Resolvent of the Laplacian near zero energy for asymptotically conic spaces

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We consider generalizations of Euclidean low energy resolvent estimates, in a Fredholm framework... relevant for e.g. wave equation asymptotics. Indeed, one motivation is understanding waves on Kerr spacetimes.

These are already interesting in explaining the Euclidean phenomena: what is *really* happening for low energies?

Structure:

- Euclidean problems
- Geometric generalization
- Zero energy
- Microlocal Fredholm analysis

Let

- g_0 be the Euclidean metric,
- g metric on $\mathbb{R}^n$ with $g - g_0 \in S^{-\delta}$, $\delta > 0$ (i.e. $g_{ij} - (g_0)_{ij} \in S^{-\delta}$), g positive definite,
- $V \in S^{-\delta}$, real.

Recall: $S^m(\mathbb{R}^n_z)$ is the symbol space: $\forall \alpha$

$$|D_z^\alpha a(z)| \leq C_\alpha \langle z \rangle^{m-|\alpha|}.$$

Different way of writing it (away from origin): $\forall k$

$$|W_1 \ldots W_k a(z)| \leq C_k \langle z \rangle^m,$$

where

$$W_r = z_{i_r} D_{z_{j_r}}.$$

Note: V non-real, but $\operatorname{Im} V \in S^{-1-\delta}$, would only cause some possible finite rank issues below.

Let

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- $V \in S^{-\delta}$, real.

Then

$$H = \Delta_g + V$$

is self-adjoint on $L^2(\mathbb{R}^n)$, so $H - \lambda$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is invertible, e.g. as a map

$$H - \lambda : H^{s+2,l} \to H^{s,l}, \;\; s, l \in \mathbb{R}.$$

Moreover, the spectrum in $(-\infty, 0)$ is discrete, with 0 a possible accummulation point (e.g. Coulomb-like potentials); $[0, \infty)$ the essential spectrum.

Here: $H^{s,l} = \langle z \rangle^{-l} H^s$, H^s standard Sobolev space.

While $H-\lambda$ will no longer be invertible between the weighted Sobolev spaces when $\lambda>0$, the limiting absorption principle states that

$$(H-(\lambda\pm i0))^{-1}=\lim_{\epsilon\to 0}(H-(\lambda\pm i\epsilon))^{-1}$$

exist e.g. as strong limits in $\mathcal{L}(H^{s,l},H^{s+2,l'})$, $l>\frac{1}{2}$, $l'<-\frac{1}{2}$ (so $l-l'>1$).

Under stronger assumptions (Coulomb!), $V\in S^{-2-\delta}$, $\delta>0$, 0 is not an accummulation point of the spectrum, and under stronger restrictions on l,l', in particular $l-l'>2$, $(H-(\lambda\pm i0))^{-1}$ is uniformly bounded between the weighted Sobolev spaces as $\lambda\to 0$ if there are no 0-energy bound states (L^2 nullspace of H) or half-bound states (to be discussed). (Jensen, Kato,...,Bony, Häfner)

What kind of structure of Euclidean space is involved? One way to address is via geometric generalizations.

Can one make the function spaces more precise? For instance, can one fit these into a Fredholm (here typically invertible) statement? Such frameworks are necessarily sharp in a sense.

A natural geometric generalization is asymptotically conic spaces.

A conic metric, with cross section a Riemannian manifold (Y, h), is the metric $g_0 = dr^2 + r^2 h$ on $\mathbb{R}_r^+ \times Y$.

In local coordinates, such a metric is a linear combination, with $C^\infty(Y)$-coefficients of dr and $r\,dy_j$, y_j local coordinates on Y, with some restrictions.

We basically want to generalize the coefficients. Identifying a coordinate chart in Y with a coordinate chart on the sphere $\mathbb{S}^{n-1}$, we are working in an open conic subset O of $\mathbb{R}^n$ near infinity. (So this is a natural generalization of the process of going from $\mathbb{R}^n$ to say compact manifolds without boundary.)

From this perspective, we replace the coefficients $C^\infty(Y)$ by a *symbolic* statement, namely coefficients in $S^0(O)$, differing from elements of $C^\infty(Y)$ by $S^{-\delta}(O)$, $\delta > 0$.

Let's rephrase this in a compactified notation.

