DREAM^{2}S: Deformable Regions driven by an Eulerian Accurate Minimization Method for image and video Segmentation

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Abstract. Originally, active contours were boundary-based methods. They were driven towards the edges of an image from the minimization of a criterion with only boundary-based terms. The main purpose of region-based active contours is to introduce information on the regions to segment, in addition to the classical boundary-based information, in order to make the active contour evolve.

We propose here a general Eulerian framework for region-based active contours named DREAM^{2}S. We introduce a general criterion including both region-based terms and boundary-based terms where the information on a region is named "descriptor". The originality of this work is twofold. First we propose to use shape optimization principles to compute the evolution equation of the active contour that will make it evolve the fastest as possible towards a minimum of the criterion. Secondly, we take into account the variation of the descripors during the propagation of the curve. Indeed, a descriptor may be globally attached to the region and so "region-dependent". This is a case for example if the mean or the variance of a region is chosen as a descriptor. We show that the dependance of the descriptors with the region induces additional terms in the evolution equation of the active contour that have not been computed so far. The DREAM^{2}S method gives an easy way to take into account such a dependance and to compute the resulting addtional terms.

Experimental results point out the importance of the additional terms to reach a true minimum of the criterion and so to obtain good results. The covariance matrix determinant appears to be a very relevant tool for homogeneous color regions segmentation and it has been applied successfully to face detection in real video sequences.

Keywords: Region-based active contours, shape optimization, unsupervised segmentation, face detection

1. Introduction

Active contours are powerful tools for image and video segmentation. Since the original work on snakes (Kass et al., 1988), an extensive research has been performed that leads today to the use of "region-based active contours".

Indeed, originally, active contours were boundary-based methods. Snakes (Kass et al., 1988), balloons (Cohen, 1991) or geodesic active contours (Caselles et al., 1997) are driven towards the edges of an image. The evolution equation was computed from a criterion that only included a local information on the boundary of the object to segment.

The key idea of region-based active contours, launched firstly by (Cohen et al., 1993; Ronfard, 1994; Zhu and Yuille, 1996; Chakraborty et al., 1996), is to introduce a global information on the different regions to segment in addition to the boundary-based information to make the active contour evolve. However, it is not trivial to compute the evolution equation of the active contour that will make it evolve towards a minimum of a criterion including both region-based terms and boundary-based terms.

Recently, many papers have addressed this problem. A complete review of these methods is proposed in section 2. Some of these works do not compute the theoretical expression of the velocity vector of the active contour but they choose the displacement that will make the criterion decrease. Other works proposed the computation of the velocity vector by reducing the whole problem to boundary integrals and then using the Euler-Lagrange equations.

However, the information on the different regions, that we call here "descriptor" of the corresponding region, may be globally attached to the region. Indeed this is the case when a descriptor is a statistical feature of a region as the mean, the variance or the histogram of the region. In the case of unsupervised segmentation, these descriptors are re-evaluated each time the active contour evolves and so they vary during the propagation of the active contour. The main drawback of all previous works on region-based active contours is that they do not take into account this possible variation of the descriptors to compute the evolution equation of the active contour.

We propose here a general Eulerian framework for region-based active contours named DREAM^{2}S (Deformable Regions driven by an Eulerian Accurate Minimization Method for Segmentation). The main contribution of our work is to build a theoretical framework based on shape optimization principles. With such an approach, we can readily take into account the variation of the descriptors that are globally attached to the evolving regions (named region-dependent descriptors). We show that the variation of these region-dependent descriptors during the evolution of the active contour induces additional terms in the evolution equation of the active contour. These additional terms have not been computed so far.

Some examples of unsupervised spatial segmentation using the variance of the regions show the importance of the additional terms for the result of the segmentation. The additional terms must not be forgotten in order to get the velocity vector of the active contour that will make it evolve the fastest as possible towards a minimum of the criterion.

For color images segmentation, we propose to use the covariance matrix determinant which appears to be a very relevant tool for homogeneous color regions segmentation. This tool is successfully applied to face detection on real video sequences.

The detailed overview on existing region-based approaches is proposed in section 2, while our Eulerian framework DREAM^{2}S is described in section 3. This framework is then applied for the unsupervised segmentation of homogeneous regions using the variance in section 4 and the covariance matrix determinant in section 5.

2. State of the art of existing region-based methods using active contours

Recently, many authors have proposed a way of adding region-based terms in the evolution equation governing active contours for segmentation issues. We present here a state-of-the-art focusing on differences and similarities between all these works.

2.1. Problem statement

We describe here the equations addressing the issue of the segmentation of an image I in two regions. Let us define Ω_{in} the region containing the objects to segment and Ω_{out} the background region. Their common boundary is noted Γ. Let us consider an active contour $\Gamma(s,\tau) = (x(s,\tau), y(s,\tau))$, where s can be the arc length of the contour and τ is an evolution parameter. Classically the active contour approach makes $\Gamma(s,\tau)$ evolve through the following partial differential equation (PDE):

$$\frac{\partial \Gamma(s,\tau)}{\partial \tau} = F\boldsymbol{N} = \boldsymbol{v} \qquad \text{with } \Gamma(\tau = 0) = \Gamma_0 \tag{1}$$

where Γ_0 is an initial curve defined by the user and $\boldsymbol{N}$ the normal vector field of $\Gamma(s,\tau)$. The velocity F is derived from a minimizing criterion and is chosen so that the solution $\Gamma(.,\tau)$ converges to $\Gamma(.)$ as $\tau \to \infty$. The velocity is along the normal vector since it has been proven that only the normal component is relevant for the evolution of curves (Epstein and Cage, 1987).

In classical boundary-based methods (Caselles et al., 1997; Malladi et al., 1995), the velocity function F is derived from an energy depending on the boundary Γ:

$$J_{boundary}(\Gamma) = \int_{\Gamma} k^{(b)} ds \tag{2}$$

where $k^{(b)}$ is a function describing the boundary of the object. Caselles et al (Caselles et al., 1997) choose the function $k^{(b)} = g(|\nabla I|)$ where $g : [0, +\infty[\to I\!R^{+}$ is a strictly decreasing function so that $g(r) \to 0$ as $r \to \infty$. The computation of the velocity F from the criterion (2) is now classical (Caselles et al., 1997) and leads to the following evolution equation:

$$\frac{\partial \Gamma}{\partial \tau} = [\ k^{(b)} \cdot \kappa - \nabla k^{(b)} \cdot \boldsymbol{N}\ ]\boldsymbol{N} \tag{3}$$

where κ is the curvature of $\Gamma(\tau)$. However, both regions Ω_{in} and Ω_{out} may have different properties of texture, homogeneity or motion that cannot be included in the function $k^{(b)}$. So the main purpose of region-based approaches is to introduce such a global information in the velocity function F in addition to the local information provided by boundary-based terms. The main question, in order to justify the introduction of these region terms, is to derive the velocity function from an energy. To set a general framework and to compare the different approaches proposed in the literature, we now introduce a general criterion from which the velocity magnitude could be derived:

$$\begin{aligned} J(\Omega_{in}, \Omega_{out}, \Gamma) \ &= \iint\limits_{\Omega_{out}} k^{(out)}(x, y, \Omega_{in}) dxdy \\ &+ \iint\limits_{\Omega_{in}} k^{(in)}(x, y, \Omega_{out}) dxdy + \int\limits_{\Gamma} k^{(b)}(x, y) ds \end{aligned} \tag{4}$$

where $k^{(out)}$ is named the "descriptor" of the background region, $k^{(in)}$ the "descriptor" of the object region and $k^{(b)}$ the "descriptor" of the contour. The region descriptors may be globally attached to their respective regions, Ω_{in} or Ω_{out}, and so depend on them (in this case they are called "region-dependent descriptors"). Indeed, this is the case when choosing for a descriptor statistical features of a region like, for example, the mean or the variance.

The purpose of all region-based active contours method is then to make the active contour evolve towards a partition of the image, $(\Omega_{out}, \Omega_{in}, \Gamma)$, that minimizes the criterion (4).

2.2. Overview of existing region-based methods using active contours

We ask three questions about the existing region-based approaches using active contours:

- What are the different "descriptors" used by the authors for the criterion (3)?
- Does the evolution equation of the active contour derive from a criterion or not, and how is it obtained from the criterion?
- How is implemented the active contour?

In an early work, Cohen et al. (Cohen et al., 1993) present a surface reconstruction method using region-based active contours. They compute the velocity vector of an active contour from the minimization of a criterion including region-based terms.

Another pioneer work has been performed in (Ronfard, 1994). Ronfard feels the need to extend the classical active contours using only edges to region-based definitions. The optimization scheme is mostly heuristic and presented without a formal proof but very interesting intuitive ideas are proposed. He assesses that the velocity function should be proportional to the difference of statistical fits to object and background: $F = k^{(in)} - k^{(out)}$, where $k^{(in)}$ is a statistical model of the object region and $k^{(out)}$ a statistical model of the background region. But Ronfard affirms that this velocity function cannot easily be derived from a potential energy because it is only defined along the contour curve. Moreover he considers only region-based terms in the velocity function and not both region-based terms and boundary-based terms.

More recently, Zhu and Yuille (Zhu and Yuille, 1996) present a statistical framework for image segmentation using an algorithm called "region competition". They derive the active contour evolution equation by minimizing a generalized Bayes/MDL criterion inspired from the Mumford and Shah energy (Mumford and Shah, 1989). They propose the following descriptors for the segmentation of still images:

$$\left\{ \begin{array}{lcl} k^{(out)} & = & -logP(I(x,y)/\alpha_{out}) \\ k^{(in)} & = & -logP(I(x,y)/\alpha_{in}) \\ k^{(b)} & = & \mu \end{array} \right. \tag{5}$$

where $P()$ is a probability density function, α_i are the parameters of the distribution for the region i and μ is a positive constant. For illustrations, the authors consider Gaussian distributions characterized by two parameters, the mean and the variance. Once the α_i fixed, the evolution equation of an active contour is computed and directly derived from the criterion. The derivative of the boundary-based term is classical (Caselles et al., 1997) and leads to the Euclidian geometric heat flow. The derivative of region-based terms is performed using the Green-Riemann theorem and the Euler-Lagrange equations. Indeed, the computation of the evolution equation is performed in three main steps:

- Transformation of domain integrals into boundary integrals using the Green-Riemann theorem.
- Computation of the Euler-Lagrange equations
- Introduction of a dynamical scheme in the Euler-Lagrange equations in order to compute the velocity vector of the active contour.

