Couette-Taylor instabilities in the small gap regime: the very counter-rotating case

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Abstract

In this paper, we study the Couette-Taylor instability of a viscous fluid between two rotating cylinders in the small-gap, slow rescaled rotation rate, high Reynolds number regime, focusing on the very counter-rotating case $\mu < \mu_c \approx -0.8$ where the primary instability is non-axisymmetric. Starting from the Navier-Stokes equations, we derive a limit system that captures the leading-order dynamics and compute the critical Taylor number $T_c(\mu)$ together with the critical axial and azimuthal wavenumbers. Near criticality, the weakly nonlinear behaviour is governed by a system of two coupled complex Ginzburg-Landau equations. All coefficients of this amplitude system including the cubic nonlinear terms are evaluated numerically from the linearised eigenfunctions and the associated adjoint problem. The reduced equations admit helicoïdal waves (travelling in both the axial and azimuthal directions) and ribbon waves (standing axially, travelling azimuthally), and their existence and stability criteria are discussed. We also examine more exotic spatially modulated solutions that satisfy a third-order dynamical system, whose complete classification remains an open challenge.

1 Introduction

In this paper, we study the classical problem of the flow of a viscous fluid between two rotating cylinders, motivated in particular by the works of M. Nagata [12], [13], [14] and we consider the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0, \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

in the domain

$$\Omega = \Big\{ (r, \varphi, z) \quad | \quad r_i < r < r_o \Big\}.$$

In these equations, $u(t, x, y, z)$ denotes the velocity of the fluid, $p(t, x, y, z)$ its pressure, and $\nu > 0$ the viscosity. Moreover, (r, φ, z) are the cylindrical coordinates. The domain Ω is the interval between two cylinders with the same axis at $r = 0$, the inner one being of radius r_i and the outer one being of radius r_o.

The inner cylinder (respectively the outer) rotates with an angular velocity ω_i (respectively, ω_o). We assume that the fluid "sticks" to the boundary, namely

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that the velocity of the fluid equals the velocity of the cylinders at $r = r_i$ and $r = r_o$.

The Couette-Taylor flow between independently rotating cylinders has long served as a canonical system for studying hydrodynamic stability and pattern formation, dating back to the classic experiments of Taylor [17] and the weakly nonlinear theories of Davey [4] and DiPrima and Eagles [6]. In the narrow-gap limit, asymptotic descriptions of the bifurcating Taylor vortices were developed for co-rotating and weakly counter-rotating cylinders by Nagata [12, 13, 14] and, in a spatial-dynamics framework, in [1, 2]. When the cylinders counter-rotate strongly ($\mu < \mu_c \approx -0.8$), the primary instability becomes non-axisymmetric, leading to travelling azimuthal waves, ribbons and more complex spatio-temporal patterns [3]. Recent large-scale numerical studies by Deguchi and Nagata [5] and Meseguer and Marques [10] have charted the stability boundaries and secondary bifurcations in this regime, but a systematic analytic treatment that resolves both the axial and azimuthal modulations in the small-gap limit has been missing. The present paper fills this gap by deriving and analysing a system of two coupled Ginzburg-Landau equations that capture the leading-order nonlinear dynamics near the non-axisymmetric onset.

We define the gap d, the radii ratio η and the ratio of the rotation rates μ by

$$d = r_o - r_i, \quad \eta = \frac{r_i}{r_o}, \quad \mu = \frac{\omega_o}{\omega_i}.$$

We work in a rotating frame, with a rotation rate Ω_{rf} defined by

$$\Omega_{rf} = \frac{\omega_i + \omega_o}{2} = \omega_i \frac{1+\mu}{2}$$

and consider the stability of the Couette flow, defined by

$$U(r) = r\,\Omega_{rot}(r) = (A - \Omega_{rf})r + \frac{B}{r}, \tag{3}$$

in this rotating frame, where A and B are constants such that the velocity of the fluid coincides with the velocity of the cylinders at the inner and outer cylinders, which leads to

$$A = \omega_i \frac{\mu - \eta^2}{1 - \eta^2}, \quad B = \omega_i r_i^2 \frac{1-\mu}{1-\eta^2}.$$

We focus on the small gap / slow rescaled rotation rate / high Reynolds regime (SGSRRHR) where

$$\eta \to 1, \qquad \widehat{\omega} = \omega_i \frac{d^2}{\nu} \to 0, \qquad \mathfrak{R} = \frac{\widehat{\omega}(1-\mu)}{1-\eta} \to \infty, \tag{4}$$

and define the Taylor number to be

$$T = 2\widehat{\omega}\mathfrak{R}.$$

The question of the linear stability of the Couette flow is very classical and it is well established [7] that there exists a critical Taylor number T_c such that the

Couette flow is linearly stable if $T < T_c$ and linearly unstable if $T > T_c$. The instability is called the Couette-Taylor instability. In the particular case $\mu = 1$, $T_c \approx 1708$. For $T > T_c$, close to T_c, the unstable modes are axisymmetric and correspond to Fourier wave numbers in the z variable close to $\alpha_c \approx 3.1$.

In [1] and [2] we have studied in detail the dependency of T_c on μ in the SGSRRHR limit with T of order 1, and numerical evidence shows that

- If $\mu > \mu_c \approx -0.8$, T_c is a decreasing function of μ and the first instability is axisymmetric.
- If $\mu < \mu_c \approx -0.8$, the first instability is non-axisymmetric.

In [1] and [2], respectively in the cases $\mu = 1$ and $-0.8 < \mu < 1$, we have described these instabilities in the SGSRRHR regime and discussed how small amplitude and slowly varying solutions of the initial Navier-Stokes equations can be approximated by solutions of a reduced Ginzburg-Landau equation. We have further described the time and space periodic solutions of this Ginzburg-Landau equation. Using the theory of spatial dynamics, this leads to the existence of non-trivial time and space periodic solutions of the full Navier-Stokes equations, which are, at leading order, described by this Ginzburg-Landau equation.

In the present article, we focus on the case $\mu < \mu_c \approx -0.8$ which turns out to be more complex since the instabilities are no longer axisymmetric, but are periodic in the orthoradial variable. We will show that the behavior of small amplitude and slowly varying solutions of the Navier-Stokes equations is described not by one, but by two coupled Ginzburg-Landau equations and numerically compute the coefficients of this reduced system.

To this end, we follow the general strategy of spatial dynamics, as developed in particular in [9]. The aim is to construct time and space periodic solutions to the full Navier-Stokes system (1)-(2) as small perturbations of time and space periodic solutions of a reduced nonlinear system which catches the main dynamics near a bifurcation point. In our case, this system consists of two coupled Ginzburg-Landau equations.

The strategy may be split into the following three steps:

- First, derivation of a reduced system which describes the long wave and small amplitude perturbations of the constant state near the bifurcation point.
- Second, construction of non-trivial periodic solutions for this reduced system.
- Third, construction of genuine solutions of the Navier-Stokes system starting from solutions of the reduced system.

This last step can be done by repeating the proofs in [9], up to minor changes. We will omit this step here and refer to [9] for more details.

In the current paper, we focus on the first two points and describe new classes of solutions of the reduced system which lead to new classes of solutions of the genuine Navier-Stokes equations.

Note that the case $\mu = -1$ is considered in [14] but without the study of non-axisymmetric bifurcating solutions. Such non-axisymmetric bifurcating solutions were studied in [3] for the full Navier-Stokes equations, but not in the small gap limit, which is our aim here.

We present three main results. First, we derive the small-gap limit system and perform a complete linear stability analysis, obtaining the critical Taylor number $T_c(\mu)$ and the corresponding non-axisymmetric wavenumbers $\alpha_c(\mu)$, $\mathfrak{B}_c(\mu)$ for the strongly counter-rotating regime $\mu < \mu_c$.

Second, using a center-manifold reduction, we derive a system of two coupled complex Ginzburg–Landau equations that govern the slow modulation of the left- and right-travelling azimuthal waves near onset. More precisely, small amplitude solutions U of the limit system are of the form

$$U = A(y,t)\zeta_1 + B(y,t)\zeta_2 + \overline{A}(y,t)\overline{\zeta_1} + \overline{B}(y,t)\overline{\zeta_2}.$$

where, up to higher order terms,

$$\partial_t A = (i\omega_0 + a_3\tau)A + b_1\Re\partial_y A - a_4\Re^2\partial_y^2 A + A(b|A|^2 + c|B|^2), \quad (5)$$

$$\partial_t B = (i\omega_0 + a_3\tau)B + b_1\Re\partial_y B - a_4\Re^2\partial_y^2 B + B(c|A|^2 + b|B|^2). \quad (6)$$

We compute all coefficients of this amplitude system, namely, the linear dispersion terms $a_1, a_2, a_3, a_4, a_5, b_1$ and the cubic nonlinearities b and c, directly from the linearised eigenfunctions and the adjoint problem by spectral Chebyshev collocation.

Third, we show that the Ginzburg-Landau systems admit particular solutions which are helicoidal and ribbon waves. More precisely, we have the following theorem.

Theorem 1 *The system (5)-(6) admits solutions of the form*

$$A(t,y) = r_0 e^{(i\omega_0 + ia_{3i}\tau + ia_{4i}\beta^2)t + i\beta(y + b_{1r}t) + i\theta_0(t)},$$
$$B(t,y) = r_1 e^{(i\omega_0 + ia_{3i}\tau + ia_{4i}\beta^2)t + i\beta(y + b_{1r}t) + i\theta_1(t)}$$

where r_0 and r_1 arer real numbers, $\theta_0(t)$ and $\theta_1(t)$ are linear in t and where $\tau = T - T_c$. The "i" and "r" indices denote the imaginary and real parts of a complex number. When $r_0 \neq 0$ and $r_1 = 0$, the solutions are helicoïdal waves, and when $r_0 = r_1 \neq 0$, they are ribbon waves.

We refer to the proof of this Theorem for the explicit expressions of r_0, r_1, θ_0 and θ_1. We will also discuss more complex solutions and discuss their stability. The complete classification of the solutions of (5)-(6) remains an open problem.

The following sections are organized as follows. In Section 2, we present the small-gap limit and linear analysis. In Section 3, we derive the Ginzburg–Landau system and its particular solutions, and in the Appendices, we detail the computations of the coefficients.

2 The small gap limit

2.1 The limit system

As shown in [1], [2] and [13], [14], in the SGSRRHR limit, the initial Navier-Stokes equations (1)-(2) reduce to the following system that we call the "limit Navier-Stokes" system

$$\partial_t U = \Delta_\perp U + x\mathfrak{R}\partial_y U - \nabla_\perp p + \begin{pmatrix} Tg(x)\hat{u}_y \\ 0 \\ u_x \end{pmatrix} - (U_\perp \cdot \nabla_\perp)U + \frac{T(1-\mu)}{2}\begin{pmatrix} \hat{u}_y^2 \\ 0 \\ 0 \end{pmatrix}, \tag{7}$$
$$\nabla_\perp \cdot U_\perp + \mathfrak{R}\partial_y \hat{u}_y = 0,$$

posed in the domain $-1/2 \leq x \leq 1/2$, $y \in \mathbb{R}$ and $z \in \mathbb{R}$. In these equations, U is the velocity of the fluid in the rotating frame, the index $\perp$ denotes the (x, z) components,

$$U = (U_\perp, \hat{u}_y), \qquad U_\perp = (u_x, u_z), \qquad u_y = \mathfrak{R}\hat{u}_y,$$

the parameters T and $\mathfrak{R}$ are defined by

$$\begin{aligned} T &= \frac{2\omega_i^2 r_0^4 (1-\eta)^3(1-\mu)}{\nu^2}, \\ \mathfrak{R} &= \frac{\omega_i r_0^2 (1-\eta)(1-\mu)}{\nu} \end{aligned}$$

and

$$g(x) = \frac{1+\mu}{2} - (1-\mu)x,$$

where x is the rescaled distance to the cylinders.

