$

For $k = k_c > 0$ the eigenvalue $\lambda_0(k)$ reaches $\lambda_0 > 0$ where $\frac{d\lambda_0}{dk}(k_c) = 0$, as this results from the analyticity of the function $\lambda_0(k)$ with $\lambda_0 \to 0$ as $k \to 0$ and as $k \to \infty$ (see [74], [75]). Our purpose is to compute $\frac{d^2\lambda_0}{dk^2}(k_c)$. We need $\frac{d^2\lambda_0}{dk^2}(k_c) \neq 0$ for establishing (6.6) since the denominator in (6.6) corresponds, up to a factor, to $\lambda(k) - \lambda_0$ in a neighborhood of k_c (notice that the function $\lambda(k)$ is even in k). In fact it is only known numerically that there is only one maximum and that the graph is concave at this point, so we intend to just give a formula for $\lambda_0'' = \frac{d^2\lambda_0}{dk^2}(k_c)$.

More precisely let us differentiate (2.17) with respect to k :

$$\begin{aligned} \lambda_0'(D^2-k^2)v_{\mathbf{k}}^{(z)} - 2\lambda_0 k v_{\mathbf{k}}^{(z)} + \lambda_0(D^2-k^2)v_{\mathbf{k}}'^{(z)} + \theta_{\mathbf{k}}' - Dq_{\mathbf{k}}' &= 0, \\ \lambda_0'(D^2-k^2)V_{\mathbf{k}}^{(H)} - 2\lambda_0 k V_{\mathbf{k}}^{(H)} - i\mathbf{e}_1 q_{\mathbf{k}} + \lambda_0(D^2-k^2)V_{\mathbf{k}}'^{(H)} - ik\mathbf{e}_1 q_{\mathbf{k}}' &= 0, \\ \lambda_0'(D^2-k^2)\theta_{\mathbf{k}} - 2\lambda_0 k\theta_{\mathbf{k}} + \lambda_0(D^2-k^2)\theta_{\mathbf{k}}' + v_{\mathbf{k}}'^{(z)} &= 0, \\ Dv_{\mathbf{k}}'^{(z)} + ik\mathbf{e}_1 \cdot V_{\mathbf{k}}'^{(H)} + i\mathbf{e}_1 \cdot V_{\mathbf{k}}^{(H)} &= 0, \end{aligned} \tag{2.18}$$

which, for $k = k_c$ gives

$$\begin{aligned} -2\lambda_0 k_c v_{\mathbf{k}}^{(z)} + \lambda_0(D^2-k_c^2)v_{\mathbf{k}}'^{(z)} + \theta_{\mathbf{k}}' - Dq_{\mathbf{k}}' &= 0, \\ -2\lambda_0 k_c V_{\mathbf{k}}^{(H)} - i\mathbf{e}_1 q_{\mathbf{k}} + \lambda_0(D^2-k_c^2)V_{\mathbf{k}}'^{(H)} - ik_c\mathbf{e}_1 q_{\mathbf{k}}' &= 0, \\ -2\lambda_0 k_c\theta_{\mathbf{k}} + \lambda_0(D^2-k_c^2)\theta_{\mathbf{k}}' + v_{\mathbf{k}}'^{(z)} &= 0, \\ Dv_{\mathbf{k}}'^{(z)} + ik_c\mathbf{e}_1 \cdot V_{\mathbf{k}}'^{(H)} + i\mathbf{e}_1 \cdot V_{\mathbf{k}}^{(H)} &= 0, \end{aligned} \tag{2.19}$$

with the same boundary conditions for $(V_{\mathbf{k}}'^{(H)}, v_{\mathbf{k}}'^{(z)}, \theta_{\mathbf{k}}')$ as for the eigenvector $U_{\mathbf{k}} = (V_{\mathbf{k}}^{(H)}, v_{\mathbf{k}}^{(z)}, \theta_{\mathbf{k}})$. Before going further we need to determine the derivative with respect to k of the eigenvector $U_{\mathbf{k}}$ in $k = k_c$. We observe that the last equation in (2.19) is not exactly as in (2.17), so we need to make a little change of notation, for being able to use the pseudo-inverse of $\lambda_0 L_{\mathbf{k}_c} + A_{\mathbf{k}_c}$ in $\mathbf{k}_c = k_c\mathbf{e}_1$.

Let us define

$$\widetilde{U}_{\mathbf{k}}' = (\widetilde{V}_{\mathbf{k}}'^{(H)}, v_{\mathbf{k}}'^{(z)}, \theta_{\mathbf{k}}'), \text{ with } \widetilde{V}_{\mathbf{k}}'^{(H)} = V_{\mathbf{k}}'^{(H)} + \frac{1}{k_c} V_{\mathbf{k}}^{(H)},$$

then (2.19) becomes

$$\begin{aligned} \lambda_0(D^2-k_c^2)v_{\mathbf{k}}^{\prime(z)}+\theta'_{\mathbf{k}}-Dq'_{\mathbf{k}} &= 2\lambda_0k_cv_{\mathbf{k}}^{(z)}, \\ \lambda_0(D^2-k_c^2)\widetilde{V}_{\mathbf{k}}^{\prime(H)}-ik_c\mathbf{e}_1q'_{\mathbf{k}} &= 2\lambda_0k_cV_{\mathbf{k}}^{(H)}+\frac{2\lambda_0}{k_c}(D^2-k_c^2)V_{\mathbf{k}}^{(H)}, \\ \lambda_0(D^2-k_c^2)\theta'_{\mathbf{k}}+v_{\mathbf{k}}^{\prime(z)} &= 2\lambda_0k_c\theta_{\mathbf{k}}, \\ Dv_{\mathbf{k}}^{\prime(z)}+ik_c\mathbf{e}_1\cdot\widetilde{V}_{\mathbf{k}}^{\prime(H)} &= 0. \end{aligned} \tag{2.20}$$

The system (2.20) holds because of the property $\lambda'_0=0$, which implies that the compatibility condition is realized for the right hand side (cancelling the scalar product of the 3 first lines resp. with $(v_{\mathbf{k}}^{(z)},V_{\mathbf{k}}^{(H)},\theta_{\mathbf{k}})$):

$$2\lambda_0k_c||U_{\mathbf{k}}||_0^2+\frac{2\lambda_0}{k_c}\int_0^1(D^2-k_c^2)V_{\mathbf{k}}^{(H)}\cdot\overline{V_{\mathbf{k}}^{(H)}}dz=0,$$

i.e. after integrating by parts

$$k_c^2(||v_{\mathbf{k}}^{(z)}||_0^2+||\theta_{\mathbf{k}}||_0^2)-||DV_{\mathbf{k}}^{(H)}||_0^2=0. \tag{2.21}$$

Notice that for $\mathbf{k}=k\mathbf{e}_1$, the functions $v_{\mathbf{k}}^{(z)},\theta_{\mathbf{k}},v_{\mathbf{k}}^{\prime(z)},\theta'_{\mathbf{k}}$ are real valued, while $V_{\mathbf{k}}^{(H)}$ and $V_{\mathbf{k}}^{\prime(H)}$ are pure imaginary.

Remark 2.1 *We can also give a formula for any* $\mathbf{k}$ *in using (2.18):*

$$\lambda'_0(k)||U_{\mathbf{k}}||_{\widetilde{1}}^2=\frac{2\lambda_0}{k}\left[||DV_{\mathbf{k}}^{(H)}||_0^2-k^2(||v_{\mathbf{k}}^{(z)}||_0^2+||\theta_{\mathbf{k}}||_0^2)\right], \tag{2.22}$$

where

$$||U_{\mathbf{k}}||_{\widetilde{1}}^2=||DU_{\mathbf{k}}||_0^2+|\mathbf{k}|^2||U_{\mathbf{k}}||^2, \tag{2.23}$$

which corresponds to the norm of the $\mathbf{k}$*-component in the definition (5.10) of norm* $||\cdot||_{\widetilde{1,s}}$.

From (2.20) we can now write

$$\widetilde{U}'_{\mathbf{k}}=(\lambda_0\widetilde{L_{\mathbf{k}_c}+A}_{\mathbf{k}_c})^{-1}\left[2\lambda_0k_cU_{\mathbf{k}}+\mathfrak{P}_k(\frac{2\lambda_0}{k_c}(D^2-k_c^2)V_{\mathbf{k}}^{(H)},0,0)\right],$$

where $(\lambda_0\widetilde{L_{\mathbf{k}_c}+A}_{\mathbf{k}_c})^{-1}$ is the pseudo-inverse of $(\lambda_0L_{\mathbf{k}_c}+A_{\mathbf{k}_c})$ taking values in the orthogonal of its kernel (selfadjoint operator) and $\mathfrak{P}_k$ is the k-component of the projection $\mathfrak{P}$ defined in section 2.4.1. Hence

$$U'_{\mathbf{k}}=(\lambda_0\widetilde{L_{\mathbf{k}_c}+A}_{\mathbf{k}_c})^{-1}\left[2\lambda_0k_cU_{\mathbf{k}}+\mathfrak{P}_k(\frac{2\lambda_0}{k_c}(D^2-k_c^2)V_{\mathbf{k}}^{(H)},0,0)^t\right]-(\frac{1}{k_c}V_{\mathbf{k}}^{(H)},0,0)^t. \tag{2.24}$$

Differentiating (2.22) with respect to k in $k=k_c$ then gives

$$\lambda''_0||U_{\mathbf{k}}||_{\widetilde{1}}^2=2\lambda_0\frac{d}{dk}\left(\frac{1}{k}||DV_{\mathbf{k}}^{(H)}||_0^2-k(||v_{\mathbf{k}}^{(z)}||_0^2+||\theta_{\mathbf{k}}||_0^2)\right)|_{k=k_c}, \tag{2.25}$$

which is the desired formula, where all terms are now known.

6.2.5 Proof of Lemma 8.13

We refer extensively to [4], pages 628-636, here adapted to an operator in an infinite-dimension space (since we do not consider the projection Π').

The operator $(\mathcal{A} - \lambda_0)$ is diagonal (all $\mathbf{k}$-th Fourier components are uncoupled for operators $\Delta, \mathfrak{P}, L, A$) as well as for orthogonal projections π_0 and Π_N. The projection $\mathbf{Q}_0 = \mathbb{I} - \mathbf{P}_0$ is also diagonal, since it just modifies each Fourier component $e^{i\mathbf{k}_j \cdot \mathbf{x}}$, $j = 1, 2, .., 2q$. Moreover, notice that $\mathbf{k}_j$ belongs to the singular set $S_{(N)}$ for any N since $|\mathbf{k}_j| = k_c$. However 0 is not an eigenvalue because of the z dependency of coefficients of $e^{i\mathbf{k}_j \cdot \mathbf{x}}$, the corresponding eigenvalues being $\lambda_j(k_c^2) - \lambda_0 < -\delta_0 < 0$, $j = 1, 2, ...$

Eigenvalues of $D_N = \Pi_N \pi_0 \mathbf{Q}_0 (\mathcal{A} - \lambda_0) \mathbf{Q}_0 \pi_0 \Pi_N$ are $\lambda_j(|\mathbf{k}|^2) - \lambda_0$, $j = 0, 1, ...$ with $||\mathbf{k}| - k_c| \leq \delta_1$, and $N_{\mathbf{k}} \leq N$, the eigenvalues close to 0 corresponding to $j = 0$, with the estimate (8.19) (notice that the operator $\mathbf{Q}_0$ eliminates the eigenvalue 0). Then, the required estimates on $(D_N)^{-1}$ restricted to the subspace corresponding to parts of $\Omega_N = R_{(N)} + S_{(N)}$ are valid. For example, since we have for $\mathbf{k} \in R_{(N)}$, $\lambda_0 - \lambda_0(|\mathbf{k}|^2) \geq \rho$, and since the operator is self adjoint in $\mathcal{K}_{0,s}$,

$$||D_R h||_{0,s} \geq \rho ||h||_{0,s} \text{ for any } h \in E_N,$$

where D_R is the operator D_N restricted to Fourier modes with $\mathbf{k} \in R_{(N)}$.

Let us now show the "multiplication property" of operator εT, where Lemma 7.2 gives, for $(\varepsilon, \widetilde{\mu}, V) \in [0, \varepsilon_1] \times [-\varepsilon, \varepsilon] \times \mathbf{Q}_0 \mathcal{K}_{0,s}, ||V||_{0,s_0} \leq 1$

$$\varepsilon T(\varepsilon, \widetilde{\mu}, V) =: \Pi_N(\widetilde{\mu} + \mathfrak{B}_\varepsilon + \varepsilon^2 \widetilde{\mu} \mathfrak{C}_{\varepsilon, \widetilde{\mu}} + \mathfrak{R}_{\varepsilon, \widetilde{\mu}, V}) \Pi_N, \tag{2.26}$$

with estimates (7.22).

First for $U \in \mathcal{K}_{1,s}$, $s \geq s_0 > d/2$ and $H \in \mathcal{K}_{1,0}$ we see with the definition 5.2 of $B(U, H)$, that for $U = (V, \theta)$ and $H = (V_H, \theta_H)$, there are functions occurring in components of

$$V \cdot \nabla V_H, V_H \cdot \nabla V, V \cdot \nabla \theta_H, V_H \cdot \nabla \theta$$

each one denoted by $T_1 H$ lies in $\mathcal{H}_{0,0}$ (see Lemmas 4.10, 4.11), satisfying a bound such that, for $A, B \subset \Omega(N)$ (see definition of T_B^A at Lemma 8.13)

$$||[T_1]_B^A H||_{0,0} \leq c(s) \frac{||U||_{1,s}}{(1 + d(A, B))^{s - d/2}} ||H||_{1,0},$$

as it is obtained by the same proof as Lemma 3.9 in [4]. We observe that the projection $\mathfrak{P}$ is diagonal in Fourier components, so that the above estimate stays valid for $B(U, H)$ in $\mathcal{K}_{0,0}$. Now the operator $(-L)^{-1/2}$ is also diagonal, and bounded from $\mathcal{K}_{0,s}$ to $\mathcal{K}_{1,s}$ for all

$s \geq 0$. It then results from the definition of $\mathcal{B}$ that we have the following generalization of (5.7) for any $V \in \mathcal{K}_{0,s}$, $s \geq s_0 > d/2$ and $h \in \mathcal{K}_{0,0}$:

$$||[\mathcal{B}(V,\cdot)]_B^A h||_{0,0} \leq c(s)\frac{||V||_{0,s}}{(1+d(A,B))^{s-d/2}}||h||_{0,0}. \tag{2.27}$$

We then look at the operator appearing in (7.11):

$$\widetilde{\mu} + \mu_\varepsilon - 2\mathbf{Q}_0\mathcal{B}(u_\varepsilon,\cdot) - 2\varepsilon^4\mathbf{Q}_0\mathcal{B}_1(V,\cdot).$$

The operator $\mathbf{Q}_0$ is diagonal, hence the above estimate (2.27) leads to a bound in $\mathcal{K}_{0,0}$ as

$$c(s)\frac{(\varepsilon+\varepsilon^4||V||_{0,s})}{(1+d(A,B))^{s-d/2}}||h||_{0,0}. \tag{2.28}$$

Now we need to track the estimate for the transformed operator after the splitting by π_0 (see section 2.7.5). For its computation we need first to look at operator $\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}$ acting in $(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,0}$. It is obtained via a Neumann series of powers of operators satisfying estimates as (2.28) provided that $||V||_{0,s_0} \leq 1$, and proofs of Lemmas 3.10, 3.11 of [4] apply analogously, leading to

$$||[\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}]_B^A h||_{0,0} \leq c(s)\frac{(1+\varepsilon^4||V||_{0,s})}{(1+d(A,B))^{s-d/2}}||h||_{0,0}.$$