The Green-Riemann theorem transforms a domain integral into a boundary integral:

$$\iint_{\Omega} f(x,y)dxdy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \int_{\partial\Omega} (Pdx + Qdy) \qquad (6)$$

where:

$$\begin{aligned} Q &= \frac{1}{2}\int_0^x f(r,y)dr \\ \text{and} \quad P &= -\frac{1}{2}\int_0^y f(x,r)dr \end{aligned}$$

Then, classically, as all terms of the criterion (3) are now boundary integrals, the Euler-Lagrange equations are computed in order to find the zero of the gradient of J with respect to the boundary. The velocity function F is then computed and it leads to the following evolution equation:

$$\frac{\partial \Gamma}{\partial \tau} = (k^{(in)} - k^{(out)} - \mu\kappa)\boldsymbol{N} \qquad (7)$$

where κ is the curvature of $\Gamma(\tau)$. In order to implement this PDE, a Lagrangian formulation is used.

Simultaneously, Chakraborty et al (Chakraborty et al., 1996) introduce a similar region-based approach for the segmentation of medical images. As input of their algorithm, they consider both the actual image I and the region-classified image I_r. The classified image I_r is computed by maximizing the posterior distribution of the region labels given the intensity image. Their region term enforces the boundary to enclose a single region in I_r. The descriptors are the following with the difference that they want to maximize the criterion J:

$$\begin{cases} k^{(out)} & = K_1 \ I_r(x, y) \\ k^{(in)} & = 0 \\ k^{(b)} & = K_2 \ I_g(x, y) \end{cases} \tag{8}$$

K_1 and K_2 are two positive constants and I_g is the gradient image. A prior shape term is also added to the criterion, which can be interesting for medical imaging where the nature of the shape does not change a lot from individual to individual. Then, as in (Zhu and Yuille, 1996), the whole problem is reduced to computing contour integrals with the Green-Riemann theorem, rather than both contour and domain integrals. The implementation of the active contour in (Chakraborty et al., 1996) is done by using a Fourier parameterization for the boundary. The expression of the velocity vector of the active contour is not computed but the criterion is introduced and evaluated numerically in order to maximize it. We can note that the authors have proposed a very interesting extension of this work using the game theory (Chakraborty and Duncan, 1999). Region and boundary criterions are separated, but each criterion contains an interaction with the other. As the game progresses, both modules improve their positions through mutual information sharing.

Recently, Paragios and Deriche (Paragios and Deriche, 1999), have proposed an extension of the work of Zhu and Yuille by improving the descriptor of the contour in order to incorporate the image gradient as in geodesic active contours (Caselles et al., 1997). Their approach is called "geodesic active regions". The descriptors used are the following:

$$\begin{cases} k^{(out)} & = & -(1-\mu) \ logP(I(x, y)/\alpha_{out}) \\ k^{(in)} & = & -(1-\mu) \ logP(I(x, y)/\alpha_{in}) \\ k^{(b)} & = & \mu \ g(p_c(I(\Gamma))) \end{cases} \tag{9}$$

where $P()$ is a probability density function, μ is a positive constant, p_c the conditional boundary distribution that measures the probability of a given pixel being at the real boundary Γ, and g a Gaussian function. The way of derivating the criterion for the region-based terms is the same as in Zhu and Yuille. They obtain the following evolution equation for the active contour:

$$\frac{\partial \Gamma}{\partial \tau} = [(1-\mu)(logP(I(x,y)/\alpha_{in}) - logP(I(x,y)/\alpha_{out})) \\ +\mu(g(p_c(I(\Gamma)))\kappa - \nabla g(p_c(I(\Gamma))).\boldsymbol{N})]\boldsymbol{N} \qquad (10)$$

This equation is implemented with an Eulerian approach using the level set method introduced by Osher and Sethian (Osher and Sethian, 1988; Malladi et al., 1995; Sethian, 1996). A very interesting part of this work concerns the introduction of some descriptors $k^{(in)}$ and $k^{(out)}$ for applications like texture segmentation and moving objects detection.

In (Mansouri and Konrad, 1999), starting from a MAP(Maximum A Posteriori) framework, the authors derive a region-based criterion using Euler-Lagrange equations and express the evolution equation of the active contour using the level set method. Their application concerns motion segmentation.

A later approach has been introduced by Samson et al (Samson et al., 2000) for classification and by Chan and Vese (Chan and Vese, 2001) for segmentation of still images. Chan and Vese propose the following descriptors:

$$\begin{cases} k^{(out)} & = \lambda_1 |I - c_1|^2 + \nu \\ k^{(in)} & = \lambda_2 |I - c_2|^2 \\ k^{(b)} & = \mu \end{cases} \qquad (11)$$

with c_1 the average of I in Ω_{in}, c_2 the average of I in Ω_{out}. These two values are re-estimated during the propagation of the curve as the regions evolve. The parameters λ_1, λ_2, ν and μ are some positive constants. Here the level set method is directly used by representing the curve $\Gamma(\tau)$ as the zero level set of a Lipschitz continuous function $U(x, y, \tau)$. The level set function U is introduced in the criterion via the Heaviside function $H(U)$. The criterion is then minimized with respect to U using a regularized version of the minimization problem and they obtain the following evolution equation for the level set function U:

$$\frac{\partial U}{\partial \tau} = \delta_\epsilon(U)[\mu \text{ div } \left(\frac{\nabla U}{|\nabla U|}\right) - \nu - \lambda_1(I - c_1)^2 + \lambda_2(I - c_2)^2] \qquad (12)$$

where $\delta_\epsilon(U)$ is a regularized version of the Dirac measure concentrated at 0. We note here that the evolution equation is directly expressed with the level set function and not with the curve Γ. They find the velocity vector of a regularized version of the minimization problem. This solution may not be the solution of the initial minimization problem.

Another different approach has been introduced by Chesnaud et al (Chesnaud et al., 1999). Their descriptors are re-evaluated each time a point of the active contour evolves since they are statistical features of the image globally attached to the evolving regions. They choose the random displacement of a considered point of the active contour so that the criterion decreases. After each displacement of a point of the active contour, the descriptors are re-computed.

Region-based active contours are also powerful tools for classification (Yezzi et al., 1999).

What emerges from this survey is that these pioneer works are complementary. However, all proofs leading to the evolution equation of the active contour are based on the derivation of the criterion using Euler-Lagrange equations and the dynamical scheme is introduced after the computation of the derivative (Chan and Vese, 2001; Paragios and Deriche, 1999; Zhu and Yuille, 1996). With such a method, the case of descriptors depending upon features globally attached to the region (region-dependent descriptors) cannot readily be taken into account.

The major contribution of our work is to compute the evolution equation of the active contour for descriptors depending upon features globally attached to the region. We propose to use a shape optimization approach to compute the velocity vector of the active contour by making the regions evolve, instead of computing Euler-Lagrange equations. Some previous works, using shape optimization and the distribution theory, have been introduced by Debreuve et al (Debreuve et al., 2001) for medical imaging and by Amadieu et al (Amadieu et al., 1999) for building detection. However, in these works, they do not take into account the fact that descriptors may themselves depend on the evolving regions.

In this paper, we propose a new formulation of the derivation using shape optimization tools. With such a method,we are able to readily take into account the case of region-dependent descriptors (descriptors depending upon features globally attached to the evolving regions) that vary during the propagation of the active contour. These descriptors have to be considered since they are widely used for unsupervised segmentation. For such region-dependent descriptors, we show that additional terms appear in the evolution equation of the curve. With these additional terms, we ensure the fastest decrease of the active contour towards a minimum of the criterion.

3. Setting a general framework for region-based active contours

The main idea lies in the elaboration of a general framework for the segmentation of an image in two different regions using region-based active contours.

Each region is described using a function that we name "descriptor of the region". A descriptor is a function that measures the homogeneity of a region. Most of relevant statistical descriptors depend themselves on the region. Some examples are given in section 3.5. For example, the region descriptor $k^{(in)}$ can be chosen as the variance σ_{in}^2 of the region Ω_{in}, which gives:

$$k^{(in)}(x, y, \Omega_{in}) = \sigma_{in}^2 = \frac{\iint\limits_{\Omega_{in}} (I - \mu_{in})^2 dxdy}{\iint\limits_{\Omega_{in}} dxdy}$$

where μ_{in} is the mean of the region Ω_{in}. Such a descriptor is named a "region-dependent descriptor" since it depends on the region Ω_{in}.

This general framework can be applied to many different applications by only changing the descriptors. This general framework has been notably applied to moving objects detection in video sequences acquired either with a static (Jehan-Besson et al., 2001c) or a mobile camera (Jehan-Besson et al., 2001b). Descriptors may also be sequentially used to improve the final result as it has been proposed in (Jehan-Besson et al., 2001a).

Once the criterion is introduced, we propose a new Eulerian proof for the computation of the evolution equation of the contour. It fundamentally differs from other approaches since we keep the formulation of regions instead of reducing the whole problem to boundary integrals. Indeed, we compute the Eulerian derivative of the criterion by using classical shape optimization principles. A dynamical scheme is introduced directly in the criterion. This method provides an Eulerian formulation of the problem and the advantage that the variation of region-dependent descriptors with the evolution of the active contour can be readily taken into account.

The computation of the evolution equation of the active contour is performed in three main steps:

1. Introduction of a dynamical scheme:

   We search for the partition of the image $(\Omega_{in}, \Omega_{out}, \Gamma)$ which minimizes the criterion $J(\Omega_{in}, \Omega_{out}, \Gamma)$ described in (4). While assuming that an optimal partition exists, the issue is to compute this solution. Indeed, the set of all domains has not a structure of vectorial space, and so it is difficult to compute the derivative of the criterion according to the domains. To circumvent this problem, we apply to each domain a transformation T, such that each domain becomes continuously dependent on an evolution parameter τ. The criterion becomes dependent on τ: $J(\tau) = J(\Omega_{in}(\tau), \Omega_{out}(\tau), \Gamma(\tau))$. In this dynamical scheme, we can readily take into account a possible variation of the descriptors with the evolving regions $T(\Omega_i) = \Omega_i(\tau)$.