The boundary conditions are

$$U = 0, x = \pm 1/2.$$

We note that T and $\mathfrak{R}$ are such that, $\mu \neq 1$ being fixed,

$$(1-\eta)^3 = \mathcal{O}\Big(\frac{\nu}{\omega_i r_0^2}\Big)^2.$$

In particular, $T = \mathcal{O}(1)$ as $\eta \to 1$, while $\mathfrak{R} = \mathcal{O}[(1-\eta)^{-1/2}]$ as in [14].

The derivation of (7) starting from (1)-(2) is detailed in [1] and [2] and its main steps are recalled in the Appendix 4.1 for the reader's convenience.

We note that we keep the parameter $\mathfrak{R}$ in this limit system, whereas $\mathfrak{R} \to \infty$ since we will consider solutions U which are slowly varying in y, with scales of variations of order $\mathfrak{R}^{-1}$ in this variable. For such solutions, the operator $\mathfrak{R}\partial_y$ remains bounded as $\mathfrak{R}$ goes to infinity. It is of course possible to remove the parameter from the limit system by rescaling U in y, but the rescaled system

is less clear than (7). Keeping $\mathfrak{R}$ also clearly shows the natural scales in the y variable.

Note that x is bounded and ranges between $-1/2$ and $+1/2$. On the contrary $y \in \mathbb{R}$. Moreover, we focus on solutions which are periodic in z with a fixed period $2\pi/\alpha_c$, where α_c will be defined below.

2.2 Linearization

As shown in [2], the linearization of (7) for general perturbations of the form

$$e^{i(\alpha z+\beta y)}\widehat{U}(x)$$

leads to the eigenvalue problem (see [2])

$$\begin{aligned}
(\lambda+\alpha^2-D^2-i\mathfrak{B}x)u_x+Dp &= Tg(x)\widehat{u}_y,\\
(\lambda+\alpha^2-D^2-i\mathfrak{B}x)\widehat{u}_y &= u_x,\\
(\lambda+\alpha^2-D^2-i\mathfrak{B}x)u_z+i\alpha p &= 0,\\
Du_x+i\mathfrak{B}\widehat{u}_y+i\alpha u_z &= 0,
\end{aligned}$$

where

$$\mathfrak{B}=\beta\mathfrak{R}=O(1),$$

and

$$u_x=\widehat{u}_y=u_z=0 \quad \text{when} \quad x=\pm 1/2.$$

We recall that the function $g(x)$ depends on the parameter μ, which is fixed. This eigenvalue problem depends on the parameters T, $\mathfrak{B}$ and α. Numerically, we observe that there exists a critical Taylor number $T_1(\alpha,\mathfrak{B})$ such that all the eigenvalues of the linearised problem have negative real parts if $T<T_1(\alpha,\mathfrak{B})$ and such that there exists one eigenvalue with a positive real part if $T>T_1(\alpha,\mathfrak{B})$.

We define the critical Taylor number $T_c(\mu)$ by

$$T_c(\mu)=\min_{\alpha}\min_{\mathfrak{B}} T_1(\alpha,\mathfrak{B}).$$

Numerically, we have observed in [2] that $T_c(\mu)$ is reached for $\mathfrak{B}=0$ provided $\mu>\mu_c$ where $\mu_c\approx -0.8$. In this regime, the first instabilities are axisymmetric. We refer to [1] and [2] for a detailed analysis of this regime.

When $\mu<\mu_c\approx -0.8$, $T_c(\mu)$ is reached for some $\alpha_c(\mu)\neq 0$ and some $\mathfrak{B}_c(\mu)>0$. It is also reached by symmetry when $\mathfrak{B}=-\mathfrak{B}_c(\mu)$. We denote by λ_0 the corresponding eigenvalue of the linearised variable, which depends on the bifurcation parameter

$$\tau=T-T_c$$

and on α and $\mathfrak{B}$. By definition of T_c, at criticality, $\lambda=i\omega_0$ where $\omega_0\in\mathbb{R}$. Moreover, $\bar{\lambda}_0=-i\omega_0$ is also an eigenvalue.

We now expand λ_0 near $T = T_c$ (namely near $\tau = 0$), $\alpha = \alpha_c(\mu)$ and $\mathfrak{B} = \mathfrak{B}_c(\mu)$. By definition, at criticality, λ_0 is of the form

$$\begin{aligned}\lambda_0 &= i\omega_0 + ia_1(\alpha^2 - \alpha_c^2) + a_2(\alpha^2 - \alpha_c^2)^2 + a_3\tau + ib_1(\mathfrak{B} - \mathfrak{B}_c) + a_4(\mathfrak{B} - \mathfrak{B}_c)^2 \\ &\quad +2a_5(\mathfrak{B} - \mathfrak{B}_c)(\alpha^2 - \alpha_c^2) + O(\tau(\alpha^2 - \alpha_c^2)) + \mathcal{O}(\tau(\mathfrak{B} - \mathfrak{B}_c)), \qquad (8)\end{aligned}$$

where

$$\omega_0, a_1, b_1 \in \mathbb{R},$$

and

$$a_{3r} > 0,$$

since $\Re\lambda_0 > 0$ for $\tau > 0$. All the coefficients of (8) depend on μ.

Let us now study the manifold $\Re\lambda_0 = 0$. If α and $\mathfrak{B}$ are given, $\Re\lambda_0 = 0$ provided

$$\begin{aligned}&a_{3r}\tau + a_{2r}(\alpha^2 - \alpha_c^2)^2 + a_{4r}(\mathfrak{B} - \mathfrak{B}_c)^2 + 2a_{5r}(\mathfrak{B} - \mathfrak{B}_c)(\alpha^2 - \alpha_c^2) \\ &+\mathcal{O}(\tau(\alpha^2 - \alpha_c^2)) + \mathcal{O}(\tau(\mathfrak{B} - \mathfrak{B}_c)) = 0.\end{aligned}$$

By definition of T_c, this equation on τ has no negative solution. This leads to the constraints

$$a_{2r} + a_{4r} < 0, \qquad a_{2r}a_{4r} > a_{5r}^2,$$

which expresses that the quadratic form giving the principal part of τ near $(\alpha_c, \mathfrak{B}_c)$ is negative definite.

The computation of the coefficients a_1, a_2, a_3, a_4 and a_5 is detailed in Appendix 4.2. In the sequel we will consider only periodic solutions in z, with period $2\pi/\alpha_c(\mu)$. Then for $\tau > 0$ close to 0 and for $\mathfrak{B}$ close to $\mathfrak{B}_c$, the eigenvalue with the largest real part of the linearised system is λ_0 which depends on τ and $\mathfrak{B}$. For $\mathfrak{B} \neq \mathfrak{B}_c$, $\Re\lambda_0 < 0$ if $\tau = 0$ and more generally if τ is small enough, and becomes positive when τ becomes larger (depending on $\mathfrak{B}$).

We will denote by ζ_1 the eigenvector corresponding to $\alpha_c > 0$ and by ζ_2 the eigenvector corresponding to $-\alpha_c$. They are of the form

$$\begin{aligned}\zeta_1 &= e^{i(\alpha_c z + \beta_c y)}U_1(x), \\ \zeta_2 &= S\zeta_1 = e^{i(-\alpha_c z + \beta_c y)}SU_1(x)\end{aligned}$$

where S is the representation of the symmetry $z \mapsto -z$.

3 The Ginzburg-Landau system

In this section, we establish the Ginzburg-Landau system which describes small amplitude solutions of the limit Navier-Stokes equation and then construct particular solutions of this system called helicoïdal waves or ribbon waves as well as more exotic solutions.

3.1 General setting of Ginzburg-Landau system

We now study solutions U of Navier-Stokes which have a small amplitude and are slowly varying in y. At leading order, we look for U under the form

$$U = A(y,t)\zeta_1 + B(y,t)\zeta_2 + \overline{A}(y,t)\overline{\zeta_1} + \overline{B}(y,t)\overline{\zeta_2}.$$

The dynamics of the amplitudes $A(y,t)$ and $B(y,t)$ is described by a Ginzburg-Landau system of the form (see [3] Chap. VIII)

$$\begin{aligned} \partial_t A &= f(A,\overline{A},B,\overline{B},\partial_y), \\ \partial_t B &= g(A,\overline{A},B,\overline{B},\partial_y), \end{aligned}$$

for some functions f and g which commute with the following symmetries (for every $h,k \in \mathbb{R}$)

$$\begin{aligned} (A,\overline{A},B,\overline{B},\partial_y) &\mapsto (Ae^{ih},\overline{A}e^{-ih},Be^{-ih},\overline{B}e^{ih},\partial_y), \\ (A,\overline{A},B,\overline{B},\partial_y) &\mapsto (Ae^{ik},\overline{A}e^{-ik},Be^{ik},\overline{B}e^{-ik},\partial_y), \\ (A,\overline{A},B,\overline{B},\partial_y) &\mapsto (B,\overline{B},A,\overline{A},\partial_y). \end{aligned}$$

We also need to recover the eigenvalues λ_0 and $\overline{\lambda_0}$ in its linear part. This implies that the principal part of the Ginzburg-Landau system is

$$\begin{aligned} \partial_t A &= (i\omega_0 + a_3\tau)A + b_1\Re\partial_y A - a_4\Re^2\partial_y^2 A + A(b|A|^2 + c|B|^2), \\ \partial_t B &= (i\omega_0 + a_3\tau)B + b_1\Re\partial_y B - a_4\Re^2\partial_y^2 B + B(c|A|^2 + b|B|^2), \end{aligned} \tag{9}$$

which generalizes the standard system of Ginzburg-Landau equations described in [3] Chap. III. The change of variable

$$\begin{aligned} y &= \Re(\widetilde{y} - b_1 t), \\ A(y,t) &= \widetilde{A}(\widetilde{y},t), \qquad B(y,t) = \widetilde{B}(\widetilde{y},t), \end{aligned}$$

leads to the system

$$\begin{aligned} \partial_t \widetilde{A} &= (i\omega_0 + a_3\tau)\widetilde{A} - a_4\partial_{\widetilde{y}}^2\widetilde{A} + \widetilde{A}(b|\widetilde{A}|^2 + c|\widetilde{B}|^2), \\ \partial_t \widetilde{B} &= (i\omega_0 + a_3\tau)\widetilde{B} - a_4\partial_{\widetilde{y}}^2\widetilde{B} + \widetilde{B}(c|\widetilde{A}|^2 + b|\widetilde{B}|^2), \end{aligned}$$

with

$$a_{3r} > 0, \qquad a_{4r} < 0.$$

The coefficients b and c are studied in Appendix 4.3.