The composition of two operators satisfying estimates as above, satisfies also the same estimate, with modified constants, so that finally for (2.26) and for any $V \in \mathcal{K}_{0,s}$, $||V||_{0,s_0} \leq 1$, $s \geq s_0 > d/2$ and $h \in \mathcal{K}_{0,0}$

$$||\varepsilon T(\varepsilon,\widetilde{\mu},V)]_B^A h||_{0,0} \leq c(s)\frac{\varepsilon(1+\varepsilon^3||V||_{0,s})}{(1+d(A,B))^{s-d/2}}||h||_{0,0}.$$

6.2.6 A C^2 property for the Nash-Moser theorem in [5]

Starting point is the Nash-Moser theorem 3 in Berti-Bolle-Procesi [5] . We want to extend this theorem from the C^1-case to the C^2-case. We assume the conditions of that theorem with $\nu = 0$ and moreover that $F(\epsilon,\lambda,u)$ is C^3 in (ϵ,λ,u) on $[0,\epsilon_0)\times\Lambda\times X_{s_0}$ and that the following conditions are fulfilled for $z := (\epsilon,\lambda) \in [0,\epsilon_0)\times\Lambda$ and $u \in X_s, s \in [s_0,S)$, with $||u||_{s_0} \leq 1$:

$$(F2)^+ \quad ||\partial_\lambda^2 F(z,u)||_s \leq C(s)(||u||_s + 1)$$

$$(F3)^+ \quad ||D_u^3 F(z,u)[v_1,v_2,v_3]||_s \leq C(s)(||u||_s||v_1||_{s_0}||v_2||_{s_0}||v_3||_{s_0}$$
$$+||v_1||_s||v_2||_{s_0}||v_3||_{s_0} + ||v_2||_s||v_1||_{s_0}||v_3||_{s_0}|| + ||v_3||_s||v_1||_{s_0}||v_2||_{s_0})$$

$$(F4)^+ \quad ||\partial_\lambda^2 D_u F(z,u)[v]||_s \le C(s)(||u||_s||v||_{s_0} + ||v||_s),$$
$$||\partial_\lambda D_u^2 F(z,u)[v_1,v_2]||_s \le C(s)(||u||_s||v_1||_{s_0}||v_2||_{s_0} + ||v_1||_s||v_2||_{s_0} + ||v_2||_s||v_1||_{s_0}).$$

Then Theorem 1 of [3] holds with $\nu = 0$ and $\partial_\lambda^2 u$ exists and belongs to $C([0,\epsilon_2)\times\Lambda, X_{s_0})$. To prove this we show that

The sequence $(\partial_\lambda^2 u_n)_{n\ge 0}$ converges in $C([0,\epsilon_2)\times\Lambda, X_{s_0})$, where u_n is as in [3]. Moreover, given $\eta\in(0,1)$ we may choose $N_0(\gamma)$ large enough such that for $\partial_\lambda^2 u_n : [0,\epsilon_2)\times\Lambda \to E_{n+1}$, the properties $(Pj)_n, j = 1,2,3,4$ are supplemented by

$$(P1)_n^+ \;\; 1 + ||\partial_\lambda^2 u_n||_{s_0} \le C(\gamma)N_0^\sigma, \tag{2.29}$$

$$(P2)_n^+ \;\; ||\partial_\lambda^2(u_{n+1}-u_n)||_{s_0} \le N_{n+1}^{-1+\eta}, \tag{2.30}$$

$$(P4)_n^+ \;\; B_n'' = 1 + ||\partial_\lambda^2 u_n||_{\overline{s}} \le 2N_{n+1}^{\sigma/2+2\mu+3\eta}. \tag{2.31}$$

Finally in $(P4)_n$ we have $B_n \le 2N_{n+1}^{\mu+\eta}, B_n' \le 2N_{n+1}^{\sigma/4+\mu+2\eta}$.

We denote formula numbers from [3] in the following by adding a zero in front of that number. So (041) corresponds to (41) in [3]. First we remark that corresponding to (034) and (038) we also have for $z\in\mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$:

$$\begin{aligned} ||\tilde h_{n+1}||_{\overline{s}} &\le N_{n+1}^{2\mu+2\eta}, && (2.32)\\ ||\partial_z \tilde h_{n+1}||_{s_0} &\le N_{n+1}^{-3\sigma/4-1+2\eta}, && (2.33)\\ ||\partial_z \tilde h_{n+1}||_{\overline{s}} &\le N_{n+1}^{\sigma/2+2\mu+3\eta}. && (2.34)\end{aligned}$$

and $||h_{n+1}||_{\overline{s}} \le N_{n+1}^{2\mu+2\eta}$ with a proof quite similar to that in [3]. Similarly it follows from this that [3,Theorem 1] holds in case $\nu = 0$.

To prove the C^2 property in λ we will follow the induction process in [3]. First functions $\tilde u_0$ and $\tilde h_n$ are constructed. Then $u_0 := \psi_0\tilde u_0, h_n := \psi_n\tilde h_n, u_{n+1} := u_n + h_{n+1}$, where the cut-off function ψ_n is defined in (050), but now with the extra property that it is C^2 and

$$|\partial_z\psi_n| \le C\gamma^{-1}N_n^{\sigma/2}, |\partial_z^2\psi_n| \le C^2\gamma^{-2}N_n^\sigma. \tag{2.35}$$

From the implicit function theorem it follows that $\tilde h_n$ is C^2 in λ and then the same follows for h_n and u_n.

Next we have to estimate the norms of these functions in order to show that the sequence $\partial_\lambda^2 u_n \in C([0,\epsilon_2)\times\Lambda, E_n)$ converges in $C([0,\epsilon_2)\times\Lambda, X_{s_0})$.

By (032) we have $\Pi_{n+1}F(z,u) = 0$ if $u = u_n + \tilde h_{n+1} =: u_n^+$ and $z\in\mathcal{N}(A_{n+1}, 2\gamma N_n^{-\sigma/2})$. This also holds for $n=-1$ with $u_{-1} = 0, u_{-1}^+ = \tilde u_0 = \tilde h_0$. Applying ∂_λ^2 to this equation leads to

$$L_{n+1}^+\partial_\lambda^2\tilde h_{n+1} + M_{n+1} = 0$$

where $L_{n+1}^{+}(z) := \Pi_{n+1} D_u F(z, u_n^+)$ which is invertible by [3, Lemma 2.3] and

$$M_{n+1} := \Pi_{n+1}[\partial_\lambda^2(F(z,u_n^+))+2\partial_\lambda D_u(F(z,u_n^+)[\partial_\lambda u_n^+]+D_u^2(F(z,u_n^+))[\partial_\lambda u_n^+,\partial_\lambda u_n^+]+D_u(F(z,u_n^+))[\partial_\lambda^2 u_n]$$

for z as above.
First let $n=-1$. Then $||M_0||_s$ may be estimated using $(F2)^+, (F3)$ and $F(4)$. Thus

$$||M_0||_s \leq C(s)[||\tilde{u_0}||_s(1+2||\partial_\lambda \tilde{u_0}||_{s_0} + ||\partial_\lambda \tilde{u_0}||_{s_0}^2) + 2||\partial_\lambda \tilde{u_0}||_s(1+||\partial_\lambda \tilde{u_0}||_{s_0}) + 1].$$

From [3, p. 385] we have

$$||\tilde{u_0}||_{s_0} \leq \rho_0 = C_0\gamma^{-1}N_0^\mu\epsilon, ||\partial_\lambda\tilde{u_0}||_{s_0} \leq K\gamma^{-1}N_0^\mu, ||\tilde{u_0}||_{\overline{s}} \leq K(\gamma)N_0^\mu\epsilon, ||\partial_\lambda\tilde{u_0}||_{\overline{s}} \leq K(\gamma)N_0^\mu.$$

Then we get

$$||M_0||_s \leq C_1(\gamma)N_0^{2\mu}$$

for both $s = s_0$ and $s = \overline{s}$. Then we apply (015) and (016) to $\partial_\lambda^2\tilde{u}_0 = -(L_0^+)^{-1}M_0$ and obtain $||\partial_\lambda^2\tilde{u_0}||_s \leq C(\gamma)N_0^{3\mu}$ for both values of s.

From $u_0 := \psi_0\tilde{u}_0$ and (2.35) we deduce $(P1)_0$ and $(P4)_0$ for $\eta > 0$ and N_0 sufficiently large.

For $n \geq 0$ we write $M_{n+1} = \Pi_{n+1}\sum_{j=0}^6 A_j$ with

$$\begin{aligned}
A_0 &= \partial_\lambda^2 F(z,u_n) + 2\partial_\lambda D_u F(z,u_n)[\partial_\lambda u_n] + D_u^2 F(z,u_n)[\partial_\lambda u_n, \partial_\lambda u_n] + D_u F(z,u_n)[\partial_\lambda^2 u_n]\\
A_1 &= \partial_\lambda^2(F(z,u_n^+) - F(z,u_n)) = \int_0^1 \partial_\lambda^2 D_u(F(z,u_n+\theta\tilde{h}_{n+1}))d\theta[\tilde{h}_{n+1}]\\
A_2 &= 2\partial_\lambda D_u(F(z,u_n^+) - F(z,u_n))[\partial_\lambda u_n^+] = 2\int_0^1 \partial_\lambda D_u^2(F(z,u_n+\theta\tilde{h}_{n+1}))d\theta[\tilde{h}_{n+1},\partial_\lambda u_n^+]\\
A_3 &= 2\partial_\lambda D_u F(z,u_n))[\partial_\lambda\tilde{h}_{n+1}]\\
A_4 &= D_u^2(F(z,u_n^+) - F(z,u_n))[\partial_\lambda u_n^+, \partial_\lambda u_n^+] = \int_0^1 D_u^3(F(z,u_n+\theta\tilde{h}_{n+1}))d\theta[\tilde{h}_{n+1},\partial_\lambda u_n^+,\partial_\lambda u_n^+]\\
A_5 &= D_u^2 F(z,u_n)([\partial_\lambda u_n^+,\partial_\lambda u_n^+] - [\partial_\lambda u_n,\partial_\lambda u_n]),\\
A_6 &= D_u(F(z,u_n^+) - F(z,u_n))[\partial_\lambda^2 u_n] = \int_0^1 D_u^2(F(z,u_n+\theta\tilde{h}_{n+1}))d\theta[\tilde{h}_{n+1},\partial_\lambda^2 u_n]
\end{aligned}$$

Similarly as in [3] using $(S1), (F4), (F4)^+, (F3)^+$ and the estimates for $||\tilde{h}_{n+1}||_s, ||\partial_z\tilde{h}_{n+1}||_s, ||u_n||_s$ we obtain that there are constants $C_1(s,\gamma)$ independent of n such that

$$||\Pi_{n+1}(A_1+A_2+A_4)||_{s_0} \leq C_1(s_0,\gamma)N_{n+1}^{-\sigma-1} \tag{2.36}$$
$$||\Pi_{n+1}(A_1+A_2+A_4)||_{\overline{s}} \leq C_1(\overline{s},\gamma)N_{n+1}^{2(\mu+\eta)}. \tag{2.37}$$

Furthermore using (F4) it follows that there exist positive constants K independent of n, which may be different in different places such that

$$||\Pi_{n+1}A_3||_{s_0} \leq KN_{n+1}^{-3\sigma/4-1+2\eta}, ||\Pi_{n+1}A_3||_{\overline{s}} \leq KN_{n+1}^{\sigma/2+2\mu+3\eta}. \tag{2.38}$$

In A_5 we may replace $[\partial_\lambda u_n^+, \partial_\lambda u_n^+] - [\partial_\lambda u_n, \partial_\lambda u_n]$ by $[\partial_\lambda \tilde{h}_{n+1}, \partial_\lambda(2u_n + \tilde{h}_{n+1})]$ and then with (F3),(2.29), (2.32), (2.33) and $(P4)_n$ we obtain

$$||\Pi_{n+1}A_5||_{s_0} \leq KN_{n+1}^{-3\sigma/4-1+2\eta}, \tag{2.39}$$

$$||\Pi_{n+1}A_5||_{\overline{s}} \leq KN_{n+1}^{\sigma/2+2\mu+3\eta}. \tag{2.40}$$

and also using (2.31)

$$||\Pi_{n+1}A_6||_{s_0} \leq KN_{n+1}^{-\sigma-1}, \tag{2.41}$$

$$||\Pi_{n+1}A_6||_{\overline{s}} \leq KN_{n+1}^{2(\mu+\eta)}. \tag{2.42}$$

Finally using $(F2)^+, (F3), (F4), (F6), (P4)_n$ and $(P4)_n^+$ we get

$$||\Pi_{n+1}A_0||_{\overline{s}} \leq KN_{n+1}^{\sigma/2+2\mu+3\eta}.$$

With [3,$(S2)$] it follows as in (047) that

$$||\Pi_{n+1}A_0||_{s_0} = ||\Pi_{n+1}(I-\Pi_n)A_0||_{s_0} \leq KN_n^{-\overline{s}+s_0}||\Pi_{n+1}A_0||_{\overline{s}} \leq K'N_{n+1}^{-\sigma/2-2+3\eta}.$$

Combining the estimates for $A_j, j = 0, ...6$ it follows that

$$||M_{n+1}||_{s_0} \leq KN_{n+1}^{-\sigma/2-2+3\eta}$$

and

$$||M_{n+1}||_{\overline{s}} \leq KN_{n+1}^{\sigma/2+2\mu+3\eta}.$$

From $(P4)_n^+$ and [3, Lemma 2.3] we obtain

$$||\partial_\lambda^2 \tilde{h}_{n+1}||_{s_0} \leq KN_{n+1}^{-\sigma/2+\mu-2+3\eta}, ||\partial_\lambda^2 \tilde{h}_{n+1}||_{\overline{s}} \leq KN_{n+1}^{\sigma/2+3(\mu+\eta)} \tag{2.43}$$

With $h_{n+1} = \psi_{n+1}\tilde{h}_{n+1}$ and (2.35) it follows that

$$||\partial_\lambda^2 h_{n+1}||_s \leq ||\partial_\lambda^2 \tilde{h}_{n+1}||_s + 2|\partial_\lambda \psi_{n+1}|||\partial_\lambda \tilde{h}_{n+1}||_s + |\partial_\lambda^2 \psi_{n+1}|||\tilde{h}_{n+1}||_s \tag{2.44}$$

and from the corresponding estimates for $\tilde{h}_{n+1}$ in (2.32),(2.33),(2.34) and (2.43) we get that

$$||\partial_\lambda^2 h_{n+1}||_{s_0} \leq N_{n+1}^{-1+\eta}, ||\partial_\lambda^2 h_{n+1}||_{\overline{s}} \leq N_{n+1}^{\sigma+2\mu+4\eta}. \tag{2.45}$$

From this and $u_{n+1} = u_n + h_{n+1}$ we deduce $(P2)_{n+1}^+$ and $(P1)_{n+1}^+$. Furthermore with $(P4)_n^+$ it follows that

$$B''_{n+1} \leq B''_n + ||\partial_\lambda^2 h_{n+1}||_{\overline{s}} \leq 2N_{n+1}^{\sigma/2+2\mu+3\eta} + N_{n+1}^{\sigma+2\mu+4\eta} \leq 2N_{n+1}^{\sigma+2\mu+4\eta} = 2N_{n+2}^{\sigma/2+\mu+2\eta}$$

and so $(P4)_{n+1}$ holds and the induction step is proven. Finally this implies as in [3, section 2.4] the statement on the convergence of the maps $\partial_\lambda^2 u_n$ in $C([0, \epsilon_2) \times \Lambda, X_{s_0})$ to $\partial_\lambda^2 u$.