2. Derivation of the criterion using theorem of shape optimization:

   The criterion $J(\tau)$ is then derived according to τ thanks to the computation of the Eulerian derivative of J using shape optimization principles, which is new in the area of active contours. As far as the derivation is concerned, the variation of the descriptors with the evolving regions $\Omega_i(\tau)$ and so with τ must be taken into account.

3. From the derivative, computation of the evolution equation:

   From the derivative, we deduce the velocity vector of the active contour that will make it evolve the fastest as possible towards a minimum of the criterion. In the case of region-dependent descriptors, the evolution equation of the active contour is not the same as the one computed for region-independent descriptors. Some additional terms appear and so it is of prime necessity to consider the two kinds of descriptors separately.

3.1. Introduction of a general criterion

An image $I(x, y)$ is a function defined for $(x, y) \in \Omega \subset \mathbb{R}^2$, where Ω is the image domain. The image domain is considered to be made up of two parts (fig.1): the foreground part, containing the objects to segment, noted Ω_{in}, and the background part noted Ω_{out}. The discontinuities set is a curve noted Γ that defines the boundary between the two domains.

We then search for the two domains Ω_{out} and Ω_{in} which minimize the criterion J given by (4) which is reminded here:

$$\begin{aligned} J(\Omega_{out}, \Omega_{in}, \Gamma) &= \iint_{\Omega_{out}} k^{(out)}(x, y, \Omega_{out}) dxdy \\ &+ \iint_{\Omega_{in}} k^{(in)}(x, y, \Omega_{in}) dxdy + \int_{\Gamma(\tau)} k^{(b)}(x, y) ds \end{aligned}$$

In criterion (4), the first two terms are region-based while the third term is boundary-based. The functions $k^{(out)}$, $k^{(in)}$ and $k^{(b)}$ are respectively the descriptors of the background part, the objects to segment and the contour. The descriptors may be globally attached to the evolving regions and so depend on Ω_{in} or Ω_{out}.

As the set of all domains has not a structure of vectorial space, we can not compute the derivative of the previous criterion according to domains. Thus, to compute an optimal solution, a dynamical scheme is introduced where each domain becomes continuously dependent on an evolution parameter τ. To formalize this idea, we may suppose that the evolution process is totally determined by the existence of a family of applications T_τ that makes the initial domains $\Omega_{in}(0)$ and $\Omega_{out}(0)$ evolve into the current domains $\Omega_{in}(\tau)$ and $\Omega_{out}(\tau)$ that is:

$$\begin{aligned} \Omega_i(0) &\xrightarrow{T_\tau} \Omega_i(\tau) \quad \text{where } \ i = in \text{ or } out \\ \Gamma(0) &\xrightarrow{T_\tau} \Gamma(\tau) \end{aligned} \tag{13}$$

Thus the evolving regions evolve towards an optimal solution. The triplet $\{\Omega_{out}(\tau), \Omega_{in}(\tau), \Gamma(\tau)\}$ have to act as a minimizing sequence for $J(\Omega_{out}, \Omega_{in}, \Gamma)$ as τ evolves. The functional $J(\Omega_{out}(\tau), \Omega_{in}(\tau), \Gamma(\tau))$ has to converge towards the minimum value of J as $\tau \to \infty$. The criterion then becomes:

$$\begin{aligned} J(\Omega_{out}(\tau), \Omega_{in}(\tau), \Gamma(\tau)) &= \iint\limits_{\Omega_{out}(\tau)} k^{(out)}(x, y, \Omega_{out}(\tau)) dxdy \\ &+ \iint\limits_{\Omega_{in}(\tau)} k^{(in)}(x, y, \Omega_{in}(\tau)) dxdy + \int\limits_{\Gamma(\tau)} k^{(b)}(x, y) ds \end{aligned} \tag{14}$$

The functional $J(\Omega_{out}(\tau), \Omega_{in}(\tau), \Gamma(\tau))$ is thereafter noted $J(\tau)$, and the descriptors $k^{(in)}(x, y, \Omega_{in}(\tau))$ and $k^{(out)}(x, y, \Omega_{out}(\tau))$ are respectively noted $k^{(in)}(x, y, \tau)$ and $k^{(out)}(x, y, \tau)$ which gives the following expression:

$$\begin{aligned} J(\tau) &= \iint\limits_{\Omega_{out}(\tau)} k^{(out)}(x, y, \tau) dxdy \\ &+ \iint\limits_{\Omega_{in}(\tau)} k^{(in)}(x, y, \tau) dxdy + \int\limits_{\Gamma(\tau)} k^{(b)}(x, y) ds \end{aligned} \tag{15}$$

In contrast with other active contours approaches, the dynamical scheme is directly introduced in the criterion. With such a scheme, the introduction of an active contour evolving with τ is straightforward. Hence we consider that $\Gamma(\tau)$ is modelled as an active contour that converges towards the final expected segmentation.
Let $\Gamma(0)$ be the initial curve defined by the user. We recall that we

search for $\Gamma(\tau)$ as a curve evolving according to the following PDE:

$$\frac{\partial \Gamma(s,\tau)}{\partial \tau} = \boldsymbol{v} \quad \text{with} \;\; \Gamma(0) = \Gamma_0 \tag{16}$$

where $\boldsymbol{v}$ is the velocity vector of the active contour. The main problem lies in finding the velocity $\boldsymbol{v}$ from the criterion (4) in order to get the fastest curve evolution towards the final segmentation.

3.2. Computation of the derivative of the criterion

In order to obtain the evolution equation, the criterion $J(\tau)$ must be differentiated with respect to τ. The integral bounds depend on τ and the descriptors $k^{(out)}()$ and $k^{(in)}()$ may also depend on τ.
Let us define the functional $k(x,y,\tau)$ such that :

$$k(x,y,\tau) = \begin{cases} k^{(out)}(x,y,\tau) & \text{if } (x,y) \in \Omega_{out}(\tau) \\ k^{(in)}(x,y,\tau) & \text{if } (x,y) \in \Omega_{in}(\tau) \end{cases} \tag{17}$$

The functional $k^{(out)}()$ and $k^{(in)}()$ are defined on the whole image domain Ω. The criterion $J(\tau)$ writes as:

$$J(\tau) = \int\limits_{\Omega} k(x,y,\tau)dxdy + \int\limits_{\Gamma(\tau)} k^{(b)}(x,y)ds \tag{18}$$

where Ω is the image domain. We define:

$$J_1(\tau) = \int\limits_{\Omega} k(x,y,\tau)dxdy$$

$$J_2(\tau) = \int\limits_{\Gamma(\tau)} k^{(b)}(x,y)ds$$

In order to compute the derivative of the criterion $J_1(\tau)$, the discontinuities must explicitly be taken into account. For such a purpose, we first recall a general theorem concerning the derivative of a time-dependent criterion.

Let us define $\Omega_i(\tau)$ as a region included into Ω.

THEOREM 1. *Let k_i be a smooth function on $\overline{\Omega} \times (0,T)$, and let $J(\tau) = \iint_{\Omega_i(\tau)} k_i(x,y,\tau)dxdy$, then:*

$$\frac{dJ}{d\tau} = J'(\tau) = \iint\limits_{\Omega_i(\tau)} \frac{\partial k_i}{\partial \tau} dxdy - \int\limits_{\partial\Omega_i(\tau)} k_i(\boldsymbol{V} \cdot \boldsymbol{N_{\partial\Omega_i}})ds$$

where $\boldsymbol{V}$ is the velocity of $\partial\Omega_i(\tau)$ and $\boldsymbol{N_{\partial\Omega_i}}$ is the unit inward normal to $\partial\Omega_i(\tau)$.

The derivative of J with respect to τ is the Eulerian derivative of $J(\Omega_i)$ in the direction of $\boldsymbol{V}$. The computation of the Eulerian derivative is given in Appendix A. The variation of J is due to the variation of the functional $k_i(x,y,\tau)$ and to the motion of the integral domain $\Omega_i(\tau)$. As a corollary of Theorem 1, we get:

COROLLARY 1. *Let us suppose that the domain $\Omega(\tau)$ is made up of two parts, $\Omega_{in}(\tau)$ and $\Omega_{out}(\tau)$ separated by a moving interface $\Gamma(\tau)$ whose velocity is $\boldsymbol{v}$. The function $k(x,y,\tau)$ is supposed to be separately continuous in $\Omega_{in}(\tau)$ and $\Omega_{out}(\tau)$ but can be discontinuous across $\Gamma(\tau)$. We note $k^{(in)}$ and $k^{(out)}$, the value of k in respectively $\Omega_{in}(\tau)$ and $\Omega_{out}(\tau)$. Thus the derivative of $J(\tau)$ writes as:*

$$J'(\tau) = \iint\limits_{\Omega(\tau)} \frac{\partial k}{\partial \tau} dxdy - \int\limits_{\partial\Omega(\tau)} k(\boldsymbol{w} \cdot \boldsymbol{N_{\partial\Omega}})ds + \int\limits_{\Gamma(\tau)} [[k]](\boldsymbol{v} \cdot \boldsymbol{N})ds$$

where $[[k]]$ represents the jump of k across $\Gamma(\tau)$: $[[k]] = k^{(out)} - k^{(in)}$, $\boldsymbol{N}$ the unit normal of $\Gamma(\tau)$ directed from $\Omega_{out}(\tau)$ to $\Omega_{in}(\tau)$, $\boldsymbol{N_{\partial\Omega}}$ the unit inward normal to $\Omega(\tau)$ and $\boldsymbol{w}$ the velocity vector of $\Omega(\tau)$.

Proof: We can apply Theorem 1 to the domain $\Omega_{in}(\tau)$ and to the domain $\Omega_{out}(\tau)$. By adding the two equations, we obtain the corollary.

