COMPUTING OPTICAL FLOW VIA VARIATIONAL TECHNIQUES

G. AUBERT †, R. DERICHE ‡, AND P. KORNPROBST †

Abstract

Defined as the apparent motion in a sequence of images, the optical flow is very important in the Computer Vision community where its accurate estimation is strongly needed for many applications. It is one of the most studied problem in Computer Vision. In spite of this, no much theoretical analysis has been done. In this article, we first present a synthetic review of existing variational methods. Then, we will propose an extended model that will be rigorously justified on the space of bounded variations. Finally, we present an algorithm whose convergence will be carefully demonstrated. Some results showing the capacities of this method will end that work.

Keywords

Space of bounded variations, Γ-convergence, half-quadratic minimization, optical flow, Computer Vision.

AMS Subject Classifications. 35J, 49J, 65N

1. Introduction.

This paper deals with the estimation of the movement in a sequence of images. This velocity field is be called the optical flow. In the Computer Vision community, it is well known that the optical flow is a rich source of information about the geometrical structure of the world. Numerous practical and theoretical studies on the optical flow estimation from image sequences and on the useful information it contains have been performed. They have clearly shown how the optical flow can be used to recover information about slant and tilt of surface elements, ego-motion, shape information, time to collision, etc [30, 31, 29, 33, 32, 28, 27, 45, 56, 20, 44, 55, 26, 8, 49, 51].

Almost all these approaches use the classical brightness constancy assumption that relates the gradient of brightness to the components of the local flow to estimate the optical flow. Because this problem is ill-posed, additional constraints are usually required. The most used one is to add a quadratic smoothness constraint as done originally by Horn and Schunk [28]. However, in order to estimate the optical flow more accurately, other constraints involving high order spatial derivatives have also been used [49]. Nevertheless, several of the proposed methods lacked robustness to the presence of occlusion, and yielded smooth optical flow. The variational approach proposed in this paper is motivated by the need to recover the optical flow while preventing the method from trying to smooth the solution across the flow discontinuities.

This article is organised as follows:

Section 2 is a general introduction to the optical flow problem. The purpose is to define properly what can we expect to find and how. A review of existing variational methods will be done. In section 3, we propose a general variational

† Laboratoire J.A. Dieudonné - UNSA - UMR n° 6621 du CNRS - 06108 Nice Cedex 2 - FRANCE

‡ INRIA - 2004 Route des Lucioles, BP 93 - 06902 Sophia-Antipolis Cedex - FRANCE

method which permits to regularize the velocity field while keeping its dicontinuities. Then, after some general recalls about the space of bounded variations, we will prove the existence and the uniqueness of the solution in that space. Section 4 aims at proposing a convergent algorithm to approximate the solution. To this end, we will use the theory of the Γ-convergence and some duality arguments. We conclude in Section 5 by giving some results showing the capacities of the approach.

2. Computing optical flow via variational techniques : an overview.

2.1. Definition. Let us consider a concrete situation : someone is shooting a scene in the street (see Figure 2.1). An easy way to understand the link between

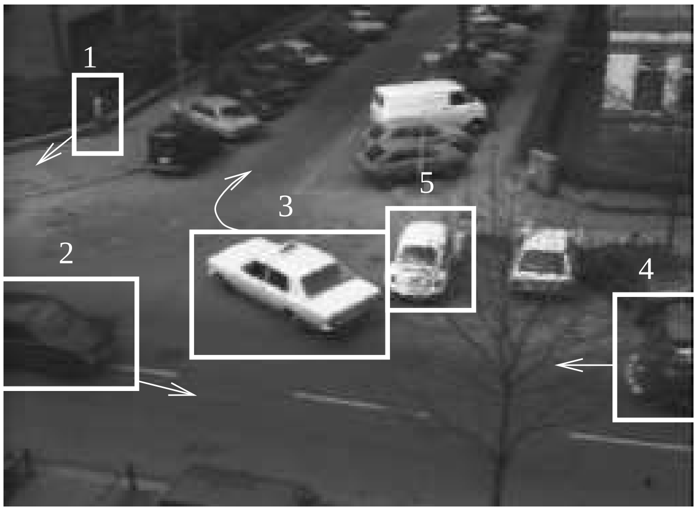


FIG. 2.1. Example of a real scene available via anonymous ftp from `ftp.csd.uwo.ca` in the directory `pub/vision`. Four objects are moving. **1** : one pedestrian, **2** and **3** : two cars **4** : a van, **5** is static but a precise observation permits to see changes of bright due to noise.

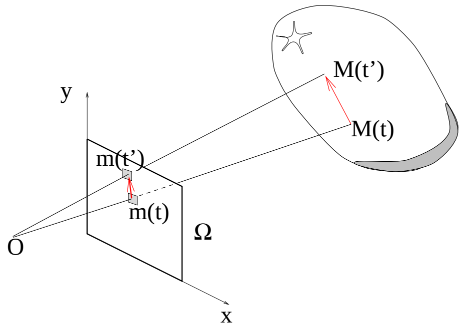


FIG. 2.2. The pinhole camera model : one of the easier one. Ω is the domain of the image and O is the optical center.

what we can observe and the real 3D movement is to modelize the camera as a simple projective model (see Figure 2.2). So the first idea is to say that the 2D velocity field in the image corresponds to the projection of the 3D velocity field of the objects. This is the case for the car **3**. However, when we look closer, we notice the shadow in the back of the car which follows it. It is clear that this apparent motion does not corresponds to any real motion. The importance of the light source can be seen from other phenomenas. For instance, if the object is sparkling, the reflected luminosity will change rapidely with the position. This is be the case for bodywork, glasses,. . . This problem is encountered for the glasses of the van **4**. Finally, notice the problem of noise in images which is unavoidable. For instance, if we look at the car **5** which is static, we will observe some random changes of brightness due to noise in the sequence. We conclude that the variations of the intensity do not always correspond to physical movements.

So, we will define the optical flow as the 2D velocity field describing the changes in intensity between images. We see in the next section how we can traduce it mathematically.

2.2. The optical flow constraint. In this last decade, numerous methods have been proposed to compute optical flow. Several ideas have been used : working with regions, curves, lines or points. There is also a wide range of methodologies : wavelets, Markov random fields, Fourier analysis and naturally partial differential equations [28, 27, 45, 56, 20, 44, 55, 26, 13, 49, 51]. We refer the interested reader to two (mainly computational) general surveys :

- Barron, Fleet and Beauchemin [8] explain the main different techniques and do numerical quantitative experiments to compare them (the database used for tests is also available).
- Orkisz and Clarysse [48] propose an "updated" version of the preceding one.

In this article we will concentrate upon the class of *differential methods* (as named by Barron, Fleet and Beauchemin) which have been proved to be among the best one [8]. Their common point is the consistency intensity hypothesis of a point during its movement. More precisely, we will assume that :

$$\text{“The intensity of a point keeps constant along its trajectory”} \tag{2.1}$$

This hypothesis is called the *optical flow constraint* (noted in the sequel OFC). We can consider it as reasonable for small displacements for which changes of the light source are small.

Let us traduce (2.1) mathematically. Let $u(x, y, t)$ denote the intensity of the pixel (x, y) at time t. Starting from a point (x_0, y_0) at the time t_0, we define the trajectory :

$$t \mapsto (x(t), y(t), t)$$

such that :

$$(x(t_0), y(t_0), t_0) = (x_0, y_0, t_0) \qquad \forall t \tag{2.2}$$

and

$$u(x(t), y(t), t) = u(x_0, y_0, t_0) \qquad \forall t \tag{2.3}$$

By differentiating (2.3) with respect to t, we obtain :

$$\frac{dx}{dt}(t_0)\frac{\partial u}{\partial x}(x_0, y_0, t_0) + \frac{dy}{dt}(t_0)\frac{\partial u}{\partial y}(x_0, y_0, t_0) + \frac{\partial u}{\partial t}(x_0, y_0, t_0) = 0 \tag{2.4}$$

So we will search the optical flow as the velocity field :

$$\sigma(x_0, y_0) = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} (x_0, y_0) \equiv \begin{pmatrix} \frac{dx}{dt}(t_0) \\ \\ \frac{dy}{dt}(t_0) \end{pmatrix}$$

such that (2.4) is true.

Recapitulation: let the sequence $u(x,y,t)$ given and t_0 the time of observation. We aim at finding the optical flow $\sigma(x,y)$ $((x,y) \in \Omega)$, that is to say the instantaneous appearent velocity at time t_0 verifying the optical flow constraint :

$$\sigma(x,y) \cdot \nabla u(x,y,t_0) + u_t(x,y,t_0) = 0 \tag{2.5}$$

Unfortunately, this equation is scalar which is not enough to find both components of the velocity field. This problem is usually designed as the *aperture problem.* Other conditions should then be found. This induces several ideas that are presented in the following section.

Remark : Is the OFC unavoidable? Even if it is widely used to compute optical flow, several reasons may force us to look for something different. Let us mention three cases :

(i) We want to take into account possible changes in the light source which may turn out the velocities to be false. The models proposed by Negahdaripour & Yu [46], or Mattavelli & Nicoulin [38] permit an affine variation of the intensity during time (and not a conservation).

(ii) We consider a special kind of movement which requires suitable conditions. For instance, Devlaminck & Dubus [17] propose an analogy with the theory of elasticity to find their constraint. Wildes, Amabile etal [57] are interested in fluid movements. Alatan & Onural [2] only consider rigid displacements...

(iii) We want to avoid to differentiate (2.3) to get the OFC. A possible solution is to say that σ should satisfy :

$$u(x_0 + \sigma_1(x_0,y_0)(t-t_0), y_0 + \sigma_2(x_0,y_0)(t-t_0), t_0) \approx u(x_0,y_0,t_0)$$

for t close to t_0. We refer to the works of Mémin, Perez etal [40] or Guichard & Rudin [24] for more details.

To conclude this remark, let us also mention the work of Willick & Yang [58], where we can find a comparison between some possible contraints. ♣

2.3. How solving the aperture problem?. As we saw in the preceding section, the optical flow constraint is not enough to compute the optical flow. Several ideas have been proposed.

2.3.1. Use second order derivatives. For instance, one could impose the conservation of $\nabla u(x,y,t)$ along trajectories that is to say :

$$\frac{d\nabla u}{dt}(x,y,t) = \mathbf{0}$$

This is a stronger restriction than (2.5) on permissible motion fields. This implies that first order deformations of intensity (eg. rotation or dilation) should not be present. This condition can be re-written in the following form :

$$\begin{bmatrix} u_{xx}(x,y,t) & u_{yx}(x,y,t) \\ u_{xy}(x,y,t) & u_{yy}(x,y,t) \end{bmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.6}$$

These conditions can be used alone or together with the optical flow constraint. Several possibilities are then proposed. We quote the works of Otte & Nagel [49] and Tistarelli [54]. However, this kind of method is often noise sensitive because we need to compute second derivatives.

2.3.2. Choosing a model of velocity. This permits to diminish the number of unknowns. We refer to the recent work of Mémin & Perez [39] for more details about the different possibilities (piecewise constant, affine,...). Several numerical experiments are also proposed to compare the models.

