Couette Taylor instabilities in the small-gap regime

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Abstract

The Couette-Taylor instability occurs in a viscous fluid confined between two coaxial rotating cylinders. When the Taylor number surpasses a critical value, the stable Couette flow destabilizes, giving way to steady Taylor vortices. As the Taylor number increases further, these vortices themselves become unstable, transitioning into wavy Taylor vortices.

In this article, we focus on the small-gap limit, where the ratio of the cylinder radii approaches unity and the rotation rates of the cylinders are nearly identical. We provide a rigorous proof of the existence of a critical Taylor number T_c, at which the Couette flow loses stability.

For Taylor numbers just above T_c —under fixed axial periodicity—the solutions to the limiting Navier-Stokes system are governed by a Ginzburg-Landau-type partial differential equation. Beyond the classical Taylor vortex flow, we demonstrate that a two-parameter family of solutions emerges at criticality for $T > T_c$. This family includes not only wavy vortices but also a variety of other exotic flow patterns, all of which remain steady in the frame rotating at the average angular velocity of the cylinders.

Contents

1 Introduction **2**

2 The Navier-Stokes equations in cylindrical coordinates **5**

3 Linear analysis in the axisymmetric case **6**
3.1 Study of the linearized operator $\mathcal{L}_{(\eta,\mu)}$ 7
3.2 Study of $\mathcal{L}_{1,\alpha}$ 8
3.3 A first bifurcation 10

4 Linear analysis in the non axisymmetric case **10**
4.1 Set up 11
4.2 Expansion of λ 12
4.3 Study of the critical Taylor number 15

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5 The Ginzburg Landau equation **16**
5.1 Limit of Navier-Stokes system 16
5.2 The amplitude equation 16
5.2.1 Mathematical justification of steady Ginzburg Landau equation 16
5.2.2 Time independent solutions of the Ginzburg Landau equation 18
5.3 Stability of bifurcating steady wavy vortices 19
5.4 Stationary solutions of the amplitude equation 20

6 Appendix A: Some computations **25**
6.1 Rescaled Navier-Stokes equations 25
6.2 Critical Taylor number in axisymmetric case 27
6.3 Coefficients of the Landau equation 30
6.4 Computation of the Ginzburg-Landau equation 35

7 Appendix B: Numerical study **38**
7.1 The axisymmetric case 38
7.2 The non axisymmetric case 38

1 Introduction

In this article, we study the classical problem of the flow of a viscous fluid between two rotating cylinders, motivated in particular by the works of M. Nagata [6], [7], [8]. More precisely, we consider the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0, \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

in the domain

$$\Omega = \Big\{ (r, \varphi, z) \quad | \quad r_i < r < r_o \Big\}.$$

In these equations, $u(t, x, y, z)$ denotes the velocity of the fluid, $p(t, x, y, z)$ the pressure, and $\nu > 0$ the viscosity. Moreover, (r, φ, z) are the cylindrical coordinates. The domain Ω is the interval between two cylinders with the same axis at $r = 0$, the inner one being of radius r_i and the outer one being of radius r_o.

The inner cylinder (respectively the outer) rotates with an angular velocity ω_i (respectively, ω_o). We assume that the fluid "sticks" to the boundary, namely that the velocity of the fluid equals the velocity of the cylinders at $r = r_i$ and $r = r_o$.

Let us define the gap d, the radii ratio η and the ratio of rotation rates μ by

$$d = r_o - r_i, \quad \eta = \frac{r_i}{r_o}, \quad \mu = \frac{\omega_o}{\omega_i}.$$

We work in a rotating frame, with a rotation rate Ω_{rf} defined by

$$\Omega_{rf} = \frac{\omega_i + \omega_o}{2} = \omega_i \frac{1 + \mu}{2}.$$

We consider the stability and bifurcations of the Couette flow, defined by

$$U(r) = r\,\Omega_{rot}(r) = (A - \Omega_{rf})r + \frac{B}{r}, \tag{3}$$

in this rotating frame, where A and B are constants such that the velocity of the fluid coincides with the velocity of the cylinders at the inner and outer cylinders

$$A = \omega_i \frac{\mu - \eta^2}{1 - \eta^2}, \quad B = \omega_i r_i^2 \frac{1 - \mu}{1 - \eta^2}.$$

We focus on the small gap / slow rotation / high Reynolds regime where

$$\eta \to 1, \qquad \mu \to 1, \qquad \hat{\omega} = \omega_i \frac{d^2}{\nu} \to 0, \qquad \mathfrak{R} = \frac{\hat{\omega}(1-\mu)}{1-\eta} \to \infty, \tag{4}$$

and we define the Taylor number to be

$$T = 2\hat{\omega}\mathfrak{R}.$$

We refer to section 2 for the adimensionalization of the Navier-Stokes equations.

We first consider axisymmetric perturbations and prove the existence of a critical Taylor number T_c.

Theorem 1.1. *For η and μ close enough to 1, there exists $T_c > 0$ such that the Couette flow (3) is linearly stable with respect to axisymmetric perturbations if $T < T_c$ and linearly unstable if $T > T_c$.*

Let α be the Fourier wave number associated with the axial coordinate z. Using the method of oscillating kernels, without taking our asymptotics, V. Yudovich [11] proved that for every $\alpha \neq 0$, there exists a critical Taylor number $T_c(\alpha) > 0$. This number defines the stability threshold for the Couette flow with respect to axisymmetric perturbations: the flow is stable for $T < T_c(\alpha)$ and unstable for $T > T_c(\alpha)$. Furthermore, V. Yudovich proved that the function $T_c(\alpha)$ is even, analytic, and goes to infinity as α approaches to 0 or to $\pm\infty$. Consequently, the critical Taylor number

$$T_c = \min_{\alpha \in \mathbb{R}} T_c(\alpha) > 0$$

is well defined. However, it remains an open question whether this minimum is achieved at only two critical wavenumbers, $\pm\alpha_c$, or if additional values of α may also yield the minimum.

Thanks to our triple limit $\eta \to 1$, $\mu \to 1$, $\hat{\omega} \to 0$, the analysis of linear stability is significantly simplified. In this regime, the theory of oscillating kernels is no longer required, and the problem reduces to determining whether a certain self-adjoint operator admits a positive eigenvalue. Specifically, it suffices to investigate whether 0 is an eigenvalue of a 6×6 system of ordinary differential equations, whose solutions can be computed explicitly. For a fixed

α, $T_c(\alpha)$ is characterized as the smallest positive root of an explicit hyperbolic-trigonometric function, ensuring its existence (see Lemma 3.2). Numerically, it is straightforward to tabulate $T_c(\alpha)$ as the zeros of this explicit function. Our calculations reveal that the global minimum $T_c \approx 1708$ is achieved at only two wavenumbers, $\alpha = \pm\alpha_c \approx \pm 3.117$, in agreement with classical results [2].

We then turn to non axisymmetric perturbations. Let β denote the azimuthal wave number. Then the detailed analysis of the eigenvalue λ_0 with the largest real part prove that λ_0 is real, and that for T close to T_c, α close to α_c, and $\beta\mathfrak{R}$ close to 0,

$$\lambda_0 = a_3(T - T_c) + a_4(\beta\mathfrak{R})^2 + a_6(\alpha^2 - \alpha_c^2)^2 + \cdots .$$

where $a_3 > 0$, $a_4 < 0$ (proved numerically) and $a_6 < 0$.

Theorem 1.2. *Assuming $a_4 < 0$ and $a_6 < 0$, then for fixed axial wavelength α, if $|\beta\mathfrak{R}|$ is small enough, then the set of eigenvalues of the linearized operator near the Couette flow is composed of a denumerable set of* real intervals. *The largest eigenvalue λ_0 is double. Moreover, for $T > T_c$, there is a full interval of quadruple real eigenvalues λ such that $0 < \lambda < \lambda_0$.*

The theorem follows from Lemmas 4.1 and 4.2. For a fixed $|\beta|$, eigenvalues are isolated and correspond to 4 eigenmodes of the form $e^{i(\pm\alpha z \pm \beta y)}$.

Remark 1.3. *It is a remarkable fact that all eigenvalues are real, despite the non-self-adjointness of the linearized operator. This is due to an additional symmetry with respect to the z axis that commutes with the system (see the proof of Lemma 4.1). This is specific to our limit case. Moreover, at criticality the eigenvalue 0 is not isolated, which prevents us from using classical reduction methods for bifurcation problems if we leave β free.*

As $\eta \to 1$, the azimuthal period tends to infinity, so the azimuthal angle φ may be treated as a real variable $y \in \mathbb{R}$ (see section 2 for the detailed computations). In this limit, the initial Navier-Stokes system turns into a new system (26) that we call the limiting Navier-Stokes system.

For $T > T_c$ with T close to T_c, the Couette flow is unstable with respect to perturbations with small azimuthal wave number β. For the limiting Navier–Stokes system (26), the dynamics of this nonlinear instability can be formally described by a Ginzburg-Landau equation, written on an amplitude $A(t, y)$, which satisfies

$$\frac{\partial A}{\partial t} = a_3(T - T_c)A - a_4\mathfrak{R}^2\frac{\partial^2 A}{\partial y^2} - cA|A|^2, \tag{5}$$

where c is a positive real number. We do not attempt to justify (5) here. Instead, we focus on the time independent Ginzburg-Landau equation

$$a_3(T - T_c)A - a_4\mathfrak{R}^2\frac{\partial^2 A}{\partial y^2} - cA|A|^2 = 0. \tag{6}$$

Using the methods of spatial dynamics, in particular the approach of [4], it can be proved that small-amplitude stationary solutions of the limiting Navier-Stokes system (26) are, at leading order, described by small solutions of (6). The argument in [4], developed for a nearby problem, carries over to our current problem with only minor modifications. To avoid reproducing a lengthy classical proof, we omit this justification and focus instead on the analysis of (6). The next Theorem describes the small-amplitude solutions of (6).

Theorem 1.4. *For $T > T_c$, assuming (following numerical results) that $a_4 < 0$ and $c > 0$, then in addition to the constant solutions, a one parameter family of solutions of (6) bifurcates supercritically. For these solutions, the amplitude A has a constant modulus $|A|$ (independent of y) and a phase that is linear in y.*

Moreover, a two-parameters family of exotic solutions of (6) bifurcates, with a modulus $|A|$ which oscillates periodically in y, and a phase that is the sum of an oscillating part and a linear part (see Figures 1 and 5).

In the theorem above, the constant solution corresponds to the steady axisymmetric Taylor vortex flow. The one-parameter family of solutions corresponds to three dimensional wavy vortices, which are steady in the rotating frame. The two-parameters family corresponds to exotic solutions of the limiting Navier-Stokes equations (26), whose amplitude oscillates periodically in y.

2 The Navier-Stokes equations in cylindrical coordinates

In this section, we adimensionalize the Navier-Stokes equations in cylindrical coordinates. In cylindrical coordinates, a (not necessarily axisymmetric) perturbation (u_r, u_φ, u_z, p) of the Couette flow (3) satisfies the system (see [7])

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t} - \nu(\Delta_{cyl} - \frac{1}{r^2})\Big)u_r + \nu\frac{2}{r^2}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial p}{\partial r} \\
&\qquad = -\Omega_{rot}\frac{\partial u_r}{\partial \varphi} + 2\Omega_{rot}u_\varphi + \frac{u_\varphi^2}{r} + 2\Omega_{rf}u_\varphi - (u\cdot\nabla)u_r, \\
&\Big(\frac{\partial}{\partial t} - \nu(\Delta_{cyl} - \frac{1}{r^2})\Big)u_\varphi - \nu\frac{2}{r^2}\frac{\partial u_r}{\partial \varphi} + \frac{1}{r}\frac{\partial p}{\partial \varphi} \\
&\qquad = -\Omega_{rot}\frac{\partial u_\varphi}{\partial \varphi} - U' u_r - \Omega_{rot}u_r - \frac{u_r u_\varphi}{r} - 2\Omega_{rf}u_r - (u\cdot\nabla)u_\varphi, \\
&\Big(\frac{\partial}{\partial t} - \nu\Delta_{cyl}\Big)u_z + \frac{\partial p}{\partial z} = -\Omega_{rot}\frac{\partial u_z}{\partial \varphi} - (u\cdot\nabla)u_z,
\end{aligned}$$

together with the incompressibility condition, which in cylindrical coordinates reads

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} = 0.$$

Here the Coriolis term is added on the right-hand side, and Δ_{cyl} is the Laplace operator in cylindrical coordinates, namely

$$\Delta_{cyl} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

We perform the change of variables

$$\hat{x} = \frac{r-\bar{r}}{d}, \quad \hat{y} = \frac{\bar{r}}{d}\varphi, \quad \hat{z} = \frac{z}{d}, \quad \hat{t} = \frac{\nu}{d^2}t,$$

where $\bar{r}$ is the average radius

$$\bar{r} = \frac{r_o + r_i}{2} = \frac{1+\eta}{2} r_o.$$

We rescale the velocity and the pressure by

$$u(t, r, \varphi, z) = \frac{\nu}{d}\hat{u}(\hat{t}, \hat{x}, \hat{y}, \hat{z}),$$
$$p(t, r, \varphi, z) = \frac{\nu^2}{d^2}\hat{p}(\hat{t}, \hat{x}, \hat{y}, \hat{z}).$$

Using the identities obtained in Appendix 6.1, we have

$$\begin{aligned}
\Big(\frac{\partial}{\partial \hat{t}} - \hat{\Delta}\Big)\hat{u}_{\hat{x}} + \frac{\partial \hat{p}}{\partial \hat{x}} &= \mathfrak{R}\hat{x}\frac{\partial \hat{u}_{\hat{x}}}{\partial \hat{y}} + \hat{\omega}(1+\mu)\hat{u}_{\hat{y}} - (\hat{u}\cdot\nabla)\hat{u}_{\hat{x}} + \mathcal{O}\Big(\hat{\omega}(1-\mu)\frac{\partial \hat{u}_{\hat{x}}}{\partial \hat{y}}\Big) \\
&\quad + \mathcal{O}(1-\eta)^2\hat{u}_{\hat{x}} - 2\Big(\hat{\omega}(1-\mu)\hat{x} + \mathcal{O}(1-\eta)\Big)\hat{u}_{\hat{y}} + \mathcal{O}(1-\eta)\frac{\partial \hat{u}_{\hat{y}}}{\partial \hat{y}} \\
&\quad + \Big((1-\eta)^2 x + \frac{(1-\eta)(1+\eta)}{2}\Big)\hat{u}_{\hat{y}}^2 \\
\Big(\frac{\partial}{\partial \hat{t}} - \hat{\Delta}\Big)\hat{u}_{\hat{y}} + \frac{\partial \hat{p}}{\partial \hat{y}} &= \mathfrak{R}\hat{x}\frac{\partial \hat{u}_{\hat{y}}}{\partial \hat{y}} + \hat{\omega}\frac{(1-\mu)(1+\eta^2)}{1-\eta^2}\hat{u}_{\hat{x}} - \hat{\omega}(1+\mu)\hat{u}_{\hat{x}} - (\hat{u}\cdot\nabla)\hat{u}_{\hat{y}} \\
&\quad + \mathcal{O}(1-\eta)\Big(\hat{u}_{\hat{x}}\hat{u}_{\hat{y}} + \frac{\partial \hat{u}_{\hat{x}}}{\partial \hat{y}}\Big) + \mathcal{O}\Big(\hat{\omega}(1-\mu)\frac{\partial \hat{u}_{\hat{y}}}{\partial \hat{y}}\Big) + \mathcal{O}(1-\eta)^2\hat{u}_{\hat{y}}, \\
\Big(\frac{\partial}{\partial \hat{t}} - \hat{\Delta}\Big)\hat{u}_{\hat{z}} + \frac{\partial \hat{p}}{\partial \hat{z}} &= \mathfrak{R}\hat{x}\frac{\partial \hat{u}_{\hat{z}}}{\partial \hat{y}} - (\hat{u}\cdot\nabla)\hat{u}_{\hat{z}} + \mathcal{O}\Big(\hat{\omega}(1-\mu)\frac{\partial \hat{u}_{\hat{z}}}{\partial \hat{y}}\Big), \\
\frac{\partial \hat{u}_{\hat{x}}}{\partial \hat{x}} + \frac{\partial \hat{u}_{\hat{y}}}{\partial \hat{y}} + \frac{\partial \hat{u}_{\hat{z}}}{\partial \hat{z}} &= \mathcal{O}(1-\eta)\Big(\hat{u}_{\hat{x}} + \frac{\partial \hat{u}_{\hat{y}}}{\partial \hat{y}}\Big).
\end{aligned}$$

In this system, $\mathcal{O}(\cdot)$ denotes various smooth functions that go to zero in the limit (4). We drop the hats and the linearized system takes the form

$$\partial_t u = \mathcal{L}_{(\eta,\mu)} u. \tag{7}$$

3 Linear analysis in the axisymmetric case

We first focus on radial, namely axisymmetric instabilities, which are thus independent of y.