It is now straightforward to get the derivative of the criterion $J_1(\tau)$: we take $\Omega(\tau) = \Omega$ the image domain (fig.2). Then, by taking explicitly into account the discontinuities thanks to the corollary, the derivative of $J_1(\tau)$ is given by:

$$J_1'(\tau) = \int\limits_{\Gamma(\tau)} [[k]](\boldsymbol{v} \cdot \boldsymbol{N})ds - \int\limits_{\partial\Omega} k(\boldsymbol{w} \cdot \boldsymbol{N_{\partial\Omega}})ds + \iint\limits_{\Omega_{in}(\tau)} \frac{\partial k^{(in)}}{\partial \tau} dxdy + \iint\limits_{\Omega_{out}(\tau)} \frac{\partial k^{(out)}}{\partial \tau} dxdy \tag{19}$$

Obviously, the second term of the derivative (19) is zero since the external boundary $\partial\Omega$ of the image is fixed. By replacing $[[k]]$ by its expression, we find:

$$J_1'(\tau) = \int\limits_{\Gamma(\tau)} (k^{(out)} - k^{(in)})(\boldsymbol{v} \cdot \boldsymbol{N})ds + \iint\limits_{\Omega_{in}(\tau)} \frac{\partial k^{(in)}}{\partial \tau} dxdy + \iint\limits_{\Omega_{out}(\tau)} \frac{\partial k^{(out)}}{\partial \tau} dxdy \tag{20}$$

The derivative of $J_2(\tau)$ is classical [2]:

$$J_2'(\tau) = \int_{\Gamma(\tau)} (-k^{(b)} \cdot \kappa + \nabla k^{(b)} \cdot \boldsymbol{N})(\boldsymbol{v} \cdot \boldsymbol{N}) ds \tag{21}$$

where $\kappa(x,y,\tau)$ is the curvature of $\Gamma(x,y,\tau)$. Thus for the derivative of the whole criterion, we find:

$$\begin{aligned} J'(\tau) = &\iint_{\Omega_{in}(\tau)} \frac{\partial k^{(in)}}{\partial \tau} dxdy + \iint_{\Omega_{out}(\tau)} \frac{\partial k^{(out)}}{\partial \tau} dxdy \\ + &\int_{\Gamma(\tau)} (k^{(out)} - k^{(in)} - k^{(b)} \cdot \kappa + \nabla k^{(b)} \cdot \boldsymbol{N})(\boldsymbol{v} \cdot \boldsymbol{N}) ds \end{aligned} \tag{22}$$

3.3. Expression of the derivative according to the unknown velocity vector $\boldsymbol{v}$

In order to compute the velocity vector $\boldsymbol{v}$, we have to make it appear in the first two domain integrals of equation (22) by expressing them as boundary integrals function of the velocity $\boldsymbol{v}$.

Here we take the general case where the descriptors may depend on features globally attached to the region and so may depend on τ. We model each descriptor as a combination of features globally attached to the evolving regions $\Omega_{in}(\tau)$ or $\Omega_{out}(\tau)$ or to the boundary $\Gamma(\tau)$:

$$\begin{aligned} k^{(in)}(x,y,\tau) &= g^{(in)}(x,y,G_1^{(in)}(\tau), G_2^{(in)}(\tau), ..., G_p^{(in)}(\tau)) \\ k^{(out)}(x,y,\tau) &= g^{(out)}(x,y,G_1^{(out)}(\tau), G_2^{(out)}(\tau), ..., G_p^{(out)}(\tau)) \end{aligned} \tag{23}$$

where:

$$G_j^{(\cdot)} = \begin{cases} \iint_{\Omega_\cdot(\tau)} \psi_j^{(\cdot)}(x,y,\tau) dxdy & \text{if} \quad i \in I. \\ \int_{\partial\Omega_\cdot(\tau)} \psi_j^{(\cdot)}(x,y) ds & \text{if} \quad i \in J. \end{cases} \tag{24}$$

with $(\cdot) = (in)$ or (out). The symbol $I.$ represents the set of indices j where $G_j^{(\cdot)}$ is a domain integral and $J.$ the set of indices j where $G_j^{(\cdot)}$ is a boundary integral. We find:

$$\frac{\partial k^{(\cdot)}}{\partial \tau} = \sum_{j=1}^{p} \frac{\partial g^{(\cdot)}}{\partial G_j^{(\cdot)}} (x,y,G_1^{(\cdot)}(\tau), ..., G_p^{(\cdot)}(\tau)) \frac{\partial G_j^{(\cdot)}}{\partial \tau}(\tau) \tag{25}$$

We have now to compute the derivative of $G_j^{(\cdot)}$ according to τ. Two cases have to be considered: the indices j where $G_j^{(\cdot)}$ is a domain integral and the indices j where $G_j^{(\cdot)}$ is a boundary integral. Let first compute these derivatives for $G_j^{(in)}$. For the indices j where $G_j^{(in)}$ is a boundary integral, the computation of the derivative is classical and leads to the following expression (we take the case where the function $\psi_j^{(in)}$ does not depend on τ for $j \in J_{in}$):

$$\frac{\partial G_j^{(in)}}{\partial \tau} = \int\limits_{\Gamma(\tau)} (-\psi_j^{(in)}\kappa + \nabla\psi_j^{(in)} \cdot \boldsymbol{N})(\boldsymbol{v} \cdot \boldsymbol{N})ds \quad \text{if } j \in J_{in} \tag{26}$$

For the indices j where $G_j^{(in)}$ is a domain integral, we apply the theorem 1 and we find:

$$\frac{\partial G_j^{(in)}}{\partial \tau} = \iint\limits_{\Omega_{in}(\tau)} \frac{\partial \psi_j^{(in)}}{\partial \tau} dxdy - \int\limits_{\Gamma(\tau)} \psi_j^{(in)}(\boldsymbol{v} \cdot \boldsymbol{N})ds \quad \text{if } j \in I_{in} \tag{27}$$

By replacing the derivatives of $G_j^{(in)}$ by their expressions (26,27) in equation (25), we obtain the following expression for the integral of the derivative of the descriptor $k^{(in)}$ according to τ:

$$\begin{aligned}\iint\limits_{\Omega_{in}(\tau)} \frac{\partial k^{(in)}}{\partial \tau} dxdy \; &= \sum_{j \in I_{in}} A_j^{(in)} \iint\limits_{\Omega_{in}(\tau)} \frac{\partial \psi_j}{\partial \tau} dxdy \\ &- \sum_{j \in I_{in}} A_j^{(in)} \int\limits_{\Gamma(\tau)} \psi_j^{(in)}(\boldsymbol{v} \cdot \boldsymbol{N})ds \\ &- \sum_{j \in J_{in}} A_j^{(in)} \int\limits_{\Gamma(\tau)} (\psi_j^{(in)}\kappa - \nabla\psi_j^{(in)} \cdot \boldsymbol{N})(\boldsymbol{v} \cdot \boldsymbol{N})ds\end{aligned} \tag{28}$$

with:

$$A_j^{(in)} = \iint\limits_{\Omega_{in}(\tau)} \frac{\partial g^{(in)}}{\partial G_j^{(in)}}(x, y, G_1^{(in)}(\tau), ..., G_p^{(in)}(\tau))dxdy \tag{29}$$

The expression for $(\cdot) = (out)$ is computed in the same manner and we thus obtain the general expression for the derivative of $J(\tau)$:

$$
\begin{aligned}
J'(\tau) \;=\; & \int_{\Gamma(\tau)} (k^{(out)} - k^{(in)} - k^{(b)} \cdot \kappa + \nabla k^{(b)} \cdot \boldsymbol{N})(\boldsymbol{v} \cdot \boldsymbol{N}) ds \\
& - \sum_{j \in I_{in}} A_j^{(in)} \int_{\Gamma(\tau)} \psi_j^{(in)} (\boldsymbol{v} \cdot \boldsymbol{N}) ds \\
& - \sum_{j \in J_{in}} A_j^{(in)} \int_{\Gamma(\tau)} (\psi_j^{(in)} \kappa - \nabla \psi_j^{(in)} \cdot \boldsymbol{N})(\boldsymbol{v} \cdot \boldsymbol{N}) ds \\
& + \sum_{j \in I_{out}} A_j^{(out)} \int_{\Gamma(\tau)} \psi_j^{(out)} (\boldsymbol{v} \cdot \boldsymbol{N}) ds \\
& + \sum_{j \in J_{out}} A_j^{(out)} \int_{\Gamma(\tau)} (\psi_j^{(out)} \kappa - \nabla \psi_j^{(out)} \cdot \boldsymbol{N})(\boldsymbol{v} \cdot \boldsymbol{N}) ds \\
& + \sum_{j \in I_{in}} A_j^{(in)} \iint_{\Omega_{in}(\tau)} \frac{\partial \psi_j^{(in)}}{\partial \tau} dx dy \\
& + \sum_{j \in I_{out}} A_j^{(out)} \iint_{\Omega_{out}(\tau)} \frac{\partial \psi_j^{(out)}}{\partial \tau} dx dy
\end{aligned}
\tag{30}
$$

3.4. From the derivative towards the evolution equation of the active contour

The goal of this part is to compute the expression of the velocity vector $\boldsymbol{v}$ which makes the curve evolve the fastest as possible towards a minimum of the criterion J.In order to deduce the velocity vector of the active contour from the derivative of the criterion, we have to make appear the velocity vector in the last two domain integrals by expressing them as boundary integrals as previously computed for $k^{(in)}$ and $k^{(out)}$. This can easily be done since domain integrals can always be expressed as boundary integrals. However, for simplicity, let us consider the cases where these last two integrals are equal to zero, which is the case for most region-dependent descriptors. In this case, according to the inequality of Cauchy-Schwartz, the fastest decrease of $J(\tau)$ is obtained by choosing $\boldsymbol{v} = F\boldsymbol{N}$ which gives the following evolution equation:

$$\begin{aligned}\frac{\partial \Gamma(\tau)}{\partial \tau} &= [\ k^{(in)} - k^{(out)} + k^{(b)} \cdot \kappa - \nabla k^{(b)} \cdot \boldsymbol{N} \\ &+ \sum_{j \in I_{in}} A_j^{(in)} \psi_j^{(in)} - \sum_{j \in I_{out}} A_j^{(out)} \psi_j^{(out)} \\ &+ \sum_{j \in J_{in}} A_j^{(in)} (\psi_j^{(in)} \kappa - \nabla \psi_j^{(in)} \cdot \boldsymbol{N}) \\ &- \sum_{j \in J_{out}} A_j^{(out)} (\psi_j^{(out)} \kappa - \nabla \psi_j^{(out)} \cdot \boldsymbol{N})\ ]\ \boldsymbol{N}\end{aligned} \tag{31}$$

If the descriptors do not depend on τ (region-independent descriptors), the PDE reduces to the following expression:

$$\frac{\partial \Gamma(\tau)}{\partial \tau} = [\ k^{(in)} - k^{(out)} + k^{(b)} \cdot \kappa - \nabla k^{(b)} \cdot \boldsymbol{N}]\ \boldsymbol{N} \tag{32}$$

We can notice that the dependance of the descriptors with τ, and so with the curve evolution, induces additional terms in the evolution equation of the active contour. The additional terms found in equation (31) cannot readily be obtained when the dynamical scheme is introduced after the computation of the derivative using Euler-Lagrange equations. Thus, it is of prime necessity to introduce the dynamical scheme into the criterion itself in order to easily get these additional terms in the case of region-dependent descriptors. Thanks to these additional terms, we ensure the fastest evolution of the curve towards a minimum of the criterion.