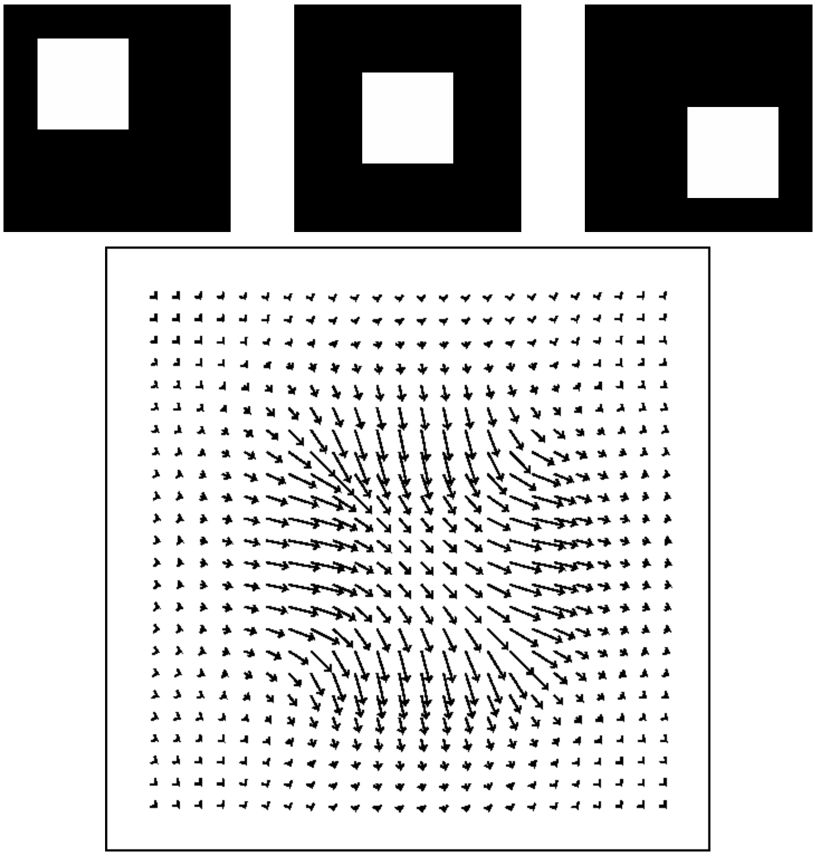


FIG. 2.3. The method of Horn & Schunk applied on a synthetic example. One may observe that the discontinuities at edge locations are lost.

2.3.3. Regularizing the velocity field. Among the first one, Horn & Schunk [28] proposed to solve the following problem :

$$\min_{\sigma} \underbrace{\int_{\Omega} (\sigma \cdot \nabla u + u_t)^2 dx}_{\mathtt{A}} + \alpha^r \underbrace{\int_{\Omega} (\|\nabla \sigma_1\|^2 + \|\nabla \sigma_2\|^2) dx}_{\mathtt{B}} \tag{2.7}$$

where α^r is a constant and $\| \cdot \|$ is the usual euclidian norm. In other words, we search for the velocity field σ fitting the best the optical flow constraint (term `A`), and so that the derivatives are low (term `B`). This kind of penalty term has been introduce by Tikhonov & Arsenin [53] and is well known to smooth isotropically without taking into account the discontinuities of the flow field (see Figure 2.3 for a typical example).

Since then, many research have been done to change the regularization term `B`. We describe below some of the most significant one :

- Modifying Horn & Schunk functional by introduction of robust norms was pioneered by Black [9]. Since then, many authors worked on that. The idea is to change the regularization term into :

$$\int_{\Omega} \phi(\|\nabla \sigma_1\|) dx + \int_{\Omega} \phi(\|\nabla \sigma_2\|) dx$$

where functions ϕ permit noise removal and edge conservation. Let us give some examples. Cohen [13] or Kumar, Tannenbaum and Balas [35] use the L^1 norm, that is to say the Total Variation ($\phi(t) = t$). Deriche, Kornprobst and Aubert [16] used more general functions to preserve discontinuities. This kind of ideas is also called robust estimators in the stochastic background. In that direction, we mention the works of Mémin, Perez etal [40] who worked with Markov random fields.

- Gupta & Prince [25] or Guichard & Rudin [24] add some penalty terms based on the divergence or the rotational of the flow field :

$$\int_\Omega \varphi(div(\sigma), rot(\sigma))dx$$

where several possibilities for φ may be proposed. For instance, in [24], the regularization term is only :

$$\int_\Omega |div(\sigma)|dx$$

In this case, the idea is to note that rigid 2D objects in 2D motions have a divergence free motion. The divergence is non zero only at the boundaries of occlusions where it looks like a concentrated measure.

- Nagel & Enkelmann [45] propose to smooth in the direction of the edges in the sequence. The idea is to penalize the variation of the displacement vector in the direction perpendicular to the gradient of the intensity. So the penalty term is of the form :

$$\int_\Omega |(u_{x_2}, -u_{x_1})^T \cdot \nabla\sigma|dx$$

- Nési [47] adapt the formulation of Horn & Schunk introducing the length of the discontinuity set of σ (noted $|S_\sigma|$). We recall that this kind of idea has been introduced by Mumford & Shah for image segmentation [43]. The regularization term is of the form :

$$\alpha^r \int_\Omega (\|\nabla\sigma_1\|^2 + \|\nabla\sigma_2\|^2)dx + \alpha^d|S_\sigma|$$

Numerically, the main difficulty is to approximate the last term. One possible solution is to use the concept of Γ-convergence (see [21, 37] for more details). We introduce a sequence of functionals so that the sequence of minimizers converge to the unique minimum of initial functional. Typically, the way to approximate the regularization term is :

$$\alpha^r \int_\Omega z^2(\|\nabla\sigma_1\|^2 + \|\nabla\sigma_2\|^2)dx + \alpha^d \int_\Omega \left(\frac{\|\nabla z\|^2}{k} + \frac{k(1-z)^2}{4}\right)$$

where z is an additional function and k is a parameter that will tend to infinity. The function z can be considered as a control variable which equals to zero near discontinuities and close to 1 in homogeneous regions.

Naturally, this list is non exhaustive. However, we can observe that not much theoretical analysis has been done. This is why we propose in the next section a model that will be rigorously justified and that is a natural extension of previous work done in [16].

3. A justified variational approach.

3.1. Construction of the model.

3.1.1. The optical flow constraint. We will choose the L^1-norm instead of the L^2 norm of the OFC as done by Horn & Schunk [28]. This choice is not fundamental in the method but it will be justified later by theoretical arguments.

3.1.2. The regularization part. To cope with discontinuities, several methods have been proposed [13, 45, 51, 26, 16]. The method presented here is inspired from a recent framework that has proven to be very useful in some image processing tasks as image restoration [52, 15, 4, 7]. The key idea is to forbid regularizing and smoothing across discontinuities. One way of taking into account these remarks is to replace $\|\nabla w\|^2$ in (2.7) (where w is σ_1 or σ_2) by $\phi(\|\nabla w\|)$ where $\phi(\cdot)$ having desired properties. To define and identify such functions, we consider the corresponding term in the Euler-Lagrange equations.

$$\begin{array}{rcl} \|\nabla w\|^2 & \rightarrow & 2\triangle w \\ \phi(\|\nabla w\|) & \rightarrow & div\left(\dfrac{\phi'(\|\nabla w\|)}{\|\nabla w\|}\nabla w\right) \end{array} \tag{3.1}$$

If we denote $\eta = \dfrac{\nabla w}{\|\nabla w\|}$ and ξ the normal direction to η, we can develope formally the divergence term in the following form :

$$div\left(\frac{\phi'(\|\nabla w\|)}{\|\nabla w\|}\nabla w\right) = \underbrace{\frac{\phi'(\|\nabla w\|)}{\|\nabla w\|}}_{c_\xi} w_{\xi\xi} + \underbrace{\phi''(\|\nabla w\|)}_{c_\eta} w_{\eta\eta} \tag{3.2}$$

where $w_{\eta\eta}$ (resp. $w_{\xi\xi}$) is the second order directional derivative of w in the direction η (resp. ξ). In order to regularize the solution and preserve optical flow discontinuities, one would like to smooth isotropically the optical flow field inside homogeneous regions and preserve the flow discontinuities in the inhomogeneous regions. Assuming that the function $\phi''(.)$ exists, the isotropic smoothing condition inside homogeneous regions can be achieved by imposing the following conditions on the $\phi(.)$ function :

$$\phi'(0) = 0, \qquad \lim_{\|\nabla w\|\to 0} \frac{\phi'(\|\nabla w\|)}{\|\nabla w\|} = \lim_{\|\nabla w\|\to 0} \phi''(\|\nabla w\|) = \phi''(0) > 0 \tag{3.3}$$

Therefore, at the points where $\|\nabla w\|$ is small, the divergence term becomes :

$$div\left(\frac{\phi'(|\nabla w|)}{\|\nabla w\|}\nabla w\right) \approx \phi''(0)(w_{\xi\xi} + w_{\eta\eta}) = \phi''(0)\triangle w. \tag{3.4}$$

In order to preserve the flow discontinuities near inhomogeneous regions presenting a strong flow gradient, one would like to smooth along the isophote (curve with constant flow) and not across them. This leads to stopping the diffusion in the gradient direction η, i.e. setting the weight $\phi''(\|\nabla w\|)$ equal to 0, while keeping a stable diffusion along the direction orthogonal ξ, i.e. setting the weight $\dfrac{\phi'(\|\nabla w\|)}{\|\nabla w\|}$ equal to some positive constant :

$$\lim_{\|\nabla w\|\to\infty} \phi''(\|\nabla w\|) = 0 \qquad \lim_{\|\nabla w\|\to\infty} \frac{\phi'(\|\nabla w\|)}{\|\nabla w\|} = \beta > 0 \tag{3.5}$$

Unfortunately, the two conditions of (3.5) cannot be satisfied simultaneously by a function $\phi(\cdot)$. However, the following conditions can be imposed in order to decrease the effects of the diffusion along the gradient more rapidly than those associated with the diffusion along the isophotes :

$$\begin{array}{l} \lim_{\|\nabla w\|\to\infty} \phi''(\|\nabla w\|) = \lim_{\|\nabla w\|\to\infty} \dfrac{\phi'(\|\nabla w\|)}{\|\nabla w\|} = 0 \\ \lim_{\|\nabla w\|\to\infty} \dfrac{\phi''(\|\nabla w\|)}{\frac{\phi'(\|\nabla w\|)}{\|\nabla w\|}} = 0 \end{array} \tag{3.6}$$

The conditions given by Equations (3.3) and (3.6) are those which one has to impose in order to deal with a regularization process which preserves the discontinuities. Several functions have been proposed in literature and we refer to table 3.1 for some examples. We notice that, among discontinuities preserving $\phi(\cdot)$ functions, there are convex and non-convex functions.

Author	$\phi(s)$
Geman et Reynolds	$\frac{s^2}{1+s^2}$
Malik et Perona	$\log(1+s^2)$
Green	$2\log[\cosh(s)]$
Aubert	$2\sqrt{1+s^2}-2$

Tab. 3.1
Some functions ϕ preserving discontinuities.

3.1.3. The homogeneous regions. When we have an homogeneous region characterized by low image gradients magnitude, no visible motion should be detected locally. This will be enforced by adding a term of the form :

$$\int_\Omega c(x)\|\sigma\|^2 dx$$

where $c(x)$ is a given function penalizing homogeneous regions. Typically, $c(x)$ is high for low spatial gradients of u (hence penalizing velocities in poor information zones) and low for high spatial gradients of u (no intervention). Precise assumptions will be given in the sequel.

