

Non-dispersive conservative regularisation of nonlinear shallow water (and isentropic Euler) equations

DIDIER CLAMOND

Université Côte d'Azur
Laboratoire J. A. Dieudonné
Parc Valrose, 06108 Nice cedex 2, France

didier.clamond@gmail.com

Denys Dutykh

LAMA, University of Chambéry, France.

Dimitrios Mitsotakis

Victoria University of Wellington, New Zealand.

Saint-Venant (and Euler) equations

Classical Saint-Venant (shallow water) equations:

$$\begin{aligned}h_t + [h\bar{u}]_x &= 0, & (\text{mass}) \\ \bar{u}_t + \left[\frac{1}{2}\bar{u}^2 + gh\right]_x &= 0. & (\text{momentum})\end{aligned}$$

Regularized momentum equation:

$$\begin{aligned}q_t + \left[\bar{u}q - \frac{1}{2}\bar{u}^2 + gh - \frac{3}{2}\epsilon h^2 \bar{u}_x^2 - \epsilon gh(hh_{xx} + h_x^2)\right]_x &= 0, \\ q &\stackrel{\text{def}}{=} \left[1 - \epsilon h^{-1} \partial_x h^3 \partial_x\right] \bar{u}.\end{aligned}$$

$q \rightarrow \bar{u}$ and rSV \rightarrow SV as $\epsilon \rightarrow 0$.

Plan

Models for water waves in shallow water

Part I. Dispersion-improved model:

Improved Serre–Green–Naghdi equations.

Part II. Dispersionless model:

Regularised Saint-Venant–Airy equations.

Motivation

Understanding water waves (in shallow water).

Analytical approximations:

- Qualitative description;
- Physical insights.

Simplified equations:

- Easier numerical resolution;
- Faster schemes.

Goal:

- Derivation of the most accurate simplest models.

Hypothesis

Physical assumptions:

- Fluid is ideal, homogeneous & incompressible;
- Flow is irrotational, i.e.,
 $\vec{V} = \text{grad } \phi$;
- Free surface is a graph;
- Atmospheric pressure is constant.

Surface tension could also be included.



Notations for 2D surface waves over a flat bottom

- x : Horizontal coordinate.
- y : Upward vertical coordinate.
- t : Time.
- u : Horizontal velocity.
- v : Vertical velocity.
- ϕ : Velocity potential.
- $y = \eta(x, t)$: Equation of the free surface.
- $y = -d$: Equation of the seabed.
- Over tildes : Quantities at the surface, e.g., $\tilde{u} = u(y = \eta)$.
- Over check : Quantities at the bottom, e.g., $\check{u} = u(y = -d)$.
- Over bar : Quantities averaged over the depth, e.g.,

$$\bar{u} = \frac{1}{h} \int_{-d}^{\eta} u \, dy, \quad h = \eta + d.$$

Mathematical formulation

- Continuity and irrotationality equations for $-d \leq y \leq \eta$

$$u_x = -v_y, \quad v_x = u_y \quad \Rightarrow \quad \phi_{xx} + \phi_{yy} = 0$$

- Bottom's impermeability condition at $y = -d$

$$\tilde{v} = 0$$

- Free surface's impermeability condition at $y = \eta(x, t)$

$$\eta_t + \tilde{u} \eta_x = \tilde{v}$$

- Dynamic free surface condition at $y = \eta(x, t)$

$$\phi_t + \frac{1}{2} \tilde{u}^2 + \frac{1}{2} \tilde{v}^2 + g \eta = 0$$

Shallow water scaling

Assumptions for large long waves in shallow water:

$$\sigma \propto \frac{\text{depth}}{\text{wavelength}} \ll 1 \quad (\text{shallowness parameter}),$$
$$\varepsilon \propto \frac{\text{amplitude}}{\text{depth}} = \mathcal{O}(\sigma^0) \quad (\text{steepness parameter}).$$