We can also remark that the velocity of the contour in (31) can be positive or negative. This an interesting property of region-based active contours. On the contrary to boundary-based active contours, region-based active contours allow some parts of the curve to shrink while others expand. Thus the initialization is less constrained.

3.5. Computation of the evolution equation for some examples of descriptors

The goal of this part is to compute the evolution equation of the active contour for some examples of region-dependent descriptors. Let us take some descriptors depending on the mean μ_{in} or on the variance σ_{in} of the evolving region $\Omega_{in}(\tau)$:

$$\begin{cases} \mu_{in}(\tau) &= \frac{1}{V_{in}} \iint\limits_{\Omega_{in}(\tau)} I dxdy \quad \text{with} \quad V_{in} = \iint\limits_{\Omega_{in}(\tau)} dxdy \\ \sigma_{in}^2(\tau) &= \frac{1}{V_{in}} \iint\limits_{\Omega_{in}(\tau)} (I - \mu_{in}(\tau))^2 dxdy \end{cases}$$

3.5.1. *Descriptors depending on the variance*
We choose the following set of descriptors:

$$\begin{cases} k^{(out)} \text{ does not depend on } \tau \\ k^{(in)} = \varphi(\sigma_{in}^2) \end{cases} \tag{33}$$

with $\varphi(r)$ a positive function of class $C^1(\mathbb{R})$.
This velocity vector is computed by replacing the terms of equation (30) by their expressions. We have:

$$k^{(in)}(x,y,\tau) = g^{(in)}(x,y,G_1^{(in)}(\tau),G_2^{(in)}(\tau)) = \varphi\left(\frac{G_1^{(in)}(\tau)}{G_2^{(in)}(\tau)}\right) \tag{34}$$

where:

$$G_j^{(in)}(\tau) = \iint\limits_{\Omega_{in}(\tau)} \psi_j^{(in)}(x,y,\tau)dxdy \tag{35}$$

with

$$\begin{cases} \psi_1^{(in)}(x,y,\tau) = (I(x,y) - \mu_{in}(\tau))^2 \\ \psi_2^{(in)}(x,y,\tau) = 1 \end{cases} \tag{36}$$

We thus obtain:

$$\begin{cases} \iint\limits_{\Omega_{in}(\tau)} \frac{\partial \psi_1^{(in)}}{\partial \tau} dxdy = -\frac{\partial \mu_{in}}{\partial \tau} \iint\limits_{\Omega_{in}(\tau)} (I(x,y) - \mu_{in}(\tau))dxdy = 0 \\ \iint\limits_{\Omega_{in}(\tau)} \frac{\partial \psi_2^{(in)}}{\partial \tau} dxdy = 0 \end{cases} \tag{37}$$

The terms $A_j^{(in)}$ can also be computed, and we find:

$$\begin{aligned} A_1^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{1}{G_2^{(in)}}\ \varphi'\left(\frac{G_1^{(in)}}{G_2^{(in)}}\right) dxdy = \varphi'(\sigma_{in}^2) \\ A_2^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{-G_1^{(in)}}{(G_2^{(in)})^2}\ \varphi'\left(\frac{G_1^{(in)}}{G_2^{(in)}}\right) dxdy = -\sigma_{in}^2\varphi'(\sigma_{in}^2) \end{aligned} \tag{38}$$

As the domain integrals of $\frac{\partial \psi_1^{(in)}}{\partial \tau}$ and $\frac{\partial \psi_2^{(in)}}{\partial \tau}$ are equal to zero, we can compute directly the velocity vector of the active contour by using equation (31) and we find:

$$\begin{aligned} \frac{\partial \Gamma(\tau)}{\partial \tau} = [\ & k^{(in)} - k^{(out)} + k^{(b)} \cdot \kappa - \nabla k^{(b)} \cdot \boldsymbol{N} \\ & +\varphi'(\sigma_{in}^2)((I - \mu_{in})^2 - \sigma_{in}^2)\ ]\boldsymbol{N} \end{aligned} \tag{39}$$

In (39), we can remark the presence of the adding term $\varphi'(\sigma_{in}^2)((I - \mu_{in})^2 - \sigma_{in}^2)\boldsymbol{N}$.

3.5.2. *Descriptors depending on the mean*
We choose the following set of descriptors:

$$\begin{cases} k^{(out)} \text{ does not depend on } \tau \\ k^{(in)} = \varphi(I - \mu_{in}(\tau)) \end{cases} \tag{40}$$

with $\varphi(r)$ a positive function of class $C^1(\mathbb{R})$.
As in the case of the descriptors depending on σ_{in}^2, the velocity vector is computed by replacing the terms of equation (30) by their expressions. We have:

$$k^{(in)}(x,y,\tau) = g^{(in)}(x,y,G_1^{(in)}(\tau),G_2^{(in)}(\tau)) = \varphi\left(I - \frac{G_1^{(in)}(\tau)}{G_2^{(in)}(\tau)}\right) \tag{41}$$

where:

$$G_j^{(in)}(\tau) = \iint\limits_{\Omega_{in}(\tau)} \psi_j^{(in)}(x,y,\tau)dxdy \tag{42}$$

with

$$\begin{cases} \psi_1^{(in)}(x,y,\tau) = I(x,y) \\ \psi_2^{(in)}(x,y,\tau) = 1 \end{cases} \tag{43}$$

As the functions $\psi_1^{(in)}$ and $\psi_2^{(in)}$ do not depend on τ, we have:

$$\begin{cases} \iint\limits_{\Omega_{in}(\tau)} \frac{\partial \psi_1^{(in)}}{\partial \tau} dxdy = 0 \\ \iint\limits_{\Omega_{in}(\tau)} \frac{\partial \psi_2^{(in)}}{\partial \tau} dxdy = 0 \end{cases} \tag{44}$$

The terms $A_j^{(in)}$ are computed, and we find:

$$\begin{aligned} A_1^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{-1}{G_2^{(in)}}\, \varphi'\left(I - \frac{G_1^{(in)}}{G_2^{(in)}}\right) dxdy = \frac{-1}{V_{in}} \iint\limits_{\Omega_{in}(\tau)} \varphi'(I - \mu_{in})dxdy \\ A_2^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{G_1^{(in)}}{(G_2^{(in)})^2}\, \varphi'\left(I - \frac{G_1^{(in)}}{G_2^{(in)}}\right) dxdy = \frac{\mu_{in}}{V_{in}} \iint\limits_{\Omega_{in}(\tau)} \varphi'(I - \mu_{in})dxdy \end{aligned}$$

We can then compute directly the velocity vector of the active contour by using the equation (31) and we find:

$$\begin{aligned}\frac{\partial \Gamma(\tau)}{\partial \tau} = [\ & k^{(in)} - k^{(out)} + k^{(b)} \cdot \kappa - \nabla k^{(b)} \cdot \boldsymbol{N} \\ & - \frac{(I - \mu_{in})}{V_{in}} \iint_{\Omega_{in}(\tau)} \varphi'(I - \mu_{in})\ ]\boldsymbol{N}\end{aligned} \tag{45}$$

In the particular case of $\varphi(r) = r^2$, we have $\varphi'(r) = 2r$ and thus, the additional terms, coming from the variation of the descriptors with τ, are zero. However, in most cases, this additional term is not zero. For example, if $\varphi(r) = \sqrt{\varepsilon + r^2}$, with ε a positive constant, we have:

$$\begin{aligned} F &= [k^{(in)} - k^{(out)} + k^{(b)} \cdot \kappa - \nabla k^{(b)} \cdot \boldsymbol{N} \\ &- \frac{(I - \mu_{in})}{V_{in}} \iint_{\Omega_{in}(\tau)} \frac{(I - \mu_{in})}{\sqrt{\varepsilon + (I - \mu_{in})^2}}]\end{aligned} \tag{46}$$

3.6. Implementation with the level set method

In this general framework, we can model the active contour by using an explicit parameterization (Lagrangian formulation) or an implicit one (Eulerian formulation) to implement the evolution equation of the active contour. See (Sethian, 1990) and (Delingette and Montagnat, 2001) for an interesting comparison between the two methods. Another interesting review on the different implementation methods of active contour is proposed in (Montagnat et al., 2001).

Here, we use the level set method approach proposed by Osher and Sethian (Osher and Sethian, 1988) but region-based active contours may also be implemented by using B-splines for an explicit parameterization (Precioso and Barlaud, 2001). The key idea of the level set method is to introduce an auxiliary function $U(x, y, \tau)$ such that $\Gamma(\tau)$ is the zero level set of U: The function U is often chosen to be the signed distance function of $\Gamma(\tau)$, and is then characterized by:

$$\Gamma(\tau) = \{(x, y)/U(x, y, \tau) = 0\} \tag{47}$$

This Eulerian formulation presents several advantages (Sethian, 1996). Firstly, the curve U may break or merge as the function U evolves, and topological changes are thus easily handled. Secondly, the evolving function $U(x, y, \tau)$ always remains a function allowing efficient numerical schemes. Thirdly, the geometric properties of the curve, like the curvature κ and the normal vector field $\boldsymbol{N}$, can be estimated directly from the level set function:

$$\kappa = div\left(\frac{\nabla U}{|\nabla U|}\right) \quad \text{and} \quad \boldsymbol{N} = \frac{\nabla U}{|\nabla U|} \tag{48}$$

The evolution equation then becomes:

$$\frac{\partial U(\tau)}{\partial \tau} = F|\nabla U| \tag{49}$$

The velocity function F is computed only on the curve $\Gamma(\tau)$ but we can extend its expression to the whole image domain Ω. To implement the level set method, solutions must be found to circumvent problems coming from the fact that the signed distance function U is not a solution of the PDE (49), see (Gomes and Faugeras, 2000) for details. In our work, the function U is reinitialized so that it remains a distance function, see (Debreuve, 2000) for details on the reinitialization equation.