3.1 Study of the linearized operator $\mathcal{L}_{(\eta,\mu)}$

For y independent functions, the linearized axisymmetric Navier-Stokes equations (7) read

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)u_x+\frac{\partial p}{\partial x}=\omega(1+\mu)u_y-2\Big(\omega(1-\mu)(x+\mathcal{O}(1-\eta))\Big)u_y+\mathcal{O}(1-\eta)^2u_x,$$

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)u_y=\Big[\omega\frac{(1-\mu)(1+\eta^2)}{1-\eta^2}-\omega(1+\mu)\Big]u_x+\mathcal{O}(1-\eta)^2u_y,$$

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)u_z+\frac{\partial p}{\partial z}=0,$$

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}=\mathcal{O}(1-\eta)u_x.$$

Let us take the Fourier transform in the z variable and denote by α the Fourier variable. We moreover take the Laplace transform in time and denote by λ the Laplace variable. Rescaling u_y with

$$u_y=\mathfrak{R}\widehat{u}_y,$$

and assuming that $\omega\mathfrak{R}$ is bounded, this leads to the system (where $D=\partial/\partial x$)

$$\begin{aligned}
(\lambda+\alpha^2-D^2)u_x+Dp-T\widehat{u}_y &= \mathcal{O}(1-\mu)\widehat{u}_y+\mathcal{O}(1-\eta)^2u_x,\\
(\lambda+\alpha^2-D^2)\widehat{u}_y-u_x &= \mathcal{O}\Big(\frac{\omega}{\mathfrak{R}}\Big)u_x+\mathcal{O}(1-\eta)^2\widehat{u}_y+\mathcal{O}(1-\eta)u_x,\\
(\lambda+\alpha^2-D^2)u_z+i\alpha p &= 0,\\
Du_x+i\alpha u_z &= \mathcal{O}(1-\eta)u_x.
\end{aligned}$$

Now, we multiply the first equation by α^2 and add $i\alpha D$ times the third one, which gives

$$(\lambda+\alpha^2-D^2)(\alpha^2u_x+i\alpha Du_z)-\alpha^2T\widehat{u}_y=\mathcal{O}(1-\mu)\widehat{u}_y+\mathcal{O}(1-\eta)^2u_x.$$

We replace $i\alpha Du_z$ by its expression from the fourth equation, and obtain

$$\begin{aligned}
(\lambda+\alpha^2-D^2)[(\alpha^2-D^2)u_x+\mathcal{O}(1-\eta)Du_x]-\alpha^2T\widehat{u}_y&=\mathcal{O}(1-\mu)\widehat{u}_y+\mathcal{O}(1-\eta)^2u_x,\\
(\lambda+\alpha^2-D^2)\widehat{u}_y-u_x&=\mathcal{O}(\frac{\omega}{\mathfrak{R}})u_x+\mathcal{O}(1-\eta)^2\widehat{u}_y.
\end{aligned}$$

We define now

$$\Gamma_x=\big[(\alpha^2-D^2)+\mathcal{O}(1-\eta)D\big]\,u_x,$$

then the system above becomes

$$(\lambda+\alpha^2-D^2)\Gamma_x-\alpha^2T\widehat{u}_y=\mathcal{O}(1-\mu)\widehat{u}_y+\mathcal{O}(1-\eta)^2\left[(\alpha^2-D^2)+\mathcal{O}(1-\eta)D\right]^{-1}\Gamma_x,$$

$$(\lambda+\alpha^2-D^2)\widehat{u}_y-\left[(\alpha^2-D^2)+\mathcal{O}(1-\eta)D\right]^{-1}\Gamma_x=\mathcal{O}(1-\eta)^2\widehat{u}_y+\mathcal{O}\Big(\frac{\omega}{\mathfrak{R}}\Big)u_x.$$

Since the operator $\left[(\alpha^2 - D^2) + \mathcal{O}(1-\eta)D\right]^{-1}$ is bounded, taking into account of the boundary conditions $u_x(\pm 1/2) = 0$ on u_x, it is clear that, for $(1-\eta)$, $(1-\mu)$ and $\omega/\Re$ close to 0, adding the boundary conditions on u_y and the complementary ones on u_x the operator on the right hand side is a small relatively compact perturbation of the operator acting on the left side on $(\Gamma_x, \widehat{u}_y)$.

This implies that the eigenvalues λ are obtained via the simple perturbation theory from the eigenvalues of the linear operator obtained for $\eta = 1, \mu = 1$, i.e.

$$\begin{aligned} (\lambda + \alpha^2 - D^2)(\alpha^2 - D^2)u_x - \alpha^2 T\widehat{u}_y &= 0, \\ (\lambda + \alpha^2 - D^2)\widehat{u}_y - u_x &= 0. \end{aligned} \tag{8}$$

3.2 Study of $\mathcal{L}_{1,\alpha}$

For $(\eta, \mu) = (1, 1)$ we consider the eigenvalue problem

$$\begin{aligned} \lambda u_x &= (D^2 - \alpha^2)u_x - Dp + T\widehat{u}_y \\ \lambda \widehat{u}_y &= (D^2 - \alpha^2)\widehat{u}_y + u_x \\ \lambda u_z &= (D^2 - \alpha^2)u_z - i\alpha p \\ 0 &= Du_x + i\alpha u_z, \end{aligned} \tag{9}$$

with the boundary conditions $u_x = \widehat{u}_y = u_z = 0$ in $x = \pm 1/2$. We denote the right-hand side of $(9)_{1,2,3}$ to be the operator $\mathcal{L}_{1,\alpha}$ acting on $u = (u_x, \widehat{u}_y, u_z)$. As seen above, this leads to the 6th-order differential equation

$$(\lambda + \alpha^2 - D^2)^2(\alpha^2 - D^2)u_x - \alpha^2 T u_x = 0, \tag{10}$$

with the boundary conditions

$$u_x = Du_x = \widehat{u}_y = 0, x = \pm 1/2.$$

Then we show the following

Lemma 3.1. *Let $\alpha > 0$. The spectrum of $\mathcal{L}_{1,\alpha}$ is only composed of real isolated (at least double) eigenvalues*

$$... < \lambda_n < .... < \lambda_2 < \lambda_1 < \lambda_0$$

where $\lambda_0 < 0$ for $T \leq \alpha^4$, and $\lambda_n \to -\infty$ as $n \to \infty$.

Proof. The linear operator $\mathcal{L}_{1,\alpha}$ is symmetric with the following suitable scalar product in $[L^2(-1/2, 1/2)]^3$:

$$\langle u, v\rangle = \int_{-1/2}^{1/2} u_x \overline{v_x} + T\hat{u}_y \overline{\hat{v}_y} + u_z \overline{v_z}\, dx.$$

Indeed we obtain, using the divergence free condition on u and v, and integration by parts

$$\langle \mathcal{L}_{1,\alpha} u, v\rangle = \langle u, \mathcal{L}_{1,\alpha} v\rangle.$$

Since it is known that the Stokes operator on

$$\left\{L^2[(-1/2,1/2)\times\mathbb{R}/(2\pi/\alpha)\mathbb{Z}]\right\}^2$$

with the above boundary conditions is self-adjoint, negative, and idem for the operator acting on u_y, we deduce that the operator $\mathcal{L}_{1,\alpha}$ is self-adjoint, hence its spectrum is real. Moreover, as for the Stokes operator, it has a compact resolvent, hence all its eigenvalues are real, discrete, with only an accumulation at $-\infty$. They all are at least double because of the symmetry $\pm\alpha$.

Moreover, the largest eigenvalue λ_0 is negative if $T \leq \alpha^4$ as can be seen by studying $\langle \mathcal{L}_{1,\alpha} u, u\rangle$. □

We now study when 0 is an eigenvalue of $\mathcal{L}_{1,\alpha}$.

Lemma 3.2. *Let $\alpha > 0$,*

$$\sigma = (T\alpha^2)^{1/3},$$

and let

$$\begin{aligned} b_1^2 &= \sigma - \alpha^2 > 0 \\ a_2 &= \frac{1}{\sqrt{2}}\sqrt{\alpha^2 + \frac{\sigma}{2} + \sqrt{\alpha^4 + \alpha^2\sigma + \sigma^2}}, \\ b_2 &= -\frac{1}{\sqrt{2}}\sqrt{\sqrt{\alpha^4 + \alpha^2\sigma + \sigma^2} - \alpha^2 - \frac{\sigma}{2}}. \end{aligned} \tag{11}$$

Then $\lambda = 0$ is an eigenvalue of $\mathcal{L}_{1,\alpha}$ if and only if $\mathfrak{F}_1 = 0$ or $\mathfrak{F}_2 = 0$ where

$$\mathfrak{F}_1 := -b_1 \tan\frac{b_1}{2}(\cosh a_2 + \cos b_2) + (\sqrt{3}b_2 - a_2)\sinh a_2 + (\sqrt{3}a_2 + b_2)\sin b_2 \tag{12}$$

and

$$\mathfrak{F}_2 := b_1(\cosh a_2 - \cos b_2) + \tan\frac{b_1}{2}[(\sqrt{3}b_2 - a_2)\sinh a_2 - (\sqrt{3}a_2 + b_2)\sin b_2]. \tag{13}$$

The solutions of $\mathfrak{F}_1 = 0$ give components u_x, u_y even in x, and u_z odd in x. The solutions of $\mathfrak{F}_2 = 0$ give components u_x, u_y odd in x and u_z even in x.

The proof of this Lemma is in Appendix 6.2. We now study the zeroes of $\mathfrak{F}_1$ and $\mathfrak{F}_2$.

Lemma 3.3. *Let $\alpha > 0$. Then $\mathfrak{F}_1(\alpha, T) = 0$ and $\mathfrak{F}_2(\alpha, T) = 0$ have infinitely many positive solutions T.*

Proof. Note that, if we define $\tilde{\sigma} = \sigma/\alpha^2$, a_2 may be rewritten

$$a_2 = \frac{|\alpha|}{\sqrt{2}}\sqrt{1 + \frac{\tilde{\sigma}}{2} + \sqrt{1 + \tilde{\sigma} + \tilde{\sigma}^2}},$$

and similarly for b_2. Moreover, we have for large $\tilde{\sigma}$

$$b_1 \sim \sqrt{\sigma}, \quad a_2 \sim \frac{\sqrt{3\sigma}}{2}, \quad b_2 \sim -\frac{\sqrt{\sigma}}{2}.$$

The end of the proof is then straightforward from expressions (12) and (13). □

Definition 3.4. *For $\alpha > 0$, we define $T_c(\alpha)$ to be the smallest positive zero of $\mathfrak{F}_1$ and $\mathfrak{F}_2$.*

Remark 3.5. *It is very easy to compute numerically $T_c(\alpha)$. Numerically, we find that $\mathfrak{F}_1 = 0$ gives $T_c(\alpha)$ minimum for $\alpha = \alpha_c \approx 3.117$ and $T_c(\alpha_c) \approx 1708$ (see Appendix 7.2), which are the classical values [2]. The equation $\mathfrak{F}_2 = 0$ gives a larger value for T_c. This is analogous to what happens (see [9]) in the convection problem, where the components u_x, u_y of the eigenvector which are even in x are more unstable that the odd ones.*

3.3 A first bifurcation

Following the numerical results, from now on we assume that there is only one couple $\pm\alpha_c$ such that $T_c(\alpha)$ reaches its minimum.

When $T < T_c$, all the eigenvalues of $\mathcal{L}_{1,\alpha}$ are real and negative. When $T = T_c$, one eigenvalue is 0. For $T > T_c$, close to T_c, one eigenvalue is positive and small. As a consequence, taking into account of the $O(2)$ symmetry, we have a pitchfork bifurcation (see [1]) at T_c.

The eigenvalues and eigenvectors of the full system are perturbations of the eigenvalues and eigenvectors of $\mathcal{L}_{1,\alpha}$. They are of the form

$$\zeta(x,z) = e^{\pm i\alpha z} \begin{pmatrix} O(\mathfrak{R}^{-1}) \\ O(1) \\ O(\mathfrak{R}^{-1}) \end{pmatrix}.$$

Let $\alpha = \alpha_c$. The eigenmodes corresponding to 0 are of the form

$$\zeta(x,z) = e^{i\alpha_c z}\widehat{v}(x)$$

with its complex conjugate for $-\alpha_c$. The bifurcating flow is called the "Taylor vortex flow" (TVF) and has the form (we choose a solution invariant under the symmetry $z \to -z$):

$$A(\zeta + \overline{\zeta}) + \mathcal{O}(|A|^2),\ A \in \mathbb{R}.$$

The Landau equation of A is described in Appendix 6.3 (see in particular formula (39)). We have a line of solutions in translating the z coordinate (changing the phase of amplitude A). On this line there is another symmetric solution obtained by shifting z to $z + \pi/\alpha_c$.

4 Linear analysis in the non axisymmetric case

We now study general, non axisymmetric perturbations.