3.1.4. The variational problem. Combining observations of the preceding sections lead us to consider the optical flow problem as the minimum of an energy. Given a sequence $u(x,y,t)$ described locally by its spatial and temporal derivatives at a fixed time t_0 (noted ∇u and u_t), we search for the velocity field σ which realizes the minimum of the energy :

$$E(\sigma) = \int_\Omega |\nabla u\cdot\sigma + u_t| dx + \alpha^r\left[\int_\Omega \phi(D\sigma_1) + \int_\Omega \phi(D\sigma_2)\right] + \alpha^h\int_\Omega c(x)\|\sigma\|^2 dx \tag{3.7}$$

where α^r, α^h are positive constants. Remark that we used the notation $D\cdot$ for the distributional derivative since we will work with functions in the space of bounded variations. The notation $\int_\Omega \phi(\nabla w)$ is formal here and will be made precise in the sequel.

3.2. The space of bouded variations. In this section we only recall main notations and definitions. We refer to [5] for more complete recalls and to [1, 18, 22, 19, 59] for the complete theory.

Let Ω be a bounded open set in R^N, with Lipschitz-regular boundary $\partial\Omega$. We denote by $\mathcal{L}^N$ or dx the N-Lebesgue dimensional measure in R^N and by $\mathcal{H}^\alpha$ the $\alpha-$dimensional Hausdorff measure. We also set $|E| = \mathcal{L}_N(E)$, the Lebesgue measure of a measurable set $E \subset R^N$. $\mathcal{B}(\Omega)$ denotes the family of the Borel subsets of Ω. We will respectively denote the strong, the weak and weak$\star$ convergences in a space $V(\Omega)$ by $\xrightarrow[V(\Omega)]{}$, $\xrightarrow[V(\Omega)]{}$, $\xrightarrow[V(\Omega)]{\star}$. Spaces of vector valued functions will be noted by bold characters.

Working with images requires that the functions that we consider can be discontinuous along curves. This is impossible with classical Sobolev spaces such as

$W^{1,1}(\Omega)$. This is why we need to use the space of bounded variations (noted $BV(\Omega)$) defined by :

$$BV(\Omega) = \left\{ u \in L^1(\Omega); sup \int_\Omega u div\varphi dx < \infty : \varphi \in \mathcal{C}_0^1(\Omega)^2, |\varphi|_\infty \leq 1 \right\}$$

where $\mathcal{C}_0^1(\Omega)$ is the set of differentiable functions with compact support in Ω. We will denote by :

$$|Du|(\Omega) = sup\{ \int_\Omega u div\varphi dx : \varphi \in \mathcal{C}_0^1(\Omega)^2, |\varphi|_\infty \leq 1\}$$

If $u \in BV(\Omega)$ and Du the gradient in the sense of distributions, then Du is a vector valued Radon measure and $|Du|(\Omega)$ is the total variation of Du on Ω. The set of Radon measure is noted $\mathcal{M}(\Omega)$

The product topology of the strong topology of $L^1(\Omega)$ for u and of the weak$\star$ topology of measures for Du will be called the weak$\star$ topology of BV, and will be denoted by $BV - w\star$.

$$u^n \underset{BV-w\star}{\longrightarrow} u \iff \left\{ \begin{array}{ccc} u^n & \underset{L^1(\Omega)}{\longrightarrow} & u \\ Du^n & \overset{\star}{\underset{\mathcal{M}(\Omega)}{\longrightarrow}} & Du \end{array} \right. \tag{3.8}$$

We recall that every bounded sequence in $BV(\Omega)$ admits a subsequence converging in $BV - w\star$.

We define by the approximate upper limit $u^+(x)$ and the approximate lower limit $u^-(x)$ by :

$$u^+(x) = inf \left\{ t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{\{u > t\} \cap B_\rho(x)}{\rho^N} = 0 \right\}$$
$$u^-(x) = sup \left\{ t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{\{u < t\} \cap B_\rho(x)}{\rho^N} = 0 \right\}$$

where $B_\rho(x)$ is the bowl of center x and radius ρ. We denote by S_u denotes the jump set, that is to say the complement of the set of Lebesgue points, *i.e.* the set of points x where $u^+(x)$ is different $u^-(x)$, namely :

$$S_u = \{x \in \Omega / u^-(x) < u^+(x)\}.$$

After choosing a normal $n_u(x)$ $(x \in S_u)$ pointing toward the largest value of u, we recall the following decompositions ([3] for more details):

$$Du = \nabla u \cdot \mathcal{L}_N + C_u + (u^+ - u^-) n_u \cdot \mathcal{H}^{N-1}_{|S_u} \tag{3.9}$$

$$|Du|(\Omega) = \int_\Omega |\nabla u| dx + \int_{\Omega \setminus S_u} |Cu| + \int_{S_u} (u^+ - u^-) d\mathcal{H}^{N-1} \tag{3.10}$$

where ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure. We then recall the definition of a convex function of measures. We refer to the works of Goffman-Serrin [23] and Demengel-Temam [14] for more details. Let ϕ be convex and finite on R with a linear growth at infinity. Let ϕ^∞ be the asymptote (or recession) function.

$$\phi^\infty(z) := \lim_{t \to \infty} \frac{\phi(tz)}{t} \in [0; +\infty).$$

Then, for $u \in BV(\Omega)$, using classical notations, we define:

$$\int_\Omega \phi(Du) = \int_\Omega \phi(|\nabla u|) dx + \phi^\infty(1) \int_{S_u} (u^+ - u^-) d\mathcal{H}^{N-1} + \phi^\infty(1) \int_{\Omega \setminus S_u} |C_u| \tag{3.11}$$

We finally mention that this function is lower semi-continuous for the $BV - w\star$-topology.

3.3. A mathematically justified model. In all this article, we will assume that the data is Lipschitz :

$$u \in W^{1,\infty}(R \times \Omega) \tag{3.12}$$

This assumption is realistic on a numerical point of view because a pre-smoothing is always necessary to diminish noise influence.

Remark : One other interesting case is to assume only :

$$u \in BV(\Omega)(R \times \Omega)$$

The consequence is that the OFC has to be interpreted as a measure. The L^1-norm is in fact a total variation. This case is treated in [6]. Roughly speaking, we need first to find an integral representation for the functional (and especially for the term $(\sigma \cdot Du)$). Unfortunately, we will prove that this functional is not lower semi continuous for the $BV - w\star$-topology so that searching for the relaxed problem is necessary. We refer to [6] for more details. ♣

We recall the energy (3.7) we want to minimize :

$$E(\sigma) = \int_\Omega |\sigma \cdot \nabla u + u_t| dx + \alpha^r \left[\int_\Omega \phi(D\sigma_1) + \int_\Omega \phi(D\sigma_2) \right] + \alpha^h \int_\Omega c(x) \|\sigma\|^2 dx$$

where we assumed :

$$\phi : R \rightarrow R^+ \text{ is an odd, convex and non-decreasing function on } R^+ \tag{3.13}$$

$$\begin{gathered}\text{There exists constants } b_1 > 0 \text{ and } b_2 \geq 0 \text{ such that} \\ b_1 x - b_2 \leq \phi(x) \leq b_1 x + b_2 \text{ for all } x \in R^+\end{gathered} \tag{3.14}$$

and $c(x)$ verifies :

$$c \in \mathcal{C}^\infty(\Omega) \tag{3.15}$$

$$\text{There exists } m_c > 0 \text{ such that } c(x) \in [m_c, 1] \text{ for all } x \in \Omega \tag{3.16}$$

We have the following theorem :

THEOREM 3.1. *Under the hypotheses (3.13)-(3.14), (3.15)-(3.16)and (3.12), the minimization problem :*

$$\inf_{\sigma \in \mathbf{BV}(\Omega)} E(\sigma) = \int_\Omega |\sigma \cdot \nabla u + u_t| dx + \sum_{j=1}^{2} \int_\Omega \phi(D\sigma_j) + \int_\Omega c(x) \|\sigma\|^2 dx \tag{3.17}$$

admits a unique solution in $\mathbf{BV}(\Omega)$.

Proof. According to (3.14), the functional E is coercive on $BV(\Omega)$. Thus, we can uniformly bound the minimizing sequences and extract a converging subsequence for the $BV - w\star$ topology. Since E is lower semi continuous for this topology, we easily deduce the existence of a minimum. □

As pointed out in the previous remark, assuming that the data is a Lipschitz function permits to justify clearly the approach. Now, the problem is to get an approximation of the solution. This is the purpose of the following section.

4. A convergent algorithm.

In this section, we propose and justify a convergent algorithm. Two steps are necessary :

- in section 4.1, we introduce a new functional noted E_ϵ so that the associated minimization problem admits a unique solution (noted σ_ϵ) in $\mathbf{W^{1,2}}(\Omega) = \{\sigma \in \mathbf{L^2}(\Omega)/\nabla\sigma_j \in \mathbf{L^2}(\Omega)\}$. We prove, via the $\Gamma-$convergence, that the solution σ_ϵ converges in $\mathbf{L^2}$-strong to the minimizer of E.

- then, for a fixed ϵ, we propose in section 4.2 a suitable numerical scheme called the half-quadratic minimization. We prove its convergence in $\mathbf{L^2}$-strong to the minimizer of E_ϵ.Consequently, merging both results permits us in fact to construct a solution that converge in $\mathbf{L^2}$-strong to the minimizer of (3.17).

4.1. A result of Γ convergence. For a function f verifying hypotheses (3.13)-(3.14), let us define the function f_ϵ by :

$$f_\epsilon(t) = \begin{cases} \dfrac{f'(\epsilon)}{2\epsilon}t^2 + f(\epsilon) - \dfrac{\epsilon f'(\epsilon)}{2} & \text{if } t \leq \epsilon \\ f(t) & \text{if } \epsilon \leq t \leq 1/\epsilon \\ \dfrac{\epsilon f'(1/\epsilon)}{2}t^2 + f(1/\epsilon) - \dfrac{f'(1/\epsilon)}{2\epsilon} & \text{if } t \geq 1/\epsilon \end{cases} \tag{4.1}$$

We have, for all ϵ, $f_\epsilon \geq f$, and for all t, $\lim_{\epsilon\to 0} f_\epsilon(t) = f(t)$. Using that definition, we denote by $\phi_{1,\epsilon}$ and $\phi_{2,\epsilon}$ the functions associated to $|t|$ and $\phi(t)$ respectively. Now, let us define the functional E_ϵ by :

$$E_\epsilon : \mathbf{BV}(\Omega) \to R \tag{4.2}$$

$$E_\epsilon(\sigma) = \begin{cases} \displaystyle\int_\Omega \phi_{1,\epsilon}(\sigma\cdot\nabla u + u_t)dx + \sum_{j=1}^{2}\int_\Omega \phi_{2,\epsilon}(\|\nabla\sigma_j\|)dx + \int_\Omega c(x)\|\sigma\|^2 dx \\ \qquad \text{if } \sigma \in \mathbf{W^{1,2}}(\Omega) \\ +\infty \quad \text{otherwise} \end{cases}$$

We are going to establish that the sequence E_ϵ Γ-converges to E. We refer the interested reader to [21, 37] for the $\Gamma-$convergence theory.