In order to improve numerical efficiency, we compute the equation on a narrowband enclosing the level 0 of the level set function. We also use multiresolution techniques by making the active contour evolve first on a low resolution image. The final contour obtained for this reduced image is then used as an initial curve for the real size image.

4. Segmentation of homogeneous regions based on a competition between the variances for greyscale images

In this section, we propose some region-dependent descriptors for the segmentation of homogeneous regions for greyscale images. These descriptors are based on the minimization of the variances of the two considered regions. We show that the additional terms have to be considered in order to make the active contour evolve towards a minimum of the criterion and so to achieve the expected segmentation. In the second part, simple examples are taken to show the importance of the additional terms coming from the variation of the descriptors with the evolving regions.

4.1. Descriptors based on the variance for greyscale images

The intensity for greyscale images is represented by a function $I : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$. We propose to minimize the variances of the two considered homogeneous regions in order to segment them. The following descriptors are proposed:

$$\begin{cases} k^{(out)} & = \varphi(\sigma^2_{out}) \\ k^{(in)} & = \varphi(\sigma^2_{in}) \\ k^{(b)} & = \lambda \end{cases} \tag{50}$$

where $\varphi(r)$ is a positive function of class $C^1(I\!R)$ and λ a positive constant.
The computation of the velocity vector is deduced from the computation of the derivative of the criterion J developed in section 3.5.1, which leads to the following evolution equation:

$$\begin{aligned} \frac{\partial \Gamma(\tau)}{\partial \tau} &= [\varphi(\sigma^2_{in}) - \varphi(\sigma^2_{out}) + \lambda\kappa + \varphi'(\sigma^2_{in})((I-\mu_{in})^2 - \sigma^2_{in}) \\ &- \varphi'(\sigma^2_{out})((I-\mu_{out})^2 - \sigma^2_{out})\ ]\boldsymbol{N} \end{aligned} \tag{51}$$

The only parameter to adjust is the smoothing parameter λ. Indeed the flow $[\lambda\kappa]$ has the properties of smoothing because it moves the curve in the direction of its length functional (Siddiqi et al., 1996).

4.2. Importance of the additional terms

In this part, we want to point out the importance of the additional terms, coming from the dependance of the descriptors with τ, for the final result of the segmentation performed using equation (51).

Let us split the velocity vector of the active contour in three parts. The first vector $\boldsymbol{v_1}$ is the velocity vector obtained without taking into account that the descriptors depend on τ and without the boundary-based term. The second vector $\boldsymbol{v_2}$ refers to the boundary-based term. And the last vector $\boldsymbol{v_{add}}$ refers to the additional terms obtained with the variation of the descriptors with τ. We have $\boldsymbol{v_1} = F_1\boldsymbol{N}$, $\boldsymbol{v_2} = F_2\boldsymbol{N}$ and $\boldsymbol{v_{add}} = F_{add}\boldsymbol{N}$ where the magnitudes of the different velocity vectors have the following expressions:

$$\begin{aligned} F_1 &= [\varphi(\sigma^2_{in}) - \varphi(\sigma^2_{out})] \qquad (52) \\ F_2 &= [\lambda\kappa] \\ F_{add} &= [\varphi'(\sigma^2_{in})((I-\mu_{in})^2 - \sigma^2_{in}) - \varphi'(\sigma^2_{out})((I-\mu_{out})^2 - \sigma^2_{out})] \end{aligned}$$

In order to evaluate the importance of the additional terms for the segmentation, we made the active contour evolve through the PDE (51), including the additional terms, which can now be expressed by:

$$\frac{\partial \Gamma(\tau)}{\partial \tau} = \boldsymbol{v_1} + \boldsymbol{v_2} + \boldsymbol{v_{add}} \tag{53}$$

and we also made it evolve with the following wrong evolution equation that does not include the additional terms:

$$\frac{\partial \Gamma(\tau)}{\partial \tau} = \boldsymbol{v_1} + \boldsymbol{v_2} \tag{54}$$

For the experiments, we took $\varphi(r) = log(1 + r)$ which gives $\varphi'(r) = 1/(1 + r)$ and we chose $\lambda = 10$.

We made the active contour evolve on a synthetic image composed by an homogeneous square of intensity 120 on a background of intensity 160. A gaussian noise of variance 20 is added to this synthetic image.

The evolution of the active contour driven by equation (53) is given in Figure 3 while the one obtained using equation (54) is given in Figure 4. We can remark that when the PDE includes the additional terms, the square is well segmented (figure 3.c), whereas when the PDE doesn't include these additional terms we obtain a circle instead of the expected square (figure 4.c). These results can be better understood while analyzing the velocity vectors of each evolution equation. We compare the two evolutions of the velocity vectors F_1 and $F_1 + F_{add}$ during the propagation of the curve. The evolution of the amplitude $(F_1 + F_{add})$ is given in figures 3.{d,e,f}, while the amplitude (F_1) is given in figures 4.{d,e,f}. We can observe that the velocity F_1 is a constant where obviously the image features do not appear. On the contrary the image features well appear in the velocity $F_1 + F_{add}$ and so the square can be well segmented.

The previous results show that it is of prime necessity to take into account the additional terms, coming from region-dependent descriptors, in order to reach a true minimum of the criterion.

5. Segmentation of homogeneous regions based on the covariance matrix for color images: application to face detection in video sequences

In this part, the segmentation of homogeneous regions in multispectral images is in concern. As an example, we show that the covariance matrix can be a very powerful tool for segmentation of homogeneous color regions. The determinant of the covariance matrix is a relevant color region homogeneity measurement (Sangwine and Horne, 1998). Minimizing this quantity means that we want to decrease the complexity of a region (Gray et al., 2001; Pateux, 2000). This tool can be used for face detection in image sequences.

5.1. Descriptors based on the covariance matrix for color images

The intensity for color images is represented by a function $\vec{I} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ where $\vec{I} = [I^1, I^2, I^3]^T$. By analogy with the study for greyscale images, we propose to minimize the determinant of the covariance matrix of the considered regions.

We can make the assumption that each color region R_k is modelled by a gaussian distribution of mean $\mu = [\mu^1, \mu^2, \mu^3]^T$ and covariance matrix Σ. The probability density function is given by the following expression:

$$p_{R_k}(I) = \frac{1}{(2\pi)^{3/2}(\det(\Sigma))^{1/2}} \exp[-\frac{1}{2}(I-\mu)^T \Sigma^{-1}(I-\mu)] \tag{55}$$

In greyscale images, the probability density function of homogeneous regions is given by:

$$p_{R_k}(I) = \frac{1}{(2\pi\sigma)^{1/2}} \exp[-\frac{(I^1-\mu^1)^2}{2\sigma^2}] \tag{56}$$

In greyscale images, minimizing the variance of a region means that we want to make this region evolve towards the closest homogeneous region. So, by comparing equations (55) and (56), we can choose to minimize the covariance matrix determinant of an evolving region to make the region converge towards the closest homogeneous color region.

Let us note Σ_{in} and Σ_{out} the covariance matrix of respectively $\Omega_{in}(\tau)$ and $\Omega_{out}(\tau)$, where:

$$\Sigma_{\cdot} = \begin{pmatrix} \sigma_{\cdot}^{11} & \sigma_{\cdot}^{12} & \sigma_{\cdot}^{13} \\ \sigma_{\cdot}^{21} & \sigma_{\cdot}^{22} & \sigma_{\cdot}^{23} \\ \sigma_{\cdot}^{31} & \sigma_{\cdot}^{32} & \sigma_{\cdot}^{33} \end{pmatrix} \tag{57}$$

where:

$$\begin{cases} \sigma_{\cdot}^{ij} &= \frac{1}{V_{\cdot}} \iint\limits_{\Omega_{\cdot}(\tau)} (I^i - \mu_{\cdot}^i)(I^j - \mu_{\cdot}^j)dxdy \quad \text{with } \cdot = in \text{ or } out \\ \mu_{\cdot}^i &= \frac{1}{V_{\cdot}} \iint\limits_{\Omega_{\cdot}(\tau)} I^i(x,y)dxdy \end{cases}$$

Obviously, we have $\sigma_{\cdot}^{ij} = \sigma_{\cdot}^{ji}$. Let us note $\det(\Sigma_{\cdot})$ the determinant of the covariance matrix whose expression is the following:

$$\begin{aligned} \det(\Sigma_{\cdot}) &= \sigma_{\cdot}^{11}\sigma_{\cdot}^{22}\sigma_{\cdot}^{33} + 2\sigma_{\cdot}^{12}\sigma_{\cdot}^{13}\sigma_{\cdot}^{23} \\ &- \sigma_{\cdot}^{11}\sigma_{\cdot}^{23}\sigma_{\cdot}^{23} - \sigma_{\cdot}^{22}\sigma_{\cdot}^{13}\sigma_{\cdot}^{13} - \sigma_{\cdot}^{33}\sigma_{\cdot}^{12}\sigma_{\cdot}^{12} \end{aligned} \tag{58}$$

As a color region homogeneity measurement, we use a function of the covariance matrix determinant for region descriptors. We choose the following set of descriptors:

$$\begin{cases} k^{(out)} & = \Phi(\det(\Sigma_{out})) \\ k^{(in)} & = \Phi(\det(\Sigma_{in})) \\ k^{(b)} & = \lambda \end{cases} \tag{59}$$

where $\Phi(r)$ is a positive function of class $C^1(\mathbb{R})$ and λ a positive constant. Let us now compute the evolution equation of the active contour by computing the different terms of equation (30). Details of the computation can be found in appendix B. The evolution equation of the active contour is the following:

$$\begin{aligned} \frac{\partial \Gamma(\tau)}{\partial \tau} &= [\ \Phi(\det(\Sigma_{in})) - \Phi(\det(\Sigma_{out})) + \lambda\kappa \\ &+ \ \Phi'(\det(\Sigma_{in}) \left[\sum_{k,l=1}^{3} (I^k - \mu_{in}^k)(I^l - \mu_{in}^l) \det(M_{in}^{kl})(-1)^{k+l} \right] \\ &- \ 3\det(\Sigma_{in})\Phi'(\det(\Sigma_{in})) \\ &+ \ \Phi'(\det(\Sigma_{out}) \left[\sum_{k,l=1}^{3} (I^k - \mu_{out}^k)(I^l - \mu_{out}^l) \det(M_{out}^{kl})(-1)^{k+l} \right] \\ &- \ 3\det(\Sigma_{out})\Phi'(\det(\Sigma_{out}))\ ]\boldsymbol{N} \end{aligned} \tag{60}$$

The matrix $M_{.}^{kl}$ is deduced from the covariance matrix $\Sigma_{.}$ by suppressing the k^{th} row and the l^{th} column. The last four lines of the equation are the additional terms coming from the variation of the descriptors with τ. As it has been previously proved for greyscale images, these terms have to be considered in order to make the active contour converge towards a minimum of the criterion.