Scale of derivatives and dependent variables:

$$\begin{aligned} \{\partial_x; \partial_t\} &= \mathcal{O}(\sigma^1), & \partial_y &= \mathcal{O}(\sigma^0), \\ \{u; v; \eta\} &= \mathcal{O}(\sigma^0), & \phi &= \mathcal{O}(\sigma^{-1}). \end{aligned}$$

Solution of the Laplace equation and bottom impermeability

Taylor expansion around the bottom (Lagrange 1791):

$$\begin{aligned}u &= \cos[(y+d) \partial_x] \check{u} \\ &= \check{u} - \frac{1}{2} (y+d)^2 \check{u}_{xx} + \frac{1}{6} (y+d)^4 \check{u}_{xxxx} + \dots \\ \Rightarrow \bar{u} &= \check{u} - \frac{1}{6} h^2 \check{u}_{xx} + \frac{1}{30} h^4 \check{u}_{xxxx} + \dots \\ \Rightarrow \check{u} &= \bar{u} + \frac{1}{6} h^2 \bar{u}_{xx} + \dots\end{aligned}$$

Low-order approximations for long waves:

$$\begin{aligned}u &= \bar{u} + \mathcal{O}(\sigma^2), && \text{(horizontal velocity)} \\ v &= -(y+d) \bar{u}_x + \mathcal{O}(\sigma^3), && \text{(vertical velocity)}.\end{aligned}$$

Energies

Kinetic energy:

$$\mathcal{K} = \int_{-d}^{\eta} \frac{u^2 + v^2}{2} dy = \frac{h\bar{u}^2}{2} + \frac{h^3\bar{u}_x^2}{6} + \mathcal{O}(\sigma^4),$$

Potential energy:

$$\mathcal{V} = \int_{-d}^{\eta} g(y+d) dy = \frac{gh^2}{2}.$$

Lagrangian density (Hamilton principle):

$$\mathcal{L} = \mathcal{K} - \mathcal{V} + \{h_t + [h\bar{u}]_x\} \phi$$

Approximate Lagrangian

$$\mathcal{L}_2 = \frac{1}{2}h\bar{u}^2 - \frac{1}{2}gh^2 + \{h_t + [h\bar{u}]_x\} \phi + \mathcal{O}(\sigma^2).$$

⇒ Saint-Venant (non-dispersive) equations.

$$\mathcal{L}_4 = \mathcal{L}_2 + \frac{1}{6}h^3\bar{u}_x^2 + \mathcal{O}(\sigma^4).$$

⇒ Serre (dispersive) equations.

$$\mathcal{L}_6 = \mathcal{L}_4 - \frac{1}{90}h^5\bar{u}_{xx}^2 + \mathcal{O}(\sigma^6).$$

⇒ Extended Serre (ill-posed) equations.

Serre equations derived from \mathcal{L}_4

Euler–Lagrange equations yield:

$$0 = h_t + \partial_x[h\bar{u}],$$

$$0 = \partial_t\left[\bar{u} - \frac{1}{3}h^{-1}(h^3\bar{u}_x)_x\right] \\ + \partial_x\left[\frac{1}{2}\bar{u}^2 + gh - \frac{1}{2}h^2\bar{u}_x^2 - \frac{1}{3}\bar{u}h^{-1}(h^3\bar{u}_x)_x\right].$$

Secondary equations:

$$\bar{u}_t + \bar{u}\bar{u}_x + gh_x + \frac{1}{3}h^{-1}\partial_x[h^2\gamma] = 0, \\ \partial_t[h\bar{u}] + \partial_x\left[h\bar{u}^2 + \frac{1}{2}gh^2 + \frac{1}{3}h^2\gamma\right] = 0, \\ \partial_t\left[\frac{1}{2}h\bar{u}^2 + \frac{1}{6}h^3\bar{u}_x^2 + \frac{1}{2}gh^2\right] + \\ \partial_x\left[\left(\frac{1}{2}\bar{u}^2 + \frac{1}{6}h^2\bar{u}_x^2 + gh + \frac{1}{3}h\gamma\right)h\bar{u}\right] = 0,$$

with

$$\gamma = h\left[\bar{u}_x^2 - \bar{u}_{xt} - \bar{u}\bar{u}_{xx}\right].$$

2D Serre's equations on flat bottom (summary)

Easy derivations via a variational principle.