Let $x = \frac{1}{r}$, and add $x = 0$ as an ideal boundary at infinity, i.e. work with $[0, \infty)_x \times Y$. The metric is then a linear combination of symmetric tensors formed from $\frac{dx}{x^2}$ and $\frac{dy_j}{x}$.

Then $C^\infty([0, \infty)_x \times Y)$ near the ideal boundary means exactly classical symbolic behavior of order 0 (on $\mathbb{R}^n$ or under the local conic identification), with the expansion being just Taylor series.

In Melrose's notation, if $\bar{X}$ a manifold with boundary which near ∂X is of the form $[0, x_0)_x \times Y$, one would say that a metric with such coefficients (sc-metric) is a C^∞ section of ${}^{\mathrm{sc}}T^*\bar{X} \otimes_s {}^{\mathrm{sc}}T^*\bar{X}$, with ${}^{\mathrm{sc}}T^*\bar{X}$ being locally spanned by $\frac{dx}{x^2}$ and $\frac{dy_j}{x}$.

Note that in general $a \in S^m$ becomes $(x\partial_x)^\alpha \partial_y^\beta a \in x^{-m} L^2$ from this perspective (cf. linear v.f. characterization!). A symbolic section of ${}^{\mathrm{sc}}T^*\bar{X}^{\otimes_s^2}$ might be denoted by $S^m(\bar{X}, {}^{\mathrm{sc}}T^*X^{\otimes_s^2})$.

Dually, sc-vector fields, $W \in \mathcal{V}_{\mathrm{sc}}(\bar{X})$ are smooth linear combinations of $x^2 D_x$ and $x D_{y_j}$. A sc-differential operator is a finite sum of products of sc-vector fields with $C^\infty(\bar{X})$-coefficients... but again, this is simply the extension, via conic coordinate charts, of classical S^0 symbol coefficients on $\mathbb{R}^n$.

The Sobolev spaces can also be extended to this setting either via the local coordinate identification, or by a direct construction. E.g. for $s \geq 0$ integer, $u \in H^{s,l}$ if for all $Q \in \mathrm{Diff}^s_{\mathrm{sc}}(\bar{X})$, $Qu \in x^l L^2$, L^2 w.r.t. any sc-metric.

In this geometric setting, we take g Riemannian metric on $X = \bar{X}^\circ$, which near $\partial\bar{X}$ is of the form $[0, x_0)_x \times Y$, with $g - g_0 \in S^{-\delta}$, g_0 conic metric.

If $V \in S^{-\delta}$ real, then again $H = \Delta_g + V$ is self-adjoint on L^2 and the limiting absorption principle holds (Melrose).

Theorem (V., cf. Guillarmou-Hassell)

If $V \in S^{-2-\delta}$ or more generally $H - \Delta_g \in S^{-2-\delta}\mathrm{Diff}^1_{\mathrm{sc}}(X)$ self-adjoint, then uniform estimates to 0 energy hold as $\lambda \to 0$ if there are no bound states or half bound states.

A further generalization is a family $P(\sigma)$, where in the above case $P(\sigma)$ is the spectral family $H - \sigma^2$.

Here we need $P(\sigma) - (\Delta_g - \sigma^2)$ to be smooth in σ with values $S^{-2-\delta}\mathrm{Diff}^2_{\mathrm{sc}}(\bar{X})$ which is elliptic in the usual sense (e.g. $S^{-2-\delta}\mathrm{Diff}^1_{\mathrm{sc}}(\bar{X})$) and is self-adjoint for real σ.

This is useful because at least near infinity, i.e. $\bar{X}$, this includes the family of operators one obtains from the Kerr wave operator by Fourier transforming in the time variable; indeed, the method of proof extends directly to the actual Kerr operator.

Before indicating the proofs, let us think about the 0-energy problem, i.e. inverting H. I'll always take $n \geq 3$.

Under the conditions discussed for the near 0 behavior, we have $H \in x^2 S^0 \mathrm{Diff}_b^2(\bar{X})$, where $\mathrm{Diff}_b^2(X)$ is locally spanned by xD_x, D_{y_j} (so e.g. the symbolic statement is $Qa \in x^{-m} L^\infty$ for all $Q \in \mathrm{Diff}_b(\bar{X})$). Notice that this needs the extra decay on V!