We first consider particular solutions of (9) which are periodic in y and only depend on the t variable, of the form

$$\widetilde{A}(\widetilde{y},t) = \widehat{A}(t)e^{i\beta\widetilde{y}}, \qquad \widetilde{B}(\widetilde{y},t) = \widehat{B}(t)e^{i\beta\widetilde{y}}.$$

For such solutions, (9) reduces to the following system of two complex ordinary differential equations

$$\begin{aligned} \partial_t \widehat{A} &= (i\omega_0 + a_3\tau + a_4\beta^2)\widehat{A} + \widehat{A}(b|\widehat{A}|^2 + c|\widehat{B}|^2), \\ \partial_t \widehat{B} &= (i\omega_0 + a_3\tau + a_4\beta^2)\widehat{B} + \widehat{B}(c|\widehat{A}|^2 + b|\widehat{B}|^2). \end{aligned}$$

If moreover we introduce the modulus and argument of $\widehat{A}$ and $\widehat{B}$ and write these two functions under the form

$$\begin{aligned}\widehat{A}(t) &= r_0(t)e^{i(\omega_0+a_{3i}\tau+a_{4i}\beta^2)t+i\theta_0(t)},\\ \widehat{B}(t) &= r_1(t)e^{i(\omega_0+a_{3i}\tau+a_{4i}\beta^2)t+i\theta_1(t)},\end{aligned}$$

we obtain the following system of four real ordinary differential equations

$$\partial_t r_0 = (a_{3r}\tau + a_{4r}\beta^2)r_0 + r_0(b_r r_0^2 + c_r r_1^2), \tag{10}$$
$$\partial_t r_1 = (a_{3r}\tau + a_{4r}\beta^2)r_1 + r_1(c_r r_0^2 + b_r r_1^2), \tag{11}$$
$$\partial_t \theta_0 = b_i r_0^2 + c_i r_1^2, \tag{12}$$
$$\partial_t \theta_1 = c_i r_0^2 + b_i r_1^2. \tag{13}$$

Note that the equations on r_0 and r_1 are decoupled from the equations on θ_0 and θ_1. Moreover, the two last equations are easily solved once r_0 and r_1 are known.

We now study the fixed points of (10)-(11), which leads to so called helicoïdal and ribbon waves. Up to symmetries, the system (10)-(11) has three fixed points:

- $r_0 = r_1 = 0$ which leads to $U = 0$.
- $r_1 = 0$ and
$$r_0^2 = -\frac{a_{3r}\tau + a_{4r}\beta^2}{b_r}$$
which leads to helicoïdal waves, described in the next section.
- $r_0 = r_1$ which leads to ribbon waves, studied in Section 3.3.

3.2 Helicoïdal waves

We start with the particular case where $r_1 = 0$ or $r_0 = 0$, following the discussion of [3] (in particular Chap. III, Section III.2.1, page 46). A first solution is obtained by choosing $r_1 = 0$ (namely $\widehat{B} = 0$), together with

$$r_0^2 = -\frac{a_{3r}\tau + a_{4r}\beta^2}{b_r}, \qquad \theta_0 = -\frac{b_i(a_{3r}\tau + a_{4r}\beta^2)}{b_r}t + \theta_0^*.$$

There exists a similar symmetric solution, with this time $r_0 = 0$. Such a solution is called an *helicoïdal waves* since it travels in both the azimuthal and the axial directions.

It exists provided $a_{3r}\tau + a_{4r}\beta^2$ and b_r have opposite signs, namely provided

$$(a_{3r}\tau + a_{4r}\beta^2)b_r < 0.$$

If $b_r < 0$, as $a_{4r} < 0$, they exist only on the supercritical side and more precisely when

$$\tau > \frac{(-a_{4r})}{a_{3r}}\beta^2.$$

If $b_r > 0$ they exist when

$$\tau < \frac{(-a_{4r})}{a_{3r}}\beta^2,$$

namely on both sides of $\tau = 0$.

The corresponding solutions for Navier-Stokes equations take the form of travelling waves (for the symmetric solution, change α_c in $-\alpha_c$ in the formula)

$$U(x, y, z, t) = V(x, \alpha_c z + \widetilde{\beta} y + \omega t),$$

where V is 2π - periodic in its second argument, with a principal part

$$r_0 e^{i(\alpha_c z + \widetilde{\beta} y + \omega t + \theta^*)} U_1(x) + c.c. \tag{14}$$

with

$$\begin{aligned} \omega &= \omega_0 + b_1\beta + a_{3i}\tau + a_{4i}\beta^2 + b_i r_0^2 + h.o.t., \\ \widetilde{\beta} &= \beta_c + \beta/\mathfrak{R}. \end{aligned}$$

It is possible to study the stability of these solutions with respect to perturbations which are $2\pi/\alpha_c$ - periodic in z, and $2\pi/\beta$ - periodic in y , as it is done for $\beta = 0$ in [3]. The result is that an helicoïdal wave (14) is stable provided

$$b_r < 0 \quad \text{and} \quad c_r - b_r < 0,$$

which means that these waves are stable only when they occur on the super-critical side, with $\tau > |a_{4r}|\beta^2/a_{3r}$.

Figure 1 shows b_r and c_r as a function of the parameter μ as μ goes from μ_c down to -1 (from right to left), together with the diagonals $b_r = c_r$ and $b_r = -c_r$. We note that the curve $\mu \to (b_r(\mu), c_r(\mu))$ is not monotonic.

We recall that $\mu_c \approx -0.785$. Numerically, we observe that

- $b_r > 0$ if $\mu > \mu_1 \approx -0.814$ and $b_r < 0$ if $\mu < \mu_1$.
- $c_r - b_r > 0$ if $\mu < \mu_2 \approx -0.848$ and $c_r - b_r < 0$ if $\mu > \mu_2$.
- $c_r + b_r > 0$ if $\mu > \mu_3 \approx -0.8$ and $c_r + b_r < 0$ if $\mu < \mu_3$.

If $b_r > 0$, helicoïdal waves exist on both sides, else they only exist on the supercritical side. If $b_r < 0$ and $c_r - b_r < 0$, they are stable.

Thus, if $\mu > \mu_1$, helicoïdal waves exist on both side. If $\mu_2 < \mu < \mu_1$, helicoïdal waves exist on the super-critical side and are stable. If $\mu < \mu_2$, they exist on the super-critical side and are unstable.

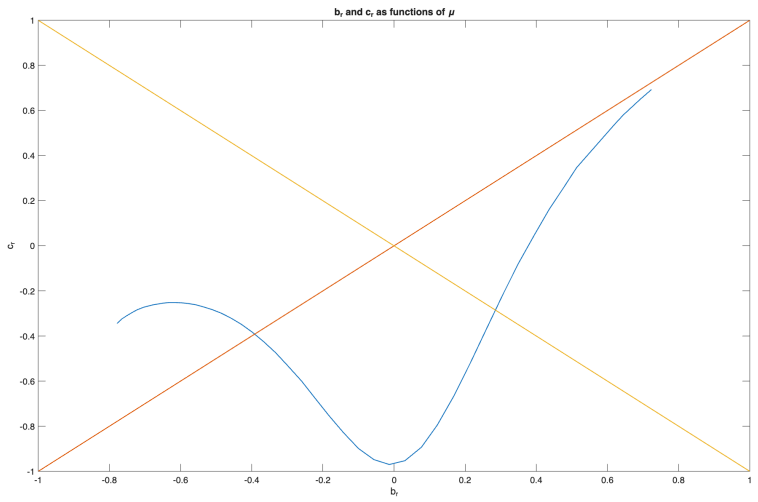


Figure 1: Evolution of the point (b_r, c_r) as a function of μ from $\mu = \mu_c$ to $\mu = -1$ (from right to left) together with the diagonals $b_r = c_r$ and $b_r = -c_r$. The blue curve starts on the upper right part of the figure at $\mu = \mu_c$ and successively crosses $c_r = -b_r$ at $\mu = \mu_3$, $b_r = 0$ at $\mu = \mu_1$ and $c_r = b_r$ at $\mu = \mu_2$. It ends on the left part of the figure at $\mu = -1$.

3.3 Ribbon waves

The *ribbon waves* correspond to

$$r_0^2 = r_1^2 = -\frac{a_{3r}\tau + a_{4r}\beta^2}{b_r + c_r},$$

$$\theta_k = -\frac{(b_i + c_i)(a_{3r}\tau + a_{4r}\beta^2)}{b_r + c_r}t + \theta_k^*.$$

They exist provided

$$(a_{3r}\tau + a_{4r}\beta^2)(b_r + c_r) < 0$$

holds true. Hence, if $b_r + c_r < 0$ they exist only on the supercritical side:

$$\tau > \frac{(-a_{4r})}{a_{3r}}\beta^2,$$

while if $b_r + c_r > 0$ they exist for

$$\tau < \frac{(-a_{4r})}{a_{3r}}\beta^2,$$

corresponding to both sides of $\tau = 0$.

The corresponding solutions of the Navier-Stokes equations take the form of standing waves in axial direction, travelling in the azimuthal direction

$$U(x, y, z, t) = V(x, \alpha_c z, \widetilde{\beta} y + \omega t),$$

where V is 2π periodic in its second and third arguments, with a principal part

$$r_0 e^{i(\alpha_c z + \widetilde{\beta} y + \omega t + \theta_0^*)} U_1(x) + r_0 e^{i(-\alpha_c z + \widetilde{\beta} y + \omega t + \theta_1^*)} S U_1(x) + c.c. \qquad (15)$$

with

$$\begin{aligned} \omega &= \omega_0 + b_1\beta + a_{3i}\tau + a_{4i}\beta^2 + (b_i + c_i)r_0^2 + h.o.t. \\ \widetilde{\beta} &= \beta_c + \beta/\mathfrak{R}. \end{aligned}$$

It is possible to study the stability of these solutions with respect to perturbations which are $2\pi/\alpha_c$ - periodic in z, and $2\pi/\beta$ - periodic in y , as it is done for $\beta = 0$ in [3]. The result is that a ribbon wave (15) is stable provided that

$$b_r + c_r < 0 \quad \text{and} \quad b_r - c_r < 0,$$

which means that this type of waves is stable only when they occur on the supercritical side and $\tau > |a_{4r}|\beta^2/a_{3r}$.

Numerically, if $\mu > \mu_3$, they exist on both side. If $\mu_2 < \mu < \mu_3$, there exist on the super-critical size and are unstable. If $\mu < \mu_2$, they exist on the super-critical size and are stable.