Non-canonical Hamiltonian structure.

(Li, J. Nonlinear Math. Phys., 2002)

Multi-symplectic structure.

(Chhay, Dutykh & Clamond, J. Phys. A, 2016)

Fully nonlinear, weakly dispersive.

(Wu, Adv. App. Mech. 37, 2001)

Can the dispersion be improved?

Modified vertical acceleration

$$\gamma = 2h\bar{u}_x^2 - h\partial_x[\bar{u}_t + \bar{u}\bar{u}_x] + \mathcal{O}(\sigma^4).$$

Horizontal momentum:

$$\underbrace{\bar{u}_t + \bar{u}\bar{u}_x}_{\mathcal{O}(\sigma)} = \underbrace{-gh_x}_{\mathcal{O}(\sigma)} - \underbrace{\frac{1}{3}h^{-1}\partial_x[h^2\gamma]}_{\mathcal{O}(\sigma^3)} + \mathcal{O}(\sigma^5).$$

Alternative vertical acceleration at the free surface:

$$\gamma = 2h\bar{u}_x^2 + gh h_{xx} + \mathcal{O}(\sigma^4).$$

Generalised vertical acceleration at the free surface:

$$\gamma = 2h\bar{u}_x^2 + \beta gh h_{xx} + (\beta - 1)h\partial_x[\bar{u}_t + \bar{u}\bar{u}_x] + \mathcal{O}(\sigma^4).$$

β : free parameter.

Modified Lagrangian

Substitute $h\bar{u}_x^2 = \gamma + h(\bar{u}_{xt} + \bar{u}\bar{u}_{xx})$:

$$\mathcal{L}_4 = \frac{h\bar{u}^2}{2} + \frac{h^2\gamma}{12} + \frac{h^3}{12} [\bar{u}_t + \bar{u}\bar{u}_x]_x - \frac{gh^2}{2} + \{h_t + [h\bar{u}]_x\} \phi.$$

Substitution of the generalised acceleration:

$$\gamma = 2h\bar{u}_x^2 + \beta gh h_{xx} + (\beta - 1)h \partial_x [\bar{u}_t + \bar{u}\bar{u}_x] + \mathcal{O}(\sigma^4).$$

Resulting Lagrangian:

$$\mathcal{L}'_4 = \mathcal{L}_4 + \underbrace{\frac{\beta h^3}{12} [\bar{u}_t + \bar{u}\bar{u}_x + gh_x]_x}_{\mathcal{O}(\sigma^4)} + \mathcal{O}(\sigma^4).$$

Reduced modified Lagrangian

After integrations by parts and neglecting boundary terms:

$$\begin{aligned}\mathcal{L}_4'' &= \frac{h\bar{u}^2}{2} + \frac{(2+3\beta)h^3\bar{u}_x^2}{12} - \frac{gh^2}{2} - \frac{\beta gh^2 h_x^2}{4} \\ &\quad + \{h_t + [h\bar{u}]_x\}\phi \\ &= \mathcal{L}_4' + \text{boundary terms.}\end{aligned}$$

Equations of motion

$$\begin{aligned}h_t + \partial_x[h\bar{u}] &= 0, \\q_t + \partial_x[\bar{u}q - \frac{1}{2}\bar{u}^2 + gh - (\frac{1}{2} + \frac{3}{4}\beta)h^2\bar{u}_x^2 - \frac{1}{2}\beta g(h^2h_{xx} + hh_x^2)] &= 0, \\\bar{u}_t + \bar{u}\bar{u}_x + gh_x + \frac{1}{3}h^{-1}\partial_x[h^2\Gamma] &= 0, \\\partial_t[h\bar{u}] + \partial_x[h\bar{u}^2 + \frac{1}{2}gh^2 + \frac{1}{3}h^2\Gamma] &= 0, \\\partial_t[\frac{1}{2}h\bar{u}^2 + (\frac{1}{6} + \frac{1}{4}\beta)h^3\bar{u}_x^2 + \frac{1}{2}gh^2 + \frac{1}{4}\beta gh^2h_x^2] + \\ \partial_x[(\frac{1}{2}\bar{u}^2 + (\frac{1}{6} + \frac{1}{4}\beta)h^2\bar{u}_x^2 + gh + \frac{1}{4}\beta gh h_x^2 + \frac{1}{3}h\Gamma)h\bar{u} + \frac{1}{2}\beta gh^3h_x\bar{u}_x] &= 0,\end{aligned}$$