Indeed, $H = x^2 \tilde{H}_{\mathrm{b}}$, where $\tilde{H}_{\mathrm{b}}$ is roughly the Laplacian of $\frac{dx^2}{x^2} + h$. It is more convenient to write $H = x^{(n+2)/2} H_{\mathrm{b}} x^{-(n-2)/2}$, where $H_{\mathrm{b}} \in S^0 \mathrm{Diff}_b^2(\bar{X})$ with

$$H_{\mathrm{b}} - \Big((xD_x)^2 + \Delta_h + \Big(\frac{n-2}{2}\Big)^2\Big) \in S^{-\delta} \mathrm{Diff}_b^2(\bar{X}).$$

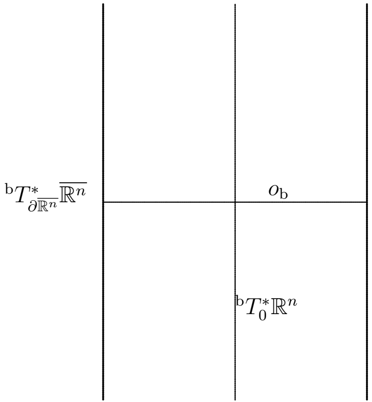
The term subtracted away should be considered an effective model at infinity, and can be easily analyzed by a Mellin transform in x.

Combined with an elliptic analysis in this b-framework ('cylindrical ends'), noting that H_{b} is *elliptic as an element of* $S^0\mathrm{Diff}_b^2(\bar{X})$, Melrose showed that it is Fredholm on weighted spaces $H_b^{s,l}$, provided that l is not an *indicial root*, which in turn arise from the eigenvalues of Δ_h.

Note that Fredholmness has two ingredients:

- gain of differentiability and
- gain in decay in errors of estimates;

the former is what is assured by the elliptic analysis, the latter by the normal operator/Mellin transform analysis. (Melrose proceeded more constructively via parametrices.)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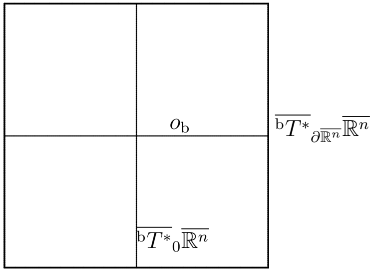
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Here let me just state the consequence:

$$H : H_b^{r,l} \to H_b^{r-2,l+2}$$

is Fredholm for $|l+1| < \frac{n-2}{2}$, with the $\frac{n-2}{2}$ corresponding to the spectrum of $\Delta_h + \left(\frac{n-2}{2}\right)^2$ starting at $\left(\frac{n-2}{2}\right)^2$. Furthermore, in this range of l and for arbitrary r, the nullspace is independent of r, l.

Non L^2-elements of the nullspace are the *half-bound states*. These cannot exist if $n \geq 5$ since l can be taken $\frac{n-2}{2} - 1 - \epsilon$, thus positive.

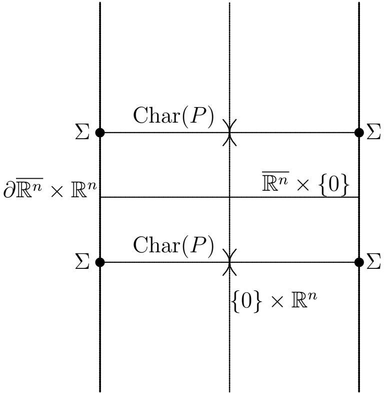
The spaces here are b-Sobolev spaces: $u \in H_b^r$ means $Qu \in L^2$ for all $Q \in \mathrm{Diff}_b^r$ if $r \geq 0$ an integer (here I am taking $L^2 = L_g^2$); $H_b^{r,l} = x^l H_b^r$.

Furthermore, it is invertible if H has no nullspace *as a map between these spaces*, which is automatically the case if $H = \Delta_g$, due to the maximum principle.

Before addressing the near 0-energy problem first consider the limiting absorption principle *as a Fredholm problem*. For this we need *microlocally weighted* ('anisotropic') Sobolev spaces.

Concretely, in the present setting, modifying the standard weighted Sobolev spaces (i.e. the scattering Sobolev spaces), it is the *decay order* r that is variable, i.e. a function on ${}^{\mathrm{sc}}T^*\bar{X}$. These are defined by variable order ps.d.o's, much like standard Sobolev spaces can be so defined. (Goes back to Unterberger, Duistermaat...)

