Brachistochrones are the shortest

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INTRODUCTION. The word *brachistochrone* (Greek for "shortest time") was created by Johann Bernoulli to qualify a problem he had solved, then submitted to the "most acute mathematicians in the entire world" in the June 1696 issue of *Acta Eruditorum* :
given two points A and B, find the curve that a point, moving from A to B in a vertical plane under its own gravity, must follow so that, starting from A with zero velocity, it reaches B in the shortest possible time.

The question actually drew attention of some of the greatest minds of the time such as Leibniz, l'Hospital, Newton and Johann's brother Jakob Bernoulli. Taken up again and generalized in the eighteenth century by Euler and Lagrange, Bernoulli's problem gave rise to the calculus of variations. More details on this fascinating chapter in the history of mathematical analysis can be found in Nahin [**3**] or Tikhomirov [**4**].

The brachistochrone problem is now a popular exercise for undergraduates, solved by application of the classical Euler-Lagrange differential equations. The purpose of this note is to solve by more elementary tools (i.e. without any knowledge of the calculus of variations) a class of problems including Bernoulli's and to establish the minimal character of the solution. The latter fact does not follow directly from the Euler-Lagrange equations, only giving a necessary condition for an extremum. We also include a short proof of a more general and slightly less elementary result (Theorem 3).

MAIN RESULT. Let $\gamma : [t_0, t_1] \to \mathbb{R}^2$, i.e. $\gamma(t) = (x(t), y(t))$, be a path in the Euclidean plane with continuous derivatives $x' = dx/dt$, $y' = dy/dt$, such that $\gamma'(t) \neq 0$ for $t_0 < t < t_1$. The parameter t can then be replaced by the arclength, denoted by s. We call this a *regular path.* If f is a continuous function on an open set containing the image of γ, we consider the integral

$$F(\gamma) := \int_{t_0}^{t_1} f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\ dt = \int_\gamma f(x, y)\ ds\ .$$

Our main result deals with the special case of a function f of one variable only.

Theorem 1 *Let f be a strictly positive continuous function on an interval I in $\mathbb{R}$. Let $\alpha : I \to \mathbb{R}$ be a continuous function such that, for all $y \in I$,*

$$f(y) \cos \alpha(y) = C\ , \tag{1}$$

where C is a constant. Let $\gamma(s) = (x(s), y(s))$ ba a solution of the differential system

$$x' = \cos \alpha(y)\ ,\ y' = \sin \alpha(y)\ . \tag{2}$$

Then γ gives the integral

$$F(\gamma) = \int_\gamma f(y)\ ds$$

a minimal value among all regular paths in $\mathbb{R} \times I$ with the same endpoints.

Proof. Let $\gamma(s) = (x(s), y(s))$, $s_0 \le s \le s_1$ such that (2) holds, and let $\delta(t) = (X(t), Y(t))$, $t_0 \le t \le t_1$, be another regular path in $\mathbb{R} \times I$ with the same endpoints as γ, i.e.

$$\delta(t_0) = \gamma(s_0) = (x_0, y_0) \ , \ \delta(t_1) = \gamma(s_1) = (x_1, y_1) \ .$$

Let

$$A = \int_{x_0}^{x_1} C \ dx + \int_{y_0}^{y_1} f(y) \sin \alpha(y) \ dy \ .$$

On the one hand, changing variables by $x = X(t)$, $y = Y(t)$ we obtain from (1) and Cauchy-Schwarz inequality

$$\begin{aligned} A &= \int_{t_0}^{t_1} (CX' + f(Y)Y' \sin \alpha(Y)) \ dt \\ &= \int_{t_0}^{t_1} f(Y)(X' \cos \alpha(Y) + Y' \sin \alpha(Y)) \ dt \\ &\le \int_{t_0}^{t_1} f(Y)\sqrt{X'^2 + Y'^2} \ dt = F(\delta) \ . \end{aligned}$$