where

$$\begin{aligned}q &= \phi_x = \bar{u} - (\frac{1}{3} + \frac{1}{2}\beta)h^{-1}[h^3\bar{u}_x]_x, \\\Gamma &= (1 + \frac{3}{2}\beta)h[\bar{u}_x^2 - \bar{u}_{xt} - \bar{u}\bar{u}_{xx}] - \frac{3}{2}\beta g[hh_{xx} + \frac{1}{2}h_x^2].\end{aligned}$$

Linearised equations

With $h = d + \eta$, η and \bar{u} small, the equations become

$$\begin{aligned}\eta_t + d \bar{u}_x &= 0, \\ \bar{u}_t - \left(\frac{1}{3} + \frac{1}{2}\beta\right) d^2 \bar{u}_{xxt} + g \eta_x - \frac{1}{2} \beta g d^2 \eta_{xxx} &= 0.\end{aligned}$$

Dispersion relation:

$$\frac{c^2}{g d} = \frac{2 + \beta (kd)^2}{2 + \left(\frac{2}{3} + \beta\right) (kd)^2} \approx 1 - \frac{(kd)^2}{3} + \left(\frac{1}{3} + \frac{\beta}{2}\right) \frac{(kd)^4}{3}.$$

Exact linear dispersion relation:

$$\frac{c^2}{g d} = \frac{\tanh(kd)}{kd} \approx 1 - \frac{(kd)^2}{3} + \frac{2(kd)^4}{15}.$$

$\beta = 2/15$ is the best choice.

Steady solitary waves

Equation:

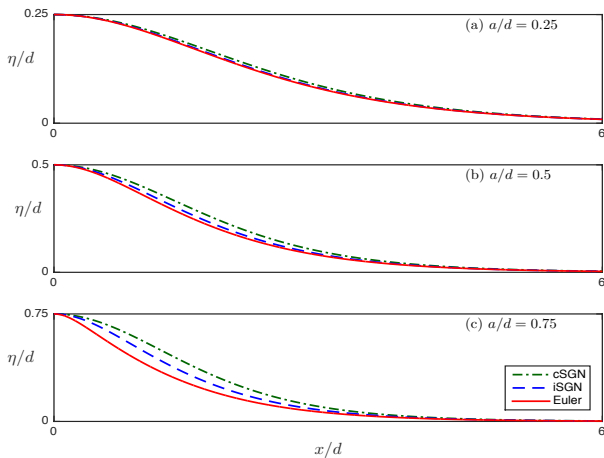
$$\left(\frac{d\eta}{dx}\right)^2 = \frac{(\mathcal{F} - 1)(\eta/d)^2 - (\eta/d)^3}{\left(\frac{1}{3} + \frac{1}{2}\beta\right)\mathcal{F} - \frac{1}{2}\beta(1 + \eta/d)^3},$$
$$\mathcal{F} = c^2 / g d.$$

Solution in parametric form:

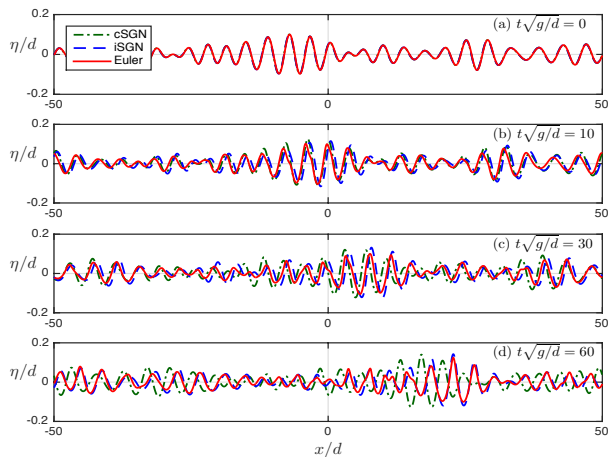
$$\frac{\eta(\xi)}{d} = (\mathcal{F} - 1) \operatorname{sech}^2\left(\frac{\kappa\xi}{2}\right), \quad (\kappa d)^2 = \frac{6(\mathcal{F} - 1)}{(2 + 3\beta)\mathcal{F} - 3\beta}.$$

$$x(\xi) = \int_0^\xi \left| \frac{(\beta + 2/3)\mathcal{F} - \beta h^3(\xi')/d^3}{(\beta + 2/3)\mathcal{F} - \beta} \right|^{1/2} d\xi',$$

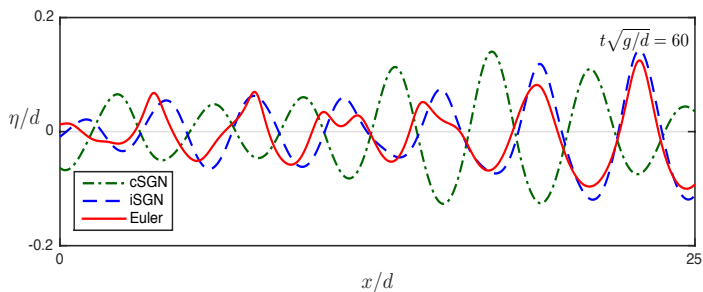
Comparisons for $\beta = 0$ and $\beta = 2/15$



Random wave field



Random wave field (zoom)



Part II

Saint-Venant equations

Non-dispersive shallow water (Saint-Venant) equations:

$$\begin{aligned}h_t + [h\bar{u}]_x &= 0, \\ \bar{u}_t + \bar{u}\bar{u}_x + gh_x &= 0.\end{aligned}$$

Shortcomings:

- No permanent regular solutions;
- Shocks appear;
- Requires ad hoc numerical schemes.

Regularisations

Add diffusion (e.g. von Neumann & Richtmayer 1950):

$$\bar{u}_t + \bar{u}\bar{u}_x + gh_x = \nu \bar{u}_{xx}.$$

⇒ Leads to dissipation of energy = bad for long time simulation.

Add dispersion (e.g. Lax & Levermore 1983):

$$\bar{u}_t + \bar{u}\bar{u}_x + gh_x = \tau \bar{u}_{xxx}.$$

⇒ Leads to spurious oscillations. Not always sufficient for regularisation.

Add diffusion + dispersion (e.g. Hayes & LeFloch 2000):

⇒ Regularises but does not conserve energy and provides spurious oscillations.

Bhat & Fetecau 2009 (J. Math. Anal. & App. 358):

$$\begin{aligned}h_t + [h\bar{u}]_x &= \epsilon^2 h \bar{u}_{xxx}, \\ \bar{u}_t + \bar{u} \bar{u}_x + g h_x &= \epsilon^2 (\bar{u}_{xxt} + \bar{u} \bar{u}_{xxx}).\end{aligned}$$

Drawbacks:

- Shocks do not propagate at the right speed;
- No equation for energy conservation.

Dispersionless model

Two-parameter Lagrangian:

$$\mathcal{L} = \frac{1}{2} h \bar{u}^2 + \left(\frac{1}{6} + \frac{1}{4} \beta_1 \right) h^3 \bar{u}_x^2 - \frac{1}{2} g h^2 \left(1 + \frac{1}{2} \beta_2 h_x^2 \right) + \{ h_t + [h \bar{u}]_x \} \phi.$$

Linear dispersion relation:

$$\frac{c^2}{g d} = \frac{2 + \beta_2 (kd)^2}{2 + \left(\frac{2}{3} + \beta_1 \right) (kd)^2}.$$