6.3 Appendix of Chapter 3

6.3.1 Adjoint operator

Denote by $\langle\cdot,\cdot\rangle$ the scalar product in $(L^2_{per}(\Omega))^8$ and consider the closed subspace

$$\mathcal{H}_0 = \left\{\mathbf{U} = (V_x, V_\perp, W_x, W_\perp, \theta, \phi) \in (L^2_{per}(\Omega))^8 \; ; \; \int_\Omega V_x \, dy \, dz = 0\right\} \subset (L^2_{per}(\Omega))^8,$$

which is the closure in $(L^2_{per}(\Omega))^8$ of both $\mathcal{X}$ and the domain of definition $\mathcal{Z}$ of the operator $\mathcal{L}_\mu$. We compute the adjoint $\mathcal{L}_\mu^*$ of $\mathcal{L}_\mu$ from the scalar product $\langle\mathcal{L}_\mu\mathbf{U}, \mathbf{U}'\rangle$, for $\mathbf{U} \in \mathcal{Z}$, and choose $\mathbf{U}' \in \mathcal{H}_0$ such that $\mathbf{U} \mapsto \langle\mathcal{L}_\mu\mathbf{U}, \mathbf{U}'\rangle$ is a linear continuous form on $\mathcal{H}_0$. We obtain the linear operator

$$\mathcal{L}_\mu^*\mathbf{U} = \begin{pmatrix} -\mu^{-1}\left(\Delta_\perp W_x - \langle\Delta_\perp W_x\rangle\right) \\ \nabla_\perp V_x - \mu^{-1}\Delta_\perp W_\perp - \mu^{-1}\nabla_\perp(\nabla_\perp\cdot W_\perp) - \mu\phi\mathbf{e}_z \\ \nabla_\perp\cdot W_\perp \\ \mu V_\perp \\ -W_z - \Delta_\perp\phi \\ \theta \end{pmatrix},$$

where

$$\langle\Delta_\perp W_x\rangle = \int_\Omega \Delta_\perp W_x(y,z)\, dy\, dz.$$

The operator $\mathcal{L}_\mu^*$ is closed in the space $\mathcal{X}^*$ defined by

$$\begin{aligned} \mathcal{X}^* = \big\{\mathbf{U} \in (L^2_{per}(\Omega))^3 \times (H^1_{per}(\Omega))^3 \times L^2_{per}(\Omega) \times H^1_{per}(\Omega) \; ; \\ W_x = W_\perp = \phi = 0 \text{ on } z = 0, 1, \text{ and } \int_\Omega V_x \, dy\, dz = 0\big\}, \end{aligned}$$

with domain

$$\begin{aligned} \mathcal{Z}^* = \big\{\mathbf{U} \in \mathcal{X}^* \cap (H^1_{per}(\Omega))^3 \times (H^2_{per}(\Omega))^3 \times H^1_{per}(\Omega) \times H^2_{per}(\Omega) \; ; \\ V_\perp = \nabla_\perp\cdot W_\perp = \theta = 0 \text{ on } z = 0, 1\big\}. \end{aligned}$$

The adjoint operator $\mathcal{L}_\mu^*$ has the same center spectrum as the operator $\mathcal{L}_\mu$. For our purposes we need to compute its kernel, an eigenvector associated with the eigenvalue $-ik$ of $\mathcal{L}^*_{\mu_0(k)}$, and one of the eigenvectors associated with the eigenvalue $-ik_x$ of $\mathcal{L}^*_{\mu_c}$.

The kernel of $\mathcal{L}_\mu^*$ is easily computed by solving the equation $\mathcal{L}_\mu^*\mathbf{U} = 0$, and we find that it is spanned by the vector

$$\boldsymbol{\varphi}_0^* = (0, 0, 0, z(1-z), 0, 0, 0, 0,)^t\,.$$

Next, for $\mu = \mu_0(k)$, the operator $\mathcal{L}^*_{\mu_0(k)}$ has the geometrically simple eigenvalues $\pm ik$, just as the operator $\mathcal{L}_{\mu_0(k)}$. In Appendix 6.3.2 we need the expression of an eigenvector $\mathbf{\Psi}^*_{k,0}$ associated with the eigenvalue $-ik$. A direct calculation gives

$$\mathbf{\Psi}^*_{k,0}(y,z) = \widehat{\mathbf{\Psi}}^*_{k,0}(z), \quad \widehat{\mathbf{\Psi}}^*_{k,0}(z) = \begin{pmatrix} -\frac{1}{\mu_0(k)k^2}\left(D^3V_k - \langle D^3V_k\rangle\right) \\ 0 \\ \frac{ik}{\mu_0(k)}V_k \\ -\frac{i}{k}DV_k \\ 0 \\ -V_k \\ -ik\phi_k \\ \phi_k \end{pmatrix}, \tag{3.1}$$

where

$$\langle D^3V_k\rangle = \int_\Omega D^3V_k(z)\,dy\,dz,$$

V_k is the solution of the boundary value problem (2.6), and ϕ_k is the unique solution of the boundary value problem

$$(D^2 - k^2)\phi_k = V_k, \quad \phi_k|_{z=0,1} = 0.$$

Finally, in the computations of next Chapters we also need an eigenvector associated with the eigenvalue $-ik_x$ of $\mathcal{L}^*_{\mu_c}$ which is of the form

$$\mathbf{\Psi}^*_+ = \widehat{\mathbf{\Psi}}^*_+(z)e^{ik_y y}.$$

We obtain that

$$\widehat{\mathbf{\Psi}}^*_+(z) = \begin{pmatrix} -\frac{1}{\mu_c k_c^2}(D^2 - k_c^2\cos^2\alpha)DV \\ -\frac{\sin\alpha\cos\alpha}{\mu_c}DV \\ \frac{ik_c\sin\alpha}{\mu_c}V \\ -\frac{i\sin\alpha}{k_c}DV \\ -\frac{i\cos\alpha}{k_c}DV \\ -V \\ -ik_c(\sin\alpha)\phi \\ \phi \end{pmatrix},$$

where V is the solution of the boundary value problem (2.8), and ϕ is the unique solution of the boundary value problem

$$(D^2 - k_c^2)\phi = V, \quad \phi|_{z=0,1} = 0.$$

Notice that the function ϕ is related to the function θ in the boundary value problem (1.16)-(1.17), taken at $k = k_c$, through the equality $\theta = -\mu_c \phi$.

6.3.2 Algebraic multiplicities of $\pm ik$ and $\pm i\omega_1(k)$

For $k_y = k_c \cos\alpha$ with $\alpha \in (0, \pi/3)$ and $k \neq k_c$, sufficiently close to k_c, consider the geometrically simple eigenvalues $\pm ik$ and the geometrically double eigenvalues $\pm i\omega_1(k)$ of the operator $\mathcal{L}_{\mu_0(k)}$ found in Lemma 2.1. We show that their algebraic multiplicities are equal to their geometric multiplicities, or equivalently, that their index is equal to 1. We prove the result for the eigenvalue ik, the arguments being the same for the eigenvalue $i\omega_1(k)$.

Assuming that the index of the eigenvalue ik is larger than 1, there exists a vector $\mathbf{\Psi}_{k,0}$ such that

$$(\mathcal{L}_{\mu_0(k)} - ik)\mathbf{\Psi}_{k,0} = \mathbf{U}_{k,0}. \tag{3.2}$$

Differentiating the eigenvalue problem

$$\mathcal{L}_{\mu_0(k)}\mathbf{U}_{k,0} = ik\mathbf{U}_{k,0},$$

with respect to k leads to the equality

$$(\mathcal{L}_{\mu_0(k)} - ik)\left(\frac{d}{dk}\mathbf{U}_{k,0}\right) = \left(i - \mu_0'(k)\,\frac{\partial}{\partial\mu}\mathcal{L}_\mu\big|_{\mu=\mu_0(k)}\right)\mathbf{U}_{k,0}.$$

Since $\mu_0'(k) \neq 0$ for $k \neq k_c$, this identity and (3.2) imply that there is a solution $\mathbf{\Phi}_{k,0}$ of the linear equation

$$(\mathcal{L}_{\mu_0(k)} - ik)\mathbf{\Phi}_{k,0} = \frac{\partial}{\partial\mu}\mathcal{L}_\mu\big|_{\mu=\mu_0(k)}\mathbf{U}_{k,0}. \tag{3.3}$$

As a consequence, the vector in the right hand side of the above equation is orthogonal to the kernel of the adjoint operator $(\mathcal{L}^*_{\mu_0(k)} + ik)$. In particular, it is orthogonal to the eigenvector $\mathbf{\Psi}^*_{k,0}$ computed in Appendix 6.3.1 and given by (3.1). A direct computation gives the term in the right hand side of (3.3),

$$\frac{\partial}{\partial\mu}\mathcal{L}_\mu\big|_{\mu=\mu_0(k)}\widehat{\mathbf{U}}_{k,0} = \begin{pmatrix} 0 \\ 0 \\ \frac{ik}{\mu_0(k)}V_k \\ \frac{i}{\mu_0^2(k)k}D^3V_k \\ 0 \\ \frac{2}{\mu_0^2(k)}D^2V_k \\ 0 \\ -V_k \end{pmatrix},$$

and taking its L^2-scalar product with the vector $\mathbf{\Psi}_{k,0}^*$ given by (3.1) we obtain

$$\frac{1}{\mu_0^2(k)k^2}\left(\|D^2V_k\|^2+2k^2\|DV_k\|^2+k^4\|V_k\|^2\right)+\|D\phi_k\|^2+k^2\|\phi_k\|^2>0.$$

The positivity of the scalar product contradicts the solvability condition for the equation (3.3), and proves that the index of the eigenvalue ik is equal to 1.

6.4 Appendix of Chapter 4

6.4.1 Proof of Lemma 3.2

Proof. The existence of the polynomial $\boldsymbol{P}_\varepsilon$ and the first two properties in Lemma 3.2 follow from the general normal form theorems in [28, Sections 3.2.1, 3.3.1, and 3.3.2]. In addition, $N(\cdot,\cdot,\varepsilon)$ is an odd polynomial of degree 3 such that $N(0,0,\varepsilon)=0$ and the identity

$$D_Z N(Z,\overline{Z},\varepsilon)L_0^*Z+D_{\overline{Z}}N(Z,\overline{Z},\varepsilon)\overline{L_0^*Z}=L_0^*N(Z,\overline{Z},\varepsilon), \tag{4.1}$$

in which L_0^* is the adjoint of L_0, holds for any $Z\in\mathbb{C}^4$ and $\varepsilon\in\mathcal{V}_2$. We write

$$N(Z,\overline{Z},\varepsilon)=N_1(Z,\overline{Z})\varepsilon+N_3(Z,\overline{Z}),$$

where N_1 and N_3 denote the linear and cubic terms, respectively, of N. It is now straightforward to check that the linear part N_1 has the form in Lemma 3.2 (*iii*), and it remains to check the cubic terms N_3.

We set $N_3=(\widetilde{N}_+,\widetilde{M}_+,\widetilde{N}_-,\widetilde{M}_-)$. Then the identity (4.1) becomes

$$(\mathcal{D}^*+ik_x)\widetilde{N}_+=0,\quad (\mathcal{D}^*+ik_x)\widetilde{M}_+=\widetilde{N}_+,$$
$$(\mathcal{D}^*+ik_x)\widetilde{N}_-=0,\quad (\mathcal{D}^*+ik_x)\widetilde{M}_-=\widetilde{N}_-,$$

in which

$$\begin{aligned}\mathcal{D}^* \;=\;& -ik_xA_+\frac{\partial}{\partial A_+}+(A_+-ik_xB_+)\frac{\partial}{\partial B_+}-ik_xA_-\frac{\partial}{\partial A_-}+(A_--ik_xB_-)\frac{\partial}{\partial B_-}\\ &+ik_x\overline{A_+}\frac{\partial}{\partial \overline{A_+}}+(\overline{A_+}+ik_x\overline{B_+})\frac{\partial}{\partial \overline{B_+}}+ik_x\overline{A_-}\frac{\partial}{\partial \overline{A_-}}+(\overline{A_-}+ik_x\overline{B_-})\frac{\partial}{\partial \overline{B_-}}.\end{aligned}$$

Due to the equivariance of the normal form under the action of the symmetry $\mathbf{S}_2$, it is enough to determine $(\widetilde{N}_+,\widetilde{M}_+)$, the components $(\widetilde{N}_-,\widetilde{M}_-)$ being obtained by switching the indices $+$ and $-$ in the expressions of $(\widetilde{N}_+,\widetilde{M}_+)$.

Cubic monomials are of the form

$$A_+^{p_+}\overline{A_+}^{q_+}B_+^{r_+}\overline{B_+}^{s_+}A_-^{p_-}\overline{A_-}^{q_-}B_-^{r_-}\overline{B_-}^{s_-},$$

with nonnegative exponents such that

$$p_+ + q_+ + r_+ + s_+ + p_- + q_- + r_- + s_- = 3. \tag{4.2}$$

We claim that the cubic monomials in $\widetilde{N}_+$ and $\widetilde{M}_+$ also satisfy

$$p_+ - q_+ + r_+ - s_+ + p_- - q_- + r_- - s_- = 1. \tag{4.3}$$

Indeed, for any monomial as above we have

$$\begin{aligned}
&\mathcal{D}^* \left(A_+^{p_+} \overline{A_+}^{q_+} B_+^{r_+} \overline{B_+}^{s_+} A_-^{p_-} \overline{A_-}^{q_-} B_-^{r_-} \overline{B_-^{s_-}} \right) = \\
&\quad -ik_x \left(p_+ - q_+ + r_+ - s_+ + p_- - q_- + r_- - s_- \right) A_+^{p_+} \overline{A_+}^{q_+} B_+^{r_+} \overline{B_+}^{s_+} A_-^{p_-} \overline{A_-}^{q_-} B_-^{r_-} \overline{B_-}^{s_-} \\
&\quad + r_+ A_+^{p_+ +1} \overline{A_+}^{q_+} B_+^{r_+ -1} \overline{B_+}^{s_+} A_-^{p_-} \overline{A_-}^{q_-} B_-^{r_-} \overline{B_-}^{s_-} \\
&\quad + s_+ A_+^{p_+} \overline{A_+}^{q_+ +1} B_+^{r_+} \overline{B_+}^{s_+ -1} A_-^{p_-} \overline{A_-}^{q_-} B_-^{r_-} \overline{B_-}^{s_-} \\
&\quad + r_- A_+^{p_+} \overline{A_+}^{q_+} B_+^{r_+} \overline{B_+}^{s_+} A_-^{p_- +1} \overline{A_-}^{q_-} B_-^{r_- -1} \overline{B_-}^{s_-} \\
&\quad + s_- A_+^{p_+} \overline{A_+}^{q_+} B_+^{r_+} \overline{B_+}^{s_+} A_-^{p_-} \overline{A_-}^{q_- +1} B_-^{r_-} \overline{B_-}^{s_- -1},
\end{aligned}$$

implying that the subspace of monomials for which the sum in the left hand side of (4.3) is constant is invariant under the action of $\mathcal{D}^*$. Ordering the monomials by decreasing exponents p_+, q_+, r_+, s_+, p_-, q_-, r_-, and s_-, this action is represented by a lower triangular matrix with equal elements on the diagonal given by