5.2. Experimental results

For the experiments, we took $\varphi(r) = log(1 + r)$ which gives $\varphi'(r) = 1/(1 + r)$ and $\Phi(r) = log(1 + r^2)$ which gives $\Phi'(r) = 2r/(1 + r^2)$. We chose $\lambda = 10$. The color space chosen is (Y, U, V), where $(I^1 = Y)$ represents the luminance and $(I^2 = U)$ and $(I^3 = V)$ represent the two chrominances. For the computation of the variance, only the luminance is used.

In order to segment an homogeneous region, we start with an initial curve inside the region of interest. The curve is then supposed to expand until it reaches the boundary of the concerned homogeneous region. The curve does not go beyond these boundaries since, over there, the criterion increases because the variance σ_{in}^2 or the covariance matrix determinant $\det(\Sigma_{in})^2$ increases.

The algorithm using the covariance matrix is performed on an image of the video sequence "akiyo" in order to detect the face of the speaker. The evolution of the curve is given in figure 5. The contour evolves to finally converge on the boundaries of the face (figure 5.c). The face is then accurately segmented. The amplitude of the velocity is also given in figures 5.{d,e,f}) and we can observe that the face region is very well delimited from the other regions. The velocity given in figures 5.{d,e,f} includes the additional terms coming from the variation of the descriptors with τ but doesn't include the boundary-based term which is a smoothing term. The amplitude of the velocity is normalized between 0 and 255, such that the negative values are lower than 128 and the positive ones are higher than 128.

We can compare the initial velocity obtained using the covariance matrix (equation 60) with the one obtained using the variance (equation 51). The face is not well delimited in the velocity obtained with the variance using equation (51) (figure 6). This velocity does not allow an efficient segmentation of the face on the contrary to the covariance matrix determinant which takes benefit of the color information.

5.3. Application to face detection in video sequences

We propose to use the descriptors based on the covariance matrix to detect human faces in video sequences. This detection can be used for video coding to encode selectively the human face. For a given compression ratio, the face can be transmitted with a higher rate to the detriment of the background which can be interesting for video-conferences where the most important and most variable information is located on the face (citer Amodou).

We first initialize the first frame by a circle inside the face to track. Then, we make the active contour evolve using equation (60), with $\Phi(r) = log(1 + r^2)$. The final contour of the previous image is then chosen as an initial curve for the next image. The algorithm has been performed on several sequences. The results for the sequence "erik" are given in figure 7. We choose $\lambda = 10$. The face is well detected and tracked on the whole sequence. The results for the sequence "akiyo" are given in figure 8. We choose $\lambda = 20$. We can see that the face is well detected even if the color of her shirt is close to the face color.

6. Conclusion

In this paper, we propose a new Eulerian minimization method to compute the velocity vector of an active contour that will make it evolve the fastest as possible towards a minimum of a criterion including both region-based and boundary-based terms. Our approach is based on shape optimization principles and is performed in three main steps:

- Introduction of a dynamical scheme in the criterion,
- Computation of the Eulerian derivative of the criterion using shape optimization tools,
- From the derivative, computation of the evolution equation of the active contour.

With our approach, we can readily take into account the case of region-dependent descriptors that are globally attached to the evolving regions and so that vary during the propagation of the curve. We show that the variation of these descriptors induce additional terms in the evolution equation of the active contour. These terms must not be forgotten in order to reach a true minimum of the criterion as it has been pointed out in experimental results. If never these terms are omitted, then the results of the segmentation are not correct since we do not reach a minimum of the criterion. As far as homogeneous color regions segmentation is concerned, we take the covariance matrix determinant as a descriptor. This descriptor appears to be a very relevant tool and so it has been applied for face detection in video sequences.

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Appendix

Appendix A.

The issue is to derive the following domain functional:

$$J(\Omega_i) = \iint_{\Omega_i} k_i(x, y, \Omega_i)\,dxdy$$

Since the set of all domains has not a structure of vectorial space, let us make the regions evolve through a family of transformations $(T(\tau, \cdot))_{\tau \geq 0}$ smooth and bijective. For a point $p = [x, y]^T$, we note:

$$\begin{aligned} p(\tau) &= T(\tau, p) \quad \text{where} \quad p(0) = p \\ \Omega_i(\tau) &= T(\tau, \Omega_i) \end{aligned}$$

Let us then define the velocity vectors field $\boldsymbol{V}$ such that:

$$\boldsymbol{V}(\tau, p(\tau)) = \frac{\partial T}{\partial \tau}(\tau, p)$$

As far as the computation of the derivative is concerned, we are interested in small deformations. We thus expand the transformation according to first order Taylor formula:

$$\begin{aligned} T(\tau, p) &= T(0, p) + \tau \frac{\partial T}{\partial \tau}(0, p) \\ &= p + \tau \boldsymbol{V}(p) \end{aligned}$$

where $\boldsymbol{V}(p) = \frac{\partial T}{\partial \tau}(0, p)$.

We then introduce three main definitions:

1. The Eulerian derivative of $J(\Omega_i)$ (in the direction of $\boldsymbol{V}$):

$$dJ(\Omega_i, \boldsymbol{V}) = \lim_{\tau \to 0} \frac{J(\Omega_i(\tau)) - J(\Omega_i)}{\tau} \tag{61}$$

2. The material derivative of $k_i(p, \Omega_i)$:

$$\dot{k}_i(p, \Omega_i, \boldsymbol{V}) = \lim_{\tau \to 0} \frac{k_i(p + \tau \boldsymbol{V}(p), \Omega_i + \tau \boldsymbol{V}(p)) - k_i(p, \Omega_i)}{\tau} \tag{62}$$

3. The shape derivative of $k_i(p, \Omega_i)$:

$$k_i'(p, \Omega_i, \boldsymbol{V}) = \lim_{\tau \to 0} \frac{k_i(p, \Omega_i + \tau \boldsymbol{V}(p)) - k_i(p, \Omega_i)}{\tau} \tag{63}$$

Obviously, by expanding equation (62) according to first order Taylor formula, we have:

$$\dot{k}_i(p, \Omega_i, \boldsymbol{V}) = k_i'(p, \Omega_i, \boldsymbol{V}) + \nabla k_i(p, \Omega_i) \cdot \boldsymbol{V}(p) \tag{64}$$

We then compute more precisely the Eulerian derivative of $J(\Omega_i)$. We have:

$$\frac{J(\Omega_i(\tau)) - J(\Omega_i)}{\tau} = \frac{1}{\tau}\left[\iint\limits_{\Omega_i(\tau)} k_i(p(\tau), \Omega_i(\tau))dp - \iint\limits_{\Omega_i} k_i(p, \Omega_i)dp\right] \tag{65}$$

In the first integral, we make the variable change $p(\tau) = p + \tau \boldsymbol{V}(p)$ where $\boldsymbol{V}(p) = [V_1(x,y), V_2(x,y)]^T$, which gives:

$$\iint\limits_{\Omega_i(\tau)} k_i(p(\tau), \Omega_i(\tau))dp =$$

$$\iint\limits_{\Omega_i} k_i(p + \tau \boldsymbol{V}(p), \Omega_i + \tau \boldsymbol{V}(p)) |\det J_\tau(p)| dp \tag{66}$$

where $J_\tau(p)$ is the following Jacobian matrix:

$$J_\tau(p) = \begin{pmatrix} 1 + \tau \frac{\partial V_1}{\partial x} & \tau \frac{\partial V_1}{\partial y} \\ \tau \frac{\partial V_2}{\partial x} & 1 + \tau \frac{\partial V_2}{\partial y} \end{pmatrix}$$

and so, we have:

$$\det J_\tau(p) = 1 + \tau \mathrm{div}(\boldsymbol{V}(p)) + \tau^2 \det(\nabla \boldsymbol{V}(p))$$

Then we obtain:

$$\lim_{\tau \to 0} \frac{\det J_\tau(p) - 1}{\tau} = \mathrm{div}(\boldsymbol{V}(p)) \tag{67}$$

The equation (65) then becomes:

$$\begin{aligned} \frac{J(\Omega_i(\tau)) - J(\Omega_i)}{\tau} = \frac{1}{\tau} \quad [& \iint\limits_{\Omega_i} k_i(p + \tau \boldsymbol{V}(p), \Omega_i + \tau \boldsymbol{V}(p)) |det J_\tau(p)| dp \\ & - \iint\limits_{\Omega_i} k_i(p, \Omega_i) dp \;] \end{aligned} \tag{68}$$

We can suppose that $\det(J_\tau(p)) \neq 0 \; \forall \tau \; \forall p$. We can take $\det(J_\tau(p)) > 0$, and we develop the expression (68) as following by adding a term in the second integral while suppressing the same term in the first integral:

$$\frac{J(\Omega_i(\tau)) - J(\Omega_i)}{\tau} = I_1 + I_2 \tag{69}$$

where:

$$I_1 = \iint_{\Omega_i} \frac{k_i(p+\tau\boldsymbol{V}(p), \Omega_i + \tau\boldsymbol{V}(p)) - k_i(p, \Omega_i)}{\tau} \det(J_\tau(p))dp \quad (70)$$

$$I_2 = \iint_{\Omega_i} k_i(p, \Omega_i) \frac{\det(J_\tau(p)) - 1}{\tau} dp \quad (71)$$

Let us make τ tend towards 0, and by using equation (64) and definitions (62,63), we get:

$$\begin{aligned}
\lim_{\tau\to 0} I_1 &= \iint_{\Omega_i} \dot{k}_i(p, \Omega_i, \boldsymbol{V})dp \\
&= \iint_{\Omega_i} k_i'(p, \Omega_i, \boldsymbol{V})dp + \iint_{\Omega_i} \nabla k_i(p, \Omega_i) \cdot \boldsymbol{V}(p)dp \\
\lim_{\tau\to 0} I_2 &= \iint_{\Omega_i} k_i(p, \Omega_i)\mathrm{div}(\boldsymbol{V})dp
\end{aligned}$$