4.1 Set up

At the leading order of the linearized system (7), we obtain the following system

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big)u_x + \frac{\partial p}{\partial x} &= \mathfrak{R}x\frac{\partial u_x}{\partial y} + 2\omega u_y,\\
\Big(\frac{\partial}{\partial t} - \Delta\Big)u_y + \frac{\partial p}{\partial y} &= \mathfrak{R}x\frac{\partial u_y}{\partial y} + \mathfrak{R}u_x,\\
\Big(\frac{\partial}{\partial t} - \Delta\Big)u_z + \frac{\partial p}{\partial z} &= \mathfrak{R}x\frac{\partial u_z}{\partial y},\\
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} &= 0.
\end{aligned}$$

We take the Fourier transform in z with dual variable α, the Fourier transform in y, with dual variable β, and the Laplace transform in time, with dual variable λ. This leads to

$$\begin{aligned}
(\lambda - \partial_x^2 + \alpha^2 + \beta^2)u_x + \frac{\partial p}{\partial x} &= i\beta\mathfrak{R}xu_x + 2\omega u_y,\\
(\lambda - \partial_x^2 + \alpha^2 + \beta^2)u_y + i\beta p &= i\beta\mathfrak{R}xu_y + \mathfrak{R}u_x,\\
(\lambda - \partial_x^2 + \alpha^2 + \beta^2)u_z + i\alpha p &= i\beta\mathfrak{R}xu_z,\\
\frac{\partial u}{\partial x} + i\beta u_y + i\alpha u_z &= 0.
\end{aligned}$$

It is thus natural to assume that

$$\mathfrak{B} = \beta\mathfrak{R}$$

remains constant as $\mathfrak{R}$ goes to infinity. As a consequence, β goes to 0 as $\mathfrak{R}$ goes to infinity. The system thus simplifies into

$$\begin{aligned}
(\lambda + \alpha^2 - D^2 - i\mathfrak{B}x)u_x + Dp &= 2\omega u_y,\\
(\lambda + \alpha^2 - D^2 - i\mathfrak{B}x)u_y + i\beta p &= \mathfrak{R}u_x,\\
(\lambda + \alpha^2 - D^2 - i\mathfrak{B}x)u_z + i\alpha p &= 0,\\
Du_x + i\beta u_y + i\alpha u_z &= 0,
\end{aligned} \tag{14}$$

where $D = \partial_x$. Applying $(\lambda + \alpha^2 - D^2 - i\mathfrak{B}x)$ to $(14)_4$ and using $(14)_{2,3}$ leads to

$$(\lambda + \alpha^2 - D^2 - i\mathfrak{B}x)Du_x + (\alpha^2 + \beta^2)p = -i\mathfrak{B}u_x. \tag{15}$$

Hence, β^2 should be neglected with respect to α^2. Applying $-D$ to (15) and adding α^2 times $(14)_1$, we have

$$(\lambda + \alpha^2 - D^2 - i\mathfrak{B}x)(\alpha^2 - D^2)u_x = 2\omega\alpha^2 u_y.$$

Finally, rescaling u_y in

$$u_y = \mathfrak{R}\widehat{u}_y,$$

it is now clear that $i\beta p$ should be neglected in the second line of (14). We obtain the system

$$\begin{aligned}(\lambda+\alpha^2-D^2-i\mathfrak{B}x)(\alpha^2-D^2)u_x &= T\alpha^2\widehat{u}_y \quad (16)\\ (\lambda+\alpha^2-D^2-i\mathfrak{B}x)\widehat{u}_y &= u_x, \quad (17)\end{aligned}$$

where the parameters are $\alpha^2, T, \mathfrak{B}$, and where the boundary conditions are

$$u_x = \widehat{u}_y = Du_x = 0, \quad x = \pm 1/2.$$

In the sequel, we will drop the hat on u_y.

4.2 Expansion of λ

In the previous section we have rescaled u_y in $u_y = \mathfrak{R}\widehat{u}_y$, and have neglected β^2 with respect to α^2 and βp in the equation on u_y. The linear system which we study (to be compared with (7)) is now

$$\begin{aligned}(\partial_t-\Delta_\perp)u_\perp+\nabla_\perp p &= \mathfrak{R}x\partial_y u_\perp + (T\widehat{u}_y,0)^t, \quad (18)\\ (\partial_t-\Delta_\perp)\widehat{u}_y &= \mathfrak{R}x\partial_y\widehat{u}_y + u_x,\\ \nabla_\perp\cdot u_\perp+\mathfrak{R}\partial_y\widehat{u}_y &= 0,\end{aligned}$$

with $2\pi/\alpha$ periodicity in z and boundary conditions

$$u_\perp = \widehat{u}_y = 0, x = \pm 1/2,$$

and where the index $\perp$ means (x,z) components. Looking for solutions of the form

$$u = e^{i(\alpha z+\beta y)+\lambda t}(u_\perp, \widehat{u}_y)^t,$$

where $u_\perp, \widehat{u}_y$ are functions of x, we obtain the system (16,17) in setting $\mathfrak{B} = \beta\mathfrak{R}$, $T = 2\omega\mathfrak{R}$.

We know that $\lambda = 0$ is the largest eigenvalue when $\mathfrak{B} = 0$, and $T = T_c(\alpha, 0)$. Moreover, $T_c(\alpha, 0)$ has a strict minimum at α_c (see figure 6 for numerical evidence), i.e.

$$\frac{dT_c}{d\alpha}(\alpha_c, 0) = 0, \qquad \frac{d^2T_c}{d\alpha^2}(\alpha_c, 0) > 0.$$

We note that, for a given $(\alpha, \mathfrak{B})$, the eigenvalue λ gives a unique eigenvector $U = (u_x, u_y)(x)$, even though λ is a double eigenvalue when we take into account of the symmetry $z \to -z$ and of the third component u_z. The analyticity of the linear operator in $(\alpha, \mathfrak{B})$ leads to the same analyticity for λ.

We first prove that the eigenvalues λ are real.

Lemma 4.1. *The eigenvalues λ are real and even functions of $(\alpha, \mathfrak{B})$ for $(\alpha, \mathcal{B})$ close to $(\alpha_c, 0)$. The largest eigenvalue λ_0 is then analytic in (α, β).*

Proof. Thanks to the form of system (16,17) we have the following symmetry properties for $\lambda(\alpha, \mathfrak{B})$:

i) changing $\mathfrak{B}$ in $-\mathfrak{B}$ changes λ in $\overline{\lambda}$, and for $\mathfrak{B}=0$, λ is real.

ii) Changing $(x,\mathfrak{B},u_x(x),u_y(x))$ into $(-x,-\mathfrak{B},u_x(-x),u_y(-x))$ shows that the same λ is valid for the eigenvector $(u_x(-x),u_y(-x))$ and $\mathcal{B}$ changed in $-\mathcal{B}$. Hence

$$\lambda(\alpha,\mathfrak{B})=\lambda(\alpha,-\mathfrak{B})=\overline{\lambda}(\alpha,\mathfrak{B}).$$

The evenness property in α is obvious. We already know that for $(\alpha,\mathcal{B})=(\alpha_c,0)$ the largest eigenvalue is simple if one considers the eigenmode $e^{i\alpha_c z}$ (double if we consider $\pm\alpha_c$). For a given couple (α,β), we need to consider real eigenvalue(s) perturbing a simple one. By perturbation theory (see [5]) there is only one perturbed eigenvalue for a mode (α,β) close to $(\alpha_c,0)$, which is still simple. So that eigenmodes in $e^{i\alpha z+i\beta y}$ are one-dimensional, moreover the corresponding eigenvalue is analytic function in $(\alpha,\mathcal{B})$ because of the analyticity of the linear operator. The largest eigenvalue is then in fact quadruple (semi-simple) because of the symmetries, with eigenmodes in $e^{i(\pm\alpha z\pm\beta y)}$. □

Let us give the Taylor expansion of the largest eigenvalue λ_0 in the neighborhood of 0, for $(\alpha,T,\mathfrak{B})$ close to $(\alpha_c,T_c,0)$. From the previous Lemma, we have

$$\begin{aligned}\lambda_0 = {} & a_2(\alpha^2-\alpha_c^2)+a_3\tau+a_4\mathfrak{B}^2+a_6(\alpha^2-\alpha_c^2)^2+a_7\mathfrak{B}^2(\alpha^2-\alpha_c^2)\\ & +a_9\tau(\alpha^2-\alpha_c^2)+a_{10}\tau^2+\cdots,\end{aligned} \tag{19}$$

where all coefficients a_j are real and

$$\tau := T-T_c.$$

Note that $\alpha^2-\alpha_c^2$ is at leading order proportional to $\alpha-\alpha_c$ since

$$\alpha^2-\alpha_c^2=(\alpha-\alpha_c)(\alpha+\alpha_c)=2\alpha_c(\alpha-\alpha_c)+\mathcal{O}\Big((\alpha-\alpha_c)^2\Big).$$

Lemma 4.2. *We have $a_2=0$, $a_3>0$, and we obtain numerically that $a_4<0$, $a_6<0$.*

Proof. Since $\tau=0$ corresponds to the strict minimum of T at $\alpha=\alpha_c$ when $\mathfrak{B}=0$, we have

$$\frac{\partial\tau}{\partial\alpha}(\alpha_c,0)=0, \tag{20}$$

$$\frac{\partial^2\tau}{\partial\alpha^2}(\alpha_c,0)>0. \tag{21}$$

Fixing $\lambda_0=\mathfrak{B}=0$ in (19), taking derivative with respect to α and evaluating at $\alpha=\alpha_c$, (20) implies $a_2=0$.

Next we prove that the coefficient a_3 is positive. Let us denote by L_0 the linear operator defined as

$$L_0U=\begin{pmatrix}-(\alpha_c^2-D^2)^2u_x+T_c\alpha_c^2u_y\\ (D^2-\alpha_c^2)u_y+u_x\end{pmatrix},$$

where

$$U = (u_x, u_y)^t, u_x = Du_x = u_y = 0, x = \pm 1/2.$$

Now the eigenvector of L_0 for the eigenvalue 0 is

$$\zeta = (u_x^0, u_y^0)^t,$$

so that

$$L_0\zeta = 0.$$

Moreover, it is seen in Lemma 3.2 (see also Remark 3.5) that u_x^0, u_y^0 are even in x. The system (16), (17) may be written as

$$\begin{aligned} \lambda_0(H_0 + \widehat{\alpha}H_{010})U &= L_0U + \mathfrak{B}L_{100}U + \widehat{\alpha}L_{010}U + \tau L_{001}U + \\ &\quad + \mathfrak{B}\widehat{\alpha}L_{110}U + \widehat{\alpha}\tau L_{011}U + \widehat{\alpha}^2 L_{020}U, \end{aligned} \tag{22}$$

with $\widehat{\alpha} = \alpha^2 - \alpha_c^2$ and

$$H_0U = \begin{pmatrix} (\alpha_c^2 - D^2)u_x \\ u_y \end{pmatrix}, H_{010}U = \begin{pmatrix} u_x \\ 0 \end{pmatrix},$$

$$L_{100}U = \begin{pmatrix} ix(\alpha_c^2 - D^2)u_x \\ ixu_y \end{pmatrix}, L_{010}U = \begin{pmatrix} -2(\alpha_c^2 - D^2)u_x + T_cu_y \\ -u_y \end{pmatrix},$$

$$L_{001}U = \begin{pmatrix} \alpha_c^2 u_y \\ 0 \end{pmatrix}, L_{110}U = \begin{pmatrix} ixu_x \\ 0 \end{pmatrix}, L_{011}U = \begin{pmatrix} u_y \\ 0 \end{pmatrix}, L_{020} = \begin{pmatrix} -u_x \\ 0 \end{pmatrix}.$$

As for the proof of selfadjointness of $\mathcal{L}_{1,\alpha}$, we may prove that L_0 is selfadjoint in $[L^2(-1/2, 1/2)]^2$ with the scalar product

$$\langle U, V\rangle = \int_{-1/2}^{1/2} (u_x\overline{v_x} + \alpha_c^2 T_c u_y \overline{v_y})dx,$$

so that

$$\langle L_0U, V\rangle = \langle U, L_0V\rangle, \text{ for any } U, V \in dom(L_0).$$

It results that ζ is orthogonal to the range of L_0. In the following computation, we use

$$\langle H_0U, U\rangle = \int_{-1/2}^{1/2} (|Du_x|^2 + \alpha_c^2|u_x|^2 + \alpha_c^2 T_c|u_y|^2)dx > 0$$

for $U \neq 0$. We also expand the eigenvector U as

$$U = \zeta + \mathfrak{B}U^{(100)} + \widehat{\alpha}U^{(010)} + \tau U^{(001)} + \mathfrak{B}\widehat{\alpha}U^{(110)} + \widehat{\alpha}\tau U^{(011)} + \widehat{\alpha}^2 U^{(020)} + \mathfrak{B}^2 U^{(200)} + ...$$

and identify the coefficients of $\widehat{\alpha}, \tau, \mathfrak{B}, \mathfrak{B}^2$ in (22) where λ is expanded in (19):

$$\begin{aligned} a_2H_0\zeta &= L_0U^{(010)} + L_{010}\zeta, \\ a_3H_0\zeta &= L_0U^{(001)} + L_{001}\zeta, \\ 0 &= L_0U^{(100)} + L_{100}\zeta, \\ a_4H_0\zeta &= L_0U^{(200)} + L_{100}U^{(100)}. \end{aligned}$$

By construction of α_c, T_c we have $a_2 = 0$, hence

$$\langle L_{010}\zeta, \zeta\rangle = 0,$$

i.e.

$$-\int_{-1/2}^{1/2}(2|Du_x^0|^2 + 2\alpha_c^2|u_x^0|^2 + \alpha_c^2 T_c|u_y^0|^2)dx + \int_{-1/2}^{1/2} T_c u_y^0 u_x^0 dx = 0 \qquad (23)$$

We also obtain

$$a_3\langle H_0\zeta, \zeta\rangle = \langle L_{001}\zeta, \zeta\rangle = \alpha_c^2\int_{-1/2}^{1/2} u_y^0 u_x^0 dx$$

hence, from (23) and $\langle H_0\zeta, \zeta\rangle > 0$, we deduce that

$$a_3 > 0.$$

The equation for $U^{(100)}$ is valid since $L_{100}\zeta$ has odd components in x, hence orthogonal to ζ which has even components. Now we obtain

$$a_4\langle H_0\zeta, \zeta\rangle = \langle L_{100}U^{(100)}, \zeta\rangle$$

which needs to be computed numerically, at least to check that $a_4 < 0$.

Fixing $\lambda_0 = \mathfrak{B} = 0$ in (19) again, and taking second derivative with respect to α and evaluating at $\alpha = \alpha_c$, we have

$$0 = a_3\frac{\partial^2\tau}{\partial\alpha^2}(\alpha_c, 0) + 8\alpha_c^2 a_6.$$

Since $a_3 > 0$ and τ has a strict minimum in α_c, this implies (numerically) $a_6 < 0$ by (21).

Finally, $a_4 < 0$ follows from the numerical evidence (see Appendix 7.2) that for $\lambda_0 = 0$ and fixed α, the critical T is an increasing convex function with respect to $\mathfrak{B} > 0$.