$$-ik_x \left(p_+ - q_+ + r_+ - s_+ + p_- - q_- + r_- - s_- \right).$$

Consequently, the polynomials $\widetilde{N}_+$ and $\widetilde{M}_+$, which belong to the kernel and generalized kernel of $\mathcal{D}_* + ik_x$, respectively, belong to the subspace for which (4.3) holds. This proves the claim. Furthermore, the commutativity of N_3 and $\boldsymbol{\tau}_a$, implies that monomials in $(\widetilde{N}_+, \widetilde{M}_+)$ also satisfy

$$p_+ - q_+ + r_+ - s_+ - p_- + q_- - r_- + s_- = 1. \tag{4.4}$$

Collecting all possible monomials in $(\widetilde{N}_+, \widetilde{M}_+)$ for which the conditions (4.2)-(4.4) hold, we compute:

$$\begin{aligned}
&(\mathcal{D}^* + ik_x)(A_+^2 \overline{A_+}) = 0, \\
&(\mathcal{D}^* + ik_x)(A_+^2 \overline{B_+}) = (\mathcal{D}^* + ik_x)(A_+ \overline{A_+} B_+) = A_+^2 \overline{A_+}, \\
&(\mathcal{D}^* + ik_x)(A_+ B_+ \overline{B_+}) = A_+^2 \overline{B_+} + A_+ \overline{A_+} B_+, \quad (\mathcal{D}^* + ik_x)(\overline{A_+} B_+^2) = 2A_+ \overline{A_+} B_+, \\
&(\mathcal{D}^* + ik_x)(B_+^2 \overline{B_+}) = 2A_+ B_+ \overline{B_+} + \overline{A_+} B_+^2,
\end{aligned}$$

and

$$\begin{aligned}
&(\mathcal{D}^* + ik_x)(A_+A_-\overline{A_-}) = 0,\\
&(\mathcal{D}^* + ik_x)(A_+A_-\overline{B_-}) = (\mathcal{D}^* + ik_x)(A_+\overline{A_-}B_-) = (\mathcal{D}^* + ik_x)(B_+A_-\overline{A_-}) = A_+A_-\overline{A_-}\\
&(\mathcal{D}^* + ik_x)(A_+B_-\overline{B_-}) = A_+A_-\overline{B_-} + A_+\overline{A_-}B_-,\\
&(\mathcal{D}^* + ik_x)(B_+A_-\overline{B_-}) = A_+A_-\overline{B_-} + B_+A_-\overline{A_-},\\
&(\mathcal{D}^* + ik_x)(B_+\overline{A_-}B_-) = A_+\overline{A_-}B_- + B_+A_-\overline{A_-},\\
&(\mathcal{D}^* + ik_x)(B_+B_-\overline{B_-}) = A_+B_-\overline{B_-} + B_+A_-\overline{B_-} + B_+\overline{A_-}B_-.
\end{aligned}$$

Since $\widetilde{N}_+$ and $\widetilde{M}_+$ are necessarily linear combinations of these 14 monomials, the equalities above imply that they are of the form

$$\begin{aligned}
\widetilde{N}_+ &= A_+\widetilde{P}_+(u_1,u_2,u_3,u_4) + A_-\widetilde{R}_+(u_5),\\
\widetilde{M}_+ &= B_+\widetilde{P}_+(u_1,u_2,u_3,u_4) + B_-\widetilde{R}_+(u_5) + A_+\widetilde{Q}_+(u_1,u_2,u_3,u_4) + A_-\widetilde{S}_+(u_5),
\end{aligned}$$

with $\widetilde{P}_+, \widetilde{R}_+, \widetilde{Q}_+, \widetilde{S}_+$ linear in their arguments, which are the quadratic expressions

$$\begin{aligned}
&u_1 = A_+\overline{A_+}, \quad u_2 = i(A_+\overline{B_+} - \overline{A_+}B_+), \quad u_3 = A_-\overline{A_-},\\
&u_4 = i(A_-\overline{B_-} - \overline{A_-}B_-), \quad u_5 = (A_+\overline{B_-} - \overline{A_-}B_+).
\end{aligned}$$

This proves the expressions of the cubic terms of N_+ and M_+ in (iii). Finally, taking into account the action of the reversibility $\mathbf{S}_1$, it is straightforward to check that the coefficients β_j, b_j, γ_5, and c_5 are real. ∎

6.4.2 Computation of the quotient $g = b_3/b_1$

For the computation of the coefficients b_1 and b_3, we follow the method in [28, Section 3.4.1]. We restrict to the 8-dimensional center manifold

$$\mathcal{M}_\pm(\varepsilon) = \{\mathbf{U}_c + \mathbf{\Phi}(\mathbf{U}_c, \varepsilon)\ ;\ \mathbf{U}_c \in E_\pm\}.$$

Recall that solutions on this submanifold are invariant under the action of $\mathbf{S}_3\boldsymbol{\tau}_\pi$. Combining the transformations from the center manifold reduction in Section 4.2.1 and the normal form in Lemma 3.2, we write

$$\begin{aligned}
\mathbf{U} \;=\;& A_+\boldsymbol{\zeta}_+ + B_+\mathbf{\Psi}_+ + A_-\boldsymbol{\zeta}_- + B_-\mathbf{\Psi}_- + \overline{A_+\boldsymbol{\zeta}_+} + \overline{B_+\mathbf{\Psi}_+} + \overline{A_-\boldsymbol{\zeta}_-} + \overline{B_-\mathbf{\Psi}_-}\\
&+ \widetilde{\mathbf{\Phi}}(A_+, B_+, A_-, B_-, \overline{A_+}, \overline{B_+}, \overline{A_-}, \overline{B_-}, \varepsilon),
\end{aligned}$$

in which $Z = (A_+, B_+, A_-, B_-)$ satisfies the normal form (3.6). Substituting $\mathbf{U}$ given by this formula in the dynamical system (2.1), and using the expressions of the derivatives of A_+, B_+, A_-, B_- given by the normal form in Lemma 3.2, we obtain an equality for the variables A_+, B_+, A_-, B_- and their complex conjugates. We find the coefficients of the normal form, and in particular b_1 and b_3, by identifying the coefficients of suitably chosen monomials in this equality.

We denote by $\mathbf{\Phi}_{rstu}$ the coefficient of the monomial $A_+^r \overline{A_+}^s A_-^t \overline{A_-}^u$ in the expansion of $\widetilde{\mathbf{\Phi}}$. Identifying successively the coefficients of the monomials $A_+^2 \overline{A_+}$, $A_+ A_- \overline{A_-}$, and then A_+^2, $A_+ \overline{A_+}$, $A_+ A_-$, $A_+ \overline{A_-}$, $A_- \overline{A_-}$, we find the equalities

$$i\beta_1 \boldsymbol{\zeta}_+ + b_1 \mathbf{\Psi}_+ = (\mathcal{L}_{\mu_c} - ik_x)\mathbf{\Phi}_{2100} + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{2000}, \overline{\boldsymbol{\zeta}_+}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1100}, \boldsymbol{\zeta}_+),$$

$$i\beta_3 \boldsymbol{\zeta}_+ + b_3 \mathbf{\Psi}_+ = (\mathcal{L}_{\mu_c} - ik_x)\mathbf{\Phi}_{1011} + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1010}, \overline{\boldsymbol{\zeta}_-}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1001}, \boldsymbol{\zeta}_-) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{0011}, \boldsymbol{\zeta}_+),$$

and

$$(\mathcal{L}_{\mu_c} - 2ik_x)\mathbf{\Phi}_{2000} = -\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \boldsymbol{\zeta}_+), \tag{4.5}$$

$$\mathcal{L}_{\mu_c}\mathbf{\Phi}_{1100} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \overline{\boldsymbol{\zeta}_+}), \tag{4.6}$$

$$(\mathcal{L}_{\mu_c} - 2ik_x)\mathbf{\Phi}_{1010} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \boldsymbol{\zeta}_-), \tag{4.7}$$

$$\mathcal{L}_{\mu_c}\mathbf{\Phi}_{1001} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \overline{\boldsymbol{\zeta}_-}), \tag{4.8}$$

$$\mathcal{L}_{\mu_c}\mathbf{\Phi}_{0011} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_-, \overline{\boldsymbol{\zeta}_-}). \tag{4.9}$$

We determine the coefficients b_1 and b_3 by taking the scalar product of the first two equalities above with the vector $\mathbf{\Psi}_+^*$ in the kernel of the adjoint operator $(\mathcal{L}_{\mu_c} - ik_x)^*$ computed in Appendix 6.3.1 of Chapter 3,

$$b_1\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^* \rangle = \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{2000}, \overline{\boldsymbol{\zeta}_+}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1100}, \boldsymbol{\zeta}_+), \mathbf{\Psi}_+^* \rangle, \tag{4.10}$$

$$b_3\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^* \rangle = \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1010}, \overline{\boldsymbol{\zeta}_-}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1001}, \boldsymbol{\zeta}_-) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{0011}, \boldsymbol{\zeta}_+), \mathbf{\Psi}_+^* \rangle, \tag{4.11}$$

where $\mathbf{\Phi}_{2000}$, $\mathbf{\Phi}_{1100}$, $\mathbf{\Phi}_{1010}$, $\mathbf{\Phi}_{1001}$, and $\mathbf{\Phi}_{0011}$ are solutions of the linear equations (4.5)-(4.9).

In the equations (4.5) and (4.7), the linear operator $(\mathcal{L}_{\mu_c} - 2ik_x)$ is invertible, except in the case $\alpha = \pi/6$ when $2k_x = k_c$. Nevertheless, we only have to solve the equations in the subspace of vectors which are invariant under the action of $\mathbf{S}_3\boldsymbol{\tau}_\pi$ and the restriction of $(\mathcal{L}_{\mu_c} - ik_c)$ to this subspace is invertible, since its two-dimensional kernel is spanned by $\boldsymbol{\zeta}_0$ and $\overline{\boldsymbol{\zeta}_0}$ which do not belong to this subspace. Consequently, $\mathbf{\Phi}_{2000}$ and $\mathbf{\Phi}_{1010}$ are uniquely determined. In the equations (4.6), (4.8) and (4.9), the linear operator $\mathcal{L}_{\mu_c}$ has a one-dimensional kernel spanned by the vector $\boldsymbol{\varphi}_0$ in Lemma 2.2 (i) of Chapter 3, and the kernel of its adjoint is spanned by the vector φ_0^* in Appendix **??** of Chapter 3. The solvability condition is easily checked in all cases, so that we can solve these equations up to an element in the kernel of $\mathcal{L}_\mu$. The choice of this element in the kernel does not influence the result from (4.10)-(4.11), since $\mathcal{B}_\mu$ is invariant upon adding a multiple of φ_0.

After long and intricate computations we obtain in the rigid-rigid case that

$$g = \frac{b_3}{b_1} = \frac{b_{31}(\sin^2\alpha) + b_{31}(\cos^2\alpha) + b_{31}(0)}{\frac{1}{2}b_{31}(1) + b_{31}(0)}, \tag{4.12}$$

in which

$$b_{31}(\Theta) = A_{31}(\Theta) + B_{31}(\Theta)\mathcal{P}^{-1} + C_{31}(\Theta)\mathcal{P}^{-2},$$

with

$$\begin{aligned} A_{31}(\Theta) &= 2\mu_c^3\langle (D^2 - 4k_c^2\Theta)^2 V_1, R_1\rangle, \\ B_{31}(\Theta) &= 4\mu_c^3\Theta\left(\langle V_1, R_2\rangle + \langle V_2, R_1\rangle\right), \\ C_{31}(\Theta) &= -\frac{2\mu_c\Theta}{k_c^2}\langle (D^2 - 4k_c^2\Theta)V_2, R_2\rangle, \end{aligned}$$

where

$$R_1 = VD\phi + (1-2\Theta)\phi DV, \quad R_2 = \left(D^2 - 4k_c^2(1-\Theta)\right)(VDV) - 4\Theta(DV)(D^2V),$$

and V_1, V_2 are the unique solutions of the boundary value problems

$$\begin{aligned} &(D^2 - 4k_c^2\Theta)^3 V_1 + 4k_c^2\mu_c^2\Theta\, V_1 = R_1, \\ &V_1 = DV_1 = (D^2 - 4k_c^2\Theta)^2 V_1 = 0 \text{ in } z = 0, 1, \end{aligned}$$

and

$$\begin{aligned} &(D^2 - 4k_c^2\Theta)^3 V_2 + 4k_c^2\mu_c^2\Theta\, V_2 = R_2, \\ &V_2 = (D^2 - 4k_c^2\Theta)V_2 = (D^2 - 4k_c^2\Theta)DV_2 = 0 \text{ in } z = 0, 1, \end{aligned}$$

respectively. Recall that V and ϕ are the unique symmetric solutions of the boundary value problems (2.6) in Chapter3, and defined in appendix 6.3.1 of Chapter 3, respectively. Notice that $g \to 2$, as $\alpha \to 0$, which was the value of g in the case of the Swift-Hohenberg equation in [32].

In the free-free case we have explicit formulas

$$\mu_0(k) = \frac{1}{|k|}\left(k^2 + \pi^2\right)^{3/2},$$

from which we easily obtain the numerical values

$$k_c = \frac{\pi}{\sqrt{2}}, \quad \mu_c = \frac{3\sqrt{3}}{2}\pi^2.$$

The solution V of the boundary value problem is also explicit as seen in section 1.3.3 of Chapter 1, $V(z) = \sin(\pi z)$. The explicit formulas above make the computation of the quotient g in Section 6.4.2 much simpler in this case. We obtain an explicit formula for b_{31} in (4.12),

$$b_{31}(\Theta) = \frac{18\sqrt{3}\pi^8(1-\Theta)^2}{\ell_\Theta}\left((\Theta+2)^2 + \frac{9}{2}\Theta\,\mathcal{P}^{-1} + 3\Theta(\Theta+2)\mathcal{P}^{-2}\right), \tag{4.13}$$

and a Maple computation of the quotient g gives the result in Figure 3.2.

Remark 4.1 *In this way we can also compute the coefficient b_0. By identifying the coefficients of the terms εA_+, and then taking the scalar product with $\mathbf{\Psi}_+^*$ we obtain*

$$b_0\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^*\rangle = \langle \mathcal{L}^{(1)}\boldsymbol{\zeta}_+, \mathbf{\Psi}_+^*\rangle,$$

in which $\mathcal{L}^{(1)}$ is the derivative with respect to μ of the operator $\mathcal{L}_\mu$ in Appendix 6.3.2 of Chapter 3 taken at $\mu = \mu_c$. A direct computation gives

$$b_0\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^*\rangle = \frac{1}{\mu_c^2 k_c^2}\left(\|D^2V\|^2 + 2k_c^2\|DV\|^2 + k_c^4\|V\|^2\right) + \|D\phi\|^2 + k_c^2\|\phi\|^2 > 0, \tag{4.14}$$

and implies that $\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^*\rangle < 0$, since $b_0 < 0$. We point out that it is not obvious to determine the sign of this scalar product directly from the explicit formulas of $\mathbf{\Psi}_+$ and $\mathbf{\Psi}_+^*$.