Proposition 4.1. *Under the hypotheses (3.13)-(3.14), (3.15)-(3.16) and (3.12), the sequence E_ϵ defined in (4.2), Γ-converges to E as ϵ tends to 0 for the $\mathbf{L^2}$-strong topology.*

Proof. Let us denote by E_3 the functional defined by :

$$E_3 : \mathbf{BV}(\Omega) \to R$$
$$E_3(\sigma) = \begin{cases} E(\sigma) & \text{if } \sigma \in \mathbf{W^{1,2}}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

By construction, we observe that E_ϵ is a decreasing sequence converging pointwise to E_3. Thanks to [37] (Proposition 5.7), we deduce that E_ϵ Γ-converges to the lower semicontinuous envelope of E_3 in $\mathbf{BV}(\Omega)$ that is to say the relaxed functional (see [11, 37] for more details). Let show that $R(E_3)(\sigma) = E(\sigma)$. Since E is lower semicontinuous for the $BV - w\star$ topology, it is enough to prove that E is the lower semicontinuous envelope of E_3. This is equivalent to say that for σ in $\mathbf{BV}(\Omega)$, there exists a sequence $(\sigma^n) \in \mathbf{W^{1,2}}(\Omega)$ such that :

$$\sigma^n \xrightarrow[L^2(\Omega)]{} \sigma \quad \text{and} \quad E(\sigma) = \liminf_{n\to\infty} E_3(\sigma^n)$$

Such a sequence can be estimated by a slightly modified version of the Theroem 2.2 in [14]. This concludes the proof. □

PROPOSITION 4.2. *Under the hypotheses (3.13)-(3.14), (3.15)-(3.16) and (3.12), the minimization problem :*

$$\underset{\sigma \in \mathbf{W^{1,2}}(\Omega)}{inf} \quad E_\epsilon(\sigma) \tag{4.3}$$

admits a unique solution noted σ_ϵ. Moreover, the sequence σ_ϵ converges for the $\mathbf{L^2}$-strong topology to the unique minimizer of E in $\mathbf{BV}(\Omega)$.

Proof. The existence and unicity of the minimization problem follows by classical techniques. As for the convergence of the sequence σ^ϵ, it is a direct application of general Γ-convergence properties [21, 37]. □

4.2. The half-quadratic minimization. Let σ_ϵ be the unique minimizer of (4.3). Our problem is to find an estimation of σ_ϵ. The main difficulty of solving directly the associated Euler-Lagrange equations is that they are nonlinear. To avoid this, the idea is to use the following "duality" result, due to Aubert etal [4] :

THEOREM 4.3. *Let $\varphi : [0,+\infty[\to [0,+\infty[$ be such that :*

$$\varphi(\sqrt{x}) \quad \textit{is concave on }]0,+\infty[. \tag{4.4}$$

Let L and M be defined as : $L = \lim_{x\to+\infty} \frac{\phi'(x)}{2x}$ and $M = \lim_{x\to 0^+} \frac{\phi'(x)}{2x}$. Then, there exists a convex and decreasing function $\psi :]L,M] \to [\beta_1, \beta_2]$ such that

$$\varphi(x) = \inf_{L\le d\le M} (dx^2 + \psi(d)) \tag{4.5}$$

where : $\beta_2 = \lim_{x\to+\infty} \left(\phi(x) - x^2 \frac{\phi'(x)}{2x}\right)$ and $\beta_1 = \lim_{x\to 0_+} \phi(x)$ Moreover, for every fixed $x \ge 0$ the value d_x for which the minimum is reached is unique and given by :

$$d_x = \frac{\phi'(x)}{2x} \tag{4.6}$$

The additional variable is usually called the *dual variable*. Let us apply Theorem 4.3 to the functions ϕ^1_ϵ and ϕ^2_ϵ which fulfil desired hypothesis. We then introduce the functional E^d_ϵ defined by :

$$E^d_\epsilon : \mathbf{W^{1,2}}(\Omega) \times L^2(\Omega) \times \mathbf{L^2}(\Omega) \to R \tag{4.7}$$

$$E^d_\epsilon(\sigma,a,b) = \int_\Omega \left(a(\sigma\cdot\nabla u + u_t)^2 + \frac{1}{a}\right)dx + \sum_{j=1}^{2}\int_\Omega \left(b_j\|\nabla\sigma_j\|^2 + \Psi^2(b_j)\right)dx$$
$$+ \int_\Omega c(x)\|\sigma\|^2 dx$$

where a is the the scalar dual function associated to the optical flow constraint and where b is the vectorial dual function whose components are associated to $\phi^2_\epsilon(\|\nabla\sigma_j\|)$. This transformation is useful because :

(i) the OFC part is now differentiable in zero,

(ii) the new regularization term permits us to consider the problem in $\mathbf{W^{1,2}}(\Omega)$.

To minimize E^d_ϵ with respect to all variables, we perform minimizations with respect to each variable alternatively. Notice that this functional, defined on an extended domain is nor convex nor quadratic with respect to σ, a, b. More precisely, for a given $(\sigma^0, a^0, b^0) \in \mathbf{W^{1,2}}(\Omega) \times L^2(\Omega) \times \mathbf{L^2}(\Omega)$, we propose the following algorithm which consists in minimizing successively with respect to each variable:

$$\sigma^{n+1} = \underset{\sigma \in \mathbf{W^{1,2}}(\Omega)}{\operatorname{argmin}} \quad E^d_\epsilon(\sigma, a^n, b^n) \tag{4.8}$$

$$a^{n+1} = \underset{\substack{a \in L^2(\Omega) \\ \epsilon \le a \le 1/\epsilon}}{\operatorname{argmin}} \quad E^d_\epsilon(\sigma^{n+1}, a, b^n) \tag{4.9}$$

$$b^{n+1} = \underset{\substack{b \in \mathbf{L^2}(\Omega) \\ \epsilon\phi'_{2,\epsilon}(1/\epsilon) \le b_j \le \phi'_{2,\epsilon}(\epsilon)/\epsilon}}{\operatorname{argmin}} \quad E^d_\epsilon(\sigma^{n+1}, a^{n+1}, b) \tag{4.10}$$

Remark that the constraints $\epsilon \le a \le 1/\epsilon$ and $\epsilon\phi'_{2,\epsilon}(1/\epsilon) \le b_j \le \phi'_{2,\epsilon}(\epsilon)/\epsilon$ in (4.9)-(4.10) come from the Theorem 4.3 and will play an important role to prove that the algorithm is convergent. Our aim is to prove that the sequence σ^n converges in $\mathbf{L^2}$-strong to σ_ϵ. Let us first write optimality conditions associated to (4.8)-(4.9)-(4.10):

$$\text{For all} \quad \varphi = (\varphi_1, \varphi_2)^T \in \mathbf{W^{1,2}}(\Omega) \tag{4.11}$$

$$\int_\Omega \left(a^n(\sigma^{n+1} \cdot \nabla u + u_t)\varphi \cdot \nabla u + c(x)\sigma^{n+1} \cdot \varphi + \sum_{j=1}^{2} b^n_j \nabla\sigma^{n+1}_j \cdot \nabla\varphi_j \right) dx = 0$$

$$a^{n+1} = \frac{{\phi_{1,\epsilon}}'(|\sigma^{n+1} \cdot \nabla u + u_t|)}{2|\sigma^{n+1} \cdot \nabla u + u_t|} = \epsilon \vee \frac{1}{|\sigma^{n+1} \cdot \nabla u + u_t|} \wedge 1/\epsilon \tag{4.12}$$

$$b^{n+1}_j = \frac{{\phi_{2,\epsilon}}'(\|\nabla\sigma^n_j\|)}{2\|\nabla\sigma^n_j\|} \tag{4.13}$$

Then we have the following proposition:

Proposition 4.4. *Let $(\sigma^0, a^0, b^0) \in \mathbf{W^{1,2}}(\Omega) \times L^2(\Omega) \times \mathbf{L^2}(\Omega)$ given. Then the sequence (σ^n, a^n, b^n) defined by (4.11)-(4.12)-(4.13) is convergent. Moreover (σ^n) converges in $\mathbf{L^2}$-strong to σ_ϵ, the unique minimum of E_ϵ.*

Before proving this proposition, let us show a preliminary result.

Lemma 4.5. *Let $(\sigma^n, a^n, b^n) \in \mathbf{W^{1,2}}(\Omega) \times L^2(\Omega) \times \mathbf{L^2}(\Omega)$ a minimizing sequence of E^d_ϵ defined by (4.8)-(4.9)-(4.10). Let Ω' an open subset strictly included in Ω. Then, there exists $p' > 2$ such that:*

$$\sigma^n \in \mathbf{W^{1,p'}}(\Omega') \qquad \forall n \tag{4.14}$$

Moreover, we can find a constant M, independant of n, such that:

$$\|\sigma^n\|_{\mathbf{W^{1,p'}}(\Omega')} \le M \qquad \forall n \tag{4.15}$$

$$\|\sigma^n\|_{\mathbf{L^\infty}(\Omega')} \le M \qquad \forall n \tag{4.16}$$

Proof. In that proof, C and M will be constants, independant of n, which may be different from one line to another.

Firstly, since $\sigma^n \in \mathbf{W^{1,2}}(\Omega')$, Sobolev embeddings (see for instance [10]) permits us to find a constant C such that:

$$\|\sigma^n\|_{\mathbf{L^q}(\Omega')} \le C\|\sigma^n\|_{\mathbf{W^{1,2}}(\Omega')} \qquad \forall q \in [2, +\infty[$$

As the sequence (σ^n, a^n, b^n) is a minimizing sequence, we can find M such that $\|\sigma^n\|_{\mathbf{W^{1,2}}(\Omega)} \leq M$ and so we have :

$$\|\sigma^n\|_{\mathbf{L^q}(\Omega')} \leq M \qquad \forall q \in [2, +\infty[\tag{4.17}$$

where M only depends on q. But what about the gradient of σ^n? The idea is to use results from Meyers [41, 42] about elliptic operators. Let σ^n the solution of Euler-Lagrange equations (equivalent to the variational form (4.11) since the functional is convex) :

$$\begin{cases} div(b_1^n \nabla \sigma_1) = a^n(\sigma \cdot \nabla u + u_t)u_{x_1} + c(x)\sigma_1 \\ div(b_2^n \nabla \sigma_2) = a^n(\sigma \cdot \nabla u + u_t)u_{x_2} + c(x)\sigma_2 \end{cases}$$

In fact, we observe that the elliptic operator (left-hand side term) is uncoupled. Moreover, the right-hand side term only depends on terms of order zero which can be frozen. This remark permit us to apply a result from Meyer in the scalar case [41] : there exists $p' > 2$ and a constant $C > 0$ such that :

$$\|\nabla \sigma_j^n\|_{L^{p'}(\Omega')} < C\|a^{n-1}(\sigma^n \cdot \nabla u + u_t)u_{x_j} + c(x)\sigma_j^n\|_{L^2(\Omega')}$$

As the functions $a^{n-1}(\cdot), \nabla u$ and $c(x)$ belong to $L^\infty(\Omega)$ and that σ^n is a minimizing sequence, we can find a constant such that :

$$\|\nabla \sigma_j^n\|_{L^{p'}(\Omega')} \leq M \qquad \forall n \tag{4.18}$$

From (4.17) and (4.18), we deduce (4.14) and (4.15). As for (4.16), we use again Sobolev embedding results. In particular, we have $\mathbf{W^{1,p'}}(\Omega') \subset \mathbf{L}^\infty(\Omega')$ with a continuous embedding. So there exists C such that :

$$\|\sigma^n\|_{\mathbf{L}^\infty(\Omega')} \leq C\|\sigma^n\|_{\mathbf{W^{1,p'}}(\Omega')}$$

Hence the result (4.16). □

Proof. **of Proposition 4.4**. It contains three main stages. The first one is essentially technical. It aims at showing that the difference between two iterates tends to zero when n goes to infinity (4.22). In the second one, we write an optimality condition for σ^{n+1} and show that it tends to zero. Finally, we show in the third stage that we can pass to the limit in each term of the optimality condition so that we will recover the optimality condition associated to the problem (4.3). This will prove that, up to a subsequence, σ^n tends to σ_ϵ in $\mathbf{L^2}$-strong. Remark that since the minimum is unique, all the sequence converges.