And so for the Eulerian derivative, we find:

$$\begin{aligned}
dJ(\Omega_i, \boldsymbol{V}) &= \iint_{\Omega_i} k_i'(p, \Omega_i, \boldsymbol{V})dp \\
&\quad + \iint_{\Omega_i} (\nabla k_i(p, \Omega_i) \cdot \boldsymbol{V}(p) + k_i(p, \Omega_i)\mathrm{div}(\boldsymbol{V}(p)))dp \\
&= \iint_{\Omega_i} k_i'(p, \Omega_i, \boldsymbol{V})dp + \iint_{\Omega_i} \mathrm{div}(k_i \ \boldsymbol{V})dp \quad (72)
\end{aligned}$$

By applying the Green-Riemann theorem in equation (72), we finally obtain:

$$dJ(\Omega_i, \boldsymbol{V}) = \iint_{\Omega_i} k_i'(p, \Omega_i, \boldsymbol{V})dp - \int_{\partial\Omega_i} k_i(\boldsymbol{V} \cdot \boldsymbol{N_{\partial\Omega_i}})ds \quad (73)$$

where $\boldsymbol{N_{\partial\Omega_i}}$ is the unit inward normal to $\partial\Omega_i$. The Eulerian derivative is noted $J'(\tau)$ in the paper and the shape derivative k_i' is noted $\frac{\partial k_i}{\partial \tau}$.

NB: For simplicity, we choose the following variation for the proof: $p(\tau) = p + \tau\boldsymbol{V}(p)$. The proof may also be developped with more general variations such as: $p(\tau) = f(p, \tau)$ with f a smooth function and $f(\cdot, \tau)$ bijective.

Appendix B.

We have to compute the different terms of equation (31) in order to obtain the evolution equation of the active contour for the descriptors based on the covariance matrix determinant (59). We make the computation for the descriptor $k^{(in)}$. We can express it as a combination of functions depending on the region Ω_{in}:

$$k^{(in)}(x,y,\tau) = g\left(\frac{H^{(in)}}{(G_7^{(in)})^2}\right) \tag{74}$$

where $g = \Phi$, with Φ the function introduced in (59), and:

$$\begin{aligned} H^{(in)} = \quad & G_1^{(in)} G_2^{(in)} G_3^{(in)} + 2 G_4^{(in)} G_5^{(in)} G_6^{(in)} \\ - \; & G_1^{(in)} (G_6^{(in)})^2 - G_2^{(in)} (G_5^{(in)})^2 - G_3^{(in)} (G_4^{(in)})^2 \end{aligned}$$

with

$$\begin{aligned} G_1^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} (I^1 - \mu_{in}^1)^2 dxdy \\ G_2^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} (I^2 - \mu_{in}^2)^2 dxdy \\ G_3^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} (I^3 - \mu_{in}^3)^2 dxdy \\ G_4^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} (I^1 - \mu_{in}^1)(I^2 - \mu_{in}^2) dxdy \\ G_5^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} (I^1 - \mu_{in}^1)(I^3 - \mu_{in}^3) dxdy \\ G_6^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} (I^2 - \mu_{in}^2)(I^3 - \mu_{in}^3) dxdy \\ G_7^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} dxdy \end{aligned}$$

And so the functions $\psi_j^{(in)}$ have the following expressions:

$$\begin{aligned}
\psi_1^{(in)} &= (I^1 - \mu_{in}^1)^2 \\
\psi_2^{(in)} &= (I^2 - \mu_{in}^2)^2 \\
\psi_3^{(in)} &= (I^3 - \mu_{in}^3)^2 \\
\psi_4^{(in)} &= (I^1 - \mu_{in}^1)(I^2 - \mu_{in}^2) \\
\psi_5^{(in)} &= (I^1 - \mu_{in}^1)(I^3 - \mu_{in}^3) \\
\psi_6^{(in)} &= (I^2 - \mu_{in}^2)(I^3 - \mu_{in}^3) \\
\psi_7^{(in)} &= 1
\end{aligned}$$

From these expressions, we can easily deduce that :

$$\iint\limits_{\Omega_{in}(\tau)} \frac{\partial \psi_j^{(in)}}{\partial \tau} dxdy = 0 \quad \forall j.$$

We then have to compute the terms $A_j^{(in)}$ from the expressions of $G_j^{(in)}$, which gives:

$$\begin{aligned}
A_1^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{\partial g}{\partial G_1^{(in)}} = \iint\limits_{\Omega_{in}(\tau)} \frac{G_2^{(in)} G_3^{(in)} - (G_6^{(in)})^2}{(G_7^{(in)})^3} \Phi'(\det(\Sigma_{in})) \\
A_2^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{\partial g}{\partial G_2^{(in)}} = \iint\limits_{\Omega_{in}(\tau)} \frac{G_1^{(in)} G_3^{(in)} - (G_5^{(in)})^2}{(G_7^{(in)})^3} \Phi'(\det(\Sigma_{in})) \\
A_3^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{\partial g}{\partial G_3^{(in)}} = \iint\limits_{\Omega_{in}(\tau)} \frac{G_1^{(in)} G_2^{(in)} - (G_4^{(in)})^2}{(G_7^{(in)})^3} \Phi'(\det(\Sigma_{in})) \\
A_4^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{\partial g}{\partial G_4^{(in)}} = \iint\limits_{\Omega_{in}(\tau)} \frac{-2(G_3^{(in)} G_4^{(in)} - G_5^{(in)} G_6^{(in)})}{(G_7^{(in)})^3} \Phi'(\det(\Sigma_{in})) \\
A_5^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{\partial g}{\partial G_5^{(in)}} = \iint\limits_{\Omega_{in}(\tau)} \frac{2(G_4^{(in)} G_6^{(in)} - G_2^{(in)} G_5^{(in)})}{(G_7^{(in)})^3} \Phi'(\det(\Sigma_{in})) \\
A_6^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{\partial g}{\partial G_6^{(in)}} = \iint\limits_{\Omega_{in}(\tau)} \frac{-2(G_1^{(in)} G_6^{(in)} - G_4^{(in)} G_5^{(in)})}{(G_7^{(in)})^3} \Phi'(\det(\Sigma_{in})) \\
A_7^{(in)} &= \iint\limits_{\Omega_{in}(\tau)} \frac{\partial g}{\partial G_7^{(in)}} = \iint\limits_{\Omega_{in}(\tau)} \frac{-3H^{(in)}}{(G_7^{(in)})^4} \Phi'(\det(\Sigma_{in}))
\end{aligned}$$

And by expressing the covariance matrix with the functions $G_j^{(in)}$ as following:

$$\Sigma_{in} = \begin{pmatrix} \frac{G_1^{(in)}}{G_7^{(in)}} & \frac{G_4^{(in)}}{G_7^{(in)}} & \frac{G_5^{(in)}}{G_7^{(in)}} \\ \frac{G_4^{(in)}}{G_7^{(in)}} & \frac{G_2^{(in)}}{G_7^{(in)}} & \frac{G_6^{(in)}}{G_7^{(in)}} \\ \frac{G_5^{(in)}}{G_7^{(in)}} & \frac{G_6^{(in)}}{G_7^{(in)}} & \frac{G_3^{(in)}}{G_7^{(in)}} \end{pmatrix}$$

We then deduce:

$$\begin{aligned} A_1^{(in)} &= (-1)^{(1+1)} \det(M_{in}^{11})\Phi'(\det(\Sigma_{in})) \\ A_2^{(in)} &= (-1)^{(2+2)} \det(M_{in}^{22})\Phi'(\det(\Sigma_{in})) \\ A_3^{(in)} &= (-1)^{(3+3)} \det(M_{in}^{33})\Phi'(\det(\Sigma_{in})) \\ A_4^{(in)} &= (-1)^{(1+2)} 2\det(M_{in}^{12})\Phi'(\det(\Sigma_{in})) \\ A_5^{(in)} &= (-1)^{(1+3)} 2\det(M_{in}^{13})\Phi'(\det(\Sigma_{in})) \\ A_6^{(in)} &= (-1)^{(2+3)} 2\det(M_{in}^{23})\Phi'(\det(\Sigma_{in})) \\ A_7^{(in)} &= -3\det(\Sigma_{in})\Phi'(\det(\Sigma_{in})) \end{aligned}$$

where M_{in}^{kl} is the matrix deduced from Σ_{in} by suppressing the k^{th} row and the l^{th} column. The computation for the descriptor $k^{(out)}$ is similar and we then have all the expressions to compute the evolution equation from the general formula (31). This gives the evolution equation (60).

Acknowledgements

Figure 1. The two regions of an image

Figure 2. The domains and the vectors involved in the derivation

a. Initial contour b. Iteration 500 c. Final contour

d. Initial velocity e. Iteration 500 f. Final velocity

Figure 3. Figures a,b,c: Evolution of the active contour managed by the PDE (**??**) including the additional terms. Figures d,e,f: Visualization of the corresponding velocity function $(F_1 + F_{add})$ (including the additional terms) see equation (52).

a. Initial contour b. Iteration 500 c. Iteration 2500

d. Initial velocity e. Iteration 3500 f. Final velocity

Figure 4. Figures a,b,c: Evolution of the active contour managed by the PDE (54), without the additional terms (figures a,b,c).
Figures d,e,f: Visualization of the corresponding velocity function F_1 (see equation (52)), without the additional terms (normalized such that $\max(F_1) = 255$).

a. Initial contour b. Iteration 500 c. Final contour

d. Initial velocity e. Iteration 500 f. Final velocity

Figure 5. Visualization of the evolution of the contour and the velocity (without the boundary-based term) for the segmentation of the face using the covariance matrix.

a. Initial curve b. Initial velocity (covariance) c. Initial velocity (variance)

Figure 6. Comparison of the initial velocity computed using the covariance matrix (figure b) to the one computed using the variance (figure c) from the same initial curve (figure a).

Final curve Extracted face

Figure 7. Detection of the face on the video sequence "erik".

Final curve Extracted face

Figure 8. Detection of the face on the video sequence "akiyo".