Wherever the problem is elliptic, the order actually does not matter; since H is elliptic in the standard sense, the sc-differential order never matters. Moreover, the sc-decay order only matters near the characteristic set, $G - \lambda = 0$ ($\lambda > 0$, $G = g^{-1}$), which is a compact subset of ${}^{\mathrm{sc}}T^*_{\partial X}\bar{X}$ (so $|\zeta|^2 - \lambda = 0$ in the Euclidean case).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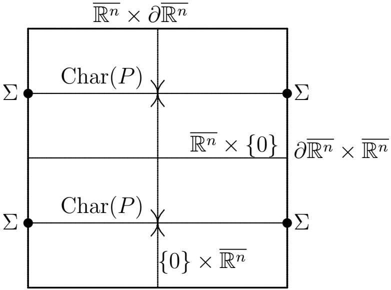
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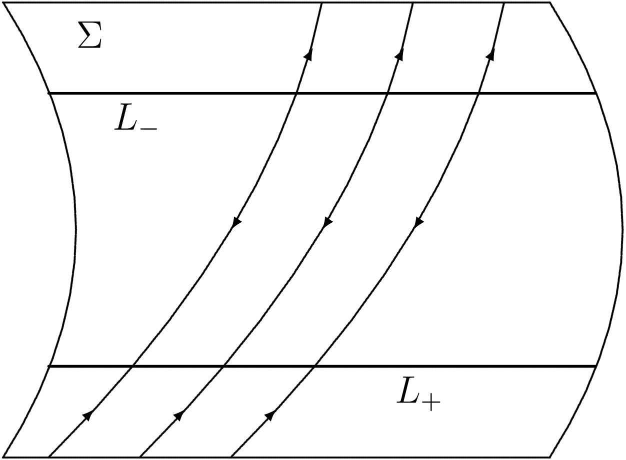
Within this characteristic set, one has propagation of singularities (here meaning microlocal lack of decay) between two radial sets. The latter are places where the Hamilton vector field H_G is *radial*, here *in the base manifold,* X (multiple of $x\partial_x$, or in Euclidean notation $z \cdot \partial_z$).

Thus, if one takes r to be monotone function along H_G, satisfying that it is above/below the threshold regularity at one of the radial sets vs. the other, in this case the threshold being $1/2$, then

$$H - \lambda : \{u \in H^{s+2,r-1} : (H-\lambda)u \in H^{s,r}\} \to H^{s,r}$$

is Fredholm, indeed in this case invertible. There are two options: which of the radial sets is the high, and which is the low regularity: incoming vs. outgoing resolvents ($\pm i0$ limits).

Note that this corresponds exactly to *global* version of the distinguished propagators (advanced, retarded, Feynman, anti-Feynman) of Duistermaat-Hörmander.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Notice that the loss of ϵ in the usual way the limiting absorption principle is stated *disappears* with these variable order spaces ($r-1$ vs. r, exactly the real principal type numerology).

Concretely, with τ, μ the fiber coordinates in the sc-notation on ${}^{\mathrm{sc}}T^*\bar{X}$, i.e. writing covectors as $\tau\,\frac{dx}{x^2}+\mu\cdot\frac{dy}{x}$, one can take for $\beta>0$

$$r=\frac{1}{2}\pm\beta\frac{\tau}{\sqrt{\tau^2+|\mu|_h^2}},$$

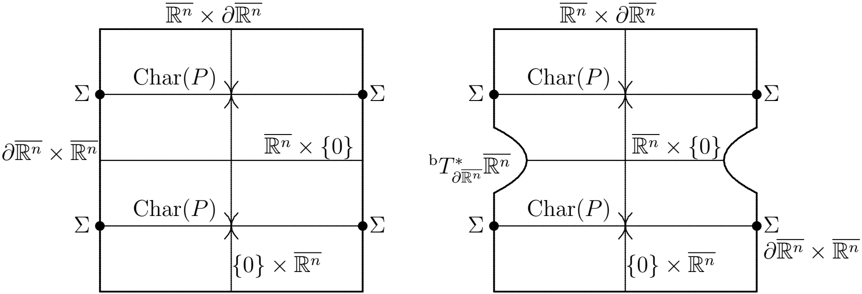
which in the Euclidean case just means

$$r=\frac{1}{2}\mp\beta\frac{z\cdot\zeta}{|z||\zeta|}.$$

These Fredholm estimates, i.e. the propagation of singularities estimates, including radial points, are proved by positive commutator estimates, which become degenerate (quadratic vanishing of symbol) at 0 energy.