`Stage 1` : Using the optimality conditions (4.11)-(4.12)-(4.13), some easy calculus permit us to establish that :

$$\begin{aligned} U^n \equiv & E_\epsilon^d(\sigma^n, a^n, b^n) - E_\epsilon^d(\sigma^{n+1}, a^n, b^n) \\ \geq & min(\epsilon, m_c)\|\sigma^n - \sigma^{n+1}\|^2_{\mathbf{W^{1,2}}(\Omega)} \end{aligned} \tag{4.19}$$

$$\begin{aligned} V^n \equiv & E_\epsilon^d(\sigma^{n+1}, a^n, b^n) - E_\epsilon^d(\sigma^{n+1}, a^{n+1}, b^n) \\ \geq & \epsilon^3\|a^n - a^{n+1}\|^2_{L^2(\Omega)} \end{aligned} \tag{4.20}$$

$$\begin{aligned} W^n \equiv & E_\epsilon^d(\sigma^{n+1}, a^{n+1}, b^n) - E_\epsilon^d(\sigma^{n+1}, a^{n+1}, b^{n+1}) \\ \geq & c(\epsilon)\|b^n - b^{n+1}\|^2_{\mathbf{L^2}(\Omega)} \end{aligned} \tag{4.21}$$

We first remark that the sequences $(U^n), (V^n)$ and (W^n) tends to zero as n goes to infinity : if we denote $T^n = E^d_\epsilon(\sigma^n, a^n, b^n)$, we have :

$$U^n + V^n + W^n = T^n - T^{n+1}$$

Since $(U^n), (V^n)$ and (W^n) are positive sequences, the sequence (T^n) is positive and non-increasing. Consequently, it is convergent and so $(U^n), (V^n)$ and (W^n) tend to zero. Moreover, as Ω is bounded, and the sequences (a^n), (b^n) are also bounded, we deduce from the preceding inequalities that :

$$\begin{cases} \|\sigma^n - \sigma^{n+1}\|_{\mathbf{W^{1,2}}(\Omega)} \to 0 & \\ \|a^n - a^{n+1}\|_{L^p(\Omega)} \to 0 \qquad \forall p & (n \to \infty) \\ \|b^n - b^{n+1}\|_{\mathbf{L^p}(\Omega)} \to 0 \qquad \forall p & \end{cases} \tag{4.22}$$

`Stage 2` : Let $\mathcal{I}^{n+1}(\Omega)$ the integral defined over Ω by :

$$\mathcal{I}^{n+1}(\Omega) = \tag{4.23}$$

$$\int_\Omega a^{n+1}\left(\sigma^{n+1}\cdot\nabla u + u_t\right)\varphi\cdot\nabla u + c(x)\sigma^{n+1}\cdot\varphi + \sum_{j=1}^{2} b_j^{n+1}\nabla\sigma_j^{n+1}\cdot\nabla\varphi_j dx \tag{4.24}$$

The aim is to show that $\lim_{n\to+\infty}\mathcal{I}^{n+1}(\Omega) = 0$. Thanks to the optimality condition (4.11), we can re-write (4.24) in the following form :

$$\mathcal{I}^{n+1}(\Omega) = \tag{4.25}$$

$$\int_\Omega \left(a^{n+1} - a^n\right)\left(\sigma^{n+1}\cdot\nabla u + u_t\right)\varphi\cdot\nabla u + \sum_{j=1}^{2}\left(b^{n+1} - b^n\right)\nabla\sigma_j^{n+1}\cdot\nabla\varphi_j dx \tag{4.26}$$

Let us now introduce an open subset Ω' strictly included in Ω. We will precise in the sequel how to choose it. Then we have :

$$\mathcal{I}^{n+1}(\Omega) = \mathcal{I}^{n+1}(\Omega/\Omega') + \mathcal{I}^{n+1}(\Omega') \tag{4.27}$$

As for the first integral, since a^{n+1}, b^{n+1} are bounded independently of n, we can find a constant C such that :

$$|\mathcal{I}^{n+1}(\Omega/\Omega')| \le C\|\varphi\|_{\mathbf{W_{1,2}}(\Omega/\Omega')} \tag{4.28}$$

As for the second one, defined over Ω', we have, for all p and p' such that :

$$\frac{1}{p} + \frac{1}{p'} + \frac{1}{2} = 1$$

the following inequality :

$$|\mathcal{I}^{n+1}(\Omega')| \le C\|\nabla u\|_{L^\infty(\Omega')}\|\varphi\|_{L^2(\Omega')}\|a^{n+1} - a^n\|_{L^p(\Omega')}\|\sigma^{n+1}\|_{L^{p'}(\Omega')}$$

$$+ \sum_{j=1}^{2}\|\nabla\varphi\|_{L^2(\Omega')}\|b_j^{n+1} - b_j^n\|_{L^p(\Omega')}\|\nabla\sigma_j^{n+1}\|_{L^{p'}(\Omega')} \equiv R(n) \tag{4.29}$$

Then, using Lemma 4.5, there exists p' $(p' > 2)$ such that $\|\nabla\sigma_j^{n+1}\|_{\mathbf{W^{1,p'}}(\Omega')}$ is finite. With that choice, we observe that the right-hand side term in (4.29) tends to zero. So, for all $\varphi \in \mathbf{W^{1,2}}(\Omega)$, we deduce from (4.28) and (4.29) that :

$$|\mathcal{I}^{n+1}(\Omega)| \le C\|\varphi\|_{\mathbf{W_{1,2}}(\Omega/\Omega')} + R(n) \tag{4.30}$$

with $\lim_{n\to+\infty} R(n) \to 0$. Consequence : Let $\eta > 0$ and $\varphi \in \mathbf{W^{1,2}}(\Omega)$ given. Firstly, we can choose Ω' close enough from Ω so that $\|\varphi\|_{\mathbf{W_{1,2}}(\Omega/\Omega')} < \eta/2$. Secondly, since $\lim_{n\to+\infty} R(n) \to 0$, there exists N such that if $n > N$, we have $R(n) < \eta/2$. As a result, for every $\eta > 0$ given, there exist N such that $n > N$ implies :

$$|\mathcal{I}^{n+1}(\Omega)| \leq \eta$$

that is to say :

$$\left(-\underbrace{div(b_j^n \nabla \sigma_j^n)}_{S_j^n} + \underbrace{a^n(\sigma^n \cdot \nabla u + u_t) u_{x_j}}_{T_j^n} + \underbrace{c(x)\sigma_j^n}_{U_j^n} \right) \xrightarrow[\mathbf{W^{1,2}}(\Omega)']{} 0 \tag{4.31}$$

Now, we are going to show that we can pass to the limit in each term of (4.31). This is proved in the next stage.

`Stage 3` : (σ^n) is bounded in $\mathbf{W^{1,2}}(\Omega)$, compact in $\mathbf{L^2}(\Omega)$. So we can extract a subsequence (again noted (σ^n)) such that $\sigma^n \xrightarrow[\mathbf{L^2}(\Omega)]{} \sigma$, $\nabla\sigma_j^n \xrightarrow[\mathbf{L^2}(\Omega)]{} \nabla\sigma_j$, that is to say $\sigma^n \xrightarrow[\mathbf{W^{1,2}}(\Omega)']{} \sigma$. So the term U_j^n converges in $\mathbf{W^{1,2}}(\Omega)'$.

As for the term T_j^n, using (4.12), we have :

$$T_j^n = \frac{{\phi_{1,\epsilon}}'(|\sigma^{n+1} \cdot \nabla u + u_t|)}{2|\sigma^{n+1} \cdot \nabla u + u_t|}(\sigma^{n+1} \cdot \nabla u + u_t)$$

The question is to know if :

$$T_j^n \xrightarrow[n\to+\infty]{} \frac{{\phi_{1,\epsilon}}'(|\sigma \cdot \nabla u + u_t|)}{2|\sigma \cdot \nabla u + u_t|}(\sigma \cdot \nabla u + u_t) \equiv T_j \quad ?$$

Since $\sigma^n \xrightarrow[\mathbf{L^2}(\Omega)]{} \sigma$, we also have the pointwise convergence. The application $t \mapsto \phi'_{1,\epsilon}(t)/2t$ being continuous, we can assert that T_j^n converges almost everywhere to T_j. Moreover, since this application is bounded and so is the L^∞-norm of (σ^n) (Lemma 4.5), there exist a constant M such that $T_j^n \leq M$. From the dominated convergence Theorem, we deduce a convergence in $\mathbf{W^{1,2}}(\Omega)'$. Until now, taking in to account (4.13) we proved :

$$-\,div\left(\frac{{\phi_{2,\epsilon}}'(\|\nabla\sigma_j^n\|)}{2\|\nabla\sigma_j^n\|}\nabla\sigma_j^n\right) \xrightarrow[\mathbf{W^{1,2}}(\Omega)']{} -\frac{{\phi_{1,\epsilon}}'(|\sigma^{n+1} \cdot \nabla u + u_t|)}{2|\sigma^{n+1} \cdot \nabla u + u_t|}(\sigma \cdot \nabla u + u_t)u_{x_j} - c(x)\sigma_j \tag{4.32}$$

as n tends to infinity. If we denote by S_j, the right-hand side term in (4.32) and if we define the operator $\mathcal{A}$ by :

$$\mathcal{A} : \mathbf{W^{1,2}}(\Omega) \to \mathbf{W^{1,2}}(\Omega)'$$
$$\mathcal{A}(\varphi) = -div\left(\frac{\phi'_{2,\epsilon}(\|\nabla\varphi\|)}{2\|\nabla\varphi\|}\nabla\varphi\right)$$

we can re-write : (4.32) under the form :

$$\mathcal{A}(\sigma_j^n) \xrightarrow[\mathbf{W^{1,2}}(\Omega)']{} S_j$$

So the problem is to show that $\mathcal{A}(\sigma_j^n) \xrightarrow[\mathbf{W^{1,2}}(\Omega)']{} \mathcal{A}(\sigma_j)$, or, equivalently :

$$\mathcal{A}(\sigma_j) = S_j?$$

Since $\mathcal{A}$ corresponds to the derivative of a convex functional, it is a monotone operator and we have:

$$< \mathcal{A}(\sigma_j^n) - \mathcal{A}(\varphi), \sigma_j^n - \varphi > \; \geq \; 0 \qquad \forall \varphi \in \mathbf{W^{1,2}}(\Omega)'$$