In conclusion, the principal part of λ becomes

$$\lambda_0 = a_3\tau + a_4\mathfrak{B}^2 + a_6(\alpha^2 - \alpha_c^2)^2 + a_7\mathfrak{B}^2(\alpha^2 - \alpha_c^2) + a_9\tau(\alpha^2 - \alpha_c^2) + a_{10}\tau^2 \quad (24)$$

where a_j are real. □

4.3 Study of the critical Taylor number

Let us now study how T_c depends on α and $\mathfrak{B}$. Let

$$b_4 = -a_4 > 0, \qquad b_6 = -a_6 > 0$$

where the strict signs of a_4 and a_6 come from numerical computation (see Appendices 7.1, and 7.2). From the expression (24) of λ, we deduce that the critical value of T for $\alpha \neq \alpha_c$ is approximately reached for

$$\alpha^2 - \alpha_c^2 \approx \Big[\frac{a_7}{2b_6} + \frac{a_9 b_4}{2b_6 a_3}\Big]\mathfrak{B}^2 \qquad (25)$$

and

$$T \approx T_c + \frac{b_4}{a_3}\mathfrak{B}^2 - \frac{1}{4b_6 a_3^3}\Big[(a_3 a_7 + a_9 b_4)^2 + 4a_{10}b_6 a_4^2\Big]\mathfrak{B}^4 > T_c.$$

5 The Ginzburg Landau equation

5.1 Limit of Navier-Stokes system

From now on we are dealing with the limit system

$$\begin{aligned}(\partial_t - \Delta_\perp)u_\perp + \nabla_\perp p &= \mathfrak{R}x\partial_y u_\perp + (T\widehat{u}_y, 0)^t - (u_\perp \cdot \nabla_\perp)u_\perp, \\ (\partial_t - \Delta_\perp)\widehat{u}_y &= \mathfrak{R}x\partial_y \widehat{u}_y + u_x - (u_\perp \cdot \nabla_\perp)\widehat{u}_y, \\ \nabla_\perp \cdot u_\perp + \mathfrak{R}\partial_y \widehat{u}_y &= 0,\end{aligned} \tag{26}$$

with $2\pi/\alpha_c$ periodicity in z and boundary conditions

$$u_\perp = \widehat{u}_y = 0, x = \pm 1/2.$$

Here the index $\perp$ means (x, z) components. The system (26) is derived from the system in section 2 by rescaling of u_y and dropping the small terms such as $\mathfrak{R}^{-1}\partial_y p$, as well as terms with factors of order $(1-\eta)$ and $(1-\mu)$. Moreover, since we are looking for slowly varying functions of y, we also suppress ∂_y^2 in linear terms. Finally, because we seek solutions close to 0, we suppress quadratic terms involving the operator $\mathfrak{R}\partial_y$, while keeping such terms in the linear part. The linear part of (26) is studied in subsection 4.2.

5.2 The amplitude equation

Heuristically, the solutions of the limit Navier-Stokes equations (26) can be described by the solutions of the following Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = a_3 \tau A + b_4 \mathfrak{R}^2 \frac{\partial^2 A}{\partial y^2} - cA|A|^2, \tag{27}$$

where the coefficients a_3, $a_4 = -b_4$ and c are computed in Appendices 6.3 and 6.4.

We do not attempt to justify this assertion in the general time-dependent case, sending the reader to the work of G. Schneider [10]. Instead we restrict our attention to steady solutions. The mathematical justification for the search of steady solutions comes back to the spatial dynamics theory (see [4]), that we detail roughly below.

5.2.1 Mathematical justification of steady Ginzburg Landau equation

Let us consider the full Navier-Stokes system (1), (2), written in rescaled cylindrical coordinates (x, y, z) as in Section 2. Then consider the steady solutions and rewrite the system as in [4] section 4 under the form

$$\frac{\partial U}{\partial y} = \mathfrak{L}_\tau U + \mathfrak{B}(U, U), \tag{28}$$

where U has 6 components:

$$\begin{aligned} U &= (u, w),\ u = (u_\perp, u_y), \\ w_y &= \nu \frac{\partial u_y}{\partial y} - p,\quad w_\perp = \nu \frac{\partial u_\perp}{\partial y}, \end{aligned}$$

and $\mathfrak{L}_\tau U$ is the linear part, $\mathfrak{B}(U, U)$ being the quadratic part, both not containing any derivatives with respect to y. Functions are supposed to be periodic in z, with period $2\pi/\alpha_c$, while $x \in [-1/2, 1/2]$. The system (28) appears as an evolution system with y playing the role of time, however it is a ill posed system, since we consider solutions satisfying a bounded condition at infinity in y.

We already know that for τ such that

$$a_3\tau - b_4\mathcal{R}^2\beta^2 + h.o.t = 0 \tag{29}$$

the eigenvalue λ_0 of $\mathcal{L}_{1,\alpha_c}$ cancels, corresponding to eigenvectors of the form

$$e^{\pm i\alpha_c z \pm i\beta y}\widehat{u}(x),$$

all other eigenvalues being on the left side of the imaginary axis. This means, for the new linear operator $\mathfrak{L}_\tau$, that if (29) is realized, we have two double eigenvalues $\pm i\beta$ on the imaginary axis for $\mathfrak{L}_\tau$ with eigenvectors of the form

$$e^{i\alpha_c z}\widehat{U}(x),\quad e^{-i\alpha_c z}S\widehat{U}(x) \text{ for } i\beta$$

$$e^{i\alpha_c z}\overline{S\widehat{U}(x)},\quad e^{-i\alpha_c z}\overline{\widehat{U}(x)} \text{ for } -i\beta.$$

Moreover there is no other eigenvalue on the imaginary axis for $\mathfrak{L}_\tau$, and a general result shown in [4] is that the rest of the spectrum of $\mathfrak{L}_\tau$ lies in a sectorial region, symmetric with respect to both real and imaginary axis in the complex plane, staying at a non zero distance from the imaginary axis. It is also shown in [4] that center manifold reduction applies in the neighborhood of criticality. Here, this means that any solutions of (28), i.e. solutions of steady Navier-Stokes system, which stay close to the origin for $y \in \mathbb{R}$ (the origin here is the Couette flow), lives on a 4-dimensional center manifold that we parameterize by a couple of complex amplitudes $(A(y), B(y))$, where solutions of (28) have a principal part given by

$$U = A(y)e^{i\alpha_c z}\widehat{U}(x) + B(y)e^{i\alpha_c z}\widehat{V}(x) + c.c.$$

The reduced ODE on the center manifold still satisfies the symmetries of the original system. In particular it commutes with the representation of the symmetries $z \mapsto z + h$, and $z \mapsto -z$, while the symmetry $(x, y) \mapsto (-x, -y)$ of the limit system (26) leads to a reversibility symmetry for the corresponding limit ODE (notice that $\widehat{U}(x)$ is even in x). Hence the limit ODE commutes with the following actions:

$$\begin{aligned} (A, B, \overline{A}, \overline{B}) &\mapsto (Ae^{ih}, Be^{ih}, \overline{A}e^{-ih}, \overline{B}e^{-ih}), \\ (A, B, \overline{A}, \overline{B}) &\mapsto (\overline{A}, \overline{B}, A, B), \end{aligned}$$

and anticommutes with

$$(A, B, \overline{A}, \overline{B}) \mapsto (\overline{A}, -\overline{B}, A, -B).$$

The linear part of this ODE needs to give the spectrum of $\mathfrak{L}_\tau$ reduced to the central subspace, the principal part of which is here necessarily

$$\begin{aligned} \mathcal{R}\frac{dA}{dy} &= B \\ \mathcal{R}\frac{dB}{dy} &= -\frac{a_3}{b_4}\tau A. \end{aligned}$$

The Jordan block is due to the fact that when $i\beta \to 0$ (as $\tau \to 0$) the two imaginary eigenvalues merge in 0 which then becomes a double non semi-simple eigenvalue for $\mathfrak{L}_0$. We can choose coordinates (with a nonlinear change named "normal form" technique (see [3])) such that the lowest order nonlinear terms in the ODE is in the second component and is proportional to $A|A|^2$. This gives a second order complex differential equation, with real coefficients, orresponding to the steady version of (27), as stated in [4] section 6.

5.2.2 Time independent solutions of the Ginzburg Landau equation

Accordingly, the search of time-independent solutions of the Navier-Stokes can thus be reduced to the search of time-independent solutions of the Ginzburg-Landau equation, which is an easier task that we now detail.

We note that, for any $\theta \in \mathbb{R}$,

$$A(y) = \sqrt{\frac{a_3\tau}{c}}e^{i\theta}$$

is a stationary solution of (27), which corresponds to the Taylor vortex flows.

There exists another family of solutions of (27) which are under the form

$$\rho e^{i(\alpha_c z + \beta y)} + c.c.,$$

where

$$\rho^2 = \frac{a_3}{c}\tau - \frac{b_4}{c}\beta^2\mathfrak{R}^2.$$

These solutions corresponds to wavy vortices.

The secondary bifurcation to such solutions occurs for (recall that $T = 2\omega\mathfrak{R}$)

$$\omega = \omega_c + \frac{b_4}{2a_3}\frac{\mathfrak{B}^2}{\mathfrak{R}}.$$

This leads in particular to a secondary bifurcation branching from Couette flow (see Figure 1), under the form of steady wavy vortices (defined up to a shift in z, y).

The branches of steady solutions obtained for $A = \rho e^{i\beta y}$ with $\beta \neq 0$ do not meet each others, since they are parallel to the TVF branch (see Figure 1). Notice that these wavy vortices are steady in the rotating frame (rotation rate Ω_{rf}), hence they would correspond to time periodic wavy vortices for an observer.

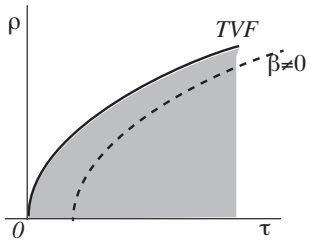


Figure 1: TVF is given by $a_3\tau = c\rho^2$. Any point (τ, ρ) in the grey region corresponds to wavy vortices with $A = \rho e^{i\beta y}$ such that $b_4\Re^2\beta^2 = a_3\tau - c\rho^2$. These solutions correspond to the boundary of the bounded region described in Figure 4.

5.3 Stability of bifurcating steady wavy vortices

Let us start with the Ginzburg-Landau equation (27) ruling the principal part of bifurcating solutions and study the stability of the steady solution

$$\begin{aligned} A &= A_0 e^{i\beta_0 y}, \\ 0 &= a_3\tau - \beta_0^2 b_4\Re^2 - c|A_0|^2 \end{aligned} \tag{30}$$

with respect to the time dependent Ginzburg-Landau equation.

Let us set

$$A = e^{i\beta_0 y}(A_0 + A'),$$

where A_0 is chosen real, and consider the linearized system for $(A', \overline{A'})$:

$$\frac{\partial A'}{\partial t} = b_4\Re^2\frac{\partial^2 A'}{\partial y^2} + 2i\beta_0 b_4\Re^2\frac{\partial A'}{\partial y} - c{A_0}^2 A' - cA_0^2\overline{A'}.$$

We only study the stability with respect to perturbations of the form $A' = a_+e^{iky} + a_-e^{-iky}$ where k is close to 0. Then we have to discuss the eigenvalues of (consider the system in $(a_+, \overline{a_-})$)

$$\begin{pmatrix} -c{A_0}^2 - b_4\Re^2k^2 - 2b_4\Re^2\beta_0 k & -cA_0^2 \\ -cA_0^2 & -c{A_0}^2 - b_4\Re^2k^2 + 2b_4\Re^2\beta_0 k \end{pmatrix},$$

The eigenvalues are

$$\lambda_\pm = -c{A_0}^2 - b_4\Re^2k^2 \pm \sqrt{c^2A_0^4 + 4b_4^2\Re^4\beta_0^2k^2},$$

which for $|k|$ small leads to

$$\begin{aligned} \lambda_+ &\simeq b_4\Re^2k^2\left(\frac{2b_4\Re^2\beta_0^2}{cA_0^2} - 1\right) \\ \lambda_- &\simeq -2cA_0^2. \end{aligned}$$

It results that the steady solution corresponding to wavy vortices (30) is unstable for

$$\mathfrak{R}^2\beta_0^2 > \frac{a_3\tau}{3b_4}$$

and stable for $\mathfrak{R}^2\beta_0^2 < \frac{a_3\tau}{3b_4}$. This result is valid for perturbations of a specific type with wave numbers close to β_0, and is related to the Eckhaus instability in hydrodynamic stability theory. Figure 2 illustrates the result.

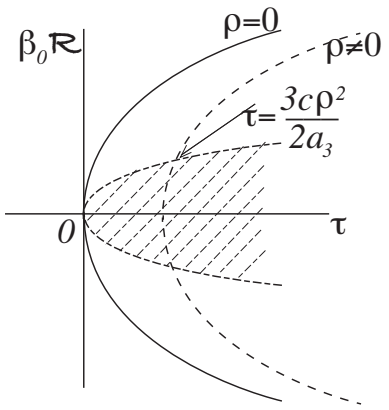


Figure 2: Hatched region of stability of Wavy vortices.

Notice that TVF corresponds to $\beta_0 = 0$, so that it is stable with respect to these perturbations, periodic in y, while for the wavy vortices with $\beta_0 \neq 0$, they are unstable with respect to such perturbations if

$$a_3\tau = cA_0^2 + b_4\mathfrak{R}^2\beta_0^2 > \frac{3}{2}cA_0^2.$$

The genuine solutions of the Navier-Stokes equations have the same stability properties as the corresponding solutions of the Ginzburg-Landau equation.

5.4 Stationary solutions of the amplitude equation

We now study all the stationary solutions of the time-independent Ginzburg-Landau equation. We recall that the principal part of a steady velocity vector field is

$$A(y)e^{i\alpha_c z}\widehat{U}(x) + \overline{A}(y)e^{-i\alpha_c z}\overline{\widehat{U}}(x)$$

where $A(y) \in \mathbb{C}$, $y \in \mathbb{R}$ and $\widehat{U}(x)$ is the eigenvector $(u_x^0, u_y^0, u_z^0)(x)$ defined in previous section. Let us now look for solutions $A(y)$ of

$$a_3\tau A + b_4\mathfrak{R}^2\frac{\partial^2 A}{\partial y^2} - cA|A|^2 = 0,$$

which are small and bounded, when τ is small. We claim that this (reversible) system is integrable.

First we change the scale in y, and define the new positive coefficients

$$\mathfrak{R}\widetilde{y} = y, \qquad c_0 = \frac{c}{2b_4}, \qquad a_0 = \frac{a_3}{b_4}. \tag{31}$$

Then the equation reads (suppressing the tilde)

$$\frac{\partial^2 A}{\partial y^2} = -a_0\tau A + 2c_0 A|A|^2. \tag{32}$$

Let us set

$$A(y) = \rho(y)e^{i\theta(y)},$$

then

$$\begin{aligned} \rho'' - \rho\theta'^2 &= 2c_0\rho^3 - a_0\tau\rho, \\ 2\rho'\theta' + \rho\theta'' &= 0, \end{aligned}$$

hence

$$\rho^2\theta' = K,$$

$$\rho'' = \frac{K^2}{\rho^3} + 2c_0\rho^3 - a_0\tau\rho,$$

where K is some constant. Now defining

$$u(\rho) = \rho'^2,$$

we finally obtain

$$u'(\rho) = \frac{2K^2}{\rho^3} + 4c_0\rho^3 - 2a_0\tau\rho$$

hence

$$u(\rho) = -\frac{K^2}{\rho^2} + c_0\rho^4 - a_0\tau\rho^2 + H,$$

where H is some other constant. It remains to look for bounded solutions of the first order differential equation

$$\rho'^2 = -\frac{K^2}{\rho^2} + c_0\rho^4 - a_0\tau\rho^2 + H.$$

The cases with $\theta' = \beta\mathfrak{R} = const$ and $\rho = const$ give

$$-\frac{K^2}{\rho^2} + c_0\rho^4 - a_0\tau\rho^2 + H = 0$$

and correspond to periodic solutions $A(y) = \rho e^{i\beta y}$.