No dispersion if $c = \sqrt{gd}$, that is $\beta_1 = \beta_2 - 2/3$, so let be

$$\beta_1 = 2\epsilon - 2/3, \quad \beta_2 = 2\epsilon.$$

Conservative regularised SV equations

Regularised Saint-Venant equations:

$$0 = h_t + \partial_x[h\bar{u}],$$

$$0 = \partial_t[h\bar{u}] + \partial_x\left[h\bar{u}^2 + \frac{1}{2}gh^2 + \epsilon R h^2\right],$$

$$R \stackrel{\text{def}}{=} h\left(\bar{u}_x^2 - \bar{u}_{xt} - \bar{u}\bar{u}_{xx}\right) - g\left(hh_{xx} + \frac{1}{2}h_x^2\right).$$

Energy equation:

$$\begin{aligned} & \partial_t\left[\frac{1}{2}h\bar{u}^2 + \frac{1}{2}gh^2 + \frac{1}{2}\epsilon h^3\bar{u}_x^2 + \frac{1}{2}\epsilon gh^2h_x^2\right] \\ & + \partial_x\left[\left\{\frac{1}{2}\bar{u}^2 + gh + \frac{1}{2}\epsilon h^2\bar{u}_x^2 + \frac{1}{2}\epsilon gh h_x^2 + \epsilon hR\right\}h\bar{u}\right. \\ & \qquad \qquad \qquad \left. + \epsilon gh^3h_x\bar{u}_x\right] = 0. \end{aligned}$$

Energies

rSV total energy density:

$$\mathcal{H}_\epsilon(x, t) \stackrel{\text{def}}{=} \underbrace{\frac{1}{2} h \bar{u}^2 + \frac{1}{2} g h^2}_{\text{SV energy}} + \underbrace{\frac{1}{2} \epsilon h^3 \bar{u}_x^2 + \frac{1}{2} \epsilon g h^2 h_x^2}_{\text{internal energy}}.$$

$\mathcal{H}_0(x, t)$: Not conserved SV energy density.

$\mathcal{H}_\epsilon(x, t) - \mathcal{H}_0(x, t)$: Not conserved density of “internal” energy or “entropy”.

Non-dispersive solitary wave:

$$\begin{aligned}x(\xi) &= \int_0^\xi \left[\frac{\mathcal{F}d^3 - h^3(\xi')}{(\mathcal{F} - 1)d^3} \right]^{\frac{1}{2}} d\xi', \\ \frac{\eta(\xi)}{d} &= (\mathcal{F} - 1) \operatorname{sech}^2\left(\frac{\kappa \xi}{2}\right), \\ (\kappa d)^2 &= \epsilon^{-1}, \\ \mathcal{F} &= 1 + a/d.\end{aligned}$$

Example: Dam break problem

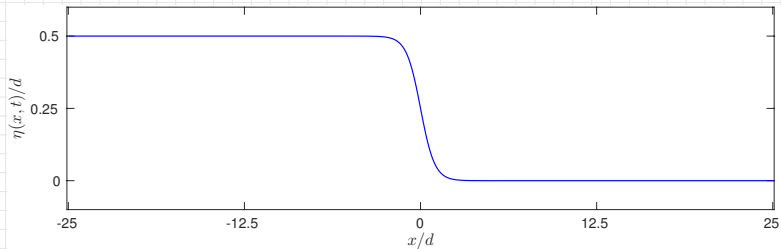
Initial condition:

$$h_0(x) = h_l + \frac{1}{2} (h_r - h_l) (1 + \tanh(\delta x)),$$
$$\bar{u}_0(x) = u_l + \frac{1}{2} (u_r - u_l) (1 + \tanh(\delta x)).$$

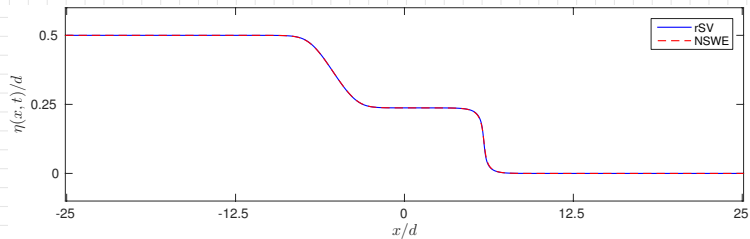
Resolution of the classical shallow water equations with finite volumes.