When n tends to infinity, two terms have to be studied more carefully:

$$< \mathcal{A}(\sigma_j^n), \sigma_j^n > \text{ and } < \mathcal{A}(\sigma_j), \sigma_j^n >$$

For the second one, we have:

$$< \mathcal{A}(\varphi), \sigma_j^n >= \int_\Omega \frac{{\phi_{2,\epsilon}}'(\|\nabla\varphi\|)}{2\|\nabla\varphi\|} \nabla\varphi \cdot \nabla\sigma_j^n dx \to < \mathcal{A}(\varphi), \sigma_j >$$

As for $< \mathcal{A}(\sigma_j^n), \sigma_j^n >$, we first use the optimality condition (4.11). So we have:

$$< \mathcal{A}(\sigma_j^n), \sigma_j^n >= \int_\Omega \frac{{\phi_{2,\epsilon}}'(\|\nabla\varphi\|)}{2\|\nabla\varphi\|} \nabla\varphi \cdot \nabla\sigma_j^n dx$$

$$= - \int_\Omega a^{n-1}(\sigma^n \cdot \nabla u + u_t)\sigma_j^n u_{x_j} + c(x){\sigma_j^n}^2 \tag{4.33}$$

$$+ \int_\Omega (b^n - b^{n-1}) \nabla {\sigma_j^n}^2 \tag{4.34}$$

$$\equiv \mathcal{J}^n(\Omega) \tag{4.35}$$

Then, we introduce an open subset Ω', strictly included in Ω and we decompose $\mathcal{J}^n(\Omega)$ in two parts:

$$\mathcal{J}^n(\Omega) = \mathcal{J}^n(\Omega/\Omega') + \mathcal{J}^n(\Omega')$$

To pass to the limit, we use same ideas as for $\mathcal{I}^n(\Omega)$ (see (4.27)): the Lemma 4.5, a subset Ω' "close enough" from Ω and the dominated convergence Theorem. So we prove that:

$$< \mathcal{A}(\sigma_j^n), \sigma_j^n > \xrightarrow[n\to\infty]{} < S_j, \sigma >$$

and so:

$$< S_j - \mathcal{A}(\varphi), \sigma_j - \varphi > \; \geq \; 0$$

Choosing φ of the form $\varphi = \sigma_j + h\vartheta$ for all $h > 0$ and $\vartheta \in \mathcal{C}_c^\infty(\overline{\Omega})$, we have:

$$< \mathcal{T} - \mathcal{A}(\sigma_j + h\vartheta), \vartheta > \leq 0 \qquad \forall \vartheta \in \mathcal{C}_c^\infty(\overline{\Omega}), \forall h > 0$$

But the function $h \to \mathcal{A}(\sigma_j + h\vartheta)$ is a continuous function which tends to $\mathcal{A}(\sigma_j)$ as h goes to zero. Moreover, it is uniformly bounded (as soon as h is bounded) by an integrable function. So we can apply the dominated convergence Theorem:

$$< \mathcal{A}(\sigma_j + h\vartheta), \vartheta > \xrightarrow[h\to 0]{} < \mathcal{A}(\sigma_j), \vartheta >$$

So, for all φ in $\mathcal{C}_c^\infty(\overline{\Omega'})$:

$$< S_j, \vartheta > \; \leq \; < \mathcal{A}(\sigma_j), \vartheta >$$

We deduce easily that $S_j = \mathcal{A}(\sigma_j)$, that is to say σ is the solution of the expected Euler-Lagrange equations. □

4.3. Some details about the algorithm. This short section is a precise description of the algorithm. It is presented in table 4.1.

```
/* Extract information from the sequence */
Compute ∇u and u_t by finite differences
(Eventual pre-smoothing with Gaussian kernel)

/* Initializiz ations */
σ^0 ≡ 0, a^0 ≡ 1, b_j^0 ≡ 1

/* General loop */
for((It=0;It≤ItNumber;It++) {

    - Compute mask discretization coefficients (p_{i+k,j+l})_{(k,l)∈D}
    corresponding to divergence terms div(b_j ∇σ_j)_{j=1,2} (see
    Appendix)

    - Find σ^{n+1} solution of the linear system:
```

$$\begin{cases} div(b_1^n \nabla \sigma_1^{n+1}) = 2a^n(\sigma^{n+1} \cdot \nabla u + u_t)u_{x_1} + 2c(x)\sigma_1^{n+1} \\ div(b_2^n \nabla \sigma_2^{n+1}) = 2a^n(\sigma^{n+1} \cdot \nabla u + u_t)u_{x_2} + 2c(x)\sigma_2^{n+1} \end{cases}$$

```
    (An iterative method like Gauss-Seidel's may be used)

    - compute a^{n+1} :
```

$$a^{n+1} = \frac{\phi_\epsilon^{1\,\prime}(|\sigma^{n+1} \cdot \nabla u + u_t|)}{|\sigma^{n+1} \cdot \nabla u + u_t|}$$

```
    - compute b_j^{n+1} :
```

$$b_j^{n+1} = \frac{\phi_\epsilon^{2\,\prime}(\|\nabla \sigma_j^{n+1}\|)}{\|\nabla \sigma_j^{n+1}\|} \qquad (j = 1, 2)$$

```
} /* Loop on It */
```

TAB. 4.1
Detailed algorithm

5. Numerical experiments.

To show the capabilities of this approach, we made numerical experiments on both synthetic (thus enabling error computations) and real images. Notice that a pre-smoothing of the data by a gaussian kernel has been done to reduce noise effects for real sequences. We also compared our method with the method of Lucas & Kanade [36] which has been designed to be the best among the class of differential techniques [8]. Roughly speaking, the idea is to choose a model of velocity (for instance constant) and to look which value best fit the OFC over a window of fixed size. It is a weighted least-square method (see [36] for more details).

We first used a synthetic sequence (Figure 5.2) The main interest is that, as the true optical flow is known, one may have a quantitative estimation of the error. To this end, we are considering the following indicators (See Figure 5.1):

$$\begin{cases} E(\theta) \\ \sigma^2(\theta) \\ SNR(\|\sigma_r\|/\|\sigma_e\|) = 10 \log \left(\frac{Var(\|\sigma_e\|)}{Var(\|\sigma_r\| - \|\sigma_e\|)} \right) \end{cases}$$

where $E(\theta)$ is the mean and $\sigma^2(\theta)$ is the variance of the angular error between the true and the estimated optical flow.

Notice that the last indicator, the SNR is usually used in image restoration

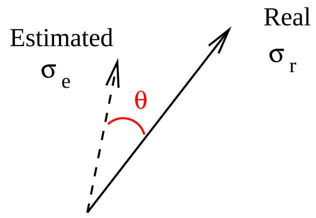


FIG. 5.1. To estimate the quality of the results, we need to estimate both the angular error and the norm differences.

and is a scale invariant indicator for errors on norms. Finally, we propose results for two real sequences (Figures 5.3 and 5.4) for which qualitative observations may be done. Particularly, this method permits an accurate reconstruction of a regularized discontinuous optical flow.

6. Conclusion.

Describing the movement in a sequence of images is very usefull for many low level vision procedures. However, it is something hard to define what we are able to find, since it is strongly related to our perception, that is to say the reflected intensity. What we see is not always in relation with a physical displacement. We first presented an overview of main existing techniques trying to emphasize on differential techniques and their variety. Secondly, we proposed a variational technique that we justified both theoretically and numerically. Some numerical experiments concluded that work showing the capabilities of this approach.

Annexe. On discretizing the divergence operator.

Let d and A given at nodes (i,j). The problem is to get an approximation of $div(d\nabla A)$ at the node (i,j). We denote by δ^{x_1} and δ^{x_2} the finite difference operators defined by :

$$\delta^{\mathrm{x}_1} A_{\mathrm{i,j}} = A_{\mathrm{i}+\frac{1}{2},\mathrm{j}} - A_{\mathrm{i}-\frac{1}{2},\mathrm{j}}$$
$$\delta^{\mathrm{x}_2} A_{\mathrm{i,j}} = A_{\mathrm{i},\mathrm{j}+\frac{1}{2}} - A_{\mathrm{i},\mathrm{j}-\frac{1}{2}}$$

Using that notation, Perona and Malik [50] proposed the following approximation :

$$div(d\nabla A)_{\mathrm{i,j}} = \frac{\partial}{\partial x_1}\left(d\frac{\partial A}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(d\frac{\partial A}{\partial x_2}\right) \approx \delta^{x_1}(d\delta^{x_1} A_{\mathrm{i,j}}) + \delta^{x_2}(d\delta^{x_2} A_{\mathrm{i,j}})$$
$$\approx \begin{pmatrix} 0 & d_{\mathrm{i},\mathrm{j}+\frac{1}{2}} & 0 \\ d_{\mathrm{i}-\frac{1}{2},\mathrm{j}} & -S^{\mathrm{P}} & d_{\mathrm{i}+\frac{1}{2},\mathrm{j}} \\ 0 & d_{\mathrm{i},\mathrm{j}-\frac{1}{2}} & 0 \end{pmatrix} \star A_{i,j} \tag{A.1}$$

where the symbol $\star$ denotes the convolution and S^{P} is the sum of the four weights in the principal directions. Notice that we need to estimate the function d at intermediate nodes. Our aim is to extend this approximation so that we could take into account the values of A at the diagonal nodes :

$$div(d\nabla A)_{\mathrm{i,j}} = \alpha_{\mathrm{P}} \begin{pmatrix} 0 & d_{\mathrm{i},\mathrm{j}+\frac{1}{2}} & 0 \\ d_{\mathrm{i}-\frac{1}{2},\mathrm{j}} & -S^{\mathrm{P}} & d_{\mathrm{i}+\frac{1}{2},\mathrm{j}} \\ 0 & d_{\mathrm{i},\mathrm{j}-\frac{1}{2}} & 0 \end{pmatrix} \star A_{\mathrm{i,j}}$$
$$+ \alpha_{\mathrm{D}} \begin{pmatrix} d_{\mathrm{i}-\frac{1}{2},\mathrm{j}+\frac{1}{2}} & 0 & d_{\mathrm{i}+\frac{1}{2},\mathrm{j}+\frac{1}{2}} \\ 0 & -S^{\mathrm{D}} & 0 \\ d_{\mathrm{i}-\frac{1}{2},\mathrm{j}-\frac{1}{2}} & 0 & d_{\mathrm{i}+\frac{1}{2},\mathrm{j}-\frac{1}{2}} \end{pmatrix} \star A_{\mathrm{i,j}} \tag{A.2}$$

where α_{P} and α_{D} are two weights to be discussed, and S^{D} is the sum of the four weights in the diagonal directions. Approximation (A.2) is consistent if and only if :

$$\alpha_{\mathrm{P}} + 2\alpha_{\mathrm{D}} = 1 \tag{A.3}$$

Now, there remains one degree of freedom. Two possibilities have been considered :

$$(\alpha_{\mathrm{P}}, \alpha_D) = \text{ constant and for instance } = \left(\frac{1}{2}, \frac{1}{4}\right) \tag{A.4}$$
$$(\alpha_{\mathrm{P}}, \alpha_D) = \text{ functions depending on } d \text{ (See Figure A.1)} \tag{A.5}$$

Qualitative and quantitatives tests have been done to estimate different possibilities. We worked on the image restoration problem which permits a good appreciation of results. More precisely, for a given noisy image I_N, the problem becomes to find I as the minimum of the following functional :

$$\inf_I \int_\Omega (I - I_N)^2 dx + \alpha_r \int_\Omega \phi(\|\nabla I\|) dx$$

where the function ϕ verify hypotheses (3.13)-(3.14). Then, using the Theorem 4.3, the problems is to find I and d_I minimizing :

$$\inf_{I, d_I} \int_\Omega (I - I_N)^2 dx + \alpha^r \int_\Omega b\|\nabla I\|^2 + \Psi(b) dx$$

It is easy to check that we have the same kind of divergence term that we need to discretize. We refer to [12, 34] for more details. The value of $d_{\mathrm{I}}^{\mathrm{n}}$ $\left(= \frac{\phi'(\|\nabla I^{\mathrm{n}}\|)}{2\|\nabla I^{\mathrm{n}}\|}\right)$ at intermediate nodes is computed by interpolation (see [50]).