There are many other solutions, and we are interested in solutions such that as y tends to $\pm\infty$, $A(y)$ goes to the same bounded limit (small for μ small) in such a way that this will respect the azimuthal periodicity, since y replaces φ.

Let us first study the case $K = 0$, where we may take A real. The system (32) leads to

$$A'^2 = -a_0\tau A^2 + c_0 A^4 + H,$$

and we can describe the phase portrait of the solutions in the plane (A, A') as indicated in Figure 3. For

$$0 < H < \frac{(a_0\tau)^2}{4c_0},$$

there is a family of closed orbits, surrounding 0 corresponding to periodic solutions $\rho(y)$, of period running from $\frac{2\pi}{\sqrt{a_0\tau}}$ (multiplied by $\mathfrak{R}$ in the original scaling), to ∞. All these solutions are eligible for our problem. The limit case where $H = \frac{(a_0\tau)^2}{4c_0}$ leads to

$$A'^2 = c_0\Big(A^2 - \frac{a_0\tau}{2c_0}\Big)^2$$

which gives a heteroclinic orbit connecting the equilibrium points $A = \pm\sqrt{\frac{a_0\tau}{2c_0}}$. These points correspond to the Taylor Vortex Flow solution (TVF). In this case, the phase θ is constant, and at each infinity the limits for A are different, corresponding to a vertical shift of TVF by π/α_c. We may consider the juxtaposition along the y coordinate of this solution with the symmetric one, then recovering the periodicity in the azimuthal coordinate.

Very close to this heteroclinic we have periodic solutions with very long periods, staying very close to TVF flow for long intervals in y. These solutions are clearly eligible.

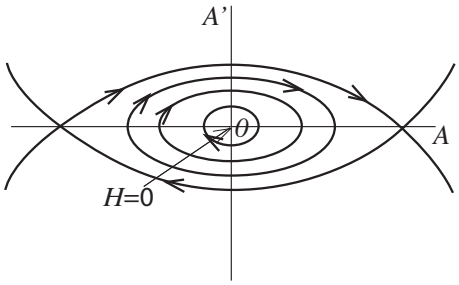


Figure 3: Bounded solutions for K=0

Let us now consider the case where $K \neq 0$. Let us make a new scaling to simplify the computations

$$\rho = k\widetilde{\rho}, \quad K = k\widetilde{K}, \quad c_0 k^4 = 1, \quad a_0\tau k^2 = \widetilde{\tau}, \quad y = k\widetilde{y}. \tag{33}$$

Then, suppressing the tildes, we have

$$\rho'^2 = -\frac{K^2}{\rho^2} + \rho^4 - \tau\rho^2 + H,$$

The TVF equilibrium corresponds to $\rho^2 = \tau/2$, $K = 0$, $H = \tau^2/4$. We now study the function $f(X)$ defined for $X > 0$ by

$$f(X) = -\frac{K^2}{X} + X^2 - \tau X + H,$$

and determine the region where $f \geq 0$. Since $f(X) \to -\infty$ as $X \to 0^+$ and $f(X) \to +\infty$ as $X \to +\infty$, the only interesting cases are when f is not monotonous, taking the value 0 for two different values of X. We show that this occurs for (H, K) in a certain bounded region of the plane. The limit cases are when the minimum or maximum of f are on the X axis, so that we satisfy the system

$$f(X) = f'(X) = 0.$$

This gives

$$\begin{aligned} H &= -3X^2 + 2\tau X \\ K^2 &= -2X^3 + \tau X^2, \end{aligned}$$

which is the equation, in parametric form, of the curve in the (H, K) plane where $f(X)$ cancels in a maximum or a minimum. We observe that we need to study this curve from $X = 0$ until $X = \tau/2$ (TVF flow) for which K cancels (since K^2 should be positive), and then take the symmetric curve through the H axis. The point obtained for $X = \tau/3$ is singular, giving maximum values for H and $K : (H, K)_{\max} = (\tau^2/3, (\tau/3)^{3/2})$. The curve reaches the H axis ($K = 0$) for $H = 0$, and $H = \tau^2/4$ (this corresponds to the origin and to the heteroclinic in the plane (A, A') found for $K = 0$). We indicate on Figure 4 the shapes of $f(X)$ all along the curve and the shapes of f inside the region delimited by the curve.

For $\rho(y)$, the discussion is the same as for the pendulum equation. Inside the region there is a family of periodic solutions, depending on (H, K), with amplitudes varying from 0 for the left boundary with lower values of H, to a limit corresponding to a family of homoclinic solutions at the right boundary with larger values of $H = H_{max}(K)$. When ρ (which stays > 0) describes one of the homoclinics, this corresponds to solutions with the argument $\theta(y)$ such that

$$\theta'(y) \underset{y \to \pm\infty}{\to} \frac{K}{\rho_\infty^2} = \beta,$$

where the limit is exponential, hence

$$\theta(y) = \beta y + \theta_0 + K \int_0^y \left(\frac{1}{\rho^2(s)} - \frac{1}{\rho_\infty^2} \right) ds.$$

A necessary condition for being an eligible solution (i.e. periodic in the azimuthal coordinate) is that, there exists $n \in \mathbb{N}$ such that

$$K \int_{-\infty}^{\infty} \left(\frac{1}{\rho^2(s)} - \frac{1}{\rho_\infty^2} \right) ds = 2n\pi. \tag{34}$$

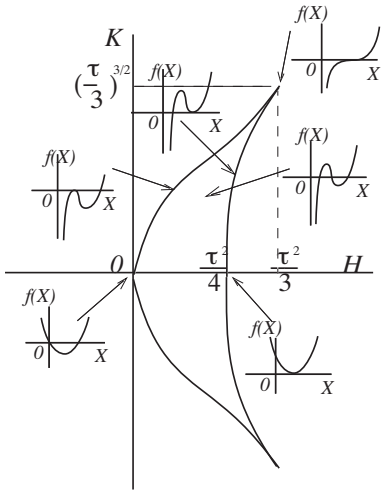


Figure 4: $f(X)$ for $K \neq 0$.

Since $\rho(y) \leq \rho_\infty$ depends on $(H_{\max}(K), K)$, the condition (34) might be realized for a discrete set of values of K. For K close to 0, $\rho(y)$ stays nearly constant close to TVF solution, except for a relatively small interval of values for y (see Figure 5), so that the corresponding flow looks like a wavy vortex flow with a small azimuthal modulation which has a little step near $y = 0$.

For the periodic solutions $\rho(y)$ of period $T(H, K)$, we obtain

$$\begin{aligned} \theta(y) &= \beta' y + \theta_0 + \phi(y), \\ \phi'(y) &= \frac{K}{\rho^2(s)} - \beta', \\ \beta' &= \frac{K}{T(H,K)} \int_0^{T(H,K)} \frac{dy}{\rho^2(y)}, \end{aligned}$$

where ϕ has the same period $T(H, K)$ as ρ. For solutions $\rho(y)$ close to some homoclinic, the amplitude stays nearly constant for long intervals of y and θ is the sum of $\beta' y$ plus periodic bumps as indicated on Figure 5.

It should be noted that all steady solutions, obtained for $H = H_{min}$ or $H = H_{max}$ for (32), correspond to bifurcating time periodic solutions of the Navier-Stokes equations, under the form of classical wavy vortices. Other solutions are new, more exotic and still time periodic in the frame of an observer.

The persistence of the above solutions under perturbation terms of higher order comes from the study (not made here) with the help of spatial dynamics (where y plays the role of time). Here we obtain a fourth-order ODE of the form (32) with higher order terms respecting the symmetries $A \to Ae^{ih}$, $A \to \overline{A}$, and reversibility symmetry $(y, A(y)) \to (-y, A(-y))$. This is analogous (however simpler) to what is done in [4], see also [3] p.222. Here the linear situation

corresponds to a quadruple 0 eigenvalue with two 2×2 Jordan blocks where the principal part of the normal form is given by (32). The periodic solutions we find, persist in using an implicit function argument in a suitable function space. The homoclinic solution also persists by the same type of geometric argument as developed in [3] p.222. More delicate is the problem of persistence under perturbation of the solutions corresponding to the region of the parameter plane (H, K) at the interior of the bounded area. We may observe that these solutions are such that, for a suitable β, the function $D(y) = A(y)e^{-i\beta y}$ is periodic, with the period of $\rho(y)$. Then, provided that in the plane (H, K) we are not too close to the boundary $H_{max}(K)$ the persistence proof applies for these periodic solutions (of a slightly modified equation), with a suitable β. By construction, the solutions we provide here (those which persist under perturbation) are principal parts of solutions of the limit Navier-Stokes equations (26).

These results are summed up in Theorem 1.4.

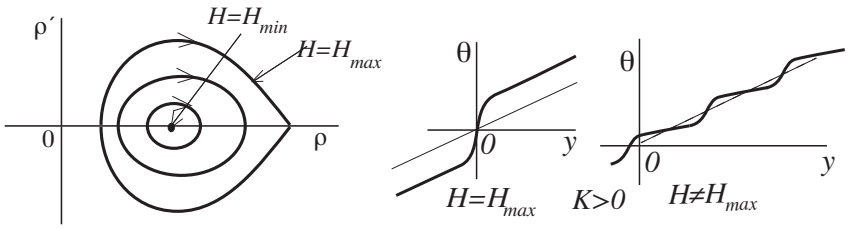


Figure 5: Bounded solutions ρ for $K > 0$ and $\theta(y)$ for $H_{min} < H \leq H_{max}$.

6 Appendix A: Some computations

6.1 Rescaled Navier-Stokes equations

In this section, we provide detail computations for the derivation of (7) in Section 2 and in particular we expand $\Delta_{cyl}u$, $r^{-2}u$, u_φ^2/r, $U' + \Omega_{rot}$, Ω_{rot} in η and μ.

- Expansion of $\Delta_{cyl}u$: we have

$$\Delta_{cyl}u = \Big(\frac{\nu}{d^3}\frac{\partial^2}{\partial\hat{x}^2} + \frac{1}{\hat{x}d+\bar{r}}\frac{\nu}{d^2}\frac{\partial}{\partial\hat{x}} + \frac{\nu}{d^3}\frac{\bar{r}^2}{(\hat{x}d+\bar{r})^2}\frac{\partial^2}{\partial\hat{y}^2} + \frac{\nu}{d^3}\frac{\partial^2}{\partial\hat{z}^2}\Big)\hat{u}.$$

We have

$$\frac{d}{\hat{x}d+\bar{r}} = \frac{1-\eta}{\hat{x}(1-\eta)+\frac{1+\eta}{2}} = \mathcal{O}(1-\eta),$$

$$\begin{aligned}\frac{\bar{r}^2}{(\hat{x}d+\bar{r})^2} =& \frac{(\frac{1+\eta}{2})^2}{(\hat{x}(1-\eta)+\frac{1+\eta}{2})^2} = 1 - \frac{\hat{x}(1-\eta)(\hat{x}(1-\eta)+(1+\eta))}{(\hat{x}(1-\eta)+\frac{1+\eta}{2})^2}\\ =& 1+\mathcal{O}(1-\eta),\end{aligned}$$

therefore,

$$\Delta_{cyl}u = \frac{\nu}{d^3}\Big(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{y}^2}+\frac{\partial^2}{\partial\hat{z}^2}\Big)\hat{u}+\mathcal{O}(1-\eta)\frac{\nu}{d^3}\Big(\frac{\partial\hat{u}}{\partial\hat{x}}+\frac{\partial^2\hat{u}}{\partial\hat{y}^2}\Big).$$

- Expansion of $r^{-2}u$:

$$\frac{u}{r^2} = \frac{\nu}{d^3}\frac{d^2}{(\hat{x}d+\bar{r})^2}\hat{u} = \mathcal{O}(1-\eta)^2\frac{\nu}{d^3}\hat{u}.$$

- Expansion of u_φ^2/r:

  Note that

$$\frac{d}{r} = \frac{d}{r_0}\frac{r_0}{r} = (1-\eta)\frac{dx\hat{+}\bar{r}}{r_0} = (1-\eta)^2\hat{x}+\frac{(1-\eta)(1+\eta)}{2}.$$

  Hence

$$\frac{u_\varphi^2}{r} = \frac{\nu^2}{d^3}\frac{d}{r}\hat{u}_{\hat{y}}^2 = \frac{\nu^2}{d^3}\Big((1-\eta)^2\hat{x}+\frac{(1-\eta)(1+\eta)}{2}\Big)\hat{u}_{\hat{y}}^2.$$

- Study of $U'+\Omega_{rot}$:

$$U'+\Omega_{rot} = 2(A-\Omega_{rf}) = 2\omega_i\Big(\frac{\mu-\eta^2}{1-\eta^2}-\frac{1+\mu}{2}\Big) = -\omega_i\frac{(1-\mu)(1+\eta^2)}{1-\eta^2}.$$

- Expansion of Ω_{rot}:

$$\begin{aligned}\Omega_{rot} =&(A-\Omega_{rf})+\frac{B}{r^2} = -\omega_i\frac{(1-\mu)(1+\eta^2)}{1-\eta^2}+\omega_i\frac{1-\mu}{1-\eta^2}\frac{r_i^2}{r^2}\\ =&-\omega_i\frac{1-\mu}{1-\eta^2}\Big(\frac{1+\eta^2}{2}-\frac{r_i^2}{r^2}\Big).\end{aligned}$$

  Note that

$$\begin{aligned}\frac{r_i^2}{r^2} =&\frac{r_i^2}{(\hat{x}d+\bar{r})^2} = \frac{\eta^2}{(\hat{x}(1-\eta)+\frac{1+\eta}{2})^2} = \Big(\frac{\eta}{(\hat{x}+\frac{1}{2})(1-\eta)+\eta}\Big)^2\\ =&\Big(1+\frac{1-\eta}{\eta}(\hat{x}+\frac{1}{2})\Big)^{-2}\\ =&1-2\Big(\frac{1-\eta}{\eta}(\hat{x}+\frac{1}{2})\Big)+3\Big(\frac{1-\eta}{\eta}(\hat{x}+\frac{1}{2})\Big)^2+\cdots.\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{1-\eta^2}\Big(\frac{1+\eta^2}{2}-\frac{r_i^2}{r^2}\Big)\\
=&\frac{1}{1-\eta^2}\Big[\frac{\eta^2-1}{2}+2\Big(\frac{1-\eta}{\eta}(\hat{x}+\frac{1}{2})\Big)-3\Big(\frac{1-\eta}{\eta}(\hat{x}+\frac{1}{2})\Big)^2+\cdots\Big]\\
=&-\frac{1}{2}+\frac{1}{\eta(1+\eta)}(2\hat{x}+1)+\mathcal{O}(1-\eta).
\end{aligned}$$

As

$$\frac{2}{\eta(1+\eta)}-1=-\frac{(\eta+2)(\eta-1)}{\eta(1+\eta)}=\mathcal{O}(1-\eta),$$

we have

$$\frac{1}{1-\eta^2}\Big(\frac{1+\eta^2}{2}-\frac{r_i^2}{r^2}\Big)=\hat{x}+\mathcal{O}(1-\eta)$$

and hence

$$\Omega_{rot}=-\omega_i(1-\mu)\Big[\hat{x}+\mathcal{O}(1-\eta)\Big].$$

6.2 Critical Taylor number in axisymmetric case

Solutions of (10) are linear combinations of $\exp(\lambda_j x)$, where the $\Lambda=\lambda_j$ are the 6 complex solutions of

$$(\lambda+\alpha^2-\Lambda^2)^2(\alpha^2-\Lambda^2)=T\alpha^2 \tag{35}$$

provided this polynomial of degree 6 has 6 different roots. If some roots are equal, then the solution involves functions of the form $x\exp(\lambda_j x)$. Note that by previous Lemma, $\sigma>\alpha^2$.