3.4 Exotic solutions

We now look for more general solutions, of the form

$$\begin{aligned}\widetilde{A} &= r_0(y,t)e^{i(\omega_0+a_{3i}\tau)t+\theta_0(y,t)},\\ \widetilde{B} &= r_1(y,t)e^{i(\omega_0+a_{3i}\tau)t+\theta_1(y,t)},\end{aligned}$$

which leads to a system of real partial differential equations of order 8:

$$\begin{aligned}
\partial_t r_0 &= a_{3r}\tau r_0 - a_{4r}[\partial_{\widetilde{y}}^2 r_0 - r_0(\partial_{\widetilde{y}}\theta_0)^2] + a_{4i}(2\partial_{\widetilde{y}} r_0\partial_{\widetilde{y}}\theta_0 + r_0\partial_{\widetilde{y}}^2\theta_0) + r_0(b_r r_0^2 + c_r r_1^2),\\
\partial_t r_1 &= a_{3r}\tau r_1 - a_{4r}[\partial_{\widetilde{y}}^2 r_1 - r_1(\partial_{\widetilde{y}}\theta_1)^2] + a_{4i}(2\partial_{\widetilde{y}} r_1\partial_{\widetilde{y}}\theta_1 + r_1\partial_{\widetilde{y}}^2\theta_1) + r_1(b_r r_1^2 + c_r r_0^2),\\
\partial_t\theta_0 &= -\frac{a_{4r}}{r_0^2}\partial_{\widetilde{y}}(r_0^2\partial_{\widetilde{y}}\theta_0) - \frac{a_{4i}}{r_0}[\partial_{\widetilde{y}}^2 r_0 - r_0(\partial_{\widetilde{y}}\theta_0)^2] + b_i r_0^2 + c_i r_1^2,\\
\partial_t\theta_1 &= -\frac{a_{4r}}{r_1^2}\partial_{\widetilde{y}}(r_1^2\partial_{\widetilde{y}}\theta_1) - \frac{a_{4i}}{r_1}[\partial_{\widetilde{y}}^2 r_1 - r_1(\partial_{\widetilde{y}}\theta_1)^2] + b_i r_1^2 + c_i r_0^2.
\end{aligned}$$

Note that this time these equations are no longer decoupled. To go further, we restrict our analysis to solutions such that r_0 or r_1 identically vanishes, and to solutions such that $r_0 = r_1, \theta_0 = \theta_1 + const$, with in both cases $\partial_t r_0 = \partial_t r_1 = 0$ and $\partial_t\theta_0 = \partial_t\theta_1 = \kappa_0$, where κ_0 is constant.

Assuming $r_1 = 0$, and defining a new function K_0 by

$$r_0^2\theta_0' = K_0,$$

the solutions satisfy the following third order ordinary differential equation

$$\begin{aligned}
\frac{a_{4r}}{r_0}\Big[r_0'' - \frac{K_0^2}{r_0^3}\Big] - \frac{a_{4i}}{r_0^2}K_0' &= a_{3r}\tau + b_r r_0^2,\\
\frac{a_{4i}}{r_0}\Big[r_0'' - \frac{K_0^2}{r_0^3}\Big] + \frac{a_{4r}}{r_0^2}K_0' &= b_i r_0^2 - \kappa_0.
\end{aligned}$$

This leads to

$$\begin{aligned}
r_0'' - \frac{K_0^2}{r_0^3} &= \frac{r_0}{|a_4|^2}\Big[a_{3r}a_{4r}\tau + (b_r a_{4r} + b_i a_{4i})r_0^2 - \kappa_0 a_{4i}\Big], \qquad (16)\\
K_0' &= \frac{r_0^2}{|a_4|^2}\Big[-a_{3r}a_{4i}\tau + (b_i a_{4r} - b_r a_{4i})r_0^2 - \kappa_0 a_{4r}\Big].
\end{aligned}$$

If we look for solutions with $r_0 = r_1$, $\theta_0 = \theta_1 + const$, $\partial_t\theta_0 = \partial_t\theta_1 = \kappa_0$ and defining K_0 as above, we obtain again the following third order ordinary differential equations

$$\begin{aligned}
\frac{a_{4r}}{r_0}\Big[r_0'' - \frac{K_0^2}{r_0^3}\Big] - \frac{a_{4i}}{r_0^2}K_0' &= a_{3r}\tau + (b_r + c_r)r_0^2,\\
\frac{a_{4i}}{r_0}\Big[r_0'' - \frac{K_0^2}{r_0^3}\Big] + \frac{a_{4r}}{r_0^2}K_0' &= (b_i + c_i)r_0^2 - \kappa_0,
\end{aligned}$$

which leads to

$$\begin{aligned} r_0'' - \frac{K_0^2}{r_0^3} &= \frac{r_0}{|a_4|^2}\Big\{a_{3r}a_{4r}\tau + \Big[(b_r+c_r)a_{4r} + (b_i+c_i)a_{4i}\Big]r_0^2 - \kappa_0 a_{4i}\Big\}, \quad (17) \\ K_0' &= \frac{r_0^2}{|a_4|^2}\Big\{-a_{3r}a_{4i}\tau + \Big[(b_i+c_i)a_{4r} - (b_r+c_r)a_{4i}\Big]r_0^2 - \kappa_0 a_{4r}\Big\}. \end{aligned}$$

Both these third order systems take the form

$$\begin{aligned} r_0'' &= \frac{K_0^2}{r_0^3} + r_0(-a_0\tau + 2c_0 r_0^2 + d_0\kappa_0), \quad (18) \\ K_0' &= r_0^2(\alpha_0\tau + \gamma_0 r_0^2 + \delta_0\kappa_0), \end{aligned}$$

where

$$a_0 = -\frac{a_{3r}a_{4r}}{|a_4|^2} > 0, \quad \alpha_0 = -\frac{a_{3r}a_{4i}}{|a_4|^2}, \quad d_0 = \frac{-a_{4i}}{|a_4|^2}, \quad \delta_0 = \frac{-a_{4r}}{|a_4|^2},$$

and for (16)

$$2c_0 = \frac{b_r a_{4r} + b_i a_{4i}}{|a_4|^2}, \quad \gamma_0 = \frac{b_i a_{4r} - b_r a_{4i}}{|a_4|^2},$$

while for (17)

$$2c_0 = \frac{(b_r+c_r)a_{4r} + (b_i+c_i)a_{4i}}{|a_4|^2}, \quad \gamma_0 = \frac{(b_i+c_i)a_{4r} - (b_r+c_r)a_{4i}}{|a_4|^2}.$$

Looking for solutions of (18) such that K_0 is independent of y gives r_0 independent of y, we come back to the known solutions seen at previous section. The existence of other kind of solutions is open.

4 Appendix

4.1 Derivation of the limit Navier-Stokes equations

In this section, we detail the computations leading to (7). Let Δ_{cyl} be the Laplace operator in cylindrical coordinates, namely

$$\Delta_{cyl} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

We perform the change of variables

$$\hat{x} = \frac{r-\bar{r}}{d}, \quad \hat{y} = \frac{\bar{r}}{d}\varphi, \quad \hat{z} = \frac{z}{d}, \quad \hat{t} = \frac{\nu}{d^2}t,$$

where $\bar{r}$ is the average radius

$$\bar{r} = \frac{r_o + r_i}{2} = \frac{1+\eta}{2}r_o.$$

We first expand $\Delta_{cyl}u$, $r^{-2}u$, u_φ^2/r, $U' + \Omega_{rot}$, Ω_{rot} in η and μ.

- Expansion of $\Delta_{cyl}u$: we have

$$\Delta_{cyl}u = \Big(\frac{\nu}{d^3}\frac{\partial^2}{\partial\hat{x}^2} + \frac{1}{\hat{x}d+\bar{r}}\frac{\nu}{d^2}\frac{\partial}{\partial\hat{x}} + \frac{\nu}{d^3}\frac{\bar{r}^2}{(\hat{x}d+\bar{r})^2}\frac{\partial^2}{\partial\hat{y}^2} + \frac{\nu}{d^3}\frac{\partial^2}{\partial\hat{z}^2}\Big)\hat{u}.$$

We have

$$\frac{d}{\hat{x}d+\bar{r}} = \frac{1-\eta}{\hat{x}(1-\eta)+\frac{1+\eta}{2}} = \mathcal{O}(1-\eta),$$

$$\begin{aligned}\frac{\bar{r}^2}{(\hat{x}d+\bar{r})^2} =& \frac{(\frac{1+\eta}{2})^2}{(\hat{x}(1-\eta)+\frac{1+\eta}{2})^2} = 1 - \frac{\hat{x}(1-\eta)(\hat{x}(1-\eta)+(1+\eta))}{(\hat{x}(1-\eta)+\frac{1+\eta}{2})^2}\\ =& 1+\mathcal{O}(1-\eta),\end{aligned}$$

therefore,

$$\Delta_{cyl}u = \frac{\nu}{d^3}\Big(\frac{\partial^2}{\partial\hat{x}^2} + \frac{\partial^2}{\partial\hat{y}^2} + \frac{\partial^2}{\partial\hat{z}^2}\Big)\hat{u} + \mathcal{O}(1-\eta)\frac{\nu}{d^3}\Big(\frac{\partial\hat{u}}{\partial\hat{x}} + \frac{\partial^2\hat{u}}{\partial\hat{y}^2}\Big).$$

- Expansion of $r^{-2}u$:

$$\frac{u}{r^2} = \frac{\nu}{d^3}\frac{d^2}{(\hat{x}d+\bar{r})^2}\hat{u} = \mathcal{O}(1-\eta)^2\frac{\nu}{d^3}\hat{u}.$$

- Expansion of u_φ^2/r: Note that

$$\frac{d}{r} = (1-\eta) + (1-\eta)^2\Big(\frac{1}{2} - x\Big) + \mathcal{O}((1-\eta)^3)$$

and

$$\frac{u_\varphi^2}{r} = \frac{\nu^2}{d^3}\frac{d}{r}\hat{u}_{\hat{y}}^2.$$

- Study of $U' + \Omega_{rot}$:

$$U' + \Omega_{rot} = 2(A - \Omega_{rf}) = 2\omega_i\Big(\frac{\mu-\eta^2}{1-\eta^2} - \frac{1+\mu}{2}\Big) = -\omega_i\frac{(1-\mu)(1+\eta^2)}{1-\eta^2}.$$

- Expansion of Ω_{rot}:

$$\begin{aligned}\Omega_{rot} =& (A-\Omega_{rf}) + \frac{B}{r^2} = -\omega_i\frac{(1-\mu)(1+\eta^2)}{1-\eta^2} + \omega_i\frac{1-\mu}{1-\eta^2}\frac{r_i^2}{r^2}\\ =& -\omega_i\frac{1-\mu}{1-\eta^2}\Big(\frac{1+\eta^2}{2} - \frac{r_i^2}{r^2}\Big).\end{aligned}$$