We tested these different discretizations on a noisy test image using quantitative measures. We checked that (A.2) permits to restore identically edges in principal or diagonal directions. Moreover, we observed that choosing α_{P} adaptatively (A.5) gave more precise results than (A.4). We used this approximation (A.5) in our experiments.

RÉFÉRENCES

[1] R. Acart and C.R. Vogel. Analysis of bounded variation penalty methods for ill-posed p roblems. *Inverse Problems*, 10:1217–1229, 1994.

[2] A.A. Alatan and L. Onural. Gibbs random field model based 3-d motion estimation by weakened rigidity. *ICIP*, II:790–794, 1994.

[3] L. Ambrosio. A compactness theorem for a new class of functions of bounded variation. *Boll. Unione Mat. Ital.*, VII(4):857–881, 1989.

[4] G. Aubert, M. Barlaud, L. Blanc-Feraud, and P. Charbonnier. Deterministic edge-preserving regularization in computed imaging. *IEEE Trans. Imag. Process.*, 5(12), February 1997.

[5] G. Aubert, R. Deriche, and P. Kornprobst. A mathematical study of the regularized optical flow problem in the space $BV(\Omega)$. Technical Report 503, Université de Nice-Sophia Antipolis, December 1997.

[6] G. Aubert and P. Kornprobst. A mathematical study in the space $BV(\Omega)$ for the relaxation of the ill posed problem in optical flow. *SIAM*, 1998. Submitted.

[7] G. Aubert and L. Vese. A variational method in image recovery. *SIAM J. Numer. Anal.*, 34(5):1948–1979, October 1997.

[8] J.L. Barron, D.J. Fleet, and S.S. Beauchemin. Performance of optical flow techniques. *The International Journal of Computer Vision*, 12(1):43–77, 1994.

[9] M.J. Black. *Robust Incremental Optical Flow*. PhD thesis, Yale University, Department of Computer Science, 1992.

[10] H. Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland Publishing Comp, Amsterdam-London, 1973.

[11] G. Buttazzo. *Semicontinuity, relaxation and integral representation in the calculus of variations*. Longman Scientific & Technical, 1989.

[12] P. Charbonnier. *Reconstruction d'image: régularisation avec prise en compte des discontinuités*. PhD thesis, Université de Nice-Sophia Antipolis, 1994.

[13] I. Cohen. Nonlinear variational method for optical flow computation. In *Proceedings 8th SCIA*, volume 1, pages 523–530, 1993.

[14] F. Demengel and R. Temam. Convex functions of a measure and applications. *Indiana University Mathematics Journal*, 33:673–709, 1984.

[15] R. Deriche and O. Faugeras. Les EDP en traitement des images et vision par ordinateur. Technical report, INRIA, November 1995. A more complete version of this Research Report has appeared in the French Revue "Traitement du Signal". Volume 13 - No 6 - Special 1996.

[16] R. Deriche, P. Kornprobst., and G. Aubert. Optical flow estimation while preserving its discontinuities: A variational approach. In *Proceedings of the 2nd Asian Conference on Computer Vision*, volume 2/3, Singapore, December 1995.

[17] V. Devlaminck and J.P. Dubus. Estimation of compressible or incompressible deformable motions for density images. *ICIP 96*, I:125–128, 1996.

[18] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC, 1992.

[19] H. Federer. *Geometric Measure Theory*. Classics in Mathematics. Springer-Verlag, Berlin . Heidelberg . New York, 1969.

[20] D.J. Fleet and A.D. Jepson. Computation of component image velocity from local phase information. *The International Journal of Computer Vision*, 5:77–104, 1990.

[21] E. De Giorgi and T. Franzoni. Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, 68:842–850, 1975.

[22] E. Giusti. *Minimal Surfaces and Functions of Bounded Variation*. Birkhäuser, 1984.

[23] C. Goffman and J. Serrin. Sublinear functions of measures and variational integrals. *Duke Math J.*, 31:159–178, 1964.

[24] F. Guichard and L. Rudin. Accurate estimation of discontinuous optical flow by minimizing divergence related functionals. *ICIP*, I:497–500, 1996.

[25] S.N. Gupta and J.L. Prince. On div-curl regularization for motion estimation in 3-d volumetric imaging. *ICIP*, pages 929–932, 1996.

[26] F. Heitz and P. Bouthemy. Multimodal estimation of discontinuous optical flow using markov random fields. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 15(12):1217–1232, December 1993.

[27] B.K. Horn. *Robot Vision*. MIT Press, 1986.

[28] B.K. Horn and B.G. Schunk. Determining Optical Flow. *Artificial Intelligence*, 17:185–203, 1981.

[29] Jan J. Koenderink. Optic Flow. *Vision Research*, 26(1):161–180, 1986.

[30] Jan J. Koenderink and A. J. van Doorn. Invariant properties of the motion parallax field due to the movement of rigid bodies relative to an observer. *Optica Acta*, 22:717–723, 1975.

[31] Jan J. Koenderink and A. J. van Doorn. How an ambulant observer can construct a model of the environment from the geometrical structure of the visual inflow. In Hauske G. and Butenandt E., editors, *Kybernetik 1978*. Oldenburg, Muenchen, 1978.

[32] Jan J. Koenderink and Andrea J. van Doorn. Affine structure from motion. *Journal of the Optical Society of America*, A8:377–385, 1991.

[33] J.J. Koenderink and A.J. van Doorn. Dynamic shape. *Biological Cybernetics*, 53:383–396, 1986.

[34] P. Kornprobst, R. Deriche, and G. Aubert. Nonlinear operators in image restoration. In *Proceedings of the International Conference on Computer Vision and Pattern Recognition*, pages 325–331, Puerto-Rico, June 1997. IEEE.

[35] A. Kumar, A. Tannenbaum, and G. Balas. Optical flow : a curve evolution approach. *IEEE Trans. Image Processing*, 5:598–611, 1996.

[36] Bruce Lucas and Takeo Kanade. An iterative image registration technique with an application to stereo vision. In *International Joint Conference on Artificial Intelligence*, pages 674–679, 1981.

[37] G. Dal Maso. *An introduction to* Γ*-convergence*. Progress in Nonlinear Differential Equations and their Applications. Birkhaeuser, 1993.

[38] M. Mattavelli and A. Nicoulin. Motion estimation relaxing the constant brightness constraint. *icip:94*, II:770–774, November 1994.

[39] E. Mémin and P. Pérez. A multigrid approach for hierarchical motion estimation. *ICCV*, pages 933–938, 1998.

[40] E. Mémin, P. Pérez, and D. Machecourt. Dense estimation and object-oriented segmentation of the optical flow with robust techniques. Technical Report 991, IRISA, March 1996.

[41] N.G. Meyers. An l^p-estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Scuola Norm. Sup. Pisa*, III(17):189–206, 1963.

[42] N.G. Meyers and A. Elcrat. Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions. *Duke math. J.*, 42:121–136, 1975.

[43] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.*, 42:577–684, 1989.

[44] H.H. Nagel. Direct estimation of optical flow and of its derivatives. In G.A. Orban and H.-H. Nagel, editors, *Artificial and Biological Vision Systems*, Basic Research Series, pages 191–224. Springer-Verlag, 1992.

[45] H.H. Nagel and W. Enkelmann. An investigation of smoothness constraint for the estimation of displacement vector fiels from images sequences. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 8:565–593, 1986.

[46] S. Negahdaripour and C.-H. Yu. A generalized brightness change model for computing optical flow. *IEEE*, 1993.

[47] P. Nési. Variational approach to optical flow estimation managing discontinuities. *Image and Vision Computing*, 11(7):419–439, September 1993.

[48] M. Orkisz and P. Clarysse. Estimation du flot optique en présence de discontinuités: une revue. *Traitement du Signal*, 13(5):489–513, 1996.

[49] M. Otte and H.H. Nagel. Optical flow estimation: Advances and comparisons. In Jan-Olof Eklundh, editor, *Proceedings of the 3rd European Conference on Computer Vision*, volume 800 of *Lecture Notes in Computer Science*, pages 51–70. Springer-Verlag, 1994.

[50] P. Perona and J. Malik. Scale-space and edge detection using anisotropic diffusion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12(7):629–639, July 1990.

[51] M. Proesmans, L. Van Gool, E. Pauwels, and A. Oosterlinck. Determination of Optical Flow and its Discontinuities using Non-Linear Diffusion. In *Proceedings of the 3rd ECCV, II*, number 801 in Lecture Notes in Computer Science, pages 295–304. Springer-Verlag, 1994.

[52] L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.

[53] A.N. Tikhonov and V.Y. Arsenin. *Solutions of Ill-posed Problems.* Winston and Sons, Washington, D.C., 1977.

[54] M. Tistarelli. Computation of coherent optical flow by using multiple constraints. *iccv:95*, pages 263–268, 1995.

[55] A. Verri, F. Girosi, and V. Torre. Differential techniques for optical flow. *Journal of the*

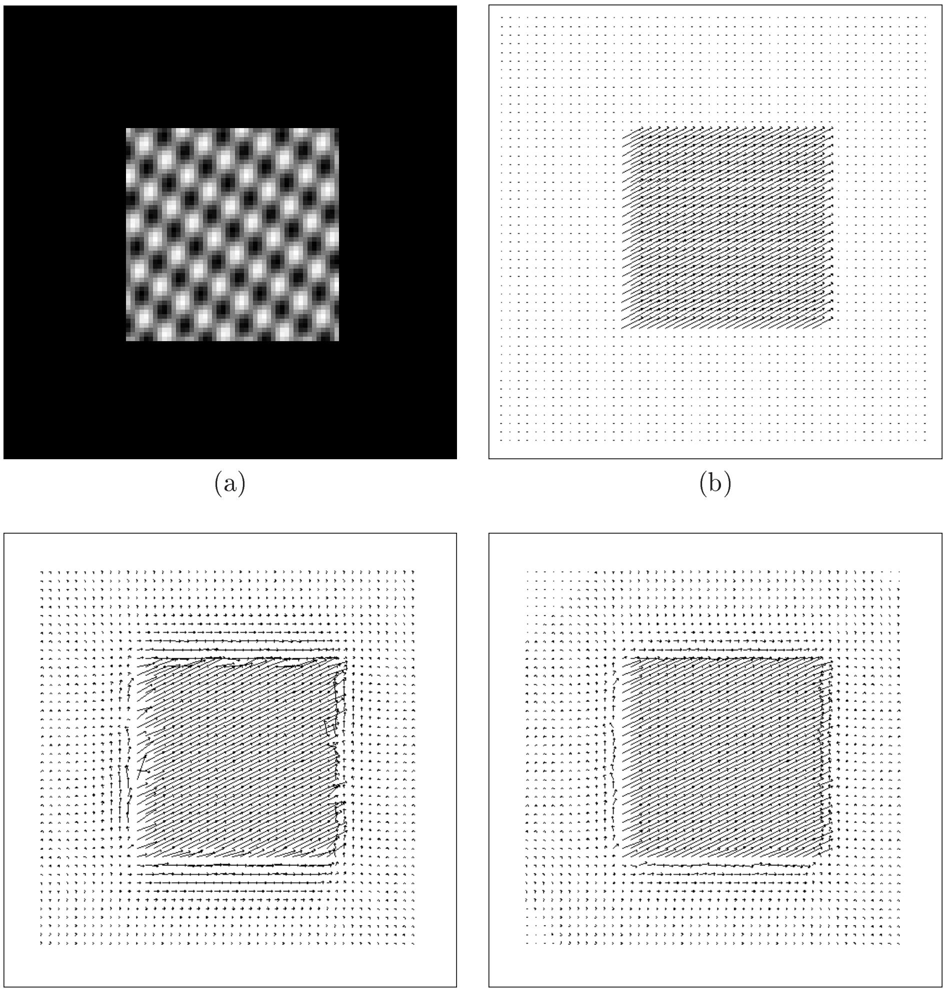
Optical Society of America A, 7:912–922, 1990.