On the other hand, changing variables by $x = x(s)$, $y = y(s)$ we obtain from (1) and (2)

$$\begin{aligned} A &= \int_{s_0}^{s_1} f(y)(x' \cos \alpha(y) + y' \sin \alpha(y)) \ ds \\ &= \int_{\gamma} f(y) \ ds = F(\gamma) \ , \end{aligned}$$

whence the theorem. ■

Motivation. To motivate the solution (1)(2) of the problem, one might of course write down the corresponding Euler-Lagrange equations. But a more instructive motivation follows from an optical analogy observed by Johann Bernoulli himself. Indeed $\int_\gamma f(y)ds$ is, up to a constant factor, the travelling time of a light ray in a nonhomogeneous optical medium with refractive index $f(y)$ at the point (x, y); this occurs for instance when studying mirages. Discretizing the problem, Bernoulli replaced this continuous medium by a large number of horizontal layers with increasing heights $y_1, ..., y_n$ and constant indices $f(y_1), ..., f(y_n)$, and the path γ by a broken line. When crossing a layer the light ray obeys Snell's law of refraction

$$f(y_i) \cos \alpha_i = f(y_{i+1}) \cos \alpha_{i+1} \ ,$$

where α_i (resp. α_{i+1}) is the angle between the ray and the horizontal below (resp. above) the layer at height y_i. Letting n go to infinity Bernoulli "proved" in this way that, calling α the angle between the path γ and the horizontal (whence the differential equations (2)), $f(y) \cos \alpha$ must remain constant (whence (1)).

EXAMPLES. Theorem 1 applies to several classical problems.

a. If $f = 1$ it gives one more proof that the shortest path between two points in the Euclidean plane is the straight line.

b. If $f(x, y) = 2\pi y$ the integral $F(\gamma)$ is the area generated by γ rotating around the x-axis; the problem is to find *minimal surfaces* of revolution.

c. If $f(x, y) = 1/y$ with $y > 0$, we obtain the *geodesics of the Poincaré upper half-plane*; the classical proof of their minimal property makes use of the transitive action of the isometry group $SL(2, \mathbb{R})$.

d. The original brachistochrone problem corresponds to $f(x, y) = 1/\sqrt{y}$ with $y > 0$; the solutions are cycloids.

e. An interesting variant of Bernoulli's question is the problem of fast tunnels through the earth (Nahin [**3**] p. 229, Tung-Po Lin [**5**]):
given two points A and B on the surface of the earth (or inside), find the shape of a tunnel from A to B such that a point starting from A with zero velocity reaches B in the shortest possible time, by a frictionless motion under its own gravity; the earth is assumed to be a sphere of homogeneous density.
Solving the corresponding Euler-Lagrange equations one finds hypocycloids ([**5**]). The problem can also be solved by the following polar coordinates variant of Theorem 1.

Corollary 2 *Let f be a strictly positive continuous function on an interval $I=]0,a[$. Let $\alpha : I \to \mathbb{R}$ be a continuous function such that, for all $r \in I$,*

$$rf(r)\cos\alpha(r) = C \ , \tag{1'}$$

where C is a constant. In polar coordinates (r,θ) let $\gamma(t) = (r(t),\theta(t))$ be a regular path satisfying the differential equations

$$r' = r\sin\alpha(r) \ , \ \theta' = \cos\alpha(r) \ . \tag{2'}$$

Then γ gives the integral

$$F(\gamma) = \int_\gamma f(r)\ ds$$

a minimal value among all regular paths with the same endpoints.

Proof. Let $\rho = \ln r$. For any path γ

$$F(\gamma) = \int_{t_0}^{t_1} f(r)\sqrt{r'^2 + r^2\theta'^2}\ dt = \int_{t_0}^{t_1} e^\rho f(e^\rho)\sqrt{\theta'^2 + \rho'^2}\ dt \ .$$

The result thus follows from Theorem 1 with (θ,ρ) and $rf(r)$ instead of (x,y) and $f(y)$. ∎
Remark. By (2') $\pi/2 - \alpha$ is the angle at the point $M = (r,\theta)$ between the vector $\overrightarrow{OM}$ and the tangent to γ at M. In the optical analogy, Snell's law (1) is now replaced by Bouguer's formula (1') for a spherically symmetric optical medium with refractive index $f(r)$ (Born and Wolf [**1**] p. 130). In the tunnel problem the function f is $f(r) = (a^2 - r^2)^{-1/2}$.

GENERALIZATION. A crucial fact in the proof of Theorem 1 is of course that the differential form

$$\omega = f(y)(\cos\alpha(y)\ dx + \sin\alpha(y)\ dy)$$

is closed if (and only if) $f(y)\cos\alpha(y)$ is constant. The method extends, under stronger regularity assumptions, by means of "Hilbert's independence integral" (Young [**6**] p. 27, Born and Wolf [**1**] Appendix 1). It leads to the following less elementary result.