6.4.3 Coefficients of the cubic normal form when S_3 does not apply

The formulas for the coefficients b_3 and b_5 found in Section 4.3 remain the same, and we compute in the same way the coefficients a_1 and a_3. We obtain

$$\begin{aligned}
a_1\langle \mathbf{\Psi}_0, \mathbf{\Psi}_0^*\rangle &= \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{200000}, \overline{\boldsymbol{\zeta}_0}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{110000}, \boldsymbol{\zeta}_0), \mathbf{\Psi}_0^*\rangle,\\
a_3\langle \mathbf{\Psi}_0, \mathbf{\Psi}_0^*\rangle &= \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{001100}, \boldsymbol{\zeta}_0) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{101000}, \overline{\boldsymbol{\zeta}_+}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{100100}, \boldsymbol{\zeta}_+), \mathbf{\Psi}_0^*\rangle,\\
b_3\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^*\rangle &= \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{002000}, \overline{\boldsymbol{\zeta}_+}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{001100}, \boldsymbol{\zeta}_+), \mathbf{\Psi}_+^*\rangle,\\
b_5\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^*\rangle &= \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{001010}, \overline{\boldsymbol{\zeta}_-}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{001001}, \boldsymbol{\zeta}_-) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{000011}, \boldsymbol{\zeta}_+), \mathbf{\Psi}_+^*\rangle.
\end{aligned}$$

where $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_\pm$ are the eigenvectors of $\mathcal{L}_{\mu_c}$ from Section 3.2.2 of Chapter 3, $\mathbf{\Psi}_0^*$ and $\mathbf{\Psi}_+^*$ are eigenvectors of the adjoint operator $\mathcal{L}_{\mu_c}^*$ associated to the eigenvalues $-ik_c$ and $-ik_x$, respectively, and the vectors $\mathbf{\Phi}_{pqrstu}$ satisfy

$$
\begin{aligned}
&(\mathcal{L}_{\mu_c} - 2ik_c)\mathbf{\Phi}_{200000} = -\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_0), \qquad \mathcal{L}_{\mu_c}\mathbf{\Phi}_{110000} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_0, \overline{\boldsymbol{\zeta}_0}),\\
&\mathcal{L}_{\mu_c}\mathbf{\Phi}_{001100} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \overline{\boldsymbol{\zeta}_+}), \qquad (\mathcal{L}_{\mu_c} - i(k_c + k_x))\mathbf{\Phi}_{101000} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_+),\\
&(\mathcal{L}_{\mu_c} - i(k_c - k_x))\mathbf{\Phi}_{100100} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_0, \overline{\boldsymbol{\zeta}_+}), \qquad (\mathcal{L}_{\mu_c} - 2ik_x)\mathbf{\Phi}_{002000} = -\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \boldsymbol{\zeta}_+),\\
&(\mathcal{L}_{\mu_c} - 2ik_x)\mathbf{\Phi}_{001010} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \boldsymbol{\zeta}_-), \qquad \mathcal{L}_{\mu_c}\mathbf{\Phi}_{001001} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \overline{\boldsymbol{\zeta}_-}),\\
&\mathcal{L}_{\mu_c}\mathbf{\Phi}_{000011} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_-, \overline{\boldsymbol{\zeta}_-}).
\end{aligned}
$$

A direct computation (see also Appendix 6.3.1 of Chapter3) gives the formulas for the eigenvectors

$$
\boldsymbol{\zeta}_0(y,z) = \begin{pmatrix} \frac{i}{k_c}DV \\ 0 \\ V \\ -\frac{1}{\mu_c k_c^2}D^3V \\ 0 \\ \frac{ik_c}{\mu_c}V \\ \frac{1}{\mu_c k_c^2}(D^2-k_c^2)^2V \\ \frac{i}{\mu_c k_c}(D^2-k_c^2)^2V \end{pmatrix}, \quad \boldsymbol{\zeta}_\pm(y,z) = e^{\pm ik_y y}\begin{pmatrix} \frac{i\sin\alpha}{k_c}DV \\ \pm\frac{i\cos\alpha}{k_c}DV \\ V \\ -\frac{1}{\mu_c k_c^2}(D^2-k_c^2\cos^2\alpha)DV \\ \mp\frac{\sin\alpha\cos\alpha}{\mu_c}DV \\ \frac{ik_c\sin\alpha}{\mu_c}V \\ \frac{1}{\mu_c k_c^2}(D^2-k_c^2)^2V \\ \frac{i\sin\alpha}{\mu_c k_c}(D^2-k_c^2)^2V \end{pmatrix},
$$

and

$$
\mathbf{\Psi}_0^*(y,z) = \begin{pmatrix} -\frac{1}{\mu_c k_c^2}\left(D^3V - \langle D^3V\rangle\right) \\ 0 \\ \frac{ik_c}{\mu_c}V \\ -\frac{i}{k_c}DV \\ 0 \\ -V \\ -ik_c\phi \\ \phi \end{pmatrix}, \quad \mathbf{\Psi}_+^*(y,z) = e^{ik_y y}\begin{pmatrix} -\frac{1}{\mu_c k_c^2}(D^2-k_c^2\cos^2\alpha)DV \\ -\frac{\sin\alpha\cos\alpha}{\mu_c}DV \\ \frac{ik_c\sin\alpha}{\mu_c}V \\ -\frac{i\sin\alpha}{k_c}DV \\ -\frac{i\cos\alpha}{k_c}DV \\ -V \\ -ik_c(\sin\alpha)\phi \\ \phi \end{pmatrix}.
$$

In these formulas, V is a real-valued solution of the boundary value problem

$$
\begin{aligned}
&(D^2-k_c^2)^3V + \mu_c^2k_c^2V = 0,\\
&V = DV = (D^2-k_c^2)^2V = 0 \text{ in } z=0,\\
&V = D^2V = D^4V = 0 \text{ in } z=1,
\end{aligned}
$$

ϕ is the unique solution of the boundary value problem

$$(D^2 - k_c^2)\phi = V, \quad \phi = 0 \text{ in } z = 0, 1,$$

and

$$\langle D^3 V \rangle = \int_{\Omega_{per}} D^3 V(z)\, dy\, dz.$$

After very long computations we obtain that

$$g_1 = \frac{a_3}{a_1} = \frac{b_{51}(\frac{1}{2}(1 + \sin\alpha)) + b_{51}(\frac{1}{2}(1 - \sin\alpha)) + b_{51}(0)}{\frac{1}{2}b_{51}(1) + b_{51}(0)}, \tag{4.15}$$

$$g_3 = \frac{b_5}{b_3} = \frac{b_{51}(\sin^2\alpha) + b_{51}(\cos^2\alpha) + b_{51}(0)}{\frac{1}{2}b_{51}(1) + b_{51}(0)}, \tag{4.16}$$

in which

$$b_{51}(\Theta) = A_{51}(\Theta) + B_{51}(\Theta)\mathcal{P}^{-1} + C_{51}(\Theta)\mathcal{P}^{-2},$$

with

$$\begin{aligned}
A_{51}(\Theta) &= 2\mu_c^3 \langle (D^2 - 4k_c^2\Theta)^2 V_1, R_1 \rangle, \\
B_{51}(\Theta) &= 4\mu_c^3 \Theta \left(\langle V_1, R_2 \rangle + \langle V_2, R_1 \rangle \right), \\
C_{51}(\Theta) &= -\frac{2\mu_c\Theta}{k_c^2} \langle (D^2 - 4k_c^2\Theta) V_2, R_2 \rangle,
\end{aligned}$$

where

$$R_1 = VD\phi + (1 - 2\Theta)\phi DV, \quad R_2 = \left(D^2 - 4k_c^2(1 - \Theta)\right)(VDV) - 4\Theta(DV)(D^2V),$$

and V_1, V_2 are the unique solutions of the boundary value problems

$$\begin{aligned}
&(D^2 - 4k_c^2\Theta)^3 V_1 + 4k_c^2\mu_c^2\Theta\, V_1 = R_1, \\
&V_1 = DV_1 = (D^2 - 4k_c^2\Theta)^2 V_1 = 0 \text{ in } z = 0, \\
&V_1 = D^2V_1 = D^4V_1 = 0 \text{ in } z = 1,
\end{aligned}$$

and

$$\begin{aligned}
&(D^2 - 4k_c^2\Theta)^3 V_2 + 4k_c^2\mu_c^2\Theta\, V_2 = R_2, \\
&V_2 = D^2V_2 = (D^2 - 4k_c^2\Theta) DV_2 = 0 \text{ in } z = 0, \\
&V_2 = D^2V_2 = D^4V_2 = 0 \text{ in } z = 1,
\end{aligned}$$

respectively.

6.5 Appendix of Chapter 5

6.5.1 Proof of Theorem 1.1

Sketch of the method

Let us consider the system (1.1). The equilibrium $(A, B) = (0, 1)$ of the system gives an approximation of convection rolls parallel to the wall (periodic in the x direction, with fixed phase) bifurcating for Rayleigh numbers $\mathcal{R} > \mathcal{R}_c$ close to $\mathcal{R}_c$, whereas the equilibrium $(A, B) = (1, 0)$ of the system (1.1) gives the same convection rolls (periodic in the y direction) rotated by an angle $\pi/2$ with the phase fixed by the imposed reflection symmetry. A heteroclinic orbit connecting these two equilibria provides then an approximation of orthogonal domain walls (see Figure 1.1).

The limit $\varepsilon \to 0$ is singular, and gives indeed a non smooth heteroclinic solution such that (see Figure 5.1)

(i) for x running from $-\infty$ to 0, then (A, B) varies from $(1, 0)$ to $(0, \frac{1}{\sqrt{g}})$ on the ellipse $A^2 + gB^2 = 1$, while

(ii) for x running from 0 to $+\infty$, then (A, B) varies from $(0, \frac{1}{\sqrt{g}})$ to $(0, 1)$, satisfying the differential equation

$$\frac{dB}{dx} = \frac{\varepsilon}{\sqrt{2}}(1 - B^2).$$

The two manifolds $A = \widetilde{A_*} = (1 - gB^2)^{1/2}$, and $A = 0$ are named "slow manifolds"

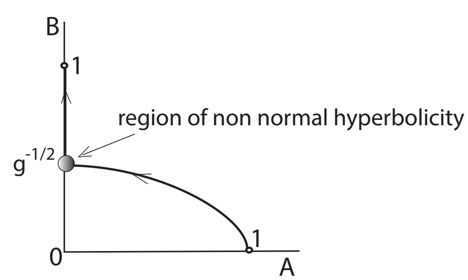

Figure 5.1: Critical manifold

in litterature (see [25],[52]). For the part (i) of the curve, where $x \in (-\infty, 0]$, the set of equilibria, here $A^2 + gB^2 = 1$, is not normally hyperbolic at the end point $(A, B) = (0, 1/\sqrt{g})$. For the second part (ii) of the curve, where $x \in [0, +\infty)$, the set of equilibria $(A, B) = (0, B)$ is also not normally hyperbolic for $B = 1/\sqrt{g}$. The normal hyperbolicity is essential in Fenichel's theorems [25], so we cannot use them directly. However we may use normal hyperbolicity up to a small neighborhood of $(A, B) = (0, 1/\sqrt{g})$, as this is done in

[39] section 3 for finding the unstable manifold of $(A, B) = (1, 0)$ in a neighborhoof of the slow manifold $A = \widetilde{A_*}$, and in [39] section 4 for finding the stable manifold of $(A, B) = (0, 1)$ in a neighborhood of the slow manifold $A = 0$.

The neighborhood of $(A, B) = (0, 1/\sqrt{g})$ not reached by the method above has a size of order $\mathcal{O}(\varepsilon^{4/5})$. We could think to use a geometric analysis, as Krupa et al did in [52], where a blow-up method is used for getting a system independent of ε. Indeed the scaling

$$\begin{aligned} A &= K^2\varepsilon^{2/5}\overline{A}, \\ B &= \frac{1}{\sqrt{g}}(1 + \frac{K^4}{2}\varepsilon^{4/5}\overline{B}), \\ x &= \frac{z}{K\varepsilon^{1/5}}, \end{aligned}$$

with $\overline{B} = z$, since at main order

$$\overline{B}'' = 0,\ \overline{B}(0) = 0,$$

leads, at main order, to

$$\frac{d^4\overline{A}}{dz^4} = -\overline{A}(\overline{A}^2 + z),\ z \in [-a_-, +a_+], \tag{5.1}$$

which is independent of ε. However, the work of [52] is made in 2 dimensions, while we have here the 6-dimensional system (1.1). It results that the nice pictures of [52] would be very hard to transpose here. In addition, we need to satisfy boundary values (also independent of ε) coming on the left side from the connection with the unstable manifold, and from the right side from the connection with the stable manifold.

There are 3 unstable eigendirections starting from $M_- = (1, 0)$, and 3 stable eigendirections in $M_+ = (0, 1)$. The heteroclinic we are looking for, results from the intersection of these two invariant manifolds. The difficulty in the proof of Theorem 1.1 is to obtain a precise estimate for the existence of the 3-dimensional unstable manifold of M_-, where the coordinate B varies from 0 to a neighborhood of $1/\sqrt{g}$, and to obtain a precise estimate for the existence of the 3-dimensional stable manifold of M_+ until B varies from 1 (backwards) to a neighborhood of $1/\sqrt{g} = 1/\sqrt{1+\delta^2}$, while A stays close to 0. For approaching the closest possible to $B = 1/\sqrt{g}$, we use the first integral of (1.1), which implies that both invariant manifolds are included in the 5-dimensional invariant manifold given by

$$0 = \varepsilon^2(A'^2)'' - 3\varepsilon^2 A''^2 - B'^2 + \frac{\varepsilon^2}{2}(A^2 + B^2 - 1)^2 + \varepsilon^2\delta^2 A^2 B^2.$$

The unstable manifold of M_- is obtained for $0 \leq B \leq \frac{1-c\varepsilon^{4/5}}{\sqrt{g}}$, while we first obtain the stable manifold of M_+ for $\frac{1+c'\varepsilon^{4/5}}{\sqrt{g}} \leq B \leq 1$. For extending the existence of the stable manifold in the gap of size of order $\varepsilon^{4/5}$, we need to solve the 4th order differential equation (5.1), independent of ε, also found in [58] and [13], after rescaling, where the boundary conditions, also independent of ε, come from the 2 times 2 parameters introduced by each invariant manifolds arriving in $\pm a_{\pm}$.

A precise estimate on a_+ is obtained in [39] for extending the domain of existence of the stable manifold, for B in the interval $\frac{1-c\varepsilon^{4/5}}{\sqrt{g}} \leq B \leq 1$. Then using results of [12] the two manifolds intersect and Theorem 1.1 with its Corollaries are proved.

Remark 5.1 *It should be noticed that in the middle of the heteroclinic, $A(0) = \mathcal{O}(\varepsilon^{2/5})$ and for $x \in (0, +\infty)$, $A(x)$ oscillates, staying of order $\mathcal{O}(\varepsilon^{2/5})$, while $B(0) = 1/\sqrt{g}$ and $B(x)$ grows monotonically until* 1.