[56] A. Verri and T. Poggio. Motion field and optical flow: qualitative properties. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 11(5):490–498, 1989.

[57] R.P. Wildes, M.J. Amabile, A.-M. Lanzillotto, and T.-S. Leu. Physically based fluid flow recovery from image sequences. *cvpr:97*, pages 969–975, 1997.

[58] D. Willick and Y.-H. Yang. Experimental evaluation of motion constraint equations. *Computer Vision, Graphics, and Image Processing*, 54(2):206–214, September 1991.

[59] W.P. Ziemer. *Weakly Differentiable Functions.* Springer-Verlag, 1989.



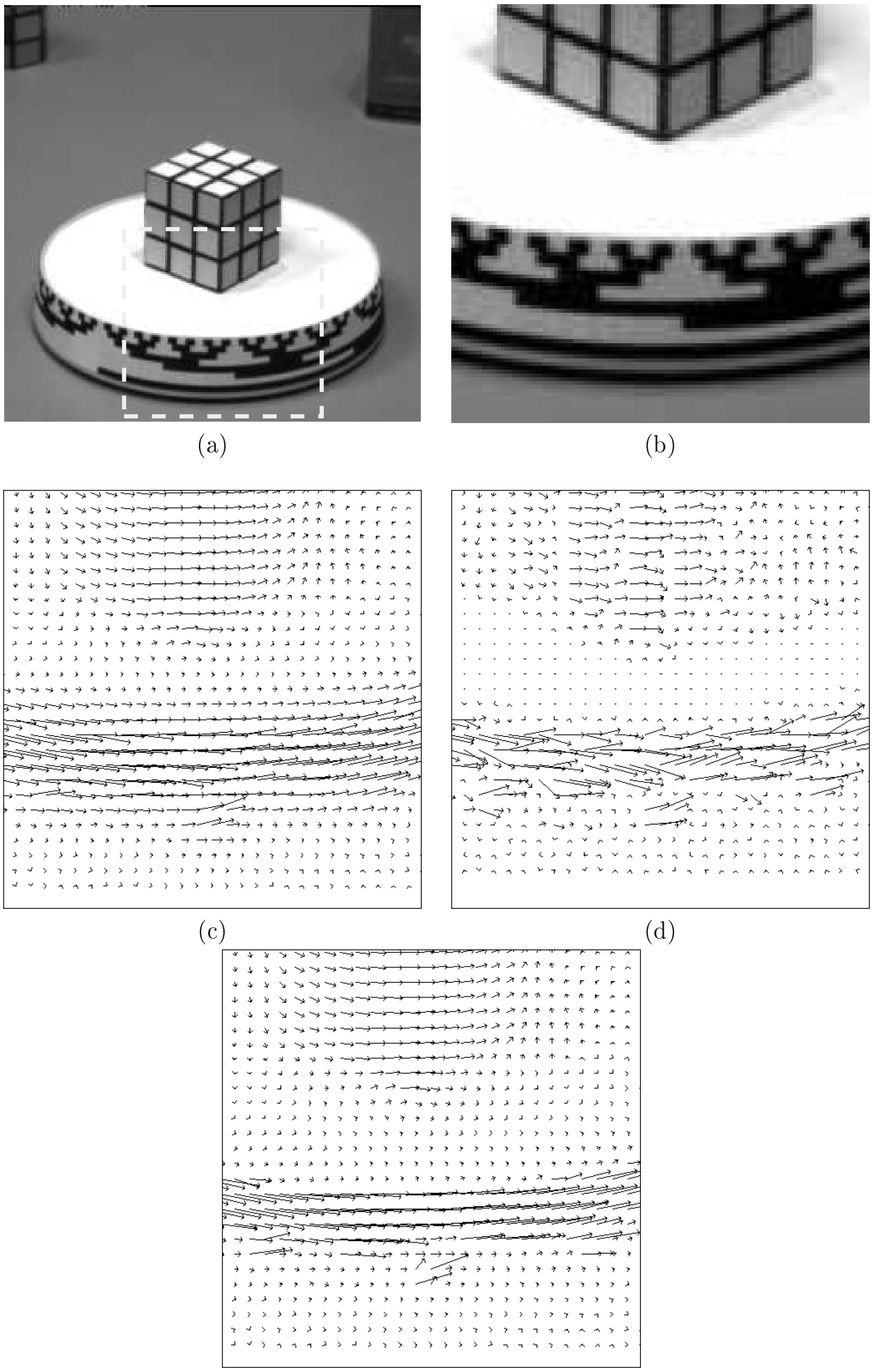
(a) (b)

(c) (d)

Method	$\phi(s)$	$E(\theta)$	$\sigma^2(\theta)$	$SNR(\|\sigma_r\|/\|\sigma_e\|)$
Horn & Schunk		8.06	12.29	8.79
Lucas & Kanade		3.94	9.15	10.23
Proposed model	$2\sqrt{1+s^2}-2$	3.87	9.20	12.27
	$log(1+s^2)$	4.32	9.24	12.51

(e)

Fig. 5.2. (a) Description: the texture is moving with the constant speed $\sigma = (1.585, 0.863)$ pixel/image (b) Real optical flow (c) Horn & Schunk (d) Proposed model (e) Error estimations



(a) (b)

(c) (d)

(e)

Fig. 5.3. (a) The rubick's cube is posed on a rotating plate. This sequence is available via anonymous ftp from `ftp.csd.uwo.ca` in the directory `pub/vision`. (b) Zoom on the plate (c) Horn & Schunk (d) Lucas & Kanade (e) Proposed model

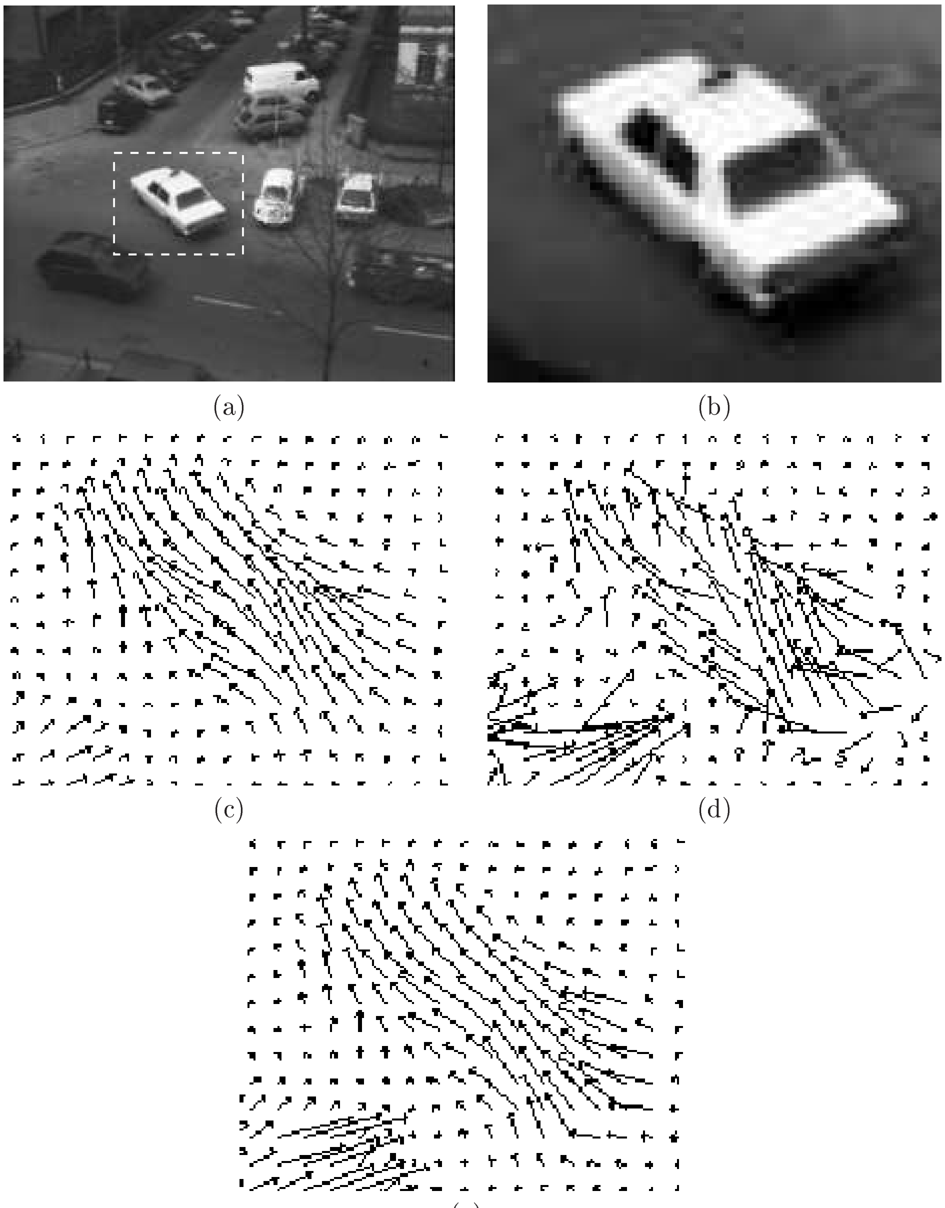


FIG. 5.4. (a) See description of the sequence in the introduction (Voir Figure 2.1) (b) Zoom on the taxi for which optical flow is observed (c) Horn & Schunk (d) Lucas & Kanade (e) Proposed model

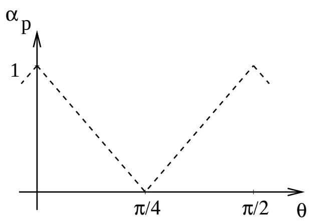


FIG. A.1. $\alpha_P = \alpha_P(\theta)$ is a $\pi/2$ periodic function where θ is the direction of the gradient of d. Notice that α_D can be deduced from the consistency condition is then computed thanks to the consistency condition.