Let T be an open interval and U, V open subsets of $\mathbb{R}^n$ with U simply connected. We denote by $\mathcal{L} : (t,x,v) \longmapsto \mathcal{L}(t,x,v)$ a real-valued C^2 function (the Lagrangian) on $T \times U \times V$ and by $p : (t,x) \mapsto p(t,x)$ a C^1 map from $T \times U$ into V. Let ω be the differential form

$$\omega := (\mathcal{L}(t,x,p) - \partial_v\mathcal{L}(t,x,p)\cdot p)\ dt + \partial_v\mathcal{L}(t,x,p)\cdot dx$$

on $T \times U$, with $p = p(t,x)$.

In Theorem 1 we had $p(t,x_1,x_2) = (\cos\alpha(x_2), \sin\alpha(x_2))$, $\mathcal{L}(t,x,v) = f(x_2)\,\|v\|$ (Euclidean norm), $T = \mathbb{R}$, $U = \mathbb{R} \times I$, $V = \mathbb{R}^2$; the convexity argument in *(ii)* below was replaced by Cauchy-Schwarz inequality.

Theorem 3 *Assume* ω is closed *in* $T \times U$.
Let γ *be any solution of the differential system* $\gamma'(t) = p(t, \gamma(t))$, C^1 *on an interval* $[a, b] \subset T$, *such that* $\gamma(t) \in U$ *and* $\gamma'(t) \in V$ *for all* $t \in [a, b]$.
(i) Then γ *satisfies the Euler-Lagrange system of differential equations*

$$\frac{d}{dt}(\partial_v \mathcal{L}(t, \gamma, \gamma')) = \partial_x \mathcal{L}(t, \gamma, \gamma') \ .$$

Thus p *is a "field of extremals".*
(ii) Assume furthermore that V *is convex and* $v \mapsto \mathcal{L}(t, x, v)$ is a convex function *on* V *for each* $(t, x) \in T \times U$. *Then*

$$F(\delta) := \int_a^b \mathcal{L}(t, \delta(t), \delta'(t)) \ dt \geq F(\gamma)$$

for all C^1 *paths* $\delta : [a, b] \to U$ *with the same endpoints as* γ *and such that* $\delta'(t) \in V$ *for all* t.

Proof. *(i)* Let $h : [a, b] \to \mathbb{R}^n$ be any C^1 path with $h(a) = h(b) = 0$ and let, for $\varepsilon \in \mathbb{R}$,

$$\gamma_\varepsilon(t) := \gamma(t) + \varepsilon h(t)$$

be a variation of γ. If the constant ε is small enough we have $(t, \gamma_\varepsilon(t), \gamma'_\varepsilon(t)) \in T \times U \times V$ for all t and

$$\gamma'_\varepsilon - p(t, \gamma_\varepsilon) = p(t, \gamma) - p(t, \gamma_\varepsilon) + \varepsilon h' = O(\varepsilon) \ ,$$

uniformly with respect to t, whence

$$\mathcal{L}(t, \gamma_\varepsilon, \gamma'_\varepsilon) = \mathcal{L}(t, \gamma_\varepsilon, p(t, \gamma_\varepsilon)) + \partial_v \mathcal{L}(t, \gamma_\varepsilon, p(t, \gamma_\varepsilon)) \cdot (\gamma'_\varepsilon - p(t, \gamma_\varepsilon)) + O(\varepsilon^2)$$

and, by integration,

$$F(\gamma_\varepsilon) = \int_{\gamma_\varepsilon} \omega + O(\varepsilon^2) \ .$$

Since ω is closed and γ_ε has the same endpoints as γ, the integral $\int_{\gamma_\varepsilon} \omega$ is independent of ε. Therefore the derivative $\frac{d}{d\varepsilon} F(\gamma_\varepsilon)$ vanishes at $\varepsilon = 0$, and the Euler-Lagrange equations follow by a classical computation.
(ii) By convexity of $\mathcal{L}$ we have

$$\mathcal{L}(t, \delta, \delta') \geq \mathcal{L}(t, \delta, p) + \partial_v \mathcal{L}(t, \delta, p) \cdot (\delta' - p)$$

with $\delta = \delta(t)$, $\delta' = \delta'(t)$, $p = p(t, \delta(t))$ and, by integration,

$$F(\delta) \geq \int_\delta \omega = \int_\gamma \omega = F(\gamma) \ ;$$

in the latter equalities we successively used the assumptions that ω is closed, δ and γ have common endpoints and $\gamma' = p(t, \gamma)$. ■
Remark. Taking $\delta = \gamma + \varepsilon h$ in *(ii)* would imply the Euler-Lagrange equations again, since F(γ) is a minimum.

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