Note that

$$\begin{aligned}\frac{r_i^2}{r^2} =& \frac{r_i^2}{(\hat{x}d+\bar{r})^2} = \frac{\eta^2}{(\hat{x}(1-\eta)+\frac{1+\eta}{2})^2} = \Big(\frac{\eta}{(\hat{x}+\frac{1}{2})(1-\eta)+\eta}\Big)^2\\ =& \Big(1+\frac{1-\eta}{\eta}(\hat{x}+\frac{1}{2})\Big)^{-2}\\ =& 1-2\Big(\frac{1-\eta}{\eta}(\hat{x}+\frac{1}{2})\Big)+3\Big(\frac{1-\eta}{\eta}(\hat{x}+\frac{1}{2})\Big)^2+\cdots.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{1}{1-\eta^2}\Big(\frac{1+\eta^2}{2}-\frac{r_i^2}{r^2}\Big)\\ =&\frac{1}{1-\eta^2}\Big[\frac{\eta^2-1}{2}+2\Big(\frac{1-\eta}{\eta}(\hat{x}+\frac{1}{2})\Big)-3\Big(\frac{1-\eta}{\eta}(\hat{x}+\frac{1}{2})\Big)^2+\cdots\Big]\\ =&-\frac{1}{2}+\frac{1}{\eta(1+\eta)}(2\hat{x}+1)+\mathcal{O}(1-\eta).\end{aligned}$$

As

$$\frac{2}{\eta(1+\eta)}-1=-\frac{(\eta+2)(\eta-1)}{\eta(1+\eta)}=\mathcal{O}(1-\eta),$$

we have

$$\frac{1}{1-\eta^2}\Big(\frac{1+\eta^2}{2}-\frac{r_i^2}{r^2}\Big)=\hat{x}+\mathcal{O}(1-\eta)$$

and hence

$$\Omega_{rot}=-\omega_i(1-\mu)\Big[\hat{x}+\mathcal{O}(1-\eta)\Big].$$

We now adimensionalize the Navier-Stokes equations written in cylindrical coordinates. In cylindrical coordinates, a (not necessarily axisymmetric) perturbation (u_r, u_φ, u_z, p) of the Couette flow (3) satisfies the system (see [13])

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}-\nu(\Delta_{cyl}-\frac{1}{r^2})\Big)u_r+\nu\frac{2}{r^2}\frac{\partial u_\varphi}{\partial\varphi}+\frac{\partial p}{\partial r}\\ &\qquad=-\Omega_{rot}\frac{\partial u_r}{\partial\varphi}+2\Omega_{rot}u_\varphi+\frac{u_\varphi^2}{r}+2\Omega_{rf}u_\varphi-(u\cdot\nabla)u_r,\\ &\Big(\frac{\partial}{\partial t}-\nu(\Delta_{cyl}-\frac{1}{r^2})\Big)u_\varphi-\nu\frac{2}{r^2}\frac{\partial u_r}{\partial\varphi}+\frac{1}{r}\frac{\partial p}{\partial\varphi}\\ &\qquad=-\Omega_{rot}\frac{\partial u_\varphi}{\partial\varphi}-U'u_r-\Omega_{rot}u_r-\frac{u_ru_\varphi}{r}-2\Omega_{rf}u_r-(u\cdot\nabla)u_\varphi,\\ &\Big(\frac{\partial}{\partial t}-\nu\Delta_{cyl}\Big)u_z+\frac{\partial p}{\partial z}=-\Omega_{rot}\frac{\partial u_z}{\partial\varphi}-(u\cdot\nabla)u_z,\end{aligned}$$

together with the incompressibility condition, which in cylindrical coordinates reads

$$\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{\partial u_z}{\partial z}=0.$$

Here the Coriolis term is added on the right-hand side. We rescale the velocity and the pressure by

$$u(t,r,\varphi,z) = \frac{\nu}{d}\hat{u}(\hat{t},\hat{x},\hat{y},\hat{z}),$$
$$p(t,r,\varphi,z) = \frac{\nu^2}{d^2}\hat{p}(\hat{t},\hat{x},\hat{y},\hat{z}).$$

This leads to

$$\begin{aligned}
\Big(\frac{\partial}{\partial\hat{t}} - \hat{\Delta}\Big)\hat{u}_{\hat{x}} + \frac{\partial\hat{p}}{\partial\hat{x}} &= \mathfrak{R}\hat{x}\frac{\partial\hat{u}_{\hat{x}}}{\partial\hat{y}} + \hat{\omega}(1+\mu)\hat{u}_{\hat{y}} - (\hat{u}\cdot\nabla)\hat{u}_{\hat{x}} + \mathcal{O}\Big(\hat{\omega}(1-\mu)\frac{\partial\hat{u}_{\hat{x}}}{\partial\hat{y}}\Big)\\
&\quad + \mathcal{O}(1-\eta)^2\hat{u}_{\hat{x}} - 2\Big(\hat{\omega}(1-\mu)\hat{x} + \mathcal{O}(1-\eta)\Big)\hat{u}_{\hat{y}} + \mathcal{O}(1-\eta)\frac{\partial\hat{u}_{\hat{y}}}{\partial\hat{y}}\\
&\quad + \Big((1-\eta)^2 x + \frac{(1-\eta)(1+\eta)}{2}\Big)\hat{u}_{\hat{y}}^2,\\
\Big(\frac{\partial}{\partial\hat{t}} - \hat{\Delta}\Big)\hat{u}_{\hat{y}} + \frac{\partial\hat{p}}{\partial\hat{y}} &= \mathfrak{R}\hat{x}\frac{\partial\hat{u}_{\hat{y}}}{\partial\hat{y}} + \hat{\omega}\frac{(1-\mu)(1+\eta^2)}{1-\eta^2}\hat{u}_{\hat{x}} - \hat{\omega}(1+\mu)\hat{u}_{\hat{x}} - (\hat{u}\cdot\nabla)\hat{u}_{\hat{y}}\\
&\quad + \mathcal{O}(1-\eta)\Big(\hat{u}_{\hat{x}}\hat{u}_{\hat{y}} + \frac{\partial\hat{u}_{\hat{x}}}{\partial\hat{y}}\Big) + \mathcal{O}\Big(\hat{\omega}(1-\mu)\frac{\partial\hat{u}_{\hat{y}}}{\partial\hat{y}}\Big) + \mathcal{O}(1-\eta)^2\hat{u}_{\hat{y}},\\
\Big(\frac{\partial}{\partial\hat{t}} - \hat{\Delta}\Big)\hat{u}_{\hat{z}} + \frac{\partial\hat{p}}{\partial\hat{z}} &= \mathfrak{R}\hat{x}\frac{\partial\hat{u}_{\hat{z}}}{\partial\hat{y}} - (\hat{u}\cdot\nabla)\hat{u}_{\hat{z}} + \mathcal{O}\Big(\hat{\omega}(1-\mu)\frac{\partial\hat{u}_{\hat{z}}}{\partial\hat{y}}\Big),\\
\frac{\partial\hat{u}_{\hat{x}}}{\partial\hat{x}} + \frac{\partial\hat{u}_{\hat{y}}}{\partial\hat{y}} + \frac{\partial\hat{u}_{\hat{z}}}{\partial\hat{z}} &= \mathcal{O}(1-\eta)\Big(\hat{u}_{\hat{x}} + \frac{\partial\hat{u}_{\hat{y}}}{\partial\hat{y}}\Big).
\end{aligned}$$

In this system, $\mathcal{O}(\cdot)$ denotes various smooth functions that go to zero in the limit (4). As η goes to 1, under the scaling (4), the limiting system is

$$\begin{aligned}
(\partial_t - \Delta_\perp)u_\perp + \nabla_\perp p &= \mathfrak{R}x\partial_y u_\perp + Tg(x)(\widehat{u}_y, 0)^t\\
&\quad -(u_\perp\cdot\nabla_\perp)u_\perp + \frac{T}{2}(1-\mu)(\widehat{u}_y^2, 0)^t, \qquad (19)\\
(\partial_t - \Delta_\perp)\widehat{u}_y &= \mathfrak{R}x\partial_y\widehat{u}_y + u_x - (u_\perp\cdot\nabla_\perp)\widehat{u}_y, \qquad (20)\\
\nabla_\perp\cdot u_\perp + \mathfrak{R}\partial_y\widehat{u}_y &= 0, \qquad (21)
\end{aligned}$$

where the subscript $\perp$ denotes the components in the (x,z) plane, $\widehat{u}_y = \mathfrak{R}u_y$, and

$$g(x) = \frac{1+\mu}{2} - (1-\mu)x.$$

In this system, we keep the linear terms with $\mathfrak{R}\partial_y$ since this operator remains bounded in the limiting regime described previously.

4.2 Range of $(i\omega_0 - L_0)$ and vector ζ_1^*

We define the scalar product $\langle U, V\rangle$ of two vector fields $U = (u_x, u_y, u_z)$ and $V = (v_x, v_y, v_z)$ by

$$\langle U, V\rangle = \int_{-1/2}^{1/2}\Big[Du_x\overline{Dv_x} + \alpha_c^2 u_x\overline{v_x} + u_y\overline{v_y}\Big]\,dx. \qquad (22)$$

We have to take care that the divergence free vector field U is no longer orthogonal to gradients. Let us solve the equation

$$(i\omega_0 - L_0)U = F$$

where

$$\begin{aligned} F &= e^{i(\alpha_c z+\beta_c y)}\Phi(x), \qquad \Phi = (\phi_x, \phi_y, \phi_z), \\ 0 &= D\phi_x + i\mathfrak{B}\phi_y + i\alpha\phi_z. \end{aligned}$$

The vector Φ satisfies the same boundary conditions as U at $x = \pm 1/2$. We look for $\widehat{U}(x)$ such that

$$\begin{aligned} U &= e^{i(\alpha_c z+\beta_c y)}\widehat{U}(x), \qquad \widehat{U}(x) = (u_x, u_y, u_z), \\ 0 &= Du_x + i\mathfrak{B}_c u_y + i\alpha u_z. \end{aligned}$$

This gives

$$\begin{aligned} (i\omega_0 + \alpha_c^2 - D^2 - i\mathfrak{B}_c x)u_x + Dp - T_c g(x)u_y &= \phi_x, \\ (i\omega_0 + \alpha_c^2 - D^2 - i\mathfrak{B}_c x)u_y - u_x &= \phi_y, \\ (i\omega_0 + \alpha_c^2 - D^2 - i\mathfrak{B}_c x)u_z + i\alpha p &= \phi_z, \\ Du_x + i\mathfrak{B}_c u_y + i\alpha u_z &= 0, \end{aligned}$$

which leads to

$$\begin{aligned} (i\omega_0 + \alpha_c^2 - D^2 - i\mathfrak{B}_c x)(\alpha_c^2 - D^2)u_x - \alpha_c^2 T_c g(x)u_y &= (\alpha_c^2 - D^2)\phi_x, \quad (23) \\ (i\omega_0 + \alpha_c^2 - D^2 - i\mathfrak{B}_c x)u_y - u_x &= \phi_y. \end{aligned}$$