Remark 5.2 *Using symmetries of the system: $A \mapsto \pm A$, $B \mapsto \pm B$ and reversibility symmetry: $(A(x), B(x)) \mapsto (A(-x), B(-x))$, we find 8 heteroclinics. Two are connecting M_- to M_+ with opposite dynamics, two others connect $-M_-$ to M_+, two connect M_- to $-M_+$, and two connect $-M_-$ to $-M_+$. The one which interests us is the only one connecting M_- to M_+ with the dynamics running from M_- to M_+.*

Remark 5.3 *It should be noticed that the study made in [58] on the heteroclinic solution for the system (1.1) uses asymptotic analysis, suggesting the existence of the heteroclinic, later proved mathematically in [12]. Contrary to these previous works, using asymptotic analysis on the full real line, the precise estimate which is obtained for a_+(see (5.1)) is essential here, for getting a rigorous result.*

6.5.2 Reduction of the normal form

We start with the 12 dimensional system (2.4) where $\tilde{\mu}$ is $\mathcal{R}^{1/2} - \mathcal{R}_c^{1/2}$. Then restricting the system to solutions symmetric in y, the full system reduces to a 8-dimensional one such as A_0 (real) and B_0 are the amplitudes of the rolls respectively at $x = -\infty$, and $x = +\infty$). Let us define (below, X is redefined as the 4 first components of X previously used in (2.4))

$$\begin{aligned} X &= (A_0, A_1, A_2, A_3)^t \in \mathbb{R}^4, \\ Y &= (B_0, B_1)^t \in \mathbb{C}^2, \\ k &= k_c(1 + \widetilde{k}), \end{aligned}$$

so that the system may now be written under normal form as (see (2.4))

$$\begin{aligned} \frac{dX}{dx} &= LX + N(X, Y, \overline{Y}, \mu, \widetilde{k}) + F(X, Y, \overline{Y}, \tilde{\mu}, \widetilde{k}), \\ \frac{dY}{dx} &= L_{k_c} Y + M(X, Y, \overline{Y}, \mu) + G(X, Y, \overline{Y}, \tilde{\mu}), \end{aligned} \tag{5.2}$$

with

$$\begin{aligned} LX &= (A_1, A_2, A_3, 0)^t, \\ L_{k_c}Y &= (ik_cB_0 + B_1, ik_cB_1)^t. \end{aligned}$$

The (reversible) system (5.2) anticomutes with the symmetry $\mathbf{S}_1$ (representing the reflection $x \mapsto -x$). and commutes with $\boldsymbol{\tau}_\pi$ (shift by half of one period in y direction):

$$\begin{aligned} (A_0, A_1, A_2, A_3, B_0, B_1) &\overset{\mathbf{S}_1}{\mapsto} (A_0, -A_1, A_2, -A_3, \overline{B_0}, -\overline{B_1}), \\ (A_0, A_1, A_2, A_3, B_0, B_1) &\overset{\boldsymbol{\tau}_\pi}{\mapsto} (-A_0, -A_1, -A_2, -A_3, B_0, B_1). \end{aligned}$$

Remark 5.4 *We don't use the vertical symmetry $z \mapsto 1 - z$ here (valid only in rigid-rigid or free-free boundaries). In the case of rigid-free boundary conditions, we have no such symmetry. The symmetry $\boldsymbol{\tau}_\pi$ implies that F is odd in X and G even in X. Moreover it can be shown that there is no term of degree 4 in $X, Y, \overline{Y}$ in the normal form.*

Then we obtain the estimates for F and G which are C^m- smooth in their arguments close to 0, with m as large as we need, and

$$\begin{aligned} |F(X, Y, \overline{Y}, \tilde{\mu}, \widetilde{k})| &\leq c|X|(|X|^2 + |Y|^2 + |\widetilde{k}| + |\tilde{\mu}|)^2 \\ |G(X, Y, \overline{Y}, \tilde{\mu})| &\leq c(|X|^2 + |Y|)(|X|^2 + |Y|^2 + |\tilde{\mu}|)^2, \end{aligned} \tag{5.3}$$

and the normal form which may be computed as in Chapter 4, is (see[28] Chapter 3)

$$N(X, Y, \overline{Y}, \tilde{\mu}) = \begin{pmatrix} 0 \\ A_0P_1 \\ A_1P_1 + c_8u_8 + c_{13}u_{13} \\ A_2P_1 + A_0P_3 + +c_8v_8 + c_{13}v_{13} + d_{14}u_{14} \end{pmatrix},$$

$$M(X, Y, \overline{Y}, \tilde{\mu}) = \begin{pmatrix} iB_0Q_0 + \alpha_{10}u_{10} \\ iB_1Q_0 + B_0Q_1 + \alpha_{10}v_{10} + i\beta_{10}u_{10} + i\beta_{12}u_{12} \end{pmatrix},$$

$$\begin{aligned} P_1 &= b_0\tilde{\mu} + b_0'\widetilde{k} + b_1u_1 + b_3u_3 + b_5u_5 + b_6u_6, \\ P_3 &= d_0\tilde{\mu} + d_0''\widetilde{k}^2 + d_1u_1 + d_1'\widetilde{k}u_1 + d_3u_3 + d_5u_5 + d_6u_6, \end{aligned}$$

$$\begin{aligned} Q_0 &= \alpha_0\tilde{\mu} + \alpha_1u_1 + \alpha_3u_3 + \alpha_5u_5 + \alpha_6u_6 \\ Q_1 &= \beta_0\tilde{\mu} + \beta_1u_1 + \beta_3u_3 + \beta_5u_5 + \beta_6u_6, \end{aligned}$$

where

$$
\begin{aligned}
u_1 &= A_0^2, \quad v_1 = A_0 A_1, \quad w_1 = \frac{1}{2} A_1^2, \\
u_3 &= 2A_0 A_2 - A_1^2, \quad v_3 = 3A_0 A_3 - A_1 A_2 \\
u_5 &= B_0 \overline{B_0}, \quad v_5 = \frac{1}{2}(B_0 \overline{B_1} + \overline{B_0} B_1), \quad w_5 = \frac{1}{2} B_1 \overline{B_1} \\
u_6 &= i(B_0 \overline{B_1} - \overline{B_0} B_1).
\end{aligned}
$$

$$
\begin{aligned}
u_8 &= A_0 v_3 - A_1 u_3, \quad v_8 = A_1 v_3 - 2A_2 u_3, \\
u_{13} &= A_0 v_5 - A_1 u_5, \quad v_{13} = A_0 w_5 - A_2 u_5, \\
u_{14} &= A_0 w_5 + A_2 u_5 - A_1 v_5,
\end{aligned}
$$

$$
\begin{aligned}
u_{10} &= B_0 v_1 - B_1 u_1, \quad v_{10} = 2B_0 w_1 - B_1 v_1 \\
u_{12} &= B_0 v_3 - B_1 u_3.
\end{aligned}
$$

Then, the X part of the system (5.2) may be written as a 4th order real ODE, while the Y part becomes a 2nd order complex ODE as

$$
\begin{aligned}
A_0^{(4)} &= A_0[d_0 \tilde{\mu} + (d_0'' - b_0'^2)\widetilde{k}^2 + d_1 A_0^2 + d_1' \widetilde{k} A_0^2 + d_5 \widetilde{B_0}\overline{\widetilde{B_0}} + d_1' \widetilde{k} A_0^2 \\
&\quad + i d_6 (\widetilde{B_0}\overline{\widetilde{B_0}}' - \overline{\widetilde{B_0}}\widetilde{B_0}')] + (a_0 \mu + 3b_0' \widetilde{k}) A_0'' + a_1 A_0^2 A_0'' + a_2 A_0 A_0'^2 \\
&\quad + a_3 A_0 \widetilde{B_0}' \overline{\widetilde{B_0}}' + a_4 A_0' (\widetilde{B_0}\overline{\widetilde{B_0}}' + \overline{\widetilde{B_0}}\widetilde{B_0}') + a_5 A_0'' \widetilde{B_0}\overline{\widetilde{B_0}} \\
&\quad + 3 i b_6 A_0'' (\widetilde{B_0}\overline{\widetilde{B_0}}' - \overline{\widetilde{B_0}}\widetilde{B_0}') + a_6 A_0 A_0' A_0''' + a_7 A_0 A_0''^2 + a_8 A_0'^2 A_0'' + \mathcal{O}_X(5),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B_0}'' &= \widetilde{B_0}[\beta_0 \tilde{\mu} + \beta_1 A_0^2 + \beta_5 \widetilde{B_0}\overline{\widetilde{B_0}}] + i c_1 \widetilde{B_0}' A_0^2 + i c_2 \widetilde{B_0}' |\widetilde{B_0}|^2 + i c_3 \overline{\widetilde{B_0}}' \widetilde{B_0}^2 \\
&\quad + 2 i \alpha_0 \mu \widetilde{B_0}' + i c_4 \widetilde{B_0} A_0 A_0' - 2\alpha_6 \widetilde{B_0}' (\widetilde{B_0}\overline{\widetilde{B_0}}' - \overline{\widetilde{B_0}}\widetilde{B_0}') \\
&\quad + c_5 \widetilde{B_0} A_0 A_0'' + c_6 \widetilde{B_0} A_0'^2 + c_7 \widetilde{B_0}' A_0 A_0' + i c_8 \widetilde{B_0} A_0 A_0''' \\
&\quad i c_9 \widetilde{B_0}' A_0 A_0'' + i c_{10} \widetilde{B_0}' A_0'^2 + i c_{11} \widetilde{B_0} A_0' A_0'' + \mathcal{O}_Y(5),
\end{aligned}
$$

with real coefficients $d_j, d_1' d_0'', a_j, b_j, b_0', c_j, \beta_j, \alpha_j$ and

$$
\widetilde{B_0} = B_0 e^{-ik_c x}, \ \widetilde{B_1} = B_1 e^{-ik_c x}, \tag{5.4}
$$

$$
\begin{aligned}
d_0 &= -4k_c^2 \beta_0 > 0, \ d_1 = -4k_c^2 \beta_5 < 0, \\
\frac{\beta_1}{\beta_5} &= \frac{d_5}{d_1} := g > 0, \ b_0' = \frac{4k_c^2}{3}, d_0'' = -\frac{20}{9} k_c^4,
\end{aligned}
$$

$$\begin{aligned}
\mathcal{O}_X(5) &= \mathcal{O}(|X|(|X|^2+|Y|^2+\widetilde{k}^2+|\mu|)^2),\\
\mathcal{O}_Y(5) &= \mathcal{O}[(|X|^2+|Y|)(|X|^2+|Y|^2+|\mu|)^2],\\
X &= (A_0, A_0', A_0'', A_0''')^t\\
Y &= (\widetilde{B_0}, \widetilde{B_0}').
\end{aligned}$$

Notice that the high order rests $\mathcal{O}_X(5)$ and $\mathcal{O}_Y(5)$ are no longer autonomous, since they are functions of $e^{\pm ik_cx}$.

Now, we make the following scaling

$$\begin{aligned}
x &= \frac{1}{2\varepsilon k_c}\widetilde{x},\ \tilde{\mu} = \frac{4k_c^2}{-\beta_0}\varepsilon^4,\ \widetilde{k} = \varepsilon^2 k_- \\
A_0(x) &= \frac{2k_c}{\sqrt{\beta_5}}\varepsilon^2\widetilde{A_0}(\widetilde{x}),\ \widetilde{B_0}(x) = \frac{2k_c}{\sqrt{\beta_5}}\varepsilon^2\widetilde{\widetilde{B_0}}(\widetilde{x}),
\end{aligned}\tag{5.5}$$

so that the system above becomes, after suppressing the tildes,

$$\begin{aligned}
A_0^{(4)} &= k_-A_0'' + A_0(1-\frac{k_-^2}{4} - A_0^2 - g|B_0|^2) + \widehat{f},\\
B_0'' &= \varepsilon^2 B_0(-1+gA_0^2+|B_0|^2) + \widehat{g},
\end{aligned}\tag{5.6}$$

with additional cubic terms of the form (changing the definitions of coefficients)

$$\begin{aligned}
\widehat{f} &= id_1\varepsilon A_0(B_0\overline{B_0}' - \overline{B_0}B_0') + \sigma_0\varepsilon^2 k_-A_0^3 + \varepsilon^2[d_3A_0'' + d_4A_0^2A_0'' + d_2A_0A_0'^2 + d_6A_0|B_0'|^2\\
&\quad + d_7A_0'(B_0\overline{B_0}' + \overline{B_0}B_0') + d_5A_0''|B_0|^2] + id_8\varepsilon^3A_0''(B_0\overline{B_0}' - \overline{B_0}B_0') + \mathcal{O}(\varepsilon^4),
\end{aligned}\tag{5.7}$$

$$\begin{aligned}
\widehat{g} &= \varepsilon^3[ic_0B_0' + ic_1B_0'|A_0|^2 + ic_2B_0'|B_0|^2 + ic_3B_0^2\overline{B_0}' + ic_9B_0A_0A_0']\\
&\quad + \varepsilon^4[c_4B_0'(B_0\overline{B_0}' - \overline{B_0}B_0') + c_5B_0A_0A_0'' + c_6B_0A_0'^2 + c_7B_0'A_0A_0']\\
&\quad + \varepsilon^5[ic_8B_0A_0A_0''' + ic_7B_0'A_0A_0'' + ic_{10}B_0'A_0'^2 + ic_{11}B_0A_0'A_0'' + \mathcal{O}(\varepsilon^6).
\end{aligned}\tag{5.8}$$

6.5.3 Equilibrium solution at $x = -\infty$

Let us look for equilibria of (2.10), which should correspond to the convective rolls at $x = -\infty$ parallel to x - axis. Cancelling all derivatives with respect to x, we obtain a system commuting with the symmetry $(A_0, B_0) \mapsto (A_0, \overline{B_0})$. It then results a system of 2 real equations for A_0, B_0 :

$$\begin{aligned}
A_0(1-\frac{k_-^2}{4} - A_0^2 + \sigma_0\varepsilon^2k_-A_0^2 - gB_0^2) + \mathcal{O}(\varepsilon^4) &= 0\\
B_0(-1+gA_0^2+B_0^2) + \mathcal{O}(\varepsilon^4) &= 0,
\end{aligned}$$

where we may observe that the terms $\mathcal{O}(\varepsilon^4)$ in the second equation contain at least terms of degree 1 in B_0, since they come from terms of order 5 in $(A_0, B_0, \overline{B_0})$. The first terms not containing B_0 may be found at order 6 in A_0, which makes order ε^6 after the scaling (5.5) in the rest (12-6=6).

It then results that the equilibrium that we are looking for satisfies (by implicit function theorem)

$$\begin{aligned} A_0^2 &= 1 - \frac{k_-^2}{4} + \sigma_0 \varepsilon^2 k_- + \mathcal{O}(\varepsilon^2 |k_-|^3 + \varepsilon^4), \\ B_0 &= \mathcal{O}(\varepsilon^6). \end{aligned}$$

Remark 5.5 *In the cases where vertical symmetry $z \mapsto 1 - z$ applies, the additional symmetry S_0 changes the signs of A_0 and B_0, implying that $Y = 0$ is an invariant subspace, so that in such cases $B_0 = 0$ for the equilibrium at $-\infty$.*

6.5.4 Periodic solution in M_+

Let us consider the 4-dimensional reversible vector field corresponding to the system (5.2) with $X = 0$ and rescaled. We intend to give precise estimates on the family of periodic bifurcating solutions $B_0^{(+\infty)}(k_+, x)$, here corresponding to the periodic convecting rolls at infinity in M_+ with wave numbers close to k_c (becomes $1/2\varepsilon$ after the scaling (5.5)).

Since we use the normal form up to cubic order, and since there is no term of order 4, it takes the form (after the scaling used in (5.5), but before we incorporate $e^{\frac{ix}{2\varepsilon}}$ in B_0, so that the system is still autonomous):

$$\begin{aligned} \frac{dB_0}{dx} &= \frac{i}{2\varepsilon} B_0 + B_1 + i\varepsilon^3 B_0 P + \varepsilon^7 g_0(\varepsilon, Y, \overline{Y}) \\ \frac{dB_1}{dx} &= \frac{i}{2\varepsilon} B_1 + \varepsilon^2 B_0 Q + i\varepsilon^3 B_1 P + \varepsilon^6 g_1(\varepsilon, Y, \overline{Y}), \end{aligned} \tag{5.9}$$

with

$$\begin{aligned} Y &= (B_0, B_1) \\ P &= \alpha + \beta |B_0|^2 + \varepsilon \gamma K \\ Q &= -1 + |B_0|^2 + \varepsilon \delta K \\ K &= \frac{i}{2}(B_0 \overline{B_1} - \overline{B_0} B_1) \end{aligned}$$

where we are looking for a periodic solution (B_0, B_1), with wave number ω close to $\frac{1+\varepsilon^2 k_+}{2\varepsilon}$.