This equation is a polynomial of degree 3 in Λ^2. It can thus be explicitly solved and has three roots $\Lambda_k(\lambda,\alpha,T)$ for $1\le k\le 3$, leading to the six roots $\pm\Lambda_k^{1/2}(\lambda,\alpha,T)$ for (35). If these 6 roots are different, u_x is of the form

$$u_x(x)=\sum_{j=1}^{6}\beta_j e^{\lambda_j x},$$

The boundary conditions give

$$\sum_{j=1}^{6}\beta_j e^{\pm\lambda_j/2}=0,$$

$$\sum_{j=1}^{6}\beta_j\lambda_j e^{\pm\lambda_j/2}=0,$$

$$\sum_{j=1}^{6}\beta_j(\lambda+\alpha^2-\lambda_j^2)(\alpha^2-\lambda_j^2)e^{\pm\lambda_j/2}=0$$

which is a 6×6 linear system. Let $\mathfrak{D}(\lambda)$ be its determinant. Then if $\mathfrak{D}(\lambda)$ vanishes and if all the roots λ_j are different, λ is an eigenvalue.

When $\lambda = 0$, equation (35) gives

$$\Lambda^2 = \alpha^2 - \sigma(1)^{1/3}, \quad \sigma = (T\alpha^2)^{1/3}$$

where $1^{1/3}$ stands for the three different cubic roots of 1. Thus we have

$$\begin{aligned} \lambda_1^2 &= \alpha^2 - \sigma, \\ \lambda_2^2 &= \alpha^2 - \sigma j, \\ \lambda_3^2 &= \alpha^2 - \sigma j^2, \end{aligned}$$

where

$$j = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

One can see that λ_1, λ_2, and λ_3 are always different. Therefore, an eigenvector u_x should take the form

$$u_x = A_1 e^{\lambda_1 x} + B_1 e^{-\lambda_1 x} + A_2 e^{\lambda_2 x} + B_2 e^{-\lambda_2 x} + A_3 e^{\lambda_3 x} + B_3 e^{-\lambda_3 x}$$

with

$$\begin{aligned} \lambda_1 &= ib_1, \\ \lambda_2 &= a_2 + ib_2, \\ \lambda_3 &= a_2 - ib_2, \end{aligned}$$

where b_1, a_2 and b_2 are given by (11).

The boundary conditions are

$$u_x(\pm 1/2) = Du_x(\pm 1/2) = (\alpha^2 - D^2)^2 u_x(\pm 1/2) = 0.$$

We may observe that

$$\begin{aligned} (\alpha^2 - \lambda_1^2)^2 &= \sigma^2, \\ (\alpha^2 - \lambda_2^2)^2 &= \sigma^2 j^2, \\ (\alpha^2 - \lambda_3^2)^2 &= \sigma^2 j, \end{aligned}$$

hence the boundary conditions become

$$\begin{aligned} A_1 e^{\lambda_1/2} + B_1 e^{-\lambda_1/2} + A_2 e^{\lambda_2/2} + B_2 e^{-\lambda_2/2} + A_3 e^{\lambda_3/2} + B_3 e^{-\lambda_3/2} &= 0, \\ A_1 e^{-\lambda_1/2} + B_1 e^{\lambda_1/2} + A_2 e^{-\lambda_2/2} + B_2 e^{\lambda_2/2} + A_3 e^{-\lambda_3/2} + B_3 e^{\lambda_3/2} &= 0, \\ \lambda_1 A_1 e^{\lambda_1/2} - \lambda_1 B_1 e^{-\lambda_1/2} + \lambda_2 A_2 e^{\lambda_2/2} - \lambda_2 B_2 e^{-\lambda_2/2} + \lambda_3 A_3 e^{\lambda_3/2} - \lambda_3 B_3 e^{-\lambda_3/2} &= 0, \\ \lambda_1 A_1 e^{-\lambda_1/2} - \lambda_1 B_1 e^{\lambda_1/2} + \lambda_2 A_2 e^{-\lambda_2/2} - \lambda_2 B_2 e^{\lambda_2/2} + \lambda_3 A_3 e^{-\lambda_3/2} - \lambda_3 B_3 e^{\lambda_3/2} &= 0, \\ A_1 e^{\lambda_1/2} + B_1 e^{-\lambda_1/2} + j^2 A_2 e^{\lambda_2/2} + j^2 B_2 e^{-\lambda_2/2} + j A_3 e^{\lambda_3/2} + j B_3 e^{-\lambda_3/2} &= 0, \\ A_1 e^{-\lambda_1/2} + B_1 e^{\lambda_1/2} + j^2 A_2 e^{-\lambda_2/2} + j^2 B_2 e^{\lambda_2/2} + j A_3 e^{-\lambda_3/2} + j B_3 e^{\lambda_3/2} &= 0. \end{aligned}$$

For obtaining a non trivial solution we need to verify that

$$\Delta_{\alpha,\tau} = \begin{vmatrix} e^{\lambda_1/2} & e^{\lambda_2/2} & e^{\lambda_3/2} & e^{-\lambda_1/2} & e^{-\lambda_2/2} & e^{-\lambda_3/2} \\ \lambda_1 e^{\lambda_1/2} & \lambda_2 e^{\lambda_2/2} & \lambda_3 e^{\lambda_3/2} & -\lambda_1 e^{-\lambda_1/2} & -\lambda_2 e^{-\lambda_2/2} & -\lambda_3 e^{-\lambda_3/2} \\ e^{\lambda_1/2} & j^2 e^{\lambda_2/2} & j e^{\lambda_3/2} & e^{-\lambda_1/2} & j^2 e^{-\lambda_2/2} & j e^{-\lambda_3/2} \\ e^{-\lambda_1/2} & e^{-\lambda_2/2} & e^{-\lambda_3/2} & e^{\lambda_1/2} & e^{\lambda_2/2} & e^{\lambda_3/2} \\ \lambda_1 e^{-\lambda_1/2} & \lambda_2 e^{-\lambda_2/2} & \lambda_3 e^{-\lambda_3/2} & -\lambda_1 e^{\lambda_1/2} & -\lambda_2 e^{\lambda_2/2} & -\lambda_3 e^{\lambda_3/2} \\ e^{-\lambda_1/2} & j^2 e^{-\lambda_2/2} & j e^{-\lambda_3/2} & e^{\lambda_1/2} & j^2 e^{\lambda_2/2} & j e^{\lambda_3/2} \end{vmatrix}$$

vanishes. We observe that $\Delta_{\alpha,\tau}$ is real since $\lambda_3 = \overline{\lambda_2}$. Now, after reorganisation, we have

$$\Delta_{\alpha,\tau} = \begin{vmatrix} e^{\lambda_1/2} & e^{\lambda_2/2} & e^{\lambda_3/2} & e^{-\lambda_1/2} & e^{-\lambda_2/2} & e^{-\lambda_3/2} \\ 0 & (1-\frac{\lambda_2}{\lambda_1})e^{\lambda_2/2} & (1-\frac{\lambda_3}{\lambda_1})e^{\lambda_3/2} & 2e^{-\lambda_1/2} & (1+\frac{\lambda_2}{\lambda_1})e^{-\lambda_2/2} & (1+\frac{\lambda_3}{\lambda_1})e^{-\lambda_3/2} \\ 0 & (1-j^2)e^{\lambda_2/2} & (1-j)e^{\lambda_3/2} & 0 & (1-j^2)e^{-\lambda_2/2} & (1-j)e^{-\lambda_3/2} \\ e^{-\lambda_1/2} & e^{-\lambda_2/2} & e^{-\lambda_3/2} & e^{\lambda_1/2} & e^{\lambda_2/2} & e^{\lambda_3/2} \\ 2e^{-\lambda_1/2} & (1+\frac{\lambda_2}{\lambda_1})e^{-\lambda_2/2} & (1+\frac{\lambda_3}{\lambda_1})e^{-\lambda_3/2} & 0 & (1-\frac{\lambda_2}{\lambda_1})e^{\lambda_2/2} & (1-\frac{\lambda_3}{\lambda_1})e^{\lambda_3/2} \\ 0 & (1-j^2)e^{-\lambda_2/2} & (1-j)e^{-\lambda_3/2} & 0 & (1-j^2)e^{\lambda_2/2} & (1-j)e^{\lambda_3/2} \end{vmatrix},$$

which takes the form

$$\Delta_{\alpha,\tau} = \begin{vmatrix} M & N \\ N & M \end{vmatrix}$$

with

$$M = \begin{pmatrix} e^{\lambda_1/2} & e^{\lambda_2/2} & e^{\lambda_3/2} \\ 0 & (1-\frac{\lambda_2}{\lambda_1})e^{\lambda_2/2} & (1-\frac{\lambda_3}{\lambda_1})e^{\lambda_3/2} \\ 0 & (1-j^2)e^{\lambda_2/2} & (1-j)e^{\lambda_3/2} \end{pmatrix},$$

$$N = \begin{pmatrix} e^{-\lambda_1/2} & e^{-\lambda_2/2} & e^{-\lambda_3/2} \\ 2e^{-\lambda_1/2} & (1+\frac{\lambda_2}{\lambda_1})e^{-\lambda_2/2} & (1+\frac{\lambda_3}{\lambda_1})e^{-\lambda_3/2} \\ 0 & (1-j^2)e^{-\lambda_2/2} & (1-j)e^{-\lambda_3/2} \end{pmatrix}.$$

We have

$$\Delta_{\alpha,\tau} = \begin{vmatrix} M+N & M+N \\ N & M \end{vmatrix} = \begin{vmatrix} M+N & 0 \\ N & M-N \end{vmatrix} = |M+N||M-N|.$$

Thus $\Delta_{\alpha,\tau} = 0$ if and only if $|M+N| = 0$ or $|M-N| = 0$.

With the "+" sign we obtain

$$\left| \begin{pmatrix} \cos\frac{b_1}{2} & \cosh\frac{\lambda_2}{2} & \cosh\frac{\lambda_3}{2} \\ e^{-ib_1/2} & \cosh\frac{\lambda_2}{2} - \frac{\lambda_2}{\lambda_1}\sinh\frac{\lambda_2}{2} & \cosh\frac{\lambda_3}{2} - \frac{\lambda_3}{\lambda_1}\sinh\frac{\lambda_3}{2} \\ 0 & (1-j^2)\cosh\frac{\lambda_2}{2} & (1-j)\cosh\frac{\lambda_3}{2} \end{pmatrix} \right| = 0$$

i.e.

$$\left| \begin{pmatrix} \cos\frac{b_1}{2} & \cosh\frac{\lambda_2}{2} & \cosh\frac{\lambda_3}{2} \\ -b_1\sin\frac{b_1}{2} & \lambda_2\sinh\frac{\lambda_2}{2} & \lambda_3\sinh\frac{\lambda_3}{2} \\ 0 & -j\cosh\frac{\lambda_2}{2} & j^2\cosh\frac{\lambda_3}{2} \end{pmatrix} \right| = 0, \tag{36}$$

where

$$\begin{aligned}\lambda_2 &= a_2 + ib_2,\\ \lambda_3 &= a_2 - ib_2,\end{aligned}$$

and where a_2, b_1, b_2 are functions of α^2 and σ defined by (11). A direct computation of this determinant leads to $\mathfrak{F}_1 = 0$.

With the "-" sign, we obtain

$$\begin{vmatrix} \sin\frac{b_1}{2} & \sinh\frac{\lambda_2}{2} & \sinh\frac{\lambda_3}{2} \\ b_1\cos\frac{b_1}{2} & \lambda_2\cosh\frac{\lambda_2}{2} & \lambda_3\cosh\frac{\lambda_3}{2} \\ 0 & -j\sinh\frac{\lambda_2}{2} & j^2\sinh\frac{\lambda_3}{2} \end{vmatrix} = 0, \tag{37}$$

which leads to $\mathfrak{F}_2 = 0$.

The equation $\mathfrak{F}_1 = 0$ comes from

$$|M + N| = 0$$

which corresponds to solutions such that

$$A_1 = B_1, A_2 = B_2, A_3 = B_3$$

in the eigenvector u_x. This means that in this case u_x and u_y are even in x. In the same way we prove that $\mathfrak{F}_2 = 0$ gives u_x and u_y odd in x. It is then clear that the solutions for $\mathfrak{F}_1 = 0$ and $\mathfrak{F}_2 = 0$ are distinct.