However, it turns out that phrasing these estimates as b-estimates one can make them uniform as $\lambda \to 0$. For this, write $\tau \frac{dx}{x^2} + \mu \cdot \frac{dy}{x} = \tau_{\mathrm{b}} \frac{dx}{x} + \mu_{\mathrm{b}} \cdot dy$, so $\tau/x = \tau_{\mathrm{b}}$, $\mu/x = \mu_{\mathrm{b}}$.

Notice that *finite non-zero* (τ, μ) such as those on the characteristic set for non-zero λ, correspond to $(\tau_{\mathrm{b}}, \mu_{\mathrm{b}}) \to \infty$, thus *it is the b-differential order that becomes variable*.



Recall $\tau/x = \tau_{\mathrm{b}}$, $\mu/x = \mu_{\mathrm{b}}$.

So we consider

$$H-\lambda:\{u\in H_b^{\tilde r,l}:\ (H-\lambda)u\in H_b^{\tilde r-1,l+2}\}\to H_b^{\tilde r-1,l+2}.$$

First of all, when $\lambda>0$ this is still a Fredholm problem if

$$\tilde r=\frac{1}{2}-(l+1)\pm\beta\frac{\tau_{\mathrm{b}}}{\sqrt{\tau_{\mathrm{b}}^2+|\mu_{\mathrm{b}}|_h^2}},$$

Here the order at the characteristic set in the sc-sense is actually the sum of the two orders, $r=\tilde r+l$, giving the same order as before!

Moreover, in this form, as λ commutes with every operator, thus drops out from positive commutator estimates, one gets a uniform estimate as $\lambda \to 0$ as $H - \lambda$ effectively acts like H, i.e. a member of $x^2 S^0 \mathrm{Diff}_b^2(\bar{X})$! The result is that

Theorem (V.)

If 0 is not a bound or half-bound state, with

$$H - \lambda : \{u \in H_b^{\tilde{r},l} : (H - \lambda)u \in H_b^{\tilde{r}-1,l+2}\} \to H_b^{\tilde{r}-1,l+2},$$

$(H - \lambda)^{-1}$ *is uniformly bounded as* $\lambda \to 0$.

Again, no ϵ loss!

One can also allow $\lambda = \sigma^2$ non-real, in the correct halfplane (depending on the choice of $\pm$ in $\tilde{r}$), and gets uniform estimates.

This can also be extended to complex scaling, showing the non-accummulation of resonances at 0.

An important point is that in order to have a Fredholm estimate we need relatively compact errors, and in the generalization of the positive commutator estimate, the usual symbolic calculation in the b-algebra gives *only the differential gain!*. This can be thought of as the a uniform version of the positive energy estimate, in which one *blows up the zero section at infinity*. This is exactly 2-microlocal analysis in this setting; one is capturing Lagrangian behavior relative to the 0-section.

In order to get the compact errors, one needs to get decay using a normal operator argument, i.e. the effective model at infinity.

Even though the normal operator family is a family of operators on the boundary, which thus do not commute, with care, namely using functions of the boundary Laplacian, and x entering *only* as an factor with a fixed weight, one can make the whole computation explicit. Then, with more care, one can actually arrange positivity even at this level, giving the theorem.

The 2-microlocal interpretation of the main theorem, writing $H^{s,r,l}$ for the 2-microlocal space resolving $H^{s,r}$ at the 0-section, with l as the weight along the zero section, arises from

$$H_b^{\tilde{r},l} \subset H^{\tilde{r},\tilde{r}+l,l},$$

and away from 'scattering fiber infinity', i.e. with (τ,μ) bounded, where the first index on the right hand side is irrelevant, the the two spaces are microlocally the same. Thus, one gets

$$H-\lambda : \{u \in H^{s+2,r-1,l} :\ (H-\lambda)u \in H^{s,r,l+2}\} \to H^{s,r,l+2},$$

Thus, as r differs from $1/2$ by $\leq \beta$, the inverse maps

$$H^{s,1/2+\beta,l+2} \to H^{s+2,-1/2-\beta,l}$$

uniformly, $l < \frac{n-2}{2}$. With $s=0$, $l=-1$, this implies for $\beta > 1/2$, that the inverse maps

$$L^{2,1/2+\beta} \to L^{2,-1/2-\beta},$$

which is the usual (lossy) estimate.