Resolution of the regularised equations with pseudo-spectral scheme.

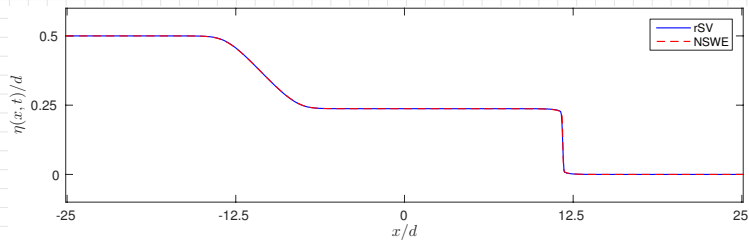
Result with $\epsilon = 0.001$ at $t \sqrt{g/d} = 0$



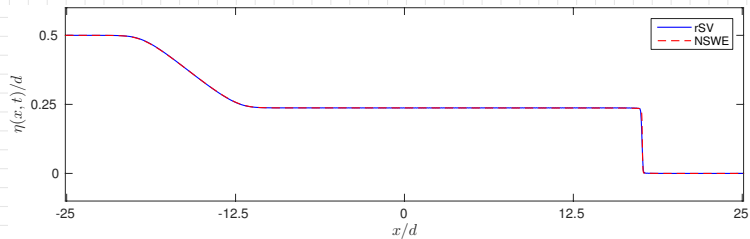
Result with $\epsilon = 0.001$ at $t \sqrt{g/d} = 5$



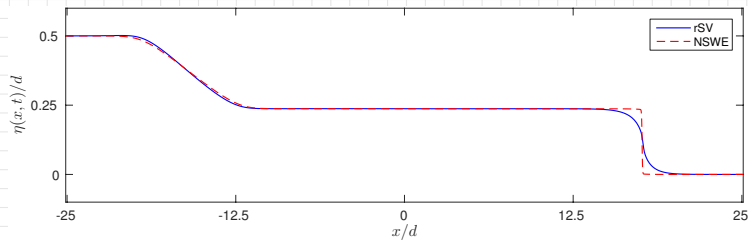
Result with $\epsilon = 0.001$ at $t \sqrt{g/d} = 10$



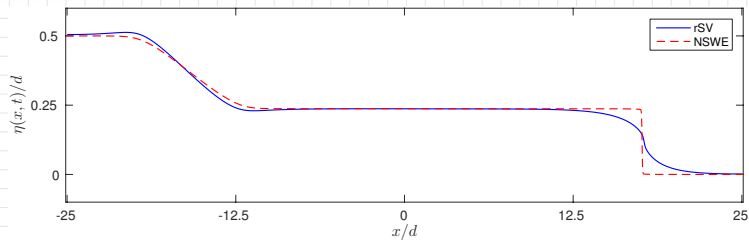
Result with $\epsilon = 0.001$ at $t \sqrt{g/d} = 15$



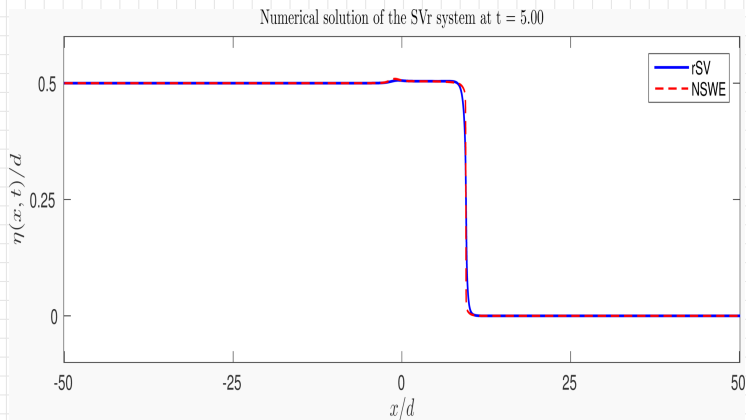
Result with $\epsilon = 1$ at $t \sqrt{g/d} = 15$