6.3 Coefficients of the Landau equation

Let us consider the system (26) ruling the velocity field $U = (u_\perp, \widehat{u}_y)$ independent of y, then suppressing the hat, we obtain (below we detail the projection Π on divergence free vector fields):

$$\begin{aligned}\partial_t U &= \Delta_\perp U - \nabla_\perp p + \begin{pmatrix} Tu_y \\ u_x \\ 0 \end{pmatrix} - (U_\perp\cdot\nabla_\perp)U,\\ \nabla_\perp\cdot U_\perp &= 0,\end{aligned}$$

where the subscipt $\perp$ means components (x, z). The field U is $2\pi/\alpha_c$ periodic in z, and satisfies the 0 boundary conditions at $x = \pm 1/2$. Let us define operators

$$\mathbf{L}_0 U = \Delta_\perp U - \nabla_\perp p_0 + \begin{pmatrix} T_c u_y \\ u_x \\ 0 \end{pmatrix},$$

where p_0 is such that

$$\begin{aligned}\nabla_\perp\cdot(\mathbf{L}_0 U)_\perp = 0,\\ (\mathbf{L}_0 U)_x = 0, \quad x = \pm 1/2;\end{aligned}$$

and

$$\mathbf{L}_1 U = -\nabla_\perp p_1 + \begin{pmatrix} u_y \\ 0 \\ 0 \end{pmatrix},$$

with p_1 such that

$$\nabla_\perp \cdot (\mathbf{L}_1 U)_\perp = 0,$$
$$(\mathbf{L}_1 U)_x = 0, \quad x = \pm 1/2;$$

and

$$\begin{aligned} \mathbf{B}(U,U) &= -(U_\perp \cdot \nabla_\perp)U + \nabla_\perp q, \\ \nabla_\perp \cdot (\mathbf{B}(U,U))_\perp &= 0, \ (\mathbf{B}(U,U))_x = 0, \ x = \pm 1/2. \end{aligned}$$

Notice that the quadratic operator $\mathbf{B}$ is symmetrized if applied to two different vector functions:

$$4\mathbf{B}(U,V)) := \{\mathbf{B}(U+V,U+V) - \mathbf{B}(U-V,U-V)\}.$$

Now our system reads as

$$\partial_t U = \mathbf{L}_0 U + \tau \mathbf{L}_1 U + \mathbf{B}(U,U), \tag{38}$$

in a space of divergence free vector fields, satisfying the boundary conditions, and where

$$\tau = T - T_c.$$

For obtaining the Landau equation describing the first bifurcation occuring for τ close to 0, we use the fact that $\mathbf{L}_0$ has a double 0 eigenvalue with eigenvectors

$$\zeta = e^{i\alpha_c z} u^0(x), \overline{\zeta} = e^{-i\alpha_c z}\overline{u^0}(x),$$

(see subsection 4.2) and the fact that the rest of the spectrum of the self adjoint operator $\mathbf{L}_0 + \tau \mathbf{L}_1$ is only composed of < 0 isolated eigenvalues, not close to 0. Moreover the center manifold reduction applies (see [3]) with a $O(2)$ symmetry, and the dynamics near 0 reduces to the 2-dimensional differential equation in $\mathbb{C}$, at main orders

$$\frac{dA}{dt} = a\tau A - cA|A|^2, \tag{39}$$

where

$$\begin{aligned} U &= A(t)\zeta + \overline{A}(t)\overline{\zeta} + \Phi(A,\overline{A},\tau), \\ \Phi(A,\overline{A},\tau) &= \tau A \Phi_{10}^{(1)} + A^2\Phi_{20} + |A|^2\Phi_{11} + \overline{A}^2\overline{\Phi_{20}} + ... \end{aligned} \tag{40}$$

coefficients Φ_{ij} being vector functions of x, z satisfying the boundary conditions. The coefficients a, c and Φ_{ij} may be computed as indicated below. Replacing

U by (40) in (38) and using (39), gives after identification of each monomial $\tau^p A^n \overline{A}^m$

$$\begin{aligned} a\zeta &= \mathbf{L}_0\Phi_{10}^{(1)} + \mathbf{L}_1\zeta \qquad (41)\\ 0 &= \mathbf{L}_0\Phi_{20} + \mathbf{B}(\zeta,\zeta),\\ 0 &= \mathbf{L}_0\Phi_{11} + 2\mathbf{B}(\zeta,\overline{\zeta}),\\ -c\zeta &= \mathbf{L}_0\Phi_{21} + 2\mathbf{B}(\zeta,\Phi_{11}) + 2\mathbf{B}(\overline{\zeta},\Phi_{20}). \qquad (42)\end{aligned}$$

Since we have

$$\mathbf{L}_0\zeta = 0,$$

with $\mathbf{L}_0$ selfadjoint, we obtain

$$\begin{aligned} a\langle\zeta,\zeta\rangle &= \langle\mathbf{L}_1\zeta,\zeta\rangle,\\ -c\langle\zeta,\zeta\rangle &= \langle 2\mathbf{B}(\zeta,\Phi_{11}) + 2\mathbf{B}(\overline{\zeta},\Phi_{20}),\zeta\rangle, \qquad (43)\end{aligned}$$

where the scalar product is defined by

$$\langle U,V\rangle = \int_0^{2\pi/\alpha_c}\int_{-1/2}^{1/2}(u_x\overline{v_x} + T_c u_y\overline{v_y} + u_z\overline{v_z})\,dxdz.$$

Now we show that a is just the coefficient a_3 computed before, and we prove below that $c>0$, without an explicit computation. Indeed we have

$$\begin{aligned} \zeta &= e^{i\alpha_c z}\begin{pmatrix} u_x^0(x)\\ u_y^0(x)\\ u_z^0(x)\end{pmatrix},\\ Du_x^0 + i\alpha_c u_z^0 &= 0,\end{aligned}$$

$$\begin{aligned} (\alpha_c^2 - D^2)^2 u_x^0 &= T_c\alpha_c^2 u_y^0,\\ (\alpha_c^2 - D^2)u_y^0 &= u_x^0,\\ u_x^0 &= Du_x^0 = u_y^0 = 0, x = \pm 1/2,\end{aligned}$$

hence

$$a\int_{-1/2}^{1/2}\left(|u_x^0|^2 + T_c|u_y^0|^2 + |u_z^0|^2\right)dx = \int_{-1/2}^{1/2} u_y^0 u_x^0 dx,$$

then noticing that (u_x^0, u_y^0) is real and $u_z^0 = (i/\alpha_c)Du_x^0$, this is exactly the coefficient $a_3 > 0$ already computed at Lemma 4.2.

Now, for the computation of c we notice that for any *real* vector field U we have

$$\langle\mathbf{B}(U,U),U\rangle = 0. \qquad (44)$$

Indeed

$$\begin{aligned} -\langle\mathbf{B}(U,U),U\rangle &= \langle(U_\perp\cdot\nabla_\perp)U,U\rangle\\ &= \int\left[\nabla_\perp\cdot(U_\perp\frac{U_\perp^2}{2}) + T_c\nabla_\perp\cdot(U_\perp\frac{U_y^2}{2})\right]dxdz = 0\end{aligned}$$

because of periodicity and boundary conditions. It then results that

$$2\langle \mathbf{B}(U_1,U_2),U_1\rangle + \langle \mathbf{B}(U_1,U_1),U_2\rangle = 0, \tag{45}$$

for any real divergence free vector fields U_1, U_2 satisfying the boundary conditions (just take in (44) $U = U_1 + tU_2$ and find the coefficient of t). Coming back to (43) we may observe that

$$-c\langle \zeta+\overline{\zeta},\zeta+\overline{\zeta}\rangle = \langle 2\mathbf{B}(\zeta+\overline{\zeta},\Phi_2),\zeta+\overline{\zeta}\rangle,$$

where Φ_2 is defined by

$$\mathbf{L}_0\Phi_2 + \mathbf{B}(\zeta+\overline{\zeta},\zeta+\overline{\zeta}) = 0.$$

Then, from (45) we obtain

$$\begin{aligned} c\langle \zeta+\overline{\zeta},\zeta+\overline{\zeta}\rangle &= \langle \mathbf{B}(\zeta+\overline{\zeta},\zeta+\overline{\zeta}),\Phi_2\rangle \\ &= -\langle \mathbf{L}_0\Phi_2,\Phi_2\rangle. \end{aligned}$$

Observing that Φ_2 is orthogonal to the kernel of $\mathbf{L}_0$, where 0 is the largest eigenvalue of the selfadjoint operator, we claim that $\langle \mathbf{L}_0\Phi_2,\Phi_2\rangle < 0$. Hence $c > 0$ (we may mention that this type of proof is due to V.Yudovich [12] for the Bénard-Rayleigh convection problem).

For an explicit calculation of c we need to compute Φ_{11} and Φ_{20}. Let us define

$$\Phi_{11} = (u_x^{11}, u_y^{11}, u_z^{11})^t,$$

and

$$(\zeta_\perp\cdot\nabla_\perp)\overline{\zeta} + (\overline{\zeta}_\perp\cdot\nabla_\perp)\zeta = \begin{pmatrix} 2D(u_x^0)^2 \\ 2D(u_x^0u_y^0) \\ 0 \end{pmatrix},$$

then we need to solve

$$\begin{aligned} D^2u_x^{11} - Dp + T_cu_y^{11} &= 2D(u_x^0)^2, \\ D^2u_y^{11} + u_x^{11} &= 2D(u_x^0u_y^0), \\ D^2u_z^{11} &= 0, \\ Du_x^{11} &= 0 \end{aligned}$$

with boundary conditions. Finally

$$\Phi_{11} = (0, u_y^{11}, 0)^t,$$

with

$$D^2u_y^{11} = 2D(u_x^0u_y^0), u_y^{11}(\pm 1/2) = 0,$$

hence u_y^{11} is odd in x, and

$$u_y^{11}(x) = 2\int_0^x u_x^0(s)u_y^0(s)ds - 4x\int_0^{1/2} u_x^0(s)u_y^0(s)ds. \tag{46}$$

Let us now define

$$\Phi_{20} = e^{2i\alpha_c z}(u_x^{20}, u_y^{20}, u_z^{20})^t$$

and compute

$$(\zeta_\perp \cdot \nabla_\perp)\zeta = e^{2i\alpha_c z} \begin{pmatrix} 0 \\ u_x^0 Du_y^0 - u_y^0 Du_x^0 \\ \frac{i}{\alpha_c} u_x^0 D^2 u_x^0 - \frac{i}{\alpha_c}(Du_x^0)^2 \end{pmatrix},$$

then we need to solve

$$\begin{aligned} (D^2 - 4\alpha_c^2)u_x^{20} - Dp + T_c u_y^{20} &= 0, \\ (D^2 - 4\alpha_c^2)u_y^{20} + u_x^{20} &= u_x^0 Du_y^0 - u_y^0 Du_x^0, \\ (D^2 - 4\alpha_c^2)u_z^{20} - 2i\alpha_c p &= \frac{i}{\alpha_c}[u_x^0 D^2 u_x^0 - (Du_x^0)^2], \\ Du_x^{20} + 2i\alpha_c u_z^{20} &= 0, \end{aligned}$$

$$u_x^{20} = Du_x^{20} = u_y^{20} = 0, x = \pm 1/2.$$

This leads to the 6th order system

$$\begin{aligned} (D^2 - 4\alpha_c^2)^2 u_x^{20} - 4\alpha_c^2 T_c u_y^{20} &= 2D[u_x^0 D^2 u_x^0 - (Du_x^0)^2], \\ (D^2 - 4\alpha_c^2)u_y^{20} + u_x^{20} &= u_x^0 Du_y^0 - u_y^0 Du_x^0, \\ u_x^{20} &= Du_x^{20} = u_y^{20} = 0, x = \pm 1/2. \end{aligned}$$

This gives u_x^{20} and u_y^{20} real and odd in x, while u_z^{20} is pure imaginary and even in x. Now we can compute

$$2\mathbf{B}(\zeta, \Phi_{11}) = -e^{i\alpha_c z} \begin{pmatrix} 0 \\ u_x^0 Du_y^{11} \\ 0 \end{pmatrix},$$

which satisfies the divergence free (in (x, z)) and boundary conditions. We also obtain

$$\begin{aligned} &2\mathbf{B}(\overline{\zeta}, \Phi_{20}) \\ &= -\Pi e^{i\alpha_c z} \begin{pmatrix} D(u_x^0 u_x^{20}) + 2u_x^{20} Du_x^0 + \frac{1}{2} u_x^0 Du_x^{20} \\ u_x^0 Du_y^{20} + 2u_y^{20} Du_x^0 + u_x^{20} Du_y^0 + \frac{1}{2} u_y^0 Du_x^{20} \\ \frac{i}{2\alpha_c}[D(u_x^0 Du_x^{20}) - 2u_x^{20} D^2 u_x^0] \end{pmatrix}. \end{aligned}$$

Hence, from (43), we finally obtain the positive number c which is given by

$$\begin{aligned} & c\int_{-1/2}^{1/2}\Big(|u_x^0|^2+T_c|u_y^0|^2+\frac{1}{\alpha_c^2}|Du_x^0|^2\Big)\,dz\,dx \\ = & \int_{-1/2}^{1/2}T_c u_y^0\Big[u_x^0Du_y^{11}+u_x^0Du_y^{20}+2u_y^{20}Du_x^0+u_x^{20}Du_y^0+\frac{1}{2}u_y^0Du_x^{20}\Big]\,dx \\ & +\int_{-1/2}^{1/2}u_x^0\Big[D(u_x^0u_x^{20})+2u_x^{20}Du_x^0+\frac{1}{2}u_x^0Du_x^{20}\Big]\,dx \\ & +\int_{-1/2}^{1/2}\frac{1}{2\alpha_c^2}Du_x^0\Big[D(u_x^0Du_x^{20})-2u_x^{20}D^2u_x^0\Big]\,dx \end{aligned}$$

since the projection Π disappears in the scalar product.

6.4 Computation of the Ginzburg-Landau equation

In this subsection we derive the partial differential equation satisfied by the amplitude $A(y,t)$, analogue to the usually called Ginzburg-Landau equation.

The perturbation U of the Couette flow is decomposed as

$$U=A\zeta+\overline{A\zeta}+\Phi(\tau,A,\overline{A},\partial_y) \tag{47}$$

where

$$A=A(y,t),\zeta=e^{i\alpha_c z}u^0(x),$$

and where we consider y as a "slow variable" in the amplitude A. Hence ∂_y is considered as a small parameter, and $A,\overline{A},\partial_yA,\partial_y\overline{A},\partial_y^2A,\partial_y^2\overline{A}...$ are considered to be independent in the expansions (see [1] and [4] for the justification of such a computation in the search of steady solutions for Navier-Stokes equations). Φ replaces the usual expression of the center manifold, which was used before for computing the coefficients in the Landau equation (see (41), (42)). The system with which we start is (26) where the hats are suppressed:

$$\partial_tU=\mathbf{L}_0U+\tau\mathbf{L}_1U+\mathbf{B}(U,U), \tag{48}$$

$$\mathbf{L}_0U=\Delta_\perp U+\nabla_\perp p_0+\mathfrak{R}x\frac{\partial U}{\partial y}+\begin{pmatrix}T_cu_y\\u_x\\0\end{pmatrix},$$

$$\mathbf{L}_1U=\begin{pmatrix}u_y\\0\\0\end{pmatrix}+\nabla_\perp p_1,$$

$$\mathbf{B}(U,U)=-(U_\perp\cdot\nabla_\perp)U+\nabla_\perp p_2,$$

where

$$\begin{aligned} \nabla_\perp\cdot U_\perp+\mathfrak{R}\partial_yu_y=0, \qquad & U|_{x=\pm1/2}=0, \\ \tau \quad = \quad & 2\mathfrak{R}(\omega-\omega_c)=T-T_c, \end{aligned}$$

the terms $\nabla p_0, \nabla p_1, \nabla p_2$ are such that the result (the left hand side) is divergence free and satisfies the boundary condition required for the projection Π. By construction, we have for the eigenvector ζ :

$$\mathbf{L}_0 \zeta = 0,$$

with

$$u^0(x) = (u_x^0, u_y^0, u_z^0),$$

where u_x^0, u_y^0 are real and even in x, while u_z^0 is pure imaginary and odd in x.

The partial differential system ruling $A(y,t)$ and $\overline{A}(y,t)$ has to commute with the symmetries commuting with the system (48). There are four important such symmetries:

i) translation $z \mapsto z + h$, represented by the operator T_h,

ii) symmetry through horizontal plane: $z \mapsto -z$ represented by the symmetry S_0,

iii) azimuthal rotation: $y \mapsto y + b$ represented by the operator R_b,

iv) symmetry with respect to z axis: $(x, y) \mapsto (-x, -y)$ represented by the symmetry S_1. It should be noted that the symmetry S_1 is new with respect to the usual Couette-Taylor system. This is due to our asymptotic geometric situation and to the fact that we sit in a suitable rotating frame.