Principal part

Let us first compute periodic solutions for $g_0 = g_1 \equiv 0$. Then these small terms will be perturbations treated by an adapted implicit function theorem.

Without g_0 and g_1, let us use polar coordinates (see [28] section 4.3.3)

$$\begin{aligned} B_0 &= r_0 e^{i\theta_0} \\ B_1 &= i r_1 e^{i\theta_1} \end{aligned}$$

then

$$\begin{aligned} K &= r_0 r_1 \cos(\theta_0 - \theta_1) = \text{const} \\ \frac{dr_0}{dx} &= r_1 \sin(\theta_0 - \theta_1) \\ \frac{dr_1}{dx} &= \varepsilon^2 r_0 \sin(\theta_0 - \theta_1) Q(\varepsilon, r_0^2, K) \\ r_0 \frac{d\theta_0}{dx} &= \frac{r_0}{2\varepsilon} + r_1 \cos(\theta_0 - \theta_1) + \varepsilon^3 r_0 P \\ r_1 \frac{d\theta_1}{dx} &= \frac{r_1}{2\varepsilon} - \varepsilon^2 r_0 \cos(\theta_0 - \theta_1) Q(\varepsilon, r_0^2, K) + \varepsilon^3 r_1 P. \end{aligned}$$

The required periodic solutions correspond to

$$\begin{aligned} & r_0 \text{ and } r_1 \text{ const} \\ \theta_0 &= \theta_1,\ \frac{d\theta_0}{dx} = \frac{1+\varepsilon^2 k_+}{2\varepsilon} \\ K &= r_0 r_1, \end{aligned}$$

hence

$$\frac{\varepsilon k_+}{2} = \frac{r_1}{r_0} + \varepsilon^3 P \tag{5.10}$$

$$\left(\frac{r_1}{r_0}\right)^2 = -\varepsilon^2 Q. \tag{5.11}$$

Solving (5.10) with respect to r_1 gives

$$\begin{aligned} r_1 &= \varepsilon r_0 \frac{k_+ - 2\varepsilon^2(\alpha + \beta r_0^2)}{2(1+\varepsilon^4 \gamma r_0^2)} \\ &= \frac{\varepsilon r_0}{2}[k_+ - 2\varepsilon^2(\alpha + \beta r_0^2)](1 + \mathcal{O}(\varepsilon^4)), \end{aligned}$$

and (5.11) leads to

$$\frac{1}{4}[k_+ - 2\varepsilon^2(\alpha + \beta r_0^2)]^2 + \frac{\varepsilon^2 \delta r_0^2}{2}[k_+ - 2\varepsilon^2(\alpha + \beta r_0^2)] = (1 - r_0^2)(1 + \gamma\varepsilon^4 r_0^2)^2$$

which is solved with respect to r_0^2, by implicit function theorem:

$$r_0^2 = 1 - \frac{k_+^2}{4} + \sigma_1 \varepsilon^2 k_+ + \sigma_2 \varepsilon^4 + \mathcal{O}[(|k_+| + \varepsilon^2)^4], \tag{5.12}$$

$$r_1 = \frac{\varepsilon r_0}{2} k_+ + \mathcal{O}(\varepsilon^3),$$

where we notice that coefficients σ_1 and σ_2 are functions of the Prandtl number. We obtain a one-parameter family of periodic solutions (parameter k_+), with only the Fourier modes $e^{\pm is}$.

Estimates of higher order terms

The proof below is self contained. There is a geometrical proof without estimates in Iooss-Pérouème [42], and a more precise proof by Horn in [34] section 3.5.

Let us define by ω the frequency of periodic solutions, where ω is close to

$$\omega_0 = \frac{1+\varepsilon^2 k_+}{2\varepsilon},$$

and set

$$\begin{aligned} s &= \omega x,\ \omega = \omega_0 + \widehat{\omega} \\ B_0(s) &= r_0 e^{is} + \widehat{B_0} \\ B_1(s) &= i r_1 e^{is} + i\widehat{B_1}, \end{aligned}$$

where B_0 and B_1 are $2\pi-$ periodic in s, and r_0, r_1 are solution of (5.10,5.11). Let us introduce the linear operator

$$L_0 = \begin{pmatrix} -(i\omega_0 \frac{d}{ds} + \frac{1}{2\varepsilon} + \varepsilon^3 P_0) & -1 \\ \varepsilon^2 Q_0 & -(i\omega_0 \frac{d}{ds} + \frac{1}{2\varepsilon} + \varepsilon^3 P_0) \end{pmatrix},$$

acting in the function space $H^1(\mathbb{R}/2\pi\mathbb{Z}) \times L^2(\mathbb{R}/2\pi\mathbb{Z})$. It appears that L_0 has a one-dimensional kernel

$$(r_0 e^{is}, r_1 e^{is}) \stackrel{def}{=} V_0 e^{is}$$

since (5.10,5.11) implies

$$\begin{aligned} [(\omega_0 - \frac{1}{2\varepsilon} - \varepsilon^3 P_0] r_0 - r_1 &= 0 \\ \varepsilon^2 Q_0 r_0 + [(\omega_0 - \frac{1}{2\varepsilon} - \varepsilon^3 P_0] r_1 &= 0, \end{aligned}$$

with

$$\begin{aligned} P_0 &= \alpha + \beta r_0^2 + \varepsilon\gamma r_0 r_1, \\ Q_0 &= -1 + r_0^2 + \varepsilon\delta r_0 r_1. \end{aligned}$$

Then the system (5.9), to be completed by its complex conjugate, becomes:

$$\begin{aligned} \widehat{\omega} V_0 e^{is} + L_0 \begin{pmatrix} \widehat{B_0} \\ \widehat{B_1} \end{pmatrix} &= i\widehat{\omega} \frac{d}{ds} \begin{pmatrix} \widehat{B_0} \\ \widehat{B_1} \end{pmatrix} + \begin{pmatrix} \varepsilon^3 r_0 P_{lin} \\ -\varepsilon^2 r_0 Q_{lin} + \varepsilon^3 r_1 P_{lin} \end{pmatrix} \\ &+ \begin{pmatrix} R_0(\widehat{Y}, \overline{\widehat{Y}}) \\ R_1(\widehat{Y}, \overline{\widehat{Y}}) \end{pmatrix}, \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} P_{lin} &= e^{2is}[\beta r_0\overline{\widehat{B_0}} + \frac{\varepsilon\gamma}{2}(r_0\overline{\widehat{B_1}} + r_1\overline{\widehat{B_0}})] \\ &\quad +[\beta r_0\widehat{B_0} + \frac{\varepsilon\gamma}{2}(r_0\widehat{B_1} + r_1\widehat{B_0})] \\ Q_{lin} &= e^{2is}[-r_0\overline{\widehat{B_0}} + \frac{\varepsilon\delta}{2}(r_0\overline{\widehat{B_1}} + r_1\overline{\widehat{B_0}})] \\ &\quad +[-r_0\widehat{B_0} + \frac{\varepsilon\delta}{2}(r_0\widehat{B_1} + r_1\widehat{B_0})], \end{aligned}$$

$$\begin{aligned} R_0(\widehat{Y}, \overline{\widehat{Y}}) &= \varepsilon^3 r_0 e^{is} P_{quad} + \varepsilon^3 \widehat{B_0}(e^{-is}P_{lin} + P_{quad}) - i\varepsilon^7 g_0, \\ R_1(\widehat{Y}, \overline{\widehat{Y}}) &= -\varepsilon^2 r_0 e^{is} Q_{quad} - \varepsilon^2 \widehat{B_0}(e^{-is}Q_{lin} + Q_{quad}) \\ &\quad +\varepsilon^3 r_1 e^{is} P_{quad} + \varepsilon^3 \widehat{B_1}(e^{-is}P_{lin} + P_{quad}) - \varepsilon^6 g_1, \end{aligned}$$

with

$$\begin{aligned} Q_{quad} &= \widehat{B_0}\overline{\widehat{B_0}} + \frac{\varepsilon\delta}{2}(\widehat{B_0}\overline{\widehat{B_1}} + \widehat{B_1}\overline{\widehat{B_0}}) \\ P_{quad} &= \beta\widehat{B_0}\overline{\widehat{B_0}} + \frac{\varepsilon\gamma}{2}(\widehat{B_0}\overline{\widehat{B_1}} + \widehat{B_1}\overline{\widehat{B_0}}). \end{aligned}$$

Let us decompose

$$\begin{pmatrix} \widehat{B_0} \\ \widehat{B_1} \end{pmatrix} = \widehat{y}\begin{pmatrix} r_1 e^{is} \\ -r_0 e^{is} \end{pmatrix} + \begin{pmatrix} \widetilde{B_0} \\ \widetilde{B_1} \end{pmatrix}$$

where $\widetilde{B_0}$ and $\widetilde{B_1}$ have no Fourier component in e^{is}, and we take the component in e^{is} orthogonal to V_0e^{is}, since adding a component proportional to (r_0, r_1) is equivalent to adapt (r_0, r_1).

We first solve (5.13) with respect to $(\widetilde{B_0}, \widetilde{B_1})$ in using the implicit function theorem, since we observe (notice the term $n\omega_0 = \frac{n}{2\varepsilon}(1 + \varepsilon^2 k_+)$ in the operator for a Fourier component e^{nis}), that the pseudo-inverse of L_0 is bounded from $H^1(\mathbb{R}/2\pi\mathbb{Z}) \times L^2(\mathbb{R}/2\pi\mathbb{Z})$ to $H^2(\mathbb{R}/2\pi\mathbb{Z}) \times H^1(\mathbb{R}/2\pi\mathbb{Z})$. Let us notice that the difference with the classical Hopf bifurcation proof is that, norms in these spaces are chosen as, for example

$$||u||_{H^2} = \frac{1}{\varepsilon^2}||u''||_{L^2} + \frac{1}{\varepsilon}||u'||_{L^2} + ||u||_{L^2},$$

and notice that $H^1(\mathbb{R}/2\pi\mathbb{Z})$ is an algebra. It results that we obtain an estimate such that

$$||(\widetilde{B_0}, \widetilde{B_1})||_{H^2\times H^1} \leq c(\varepsilon^2|\widehat{y}| + \varepsilon^6).$$

It then remains to solve the 2-dimensional system in $(\widehat{\omega}, \widehat{y})$ which is a real system, due to the reversibility symmetry:

$$\begin{aligned} \widehat{\omega}r_0 + \widehat{y}r_1 &= -\widehat{\omega}\widehat{y}r_1 + \mathcal{O}(\varepsilon^4|\widehat{y}| + \varepsilon^3|\widehat{y}| + \varepsilon^7) \\ \widehat{\omega}r_1 - \widehat{y}r_0 &= \widehat{\omega}\widehat{y}r_0 + \mathcal{O}(\varepsilon^3|\widehat{y}| + \varepsilon^2|\widehat{y}| + \varepsilon^6), \end{aligned}$$

which gives

$$\begin{aligned}\widehat{\omega} &= \mathcal{O}(\varepsilon^7)\\ \widehat{y} &= \mathcal{O}(\varepsilon^6).\end{aligned}$$

It results finally that the family of periodic solutions at M_+ are such that

$$\begin{aligned}B_0 &= r_0 e^{i\omega x} + \mathcal{O}(\varepsilon^6),\\ B_1 &= i r_1 e^{i\omega x} + \mathcal{O}(\varepsilon^6),\\ \omega &= \frac{1}{2\varepsilon} + \frac{\varepsilon k_+}{2} + \mathcal{O}(\varepsilon^7).\end{aligned} \tag{5.14}$$

6.5.5 Proof of Lemma 3.4

Let us define the heteroclinic connection we found at Theorem 1.1 as

$$(A_*(x), B_*(x)) \subset \mathbb{R}^2$$

with

$$1 + 1/9 \leq g = 1 + \delta^2 \leq 2,$$

and where we know that, for ε small enough

$$\begin{aligned}B_*(x) &> 0,\ B_*'(x) > 0\\ (A_*(x), B_*(x)) &\to \begin{cases}(1,0) \text{ as } x \to -\infty\\ (0,1) \text{ as } x \to +\infty\end{cases},\end{aligned}$$

at least as $e^{\varepsilon\delta x}$ for $x \to -\infty$, and at least as $e^{-\sqrt{2}\varepsilon x}$ for $x \to +\infty$.

The perturbed system (2.10) leading to system (1.1) is now considered with B_0 complex valued, so in (1.1) B^2 is replaced by $|B|^2$.

For being able to prove any persistence result under reversible perturbations of system (1.1) in $\mathbb{R}^4 \times \mathbb{C}^2$, as it appears in (5.6), we need to study the linearized operator at the above heteroclinic solution. We follow the lines of Chapter 4.

The linearized operator is given by

$$\begin{aligned}A^{(4)} &= (1 - 3A_*^2 - gB_*^2)A - gA_*B_*(B + \overline{B}),\\ B'' &= \varepsilon^2(-1 + gA_*^2 + 2B_*^2)B + 2\varepsilon^2 g A_* B_* A + \varepsilon^2 B_*^2 \overline{B}.\end{aligned}$$

Taking real and imaginary parts for B :

$$B = C + iD,$$

we then obtain the linearized system

$$\begin{aligned}
-A^{(4)} + (1 - 3A_*^2 - gB_*^2)A - 2gA_*B_*C &= 0,\\
\frac{1}{\varepsilon^2}C'' + (1 - gA_*^2 - 3B_*^2)C - 2gA_*B_*A &= 0,\\
\frac{1}{\varepsilon^2}D'' + (1 - gA_*^2 - B_*^2)D &= 0.
\end{aligned}$$

Notice that the equation for D decouples, so that we can split the linear operator in an operator $\mathcal{M}_g$ acting on (A, C) and an operator $\mathcal{L}_g$ acting on D :

$$\mathcal{M}_g \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} + (1 - 3A_*^2 - gB_*^2)A - 2gA_*B_*C \\ \frac{1}{\varepsilon^2}C'' + (1 - gA_*^2 - 3B_*^2)C - 2gA_*B_*A \end{pmatrix},$$

$$\mathcal{L}_g D = \frac{1}{\varepsilon^2}D'' + (1 - gA_*^2 - B_*^2)D.$$

Let us define the Hilbert spaces

$$L_\eta^2 = \{u; u(x)e^{\eta|x|} \in L^2(\mathbb{R})\},$$

$$\begin{aligned}
\mathcal{D}_0 &= \{(A, C) \in H_\eta^4 \times H_\eta^2; A \in H_\eta^4, C \in \mathcal{D}_1\}\\
\mathcal{D}_1 &= \{C \in H_\eta^2; \varepsilon^{-2}||C''||_{L_\eta^2} + \varepsilon^{-1}||C'||_{L_\eta^2} + ||C||_{L_\eta^2} \overset{def}{=} ||C||_{\mathcal{D}_1} < \infty\}
\end{aligned}$$

equiped with natural scalar products. Below, we prove Lemma 3.4.