We then define the eigenvector belonging to $i\omega_0$

$$\zeta_1 = e^{i(\alpha_c z+\beta_c y)}(u_x^0, u_y^0, u_z^0)^t,$$

with

$$\begin{aligned} (i\omega_0 + \alpha_c^2 - D^2 - i\mathfrak{B}_c x)(\alpha_c^2 - D^2)u_x^0 - \alpha_c^2 T_c g(x)u_y^0 &= 0, \qquad (24) \\ (i\omega_0 + \alpha_c^2 - D^2 - i\mathfrak{B}_c x)u_y^0 - u_x^0 &= 0, \end{aligned}$$

and

$$u_x^0 = Du_x^0 = u_y^0 = 0, x = \pm 1/2.$$

Let us look for *the compatibility condition* for solving (23). We introduce the vector field

$$\begin{aligned} \zeta_1^* &= e^{i(\alpha_c z+\beta_c y)}(v_x^0, v_y^0, v_z^0), \\ 0 &= Dv_x^0 + i\mathfrak{B}_c v_y^0 + i\alpha v_z^0, \\ v_x^0 &= Dv_x^0 = v_y^0 = 0, x = \pm 1/2, \end{aligned}$$

satisfying the adjoint system of (24)

$$\begin{aligned}(\alpha_c^2 - D^2)(-i\omega_0 + \alpha_c^2 - D^2 + i\mathfrak{B}_c x)v_x^0 - v_y^0 &= 0, \\ (-i\omega_0 + \alpha_c^2 - D^2 + i\mathfrak{B}_c x)v_y^0 - \alpha_c^2 T_c g(x) v_x^0 &= 0,\end{aligned} \tag{25}$$

then it is easy to check for (23) that (ϕ_x, ϕ_y) needs to satisfy

$$\int_{-1/2}^{1/2} \left\{ [(\alpha_c^2 - D^2)\phi_x]\overline{v_x^0} + \phi_y \overline{v_y^0} \right\} dx = 0,$$

i.e.

$$\int_{-1/2}^{1/2} \left[D\phi_x D\overline{v_x^0} + \alpha_c^2 \phi_x \overline{v_x^0} + \phi_y \overline{v_y^0} \right] dx = 0, \tag{26}$$

which may be taken as a definition of the condition

$$\langle F, \zeta_1^* \rangle = 0.$$

We notice that (with the definition above)

$$\langle \zeta_1, \zeta_1^* \rangle = \int_{-1/2}^{1/2} \left[Du_x^0 D\overline{v_x^0} + \alpha_c^2 u_x^0 \overline{v_x^0} + u_y^0 \overline{v_y^0} \right] dx \tag{27}$$

which should be non zero in the next calculations.

4.3 Computation of a_1, a_3, b_1, a_4

To compute the coefficients a_1, a_2, a_3, a_4 and a_5, we rewrite

$$\begin{aligned}(\lambda_0 + \alpha^2 - D^2 - i\mathfrak{B}x)(\alpha^2 - D^2)u_x - \alpha^2 T g(x) u_y &= 0, \\ (\lambda_0 + \alpha^2 - D^2 - i\mathfrak{B}x)u_y - u_x &= 0,\end{aligned} \tag{28}$$

under the form

$$\begin{aligned}\lambda_0(H_0 + \widehat{\alpha} H_{010})U &= L_0 U + \widehat{\mathfrak{B}} L_{100} U + \widehat{\alpha} L_{010} U + \widehat{T} L_{001} U \\ &\quad + \widehat{\alpha}\widehat{\mathfrak{B}} L_{110} U + \widehat{\alpha}\widehat{T} L_{011} U + \widehat{\alpha}^2 L_{020} U,\end{aligned} \tag{29}$$

where

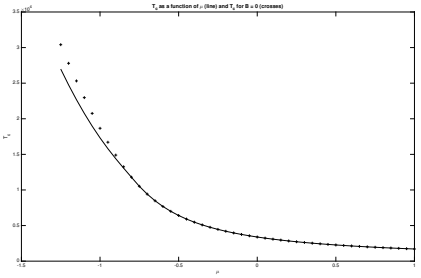
$$\widehat{T} = T - T_c, \qquad \widehat{\alpha} = \alpha^2 - \alpha_c^2, \qquad \widehat{\mathfrak{B}} = \mathfrak{B} - \mathfrak{B}_{\mathfrak{c}}$$

and where we have defined

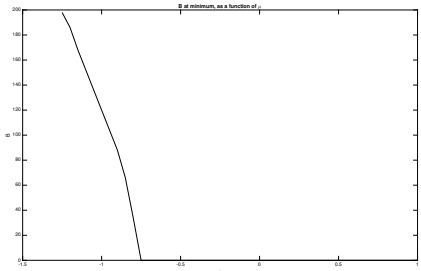
$$H_0 U = \begin{pmatrix} (\alpha_c^2 - D^2)u_x \\ u_y \end{pmatrix}, \qquad H_{010} U = \begin{pmatrix} u_x \\ 0 \end{pmatrix},$$

$$L_0 U = \begin{pmatrix} -(\alpha_c^2 - D^2 - i\mathfrak{B}_c x)(\alpha_c^2 - D^2)u_x + \alpha_c^2 T_c g(x) u_y \\ -(\alpha_c^2 - D^2 - i\mathfrak{B}_c x)u_y + u_x \end{pmatrix},$$

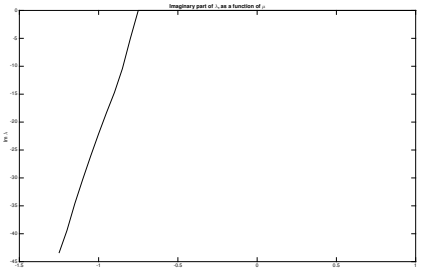
$$L_{001} U = \begin{pmatrix} \alpha_c^2 g(x) u_y \\ 0 \end{pmatrix}, \quad L_{100} U = \begin{pmatrix} ix(\alpha_c^2 - D^2)u_x \\ ixu_y \end{pmatrix},$$



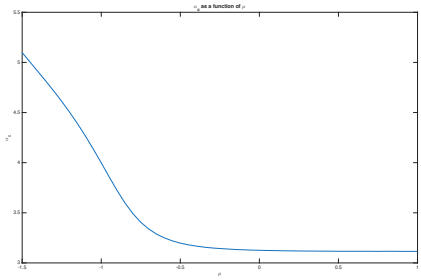
(a) Critical Taylor number as a function of μ (solid line) and critical Taylor number corresponding to $\mathfrak{B} = 0$



(b) $\mathfrak{B}$ at the critical point as a function of μ



(c) The imaginary part of the eigenvalue at the critical point as a function of μ



(d) α at the critical point as a function of μ

Figure 2: Numerical study of the critical point as a function of μ. We observe that when $\mu < \mu_c \approx -0,8$, the critical point is no longer axisymmetric ($\mathfrak{B} \neq 0$).

$$L_{010}U = \begin{pmatrix} -2(\alpha_c^2 - D^2)u_x + i\mathfrak{B}_c x u_x + T_c g(x) u_y \\ -u_y \end{pmatrix},$$

$$L_{110}U = \begin{pmatrix} ixu_x \\ 0 \end{pmatrix}, \qquad L_{011}U = \begin{pmatrix} g(x)u_y \\ 0 \end{pmatrix}, \qquad L_{020}U = \begin{pmatrix} -u_x \\ 0 \end{pmatrix}.$$

We look for an expansion of U under the form

$$U = U^0 + \widehat{\mathfrak{B}}U^{100} + \widehat{\alpha}U^{010} + \widehat{T}U^{001} + \widehat{\alpha}\widehat{\mathfrak{B}}U^{110} + \widehat{T}\widehat{\alpha}U^{011} + \widehat{\alpha}^2 U^{020} + \cdots.$$

Replacing λ_0 by (8) in (29) and identifying monomials leads to

$$\begin{aligned}
ia_1 H_0 U^0 &= (L_0 - i\omega_0 H_0)U^{010} + L_{010}U^0 - i\omega_0 H_{010}U^0, \\
a_3 H_0 U^0 &= (L_0 - i\omega_0 H_0)U^{001} + L_{001}U^0, \\
ib_1 H_0 U^0 &= (L_0 - i\omega_0 H_0)U^{100} + L_{100}U^0, \\
a_4 H_0 U^0 + ib_1 H_0 U^{100} &= (L_0 - i\omega_0 H_0)U^{200} + L_{100}U^{100}, \\
a_2 H_0 U^0 + ia_1 H_0 U^{010} &= (L_0 - i\omega_0 H_0)U^{020} + L_{010}U^{010} + L_{020}U^0 \\
&\quad -ia_1 H_{010}U^0 - i\omega_0 H_{010}U^{010}
\end{aligned}$$

and

$$\begin{aligned}
&2a_5 H_0 U^0 + ia_1 H_0 U^{100} + ib_1 H_0 U^{010} + i\omega_0 H_{010}U^{100} + ib_1 H_{010}U^0 \\
&= (L_0 - i\omega_0 H_0)U^{110} + L_{010}U^{100} + L_{100}U^{010} + L_{110}U^0,
\end{aligned}$$

where $U^0 = (u_x^0, u_y^0)^t$ is the eigenvector satisfying

$$(i\omega_0 H_0 - L_0)U^0 = 0.$$

Let $V^0 = (v_x^0, v_y^0)$ be the adjoint vector, which satisfies

$$\begin{aligned}
(\alpha_c^2 - D^2)(-i\omega_0 + \alpha_c^2 - D^2 + i\mathfrak{B}_c x)v_x^0 - v_y^0 &= 0, \qquad (30) \\
(-i\omega_0 + \alpha_c^2 - D^2 + i\mathfrak{B}_c x)v_y^0 - \alpha_c^2 T_c g(x) v_x^0 &= 0.
\end{aligned}$$

Then taking the scalar product of the previous equation with V^0, we obtain

$$\begin{aligned}
ia_1\langle H_0 U^0, V^0\rangle &= \langle L_{010}U^0, V^0\rangle - i\omega_0\langle H_{010}U^0, V^0\rangle, \\
a_3\langle H_0 U^0, V^0\rangle &= \langle L_{001}U^0, V^0\rangle, \\
ib_1\langle H_0 U^0, V^0\rangle &= \langle L_{100}U^0, V^0\rangle, \\
a_4\langle H_0 U^0, V^0\rangle &= -ib_1\langle H_0 U^{100}, V^0\rangle + \langle L_{100}U^{100}, V^0\rangle, \\
a_2\langle H_0 U^0, V^0\rangle &= -ia_1\langle H_0 U^{010}, V^0\rangle + \langle L_{010}U^{010} + L_{020}U^0, V^0\rangle \\
&\quad -ia_1\langle H_{010}U^0, V^0\rangle - i\omega_0\langle H_{010}U^{010}, V^0\rangle, \\
2a_5\langle H_0 U^0, V^0\rangle &= -i\langle a_1 H_0 U^{100} + b_1 H_0 U^{010} + \omega_0 H_{010}U^{100}, V^0\rangle \\
&\quad +\langle L_{010}U^{100} + L_{100}U^{010}, V^0\rangle - ib_1\langle H_{010}U^0, V^0\rangle + \langle L_{110}U^0, V^0\rangle,
\end{aligned}$$

where, by construction, a_1 and b_1 are real. We observe that

$$\langle H_0 U^0, V^0\rangle = \langle \zeta_1, \zeta_1^*\rangle$$

which is given by (27).