Then, we have

$$\begin{aligned} T_h U(x,y,z) &= U(x,y,z+h), \\ S_0 U(x,y,z) &= (u_x, u_y, -u_z)(x,y,-z), \\ R_b U(x,y,z) &= U(x,y+b,z), \\ S_1 U(x,y,z) &= (-u_x, -u_y, u_z)(-x,-y,z), \end{aligned}$$

which corresponds to require the commutation of the amplitude system respectively with the following actions:

$$\begin{aligned} A &\mapsto A e^{i\alpha h}, \\ A &\mapsto \overline{A}, \\ A(y) &\mapsto A(y+b), \\ A(y) &\mapsto -A(-y). \end{aligned}$$

This explains why the principal terms in the PDE system are as

$$\frac{\partial A}{\partial t} = a\tau A + b\frac{\partial^2 A}{\partial y^2} - cA|A|^2 + h.o.t. \tag{49}$$

where coefficients are real. Higher order terms would be in $\tau^2 A$, $\frac{\partial^4 A}{\partial y^4}$, $\tau\frac{\partial^2 A}{\partial y^2}$, $A|A|^4$, $\tau A|A|^2$, $|A|^2\frac{\partial^2 A}{\partial y^2}$, ... Below we explain how to compute coefficients containing y derivatives in (49)), here only coefficient b since we already gave the

method for computing the coefficients with no y derivative. It should be noticed that if we consider the linearized system near the Couette flow, we need to recover the most at right eigenvalues λ and $\overline{\lambda}$ computed before where we assume a dependency in $e^{i\beta y}$ for A. We then obtain

$$\lambda = a\tau - b\beta^2 + h.o.t.$$

and $\overline{\lambda}$. This is exactly what we already obtained at Lemma 4.2, with the explicit dependency of coefficient b on the Reynolds number ($a = a_3, b = b_4\mathfrak{R}^2$).

For the computation let us define the coefficients as

$$\Phi = \sum_{m\geq 0, |p|+|q|\geq 1} \Phi^{(m)}_{pq}(x,z)\tau^m A^{p_0}\overline{A}^{q_0}(\partial_y A)^{p_1}(\partial_y\overline{A})^{q_1}(\partial_y^2 A)^{p_2}(\partial_y^2\overline{A})^{q_2}..$$

where

$$\begin{aligned} p &= (p_0, p_1, p_2, ...) \\ q &= (q_0, q_1, q_2, ...) \end{aligned}$$

are multi-indices, and

$$\Phi^{(m)}_{pq} = \overline{\Phi^{(m)}_{qp}}$$

since U and Φ are real vector fields, and

$$\Phi^{(0)}_{10} = \Phi^{(0)}_{01} = 0$$

since the corresponding linear terms are already taken into account in (47). The method consists in identifying monomials when we write (48) using the expression (47) and replace $\frac{\partial A}{\partial t}$ by (49). For the derivation we need to rewrite the operator $\mathbf{L}_0$ as follows

$$\begin{aligned} \mathbf{L}_0 U &= \Delta_\perp U + \nabla_\perp p + \mathfrak{R} x\partial_y U + \begin{pmatrix} T_c u_y \\ u_x \\ 0 \end{pmatrix}, \\ 0 &= \nabla_\perp \cdot U_\perp + \mathfrak{R}\partial_y u_y, \end{aligned}$$

where the subscript $\perp$ means components (x, z). Then, using $\mathbf{L}_0\zeta = 0$, $\mathbf{L}_0(A\zeta)$ becomes

$$\mathbf{L}_0(A\zeta) = \mathfrak{R} x(\partial_y A)\zeta + \nabla_\perp p.$$

Now, as an example, let us consider the coefficients $\Phi^{(0)}_{(01)(00)}(x,z)$ of $\partial_y A$ and $\Phi^{(0)}_{(002)(000)}(x,z)$ of $\partial_y^2 A$ in Φ. Then identifying the coefficients of $\partial_y A$ and defining

$$p = Ap_0 + \overline{A}\overline{p_0} + (\partial_y A)p_{(01)(00)} + ...$$

where (p_0, ζ) satisfy

$$\begin{aligned} \Delta_\perp\zeta + \nabla_\perp p_0 + e^{i\alpha_c z}\begin{pmatrix} T_c u^0_y \\ u^0_x \\ 0 \end{pmatrix} &= 0 \\ \nabla_\perp \cdot \zeta_\perp &= 0, \end{aligned}$$

we obtain

$$\begin{aligned} 0 &= \Delta_\perp \Phi_{(01)(00)} + \begin{pmatrix} T_c u_{y(01)(00)} \\ u_{x(01)(00)} \\ 0 \end{pmatrix} + \mathfrak{R} x \zeta + \nabla_\perp p_{(01)(00)}, \qquad (50) \\ 0 &= \nabla_\perp \cdot \Phi_{\perp(01)(00)} + \mathfrak{R} u_y^0, \ \Phi_{(01)(00)}|_{x=\pm 1/2} = 0. \end{aligned}$$

For obtaining the compatibility condition we take the scalar product of (50) with $e^{i\alpha_c z}(u_x^0, T_c u_y^0, u_z^0)^t$. We observe that $\Phi_{(01)(00)}$ is uniquely determined in assuming its orthogonality with ζ, where the scalar product $\langle U, V\rangle$ is understood as

$$\langle U, V\rangle = \langle u_x, v_x\rangle + \langle T_c u_y, v_y\rangle + \langle u_z, v_z\rangle.$$

We may observe that the coefficient of $e^{i\alpha_c z}$ for $u_{x(01)(00)}$ and $u_{y(01)(00)}$ are real and odd in x, while for $u_{z(01)(00)}$ it is pure imaginary and even in x and for $p_{(01)(00)}$ it is real and even in x.

Now the identification of the coefficients of $\partial_y^2 A$ leads to

$$\begin{aligned} -b\zeta =& \Delta_\perp \Phi_{(001)(000)} + \begin{pmatrix} T_c u_{y(001)(000)} \\ u_{x(001)(000)} \\ 0 \end{pmatrix} + \mathfrak{R} x \Phi_{(01)(00)} \qquad (51) \\ &+ \nabla_\perp p_{(001)(000)}, \\ 0 = \nabla_\perp \cdot \Phi_{\perp(001)(000)} &+ \mathfrak{R} u_{y(01)(00)}, \ \Phi_{(001)(000)}|_{x=\pm 1/2}. \end{aligned}$$

As above we take the scalar product of (51) with $e^{i\alpha_z z}(u_x^0, T_c u_y^0, u_z^0)^t$, then we obtain b in terms of $\Phi_{(01)(00)}$ given by (50). An exercise left to the reader, consists in verifying that we indeed obtain

$$b = \mathfrak{R}^2 b_4.$$

7 Appendix B: Numerical study

7.1 The axisymmetric case

This paragraph is devoted to the numerical study of the functions $\mathfrak{F}_1$ and $\mathfrak{F}_2$. The figure 6 shows the value of the smallest positive zero $T_c(\alpha)$ of $\mathfrak{F}_1(\alpha, T) = 0$ as a function of α. For positive α, we see that the function $T(\alpha)$ is convex and has a unique minimum at $\alpha_c \approx 3.117$.

The zeros of $\mathfrak{F}_2(\alpha, T) = 0$ are always larger than the zeros of $\mathfrak{F}_1(\alpha, T) = 0$, α being fixed.

7.2 The non axisymmetric case

In this section, we discuss the numerical study of the following system

$$\begin{aligned} (\lambda + \alpha^2 - D^2)(\alpha^2 - D^2)u_x + i\mathfrak{B} x(D^2 - \alpha^2)u_x &= T\alpha^2 u_y, \\ (\lambda + \alpha^2 - D^2)u_y - i\mathfrak{B} x u_y &= u_x, \end{aligned}$$

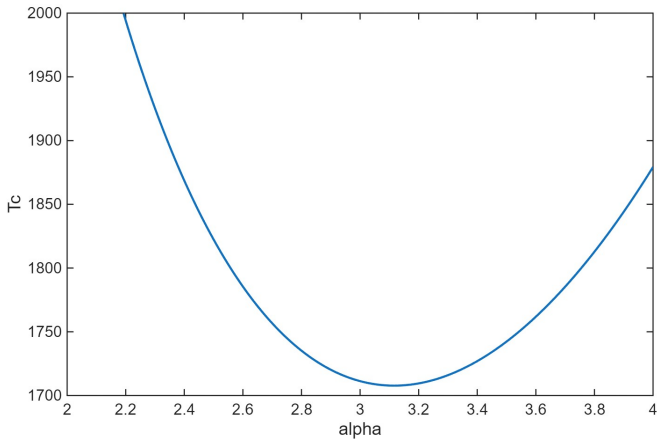


Figure 6: Value of the smallest positive zero of $T \mapsto \mathfrak{F}_1(\alpha, T)$ as a function of α

where the parameters are $\alpha^2, T, \mathfrak{B}$, and where the boundary conditions are

$$u_x = u_y = Du_x = 0, x = \pm 1/2.$$

First, we rewrite this system as a first order system on

$$U = (u_x, Du_x, D^2u_x, D^3u_x, u_y, Du_y)$$

which leads to

$$\begin{aligned} D^4u_x &= -(\lambda + \alpha^2 - i\mathfrak{B}x)(\alpha^2 - D^2)u_x + D^2\alpha^2 u_x + T\alpha^2 u_y, \\ D^2u_y &= (\lambda + \alpha^2 - i\mathfrak{B}x)u_y - u_x, \end{aligned}$$

a system that we shorten in

$$DU = AU$$

where A is the corresponding 6×6 matrix.

We consider this equation as a Cauchy problem starting at $x = -1/2$. We construct three independent solutions U^1, U^2 and U^3 defined by

$$U^1(-1/2) = (0, 0, 1, 0, 0, 0), \quad U^2(-1/2) = (0, 0, 0, 1, 0, 0)$$

and

$$U^3(-1/2) = (0, 0, 0, 0, 0, 1).$$

Any solution to our problem is a linear combination of u^1, u^2 and u^3 which satisfies

$$\delta(\lambda, \alpha, B, T) := \begin{vmatrix} u_x^1(1/2) & u_x^2(1/2) & u_x^3(1/2) \\ Du_x^1(1/2) & Du_x^2(1/2) & Du_x^3(1/2) \\ u_y^1(1/2) & u_y^2(1/2) & u_y^3(1/2) \end{vmatrix} = 0.$$

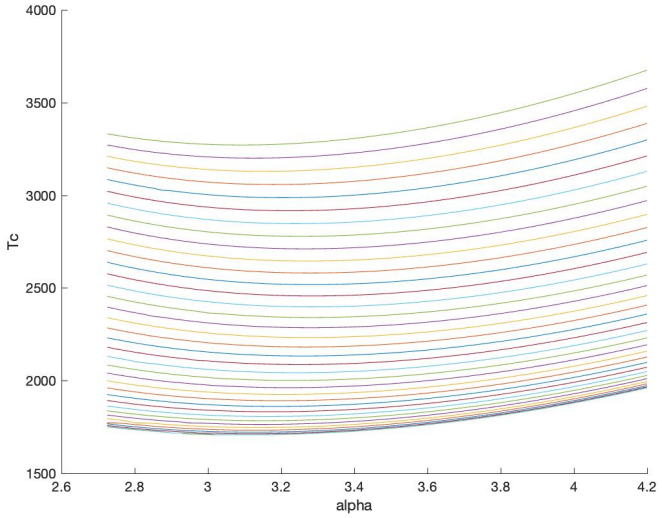


Figure 7: The critical Taylor number $T_c(\alpha, \beta)$ as a function of α for various values of $\mathfrak{B}$.

It is very easy to approximate numerically u^1, u^2 and u^3 and to compute δ.

The first step is to solve $\delta(T) = 0$, α and $\mathfrak{B}$ being given. This gives the critical Taylor number $T_c(\alpha, \mathfrak{B})$. The second step is to compute the minimum $T_c(\mathfrak{B})$ of $T_c(\alpha, \mathfrak{B})$ over α.

We note that $T_c(\mathfrak{B})$ is minimum when $\mathfrak{B} = 0$, where it equals the classical value of 1707 for $\alpha \approx 3.117$. Moreover, on the range of α which is studied on this figure, T_c is a convex function of α when $\mathfrak{B}$ is fixed.

Figure 9 displays $T_c(\alpha_c, \mathfrak{B})$ as a function of $\mathfrak{B}$, together with its quadratic approximation at $\mathfrak{B} = 0$. This in particular shows, numerically, that $a_4 < 0$.

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Conflict of interest

The authors state that there is no conflict of interest.

Data availability

Data are not involved in this research paper.

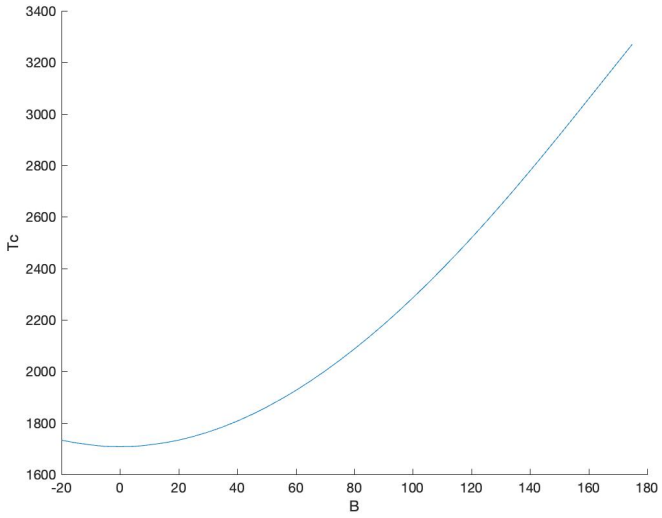


Figure 8: The critical Taylor number T_c as a function of $\mathfrak{B}$

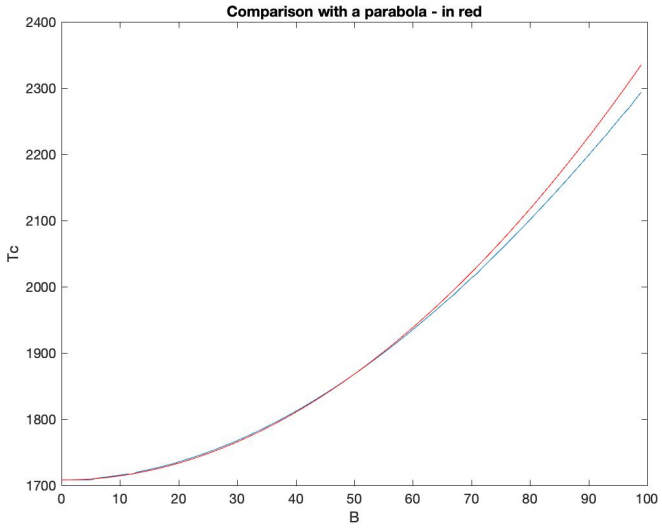


Figure 9: The critical Taylor number $T_c(\alpha_c, \mathfrak{B})$ as a function of $\mathfrak{B}$ (blue curve) and its quadratic approximation by $T_c + 0.064\mathfrak{B}^2$.

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