Remark 5.6 *For proving the Lemma, we use the uniqueness (resulting from the transversality of manifolds $\mathcal{W}_{\varepsilon,\delta}^{(u)}$ and $\mathcal{W}_{\varepsilon,\delta}^{(s)}$) and analyticity in δ (i.e. g) of the heteroclinic, proved in Theorem 1.1 (see subsection 6.5.1).*

Asymptotic operators

Let us define the operators obtained when $x = \pm\infty$:

$$\mathcal{M}_\infty^- \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} - 2A \\ \varepsilon^{-2}C'' - (g-1)C \end{pmatrix},$$

$$\mathcal{M}_\infty^+ \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} - (g-1)A \\ \varepsilon^{-2}C'' - 2C \end{pmatrix},$$

$$\begin{aligned}
\mathcal{L}_\infty^- D &= \varepsilon^{-2}D'' - (g-1)D,\\
\mathcal{L}_\infty^+ D &= \varepsilon^{-2}D''.
\end{aligned}$$

Notice that all these operators are negative. Furthermore, their spectra in $L^2(\mathbb{R})$ are such that

$$\begin{aligned}\sigma(\mathcal{M}_\infty^-) &= (-\infty, -c_-],\ c_- = \max\{2, (g-1)\} > 0,\\ \sigma(\mathcal{M}_\infty^+) &= (-\infty, -c_+],\ c_+ = c_-,\\ \sigma(\mathcal{L}_\infty^-) &= (-\infty, -(g-1)],\\ \sigma(\mathcal{L}_\infty^+) &= (-\infty, 0].\end{aligned}$$

Operators $\mathcal{M}_g$ and $\mathcal{L}_g$ are respectively relatively compact perturbations of the corresponding asymptotic operators $\mathcal{M}_\infty$ and $\mathcal{L}_\infty$ defined as

$$\mathcal{M}_\infty = \begin{cases} \mathcal{M}_\infty^-, & x < 0 \\ \mathcal{M}_\infty^+, & x > 0 \end{cases}, \quad \mathcal{L}_\infty = \begin{cases} \mathcal{L}_\infty^-, & x < 0 \\ \mathcal{L}_\infty^+, & x > 0 \end{cases},$$

Their essential spectrum, i.e. the set of $\lambda \in \mathbb{C}$ for which $\lambda - \mathcal{M}_g$ (resp. $\lambda - \mathcal{L}_g$) is not Fredholm with index 0, is equal to the essential spectrum of $\mathcal{M}_\infty$ (resp. $\mathcal{L}_\infty$) (see [48]). The latter spectra are found from the spectra of $\mathcal{M}_\infty^\pm$ and $\mathcal{L}_\infty^\pm$:

$$\begin{aligned}\sigma_{ess}(\mathcal{M}_\infty) &= (-\infty, -c_+],\\ \sigma_{ess}(\mathcal{L}_\infty) &= (-\infty, 0].\end{aligned}$$

In particular, this implies that 0 does not belong to the essential spectrum of $\mathcal{M}_g$, so that the operator $\mathcal{M}_g$ is Fredholm with index 0. Moreover operators $\mathcal{M}_\infty$ and $\mathcal{L}_\infty$ are self adjoint negative operators in L^2, and $\mathcal{M}_\infty$ has a bounded inverse [48].

$$||\mathcal{M}_\infty^{-1}||_{L^2} \leq \frac{1}{c_+}.$$

This last property remains valid in exponentially weighted spaces, with weights $e^{\eta|x|}$, and η sufficiently small, since this acts as a small perturbation of the differential operator (see [47] section 3.1).

Properties of $\mathcal{L}_g$

Notice that $\mathcal{L}_g$ is self adjoint in $L^2(\mathbb{R})$ and that

$$\mathcal{L}_g B_* = 0, \quad \text{but } B_* \notin L^2(\mathbb{R}).$$

This property allows to solve explicitely the equation $\mathcal{L}_g u = f \in L^2_\eta$ with respect to $u \in L^2_\eta$ (using variation of constants method), and shows that it has a unique solution, provided that

$$\int_{\mathbb{R}} f B_* dx = 0.$$

We obtain

$$\begin{aligned} u(x) &= \int_x^\infty \frac{\varepsilon^2 B_*(x)}{B_*^2(s)} F(s)ds \\ \text{with } F(s) &= \int_s^\infty f(\tau)B_*(\tau)d\tau \text{ for } s \geq 0 \\ &= -\int_{-\infty}^s f(\tau)B_*(\tau)d\tau \text{ for } s \leq 0. \end{aligned}$$

By Fubini's theorem we can write for $x \geq 0$

$$u(x) = \varepsilon^2 B_*(x) \int_x^\infty f(\tau)B_*(\tau) \left(\int_x^\tau \frac{ds}{B_*^2(s)} \right) d\tau$$

and, for $x \leq 0$

$$\begin{aligned} u(x) &= -\varepsilon^2 B_*(x) \int_{-\infty}^x f(\tau)B_*(\tau) \left(\int_x^0 \frac{ds}{B_*^2(s)} \right) d\tau \\ &\quad -\varepsilon^2 B_*(x) \int_x^0 f(\tau)B_*(\tau) \left(\int_\tau^0 \frac{ds}{B_*^2(s)} \right) d\tau. \end{aligned}$$

The asymptotic properties of $B_*(x)$ at $\pm\infty$ imply, for $x \geq 0$

$$|u(x)|e^{\eta x} \leq C\varepsilon^2 \int_x^\infty |f(\tau)e^{\eta\tau}|(\tau - x)e^{-\eta(\tau - x)}d\tau,$$

and for $x \leq 0$

$$\begin{aligned} |u(x)|e^{-\eta x} &\leq \frac{C\varepsilon^2}{2\varepsilon\delta} \int_{-\infty}^x |f(\tau)e^{-\eta\tau}|e^{-(\eta+\varepsilon\delta)(x-\tau)}d\tau \\ &\quad + \frac{C\varepsilon^2}{2\varepsilon\delta} \int_x^0 |f(\tau)e^{-\eta\tau}|e^{(\eta-\varepsilon\delta)(\tau-x)}d\tau. \end{aligned}$$

The bound

$$||u||_{L^2_\eta} \leq c_2||f||_{L^2_\eta}$$

follows from classical convolution results between functions in L^2 and functions in L^1, since

$$\begin{aligned} \int_{-\infty}^0 e^{(\eta-\varepsilon\delta)\tau}d\tau &= \frac{1}{\eta - \varepsilon\delta}, \\ \int_0^\infty \tau e^{-\eta\tau}d\tau &= \frac{1}{\eta^2}. \end{aligned}$$

Then, we choose $\eta = \frac{1}{2}\varepsilon\delta$, so that the pseudo-inverse of $\mathcal{L}_g$ has a bounded inverse in L^2_η :

$$||\widetilde{\mathcal{L}_g}^{-1}|| \leq c_2,$$

where c_2 is independent of ε. Using the form of $\mathcal{L}_g$ we obtain easily

$$||u||_{\mathcal{D}_1} \leq c_3||f||_{L^2_\eta}$$

with c_3 independent of ε.

Remark 5.7 *The choice made for η is such that*

$$\eta < \varepsilon\delta, \quad \eta < \varepsilon\sqrt{2},$$

for values of δ for which Theorem 1.1 is valid. This means that as $x \to -\infty$ $(A_* - 1,\ B_*)$, and, as $x \to +\infty$ $(A_*, B_* - 1)$ tend exponentially to 0 faster than $e^{-\eta|x|}$.

$\mathcal{L}_g$ is Fredholm with index -1, when acting in L^2_η, for η small enough. $\mathcal{L}_g$ has a trivial kernel, and its range is orthogonal to B_*, with the scalar product of $L^2(\mathbb{R})$.

Properties of $\mathcal{M}_g$

We saw that $\mathcal{M}_g$ is Fredholm with index 0. Furthermore the derivative of the heteroclinic solution belongs to its kernel:

$$\begin{aligned} \mathcal{M}_g \begin{pmatrix} A'_* \\ B'_* \end{pmatrix} &= \begin{pmatrix} -A_*^{(5)} + A'_* - (A_*^3)' - gB_*^2 A'_* - gA_*(B_*^2)' \\ \varepsilon^{-2} B'''_* + [B'_* - gA_*^2 B'_* - (B_*^3)' - gB_*(A_*^2)'] \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{5.15}$$

The part of the proof which differs from the proof made in Chapter 4, where the symmetry play an essential role, consists in showing below in this section that the kernel of $\mathcal{M}_g$ is one-dimensional (except for a finite set of values of g), spanned by $(A'_*, B'_*) \overset{def}{=} U_*$ with a range orthogonal to U_* in L^2. Let us admit this result for the moment, and define the projections Q_0 on $U_*^\perp$ and P_0 on U_* , which are orthogonal projections in L^2, then we need to solve in L^2_η

$$\mathcal{M}_g u = f$$

in decomposing

$$u = zU_* + v, \quad v = Q_0 u,$$

$$\mathcal{M}_g v = (\mathcal{M}_\infty + \mathcal{A}_g)v = Q_0 f$$

and we need to satisfy the compatibility condition

$$\langle f, U_* \rangle = 0,$$

while z is arbitrary and we obtain for v :

$$(\mathbb{I} + \mathcal{M}_\infty^{-1}\mathcal{A}_g)v = \mathcal{M}_\infty^{-1} Q_0 f,$$

where the operator $\mathcal{M}_\infty^{-1}\mathcal{A}_g$ is now a compact operator for which -1 is not an eigenvalue, since $v \in U_*^\perp$. It results that there is a number c independent of ε such that

$$||v||_{L^2_\eta} \leq c||f||_{L^2_\eta}.$$

From the form of operator $\mathcal{M}_g$ and using interpolation properties, we obtain for $v = (A, C)$

$$||(A, C)||_{\mathcal{D}_0} \leq c||f||_{L^2_\eta}$$

with a certain c independent of ε.

We show below that the kernel of $\mathcal{M}_g$, is one dimensional, then this implies that the range of $\mathcal{M}_g$ needs satisfy the orthogonality with only one element. In fact, because of selfadjointness in L^2, the range of $\mathcal{M}_g$ is orthogonal in $L^2(\mathbb{R})$ to

$$(A'_*, B'_*) \in L^2_\eta.$$

Dimension of $\ker \mathcal{M}_g$

Any element $\zeta(x)$ in the kernel lies, by definition, in L^2_η, hence $\zeta(x)$ tends towards 0 exponentially at $\pm\infty$. Near $x = \pm\infty$ the vector $\zeta(x) \sim \zeta_\pm(x)$ should verify

$$\mathcal{M}^\pm_\infty \zeta_\pm(x) = 0$$

where there are only 2 possible good dimensions (on each side). This gives a bound $= 2$ to the dimension of the kernel of $\mathcal{M}_g$. Let us show that *dimension 2 of* $\ker \mathcal{M}_g$ *implies non uniqueness of the heteroclinic*, which contradicts Theorem 1.1, hence the only possibility is that the dimension is one.

Let us choose arbitrarily g_0 and assume that the kernel of $\mathcal{M}_{g_0}$ consists in

$$\zeta_0(x), \zeta_1(x)$$

where $\zeta_0 = (A'_*, B'_*)|_{g_0}$ and let us decompose a solution of (1.1) in the neighborhood of g_0 as

$$U = \mathbf{T}_a(U_*^{(g_0)} + a_1\zeta_1 + Y), \tag{5.16}$$

where $\mathbf{T}_a$ represents the shift $x \mapsto x + a$, where a, $a_1 \in \mathbb{R}$, and Y belongs to a subspace transverse to $\ker \mathcal{M}_{g_0}$. Let us denote by $\mathbf{Q}_0$ and $\mathbf{P}_0 = \mathbb{I} - \mathbf{Q}_0$, projections, respectively on the range of $\mathcal{M}_{g_0}$, and on a complementary subspace ($\mathbf{Q}_0$ may be built in using the eigenvectors ζ_0^*, ζ_1^* of the adjoint operator $\mathcal{M}_{g_0}^*$). Let us denote by

$$\mathcal{F}(U, g) = 0$$

the system (1.1) where we look for an heteroclinic U for $g \neq g_0$. Then, we have

$$\begin{aligned} \mathcal{F}(U_*^{(g_0)}, g_0) &= 0, \\ D_U\mathcal{F}(U_*^{(g_0)}, g_0) &= \mathcal{M}_{g_0}, \end{aligned}$$

and since

$$\mathcal{M}_{g_0}\zeta_j = 0,\ \ j = 0, 1,$$

using the equivariance under operator $\mathbf{T}_a$, we obtain (denoting $\mathcal{F}_0 = \mathcal{F}(U_*^{(g_0)}, g_0)$ and $[..]^{(2)}$ the argument of a quadratic operator)

$$\begin{aligned} 0 \ &= \ \mathcal{M}_{g_0}Y + (g-g_0)\partial_g\mathcal{F}_0 + \frac{1}{2}D^2_{UU}\mathcal{F}_0[a_1\zeta_1 + Y]^{(2)} + \\ &\quad +\mathcal{O}(|g-g_0|[|g-g_0| + |a_1| + ||Y||] + ||Y||^3). \end{aligned}$$

The projection Q_0 of this equation allows to use the implicit function theorem to solve with respect to Y and then obtain a unique solution

$$Y = \mathcal{Y}(a_1, g),$$

with

$$\begin{aligned} \mathcal{Y} \ &= \ -(g-g_0)\widetilde{\mathcal{M}_{g_0}}^{-1}\mathbf{Q}_0\partial_g\mathcal{F}_0 - \frac{1}{2}\widetilde{\mathcal{M}_{g_0}}^{-1}\mathbf{Q}_0D^2_{UU}\mathcal{F}_0[a_1\zeta_1]^{(2)} + \\ &\quad +\mathcal{O}(|g-g_0|(|g-g_0| + |a_1|) + |a_1|^3)). \end{aligned}$$

Then projecting on the complementary space, (only one equation since we work in the subspace orthogonal to ζ_0^*), we may observe (see the proof in Appendix 6.5.3) that $\mathbf{P}_0\partial_{g_0}\mathcal{F}_0 = 0$ and then obtain the "bifurcation" equation as

$$q(a_1, g-g_0) = \mathcal{O}((|g-g_0| + |a_1|)^3),$$

where the function q is quadratic in its arguments and

$$q|_{g=g_0}\zeta_1 = \frac{1}{2}\mathbf{P}_0D^2_{UU}\mathcal{F}_0[a_1\zeta_1]^{(2)}.$$

This equation is just at main order a second degree equation in a_1 depending on $g-g_0$. Provided that the discriminant is not 0, the generic number of solutions is 2 or 0. If the discriminant is 0 for $g = g_0$, we just go a little farther in g, and obtain a non zero discriminant, since the discriminant cannot stay $= 0$. Indeed the heteroclinic is analytic in g and if the discriminant were identically 0, this would mean that we have a double root for any g, contradicting the transversality for all g, except a finite number, of the intersection of the two manifolds (unstable one of M_-, stable one of M_+). Hence, this is true except for a set of isolated values of g. We can then use the implicit function theorem for finding corresponding solutions for the system with higher order terms. In fact we already know a solution, corresponding to $U_*^{(g)} = U_*^{(g_0)} + (g-g_0)\partial_gU_*^{(g_0)} + h.o.t.$ which corresponds to specific values for a_1 and Y, of order $\mathcal{O}(g-g_0)$. It then results that there is at least another solution of order $\mathcal{O}(g-g_0)$, so that there exists another heteroclinic, in the neighborhood of the known one (then in contradiction with Theorem 1.1).

Remark 5.8 *The above proof with only 1 dimension in the Kernel, provides* $Y = -(g - g_0)\widetilde{\mathcal{M}_{g_0}}^{-1}\partial_g \mathcal{F}_0 + \mathcal{O}((g-g_0)^2)$*, which gives a unique heteroclinic. Since we found only one heteroclinic, this shows that the kernel is of dimension 1.*

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