Once the eigenvectors U^0 and V^0 are computed, we can compute a_1, a_3 and b_1. We can then evaluate U^{010}, U^{001} and U^{100}, which leads to the computation of a_2, a_4 and a_5.

Numerically, we check that, as expected, $a_{3r} > 0$, $a_{4r} < 0$, $a_{2r} < 0$ and that $a_{4r}a_{2r} > a_{5r}^2$.

4.4 Computation of b and c

We start with the limit Navier-Stokes system (7), where we look for solutions which are $2\pi/\beta_c$ periodic in y. Below, we suppress the hat on u_y. Let us define the operator L_0 as

$$L_0 U = \Delta_\perp U + x\mathfrak{R}\partial_y U - \nabla_\perp p_0 + \begin{pmatrix} T_c g(x) u_y \\ 0 \\ u_x \end{pmatrix}, \tag{31}$$

with

$$\nabla_\perp \cdot U_\perp + \mathfrak{R}\partial_y u_y = 0, \tag{32}$$

where p_0 is such that $L_0 U$ satisfies the boundary conditions

$$L_0 U|_x = 0, \quad \text{when} \quad x = \pm 1/2, \tag{33}$$

and the modified divergence free condition

$$\nabla_\perp \cdot (L_0 U)_\perp + \mathfrak{R}\partial_y (L_0 U)_y = 0.$$

Now we define the symmetric quadratic operator B as

$$B(U,V) = -\frac{1}{2}[(U_\perp \cdot \nabla_\perp)V + (V_\perp \cdot \nabla_\perp)U] + \frac{T(1-\mu)}{2}\begin{pmatrix} u_y v_y \\ 0 \\ 0 \end{pmatrix} + \nabla_\perp q, \tag{34}$$

with the boundary conditions

$$B(U,V)|_x = 0, \quad \text{when} \quad x = \pm 1/2,$$

and the modified divergence free condition

$$\nabla_\perp \cdot (B(U,V))_\perp + \mathfrak{R}\partial_y B(U,V)_y = 0.$$

Now, our system for $T = T_c$ may be written as

$$\partial_t U = L_0 U + B_c(U,U), \tag{35}$$

where

$$B_c = B|_{T_c}.$$

To obtain the coefficients b and c of the amplitude equations, we first decompose U as

$$U = A\zeta_1 + B\zeta_2 + \overline{A\zeta_1} + \overline{B\zeta_2} + \Phi(A, \overline{A}, B, \overline{B}), \tag{36}$$

where we expand Φ as

$$\Phi = \sum_{p+q+r+s\geq 2} \Phi_{pqrs} A^p \overline{A}^q B^r \overline{B}^s,$$

$$\begin{aligned} \zeta_1 &= e^{i(\alpha_c z+\beta_c y)} U_1(x), \zeta_2 = e^{i(-\alpha_c z+\beta_c y)} S U_1(x), \\ (L_0 - i\omega_0)\zeta_1 &= 0, (L_0 - i\omega_0)\zeta_2 = 0, \end{aligned}$$

where coefficients Φ_{pqrs} are divergence free and satisfy the same boundary conditions as U. Now the principal part of the amplitude equations reduces to

$$\begin{aligned} \frac{dA}{dt} &= i\omega_0 A + bA|A|^2 + cA|B|^2, \\ \frac{dB}{dt} &= i\omega_0 B + cB|A|^2 + bB|B|^2, \end{aligned} \tag{37}$$

and the coefficients b and c are deduced from an identification of the monomials, first in computing $\partial_t U$ in (36) in using (37), second in replacing U in (35). The identification of the monomials $A^2, |A|^2, |B|^2, A|A|^2, A|B|^2$ lead to

$$\begin{aligned} 0 &= (L_0 - 2i\omega_0)\Phi_{2000} + B(\zeta_1, \zeta_1), \\ 0 &= L_0\Phi_{1100} + 2B(\zeta_1, \overline{\zeta_1}), \\ 0 &= L_0\Phi_{0011} + 2B(\zeta_2, \overline{\zeta_2}), \\ 0 &= (L_0 - 2i\omega_0)\Phi_{1010} + 2B(\zeta_1, \zeta_2), \\ 0 &= L_0\Phi_{1001} + 2B(\zeta_1, \overline{\zeta_2}), \end{aligned}$$

$$\begin{aligned} b\zeta_1 &= (L_0 - i\omega_0)\Phi_{2100} + 2B(\zeta_1, \Phi_{1100}) + 2B(\overline{\zeta_1}, \Phi_{2000}), \\ c\zeta_1 &= (L_0 - i\omega_0)\Phi_{1011} + 2B(\zeta_1, \Phi_{0011}) + 2B(\overline{\zeta_2}, \Phi_{1010}) + 2B(\zeta_2, \Phi_{1001}), \end{aligned}$$

where we notice that L_0 and $(L_0 - 2i\omega_0)$ are invertible, while $(L_0 - i\omega_0)$ has a two-dimensional kernel spanned by (ζ_1, ζ_2), so that Fredholm alternative gives coefficients b and c in deriving the compatibility condition for each of the two last equations. Using the results of section 4.2, we obtain that the coefficients b and c are determined by

$$b\langle \zeta_1, \zeta_1^* \rangle = \langle 2B(\zeta_1, \Phi_{1100}) + 2B(\overline{\zeta_1}, \Phi_{2000}), \zeta_1^* \rangle, \tag{38}$$

$$c\langle \zeta_1, \zeta_1^* \rangle = \langle 2B(\zeta_1, \Phi_{0011}) + 2B(\overline{\zeta_2}, \Phi_{1010}) + 2B(\zeta_2, \Phi_{1001}), \zeta_1^* \rangle, \tag{39}$$

with

$$\begin{aligned} \Phi_{2000} &= (2i\omega_0 - L_0)^{-1} B(\zeta_1, \zeta_1), \\ \Phi_{1100} &= -2L_0^{-1} B(\zeta_1, \overline{\zeta_1}), \\ \Phi_{0011} &= -2L_0^{-1} B(\zeta_2, \overline{\zeta_2}) = S\Phi_{1100}, \\ \Phi_{1010} &= 2(2i\omega_0 - L_0)^{-1} B(\zeta_1, \zeta_2), \\ \Phi_{1001} &= -2L_0^{-1} B(\zeta_1, \overline{\zeta_2}). \end{aligned}$$

4.4.1 Computation of the various Φ

We now successively compute Φ_{2000}, Φ_{1100}, Φ_{0011}, Φ_{1010}, Φ_{1001}. We have

$$B(\zeta_1,\zeta_1)=e^{2i(\alpha_c z+\beta_c y)}(B_x^{2000},B_y^{2000},B_z^{2000})^t+e^{2i(\alpha_c z+\beta_c y)}(Dp,0,2i\alpha_c p)^t,$$

with

$$B_x^{2000}=i\mathfrak{B}_c u_x^0 u_y^0+T_c\frac{(1-\mu)}{2}(u_y^0)^2,$$

$$B_y^{2000}=-u_x^0Du_y^0+u_y^0Du_x^0+i\mathfrak{B}_c(u_y^0)^2,$$

$$B_z^{2000}=\frac{i}{\alpha_c}\Big[(Du_x^0)^2-u_x^0D^2u_x^0-\mathfrak{B}_c^2(u_y^0)^2\Big]+\frac{\mathfrak{B}_c}{\alpha_c}\Big(u_x^0Du_y^0-2Du_x^0u_y^0\Big).$$

Thus

$$\Phi_{2000}=e^{2i(\alpha_c z+\beta_c y)}(\phi_x^{2000},\phi_y^{2000},\phi_z^{2000})^t,$$

with

$$\begin{aligned}&(2i\omega_0+4\alpha_c^2-D^2-2i\mathfrak{B}_c x)(4\alpha^2-D^2)\phi_x^{2000}-4\alpha^2T_cg(x)\phi_y^{2000}\\&\quad=4\alpha^2B_x^{2000}+2i\mathfrak{B}_cDB_y^{2000}+2i\alpha_cDB_z^{2000},\end{aligned}$$

$$(2i\omega_0+4\alpha_c^2-D^2-2i\mathfrak{B}_c x)\phi_y^{2000}-\phi_x^{2000}=B_y^{2000},$$

$$\phi_z^{2000}=\frac{i}{2\alpha_c}(D\phi_x^{2000}+2i\mathfrak{B}_c\phi_y^{2000}),$$

$$\phi_x^{2000}=D\phi_x^{2000}=\phi_y^{2000}=0,x=\pm1/2.$$

We now turn to the computation of Φ_{1100}. We have

$$2B(\zeta_1,\overline{\zeta_1})=(B_x^{1100},B_y^{1100},B_z^{110})^t+(Dp,0,0)^t,$$

with

$$B_x^{1100}=-2D(u_x^0\overline{u_x^0})+i\mathfrak{B}_c(u_x^0\overline{u_y^0}-\overline{u_x^0}u_y^0)+T_c(1-\mu)u_y^0\overline{u_y^0},$$

$$B_y^{1100}=-D(u_x^0\overline{u_y^0}+\overline{u_x^0}u_y^0),$$

$$B_z^{1100}=\frac{i}{\alpha_c}(u_x^0D^2\overline{u_x^0}-\overline{u_x^0}D^2u_x^0)+\frac{\mathfrak{B}_c}{\alpha_c}(u_x^0D\overline{u_y^0}+\overline{u_x^0}Du_y^0).$$

Thus

$$\Phi_{1100}=(0,\phi_y^{1100},\phi_z^{1100})^t,$$

with

$$\begin{aligned}-T_cg(x)\phi_y^{1100}&=B_x^{1100}-Dp,\\-D^2\phi_y^{1100}&=B_y^{1100},\\-D^2\phi_z^{1100}&=B_z^{1100},\end{aligned}$$

$$\phi_y^{1100}=\phi_z^{1100}=0,x=\pm1/2.$$

We observe that ϕ_y^{1100} is real while ϕ_z^{1100} is pure imaginary. We now compute Φ_{0011}. We have

$$\Phi_{0011} = S\Phi_{1100},$$

hence

$$\Phi_{0011} = (0, \phi_y^{1100}, -\phi_z^{1100})^t = \overline{\Phi_{1100}}.$$

Next to compute Φ_{1010}, we observe that

$$2B(\zeta_1, \zeta_2) = e^{2i\beta_c y}(B_x^{1010}, B_y^{1010}, B_z^{1010})^t + e^{2i\beta_c y}(Dp, 0, 0)^t,$$

with

$$B_x^{1010} = -4u_x^0 Du_x^0 - 2i\mathfrak{B}_c u_x^0 u_y^0 + T_c(1-\mu)(u_y^0)^2,$$
$$B_y^{1010} = -2(u_y^0 Du_x^0 + u_x^0 Du_y^0) - 2i\mathfrak{B}_c (u_y^0)^2,$$
$$B_z^{1010} = 0.$$