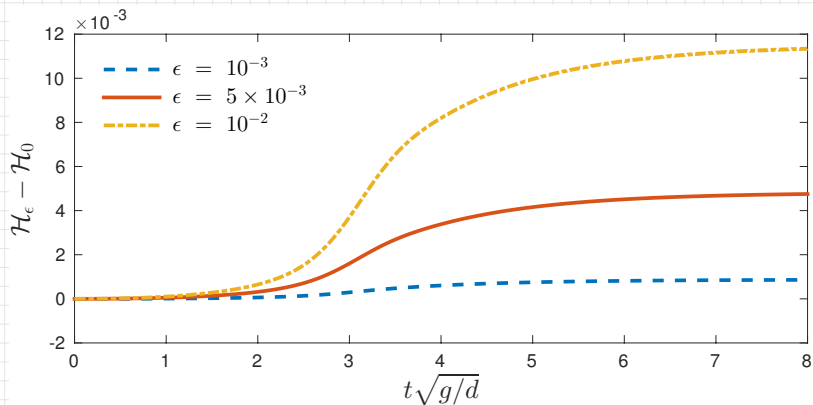
Result with $\epsilon = 5$ at $t\sqrt{g/d} = 15$



Example 2: Shock ($\epsilon = 0.1$)



Example 2: Internal energy



Rankine–Hugoniot conditions

Assuming discontinuities in second (or higher) derivatives:

$$(u - \dot{s}) \llbracket h_{xx} \rrbracket + h \llbracket \bar{u}_{xx} \rrbracket = 0,$$

$$(u - \dot{s}) \llbracket \bar{u}_{xx} \rrbracket + g \llbracket h_{xx} \rrbracket = 0,$$

$$\Rightarrow \quad \dot{s}(t) = \bar{u}(x, t) \pm \sqrt{g h(x, t)} \quad \text{at } x = s(t).$$

The regularised shock speed is **independent** of ϵ and it propagates **exactly** along the **characteristic lines** of the Saint-Venant equations!

Remark on momentum

Regularized momentum equation:

$$q_t + \left[\bar{u} q - \frac{1}{2} \bar{u}^2 + g h - \frac{3}{2} \epsilon h^2 \bar{u}_x^2 - \epsilon g h (h h_{xx} + h_x^2) \right]_x = 0,$$
$$q = \mathfrak{H}\{\bar{u}\} \stackrel{\text{def}}{=} [1 - \epsilon h^{-1} \partial_x h^3 \partial_x] \bar{u}.$$

$q \rightarrow \bar{u}$ and rSV \rightarrow SV as $\epsilon \rightarrow 0$.

If $h = \text{constant}$, $\mathfrak{H} = 1 - \epsilon h^2 \partial_x^2$ (Helmholtz operator) and

$$\bar{u}(x) = \mathfrak{H}^{-1}\{q\} = \int_{-\infty}^{\infty} \frac{\exp(-|x-y|/\sqrt{\epsilon h})}{2\sqrt{\epsilon h}} q(y) dy.$$

Summary

Variational principle yields:

- Easy derivations;
- Structure preservation;
- Suitable for enhancing models in a “robust way”.

Straightforward generalisations:

- 3D;
- Variable bottom;
- Stratification.

References

CLAMOND, D., DUTYKH, D. & MITSOTAKIS, D. 2017. Conservative modified Serre–Green–Naghdi equations with improved dispersion characteristics. *Communications in Nonlinear Science and Numerical Simulation* 45, 245–257.

CLAMOND, D. & DUTYKH, D. 2018. Non-dispersive conservative regularisation of nonlinear shallow water and isothermal Euler equations. *Communications in Nonlinear Science and Numerical Simulation* 55, 237–247.