Thus

$$\Phi_{1010} = e^{2i\beta_c y}(\phi_x^{1010}, \phi_y^{1010}, 0)^t,$$

with

$$(2i\omega_0 - D^2 - 2i\mathfrak{B}_c x)D\phi_x^{1010} + 2i\mathfrak{B}_c \phi_x^{1010} = -2i\mathfrak{B}_c B_y^{1010},$$
$$D\phi_x^{1010} + 2i\mathfrak{B}_c \phi_y^{1010} = 0,$$
$$\phi_x^{1010} = D\phi_x^{1010} = \phi_y^{1010} = 0, x = \pm 1/2.$$

It remains to compute Φ_{1001}. Note that

$$2B(\zeta_1, \overline{\zeta_2}) = e^{2i\alpha_c z}(B_x^{1001}, B_y^{1001}, B_z^{1001})^t + e^{2i\alpha_c z}(Dp, 0, 2i\alpha_c p)^t,$$

with

$$B_x^{1001} = i\mathfrak{B}_c(\overline{u_x^0}u_y^0 - u_x^0\overline{u_y^0}) + T_c(1-\mu)u_y^0\overline{u_y^0},$$
$$B_y^{1001} = u_y^0 D\overline{u_x^0} + \overline{u_y^0}Du_x^0 - u_x^0 D\overline{u_y^0} - \overline{u_x^0}Du_y^0,$$

$$\begin{aligned} B_z^{1001} &= -\frac{i}{\alpha_c}[u_x^0 D^2\overline{u_x^0} + \overline{u_x^0}D^2 u_x^0 - 2Du_x^0 D\overline{u_x^0} - 2\mathfrak{B}_c^2 u_y^0\overline{u_y^0}] \\ &\quad -\frac{\mathfrak{B}_c}{\alpha_c}[u_x^0 D\overline{u_y^0} - \overline{u_x^0}Du_y^0 + 2u_y^0 D\overline{u_x^0} - 2\overline{u_y^0}Du_x^0]. \end{aligned}$$

We notice that B_x^{1001} and B_y^{1001} are real, while B_z^{1001} is imaginary. Then we get

$$\Phi^{1001} = e^{2i\alpha_c z}(\phi_x^{1001}, \phi_y^{1001}, \phi_z^{1001})^t,$$

with

$$(4\alpha_c^2 - D^2)^2\phi_x^{1001} - 4\alpha_c^2 T_c g(x)\phi_y^{1001} = 4\alpha_c^2 B_x^{1001} + 2i\alpha_c DB_z^{1001},$$
$$(4\alpha_c^2 - D^2)\phi_y^{1001} - \phi_x^{1001} = B_y^{1001},$$
$$\phi_z^{1001} = \frac{i}{2\alpha_c}D\phi_x^{1001},$$
$$\phi_x^{1001} = D\phi_x^{1001} = \phi_y^{1001} = 0, x = \pm 1/2.$$

We notice that ϕ_x^{1001} and ϕ_y^{1001} are real, while ϕ_z^{1001} is imaginary (this is due to symmetry S).

4.4.2 Computation of the terms involving B

We have

$2B(\zeta_1, \Phi_{1100}) = e^{i(\alpha_c z+\beta_c y)}(B_x^{1,1100}, B_y^{1,1100}, B_z^{1,1100})^t + e^{i(\alpha_c z+\beta_c y)}(Dp_1, 0, i\alpha_c p_1)^t,$

with

$$\begin{aligned} B_x^{1,1100} &= -i\alpha_c u_x^0 \phi_z^{1100} + T_c(1-\mu) u_y^0 \phi_y^{1100}, \\ B_y^{1,1100} &= -u_x^0 D\phi_y^{1100} - i\alpha_c u_y^0 \phi_z^{1100}, \\ B_z^{1,1100} &= -D(u_x^0 \phi_z^{1100}) - i\mathfrak{B}_c u_y^0 \phi_z^{1100}. \end{aligned}$$

Next,

$2B(\overline{\zeta_1}, \Phi_{2000}) = e^{i(\alpha_c z+\beta_c y)}(B_x^{\overline{1},2000}, B_y^{\overline{1},2000}, B_z^{\overline{1},2000})^t + e^{i(\alpha_c z+\beta_c y)}(Dp_1, 0, i\alpha_c p_1)^t,$

with

$$\begin{aligned} B_x^{\overline{1},2000} &= -\overline{u_x^0} D\phi_x^{2000} - 3D\overline{u_x^0}\phi_x^{2000} + 2i\mathfrak{B}_c \overline{u_y^0}\phi_x^{2000} + i\alpha_c \overline{u_x^0}\phi_z^{2000} + T_c(1-\mu)\overline{u_y^0}\phi_y^{2000}, \\ B_y^{\overline{1},2000} &= -\overline{u_x^0} D\phi_y^{2000} - 2D\overline{u_x^0}\phi_y^{2000} + 2i\mathfrak{B}_c \overline{u_y^0}\phi_y^{2000} - D\overline{u_y^0}\phi_x^{2000} + i\alpha_c \overline{u_y^0}\phi_z^{2000}, \\ B_z^{\overline{1},2000} &= D\overline{u_x^0}\phi_z^{2000} - \overline{u_x^0} D\phi_z^{2000} - i\mathfrak{B}_c \overline{u_y^0}\phi_z^{2000} + \frac{i}{\alpha_c} D^2 \overline{u_x^0}\phi_x^{2000} + \frac{\mathfrak{B}}{\alpha_c} D\overline{u_y^0}\phi_x^{2000}. \end{aligned}$$

We also have

$2B(\zeta_1, \Phi_{0011}) = e^{i(\alpha_c z+\beta_c y)}(B_x^{1,0011}, B_y^{1,0011}, B_z^{1,0011})^t + e^{i(\alpha_c z+\beta_c y)}(Dp_2, 0, i\alpha_c p_2)^t,$

with

$$\begin{aligned} B_x^{1,0011} &= i\alpha_c u_x^0 \phi_z^{1100} + T_c(1-\mu) u_y^0 \phi_y^{1100}, \\ B_y^{1,0011} &= -u_x^0 D\phi_y^{1100} + i\alpha_c u_y^0 \phi_z^{1100}, \\ B_z^{1,0011} &= D(u_x^0 \phi_z^{1100}) + i\mathfrak{B}_c u_y^0 \phi_z^{1100}. \end{aligned}$$

Moreover,

$2B(\overline{\zeta_2}, \Phi_{1010}) = e^{i(\alpha_c z+\beta_c y)}(B_x^{\overline{2},1010}, B_y^{\overline{2},1010}, B_z^{\overline{2},1010})^t + e^{i(\alpha_c z+\beta_c y)}(Dp_3, 0, i\alpha_c p_3)^t,$

with

$$\begin{aligned} B_x^{\overline{2},1010} &= -D(\overline{u_x^0}\phi_x^{1010}) + T_c(1-\mu)\overline{u_y^0}\phi_y^{1010}, \\ B_y^{\overline{2},1010} &= -\overline{u_x^0} D\phi_y^{1010} - D\overline{u_y^0}\phi_x^{1010}, \\ B_z^{\overline{2},1010} &= (-\frac{i}{\alpha_c} D^2 \overline{u_x^0} + \frac{\mathfrak{B}_c}{\alpha_c} \overline{Du_y^0})\phi_x^{1010}. \end{aligned}$$

The last computation is

$2B(\zeta_2, \Phi_{1001}) = e^{i(\alpha_c z+\beta_c y)}(B_x^{2,1001}, B_y^{2,1001}, B_z^{2,1001})^t + e^{i(\alpha_c z+\beta_c y)}(Dp_4, 0, i\alpha_c p_4)^t,$

with

$$\begin{aligned} B_x^{2,1001} &= -u_x^0 D\phi_x^{1001} - 3Du_x^0 \phi_x^{1001} - 2i\mathfrak{B}_c u_y^0 \phi_x^{1001} + i\alpha_c u_x^0 \phi_z^{1001} + T_c(1-\mu) u_y^0 \phi_y^{1001}, \\ B_y^{2,1001} &= -u_x^0 D\phi_y^{1001} - 2Du_x^0 \phi_y^{1001} - Du_y^0 \phi_x^{1001} - 2i\mathfrak{B}_c u_y^0 \phi_y^{1001} + i\alpha_c u_y^0 \phi_z^{1001}, \\ B_z^{2,1001} &= -D(u_x^0 \phi_z^{1001}) + \frac{i}{\alpha_c} D^2 u_x^0 \phi_x^{1001} - \frac{\mathfrak{B}_c}{\alpha_c} Du_y^0 \phi_x^{1001} - i\mathfrak{B}_c u_y^0 \phi_z^{1001}. \end{aligned}$$

4.4.3 Study of b and c

We finally give the explicit expressions of b and c. Using (38), (39) and (27), we obtain

$$b\langle\zeta_1,\zeta_1^*\rangle = \int_{-1/2}^{1/2}\Big[(B_x^{1,1100}+B_x^{\bar{1},2000})(\alpha_c^2-D^2)\overline{v_x^0}+(B_y^{1,1100}+B_y^{\bar{1},2000})\overline{v_y^0}\Big]\,dx, \tag{40}$$

$$\begin{aligned} c\langle\zeta_1,\zeta_1^*\rangle = \int_{-1/2}^{1/2}\Big[&(B_x^{1,0011}+B_x^{\bar{2},1010}+B_x^{2,1001})(\alpha_c^2-D^2)\overline{v_x^0} \\ &+(B_y^{1,0011}+B_y^{\bar{2},1010}+B_y^{2,1001})\overline{v_y^0}\Big]\,dx, \end{aligned} \tag{41}$$

with

$$\langle\zeta_1,\zeta_1^*\rangle = \int_{-1/2}^{1/2}\Big[Du_x^0D\overline{v_x^0}+\alpha_c^2u_x^0\overline{v_x^0}+u_y^0\overline{v_y^0}\Big]\,dx,$$

where $u_x^0, u_y^0, v_x^0, v_y^0$ are defined by (24) and (30).

The numerical evaluation of the coefficients b and c follows the algebraic procedure detailed in this appendix. The interval $x\in[-1/2,1/2]$ is discretised with N Chebyshev collocation points and the differential operators D_x and D_x^2 are replaced by their spectral differentiation matrices. The eigenvalue problem (24) is solved together with the boundary conditions to obtain the direct eigenvector (u_x^0,u_y^0). The adjoint eigenvector (v_x^0,v_y^0) is obtained from the discretised adjoint problem (30), and its normalisation is chosen so that the scalar product $\langle\zeta_1,\zeta_1^*\rangle$ equals unity. The linear systems that define the second-order harmonics $\Phi_{2000},\Phi_{1100},\Phi_{0011},\Phi_{1010},\Phi_{1001}$ are then assembled. Except for the Poisson problems for Φ_{1100} and Φ_{1001}, each system is of the same structure as (24) and is solved with the appropriate factor $(\alpha,\mathfrak{B}_c,\omega_0)$. The quadratic interaction terms are built from the discretised eigenfunctions and harmonics, projected onto the adjoint eigenvector, and integrated using Clenshaw-Curtis quadrature. The values of b and c are finally obtained from formulas (40). All computations are performed with extended precision to guarantee well-converged results.

For $\mu=-1$, we obtain

$$b=-0.3931+1.6743i, \qquad c=-0.3846-0.5753i.$$

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