Attractors for 2D-internal waves

in progress with Laure St-Raymond

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Topics:

1. Internal waves in physics

2. Linearization of Navier-Stokes near equilibrium

3. Classical dynamics

4. Wave dynamics: absolutely continuous spectra using Mourre theory

5. Forcing for frequencies in the continuous spectrum

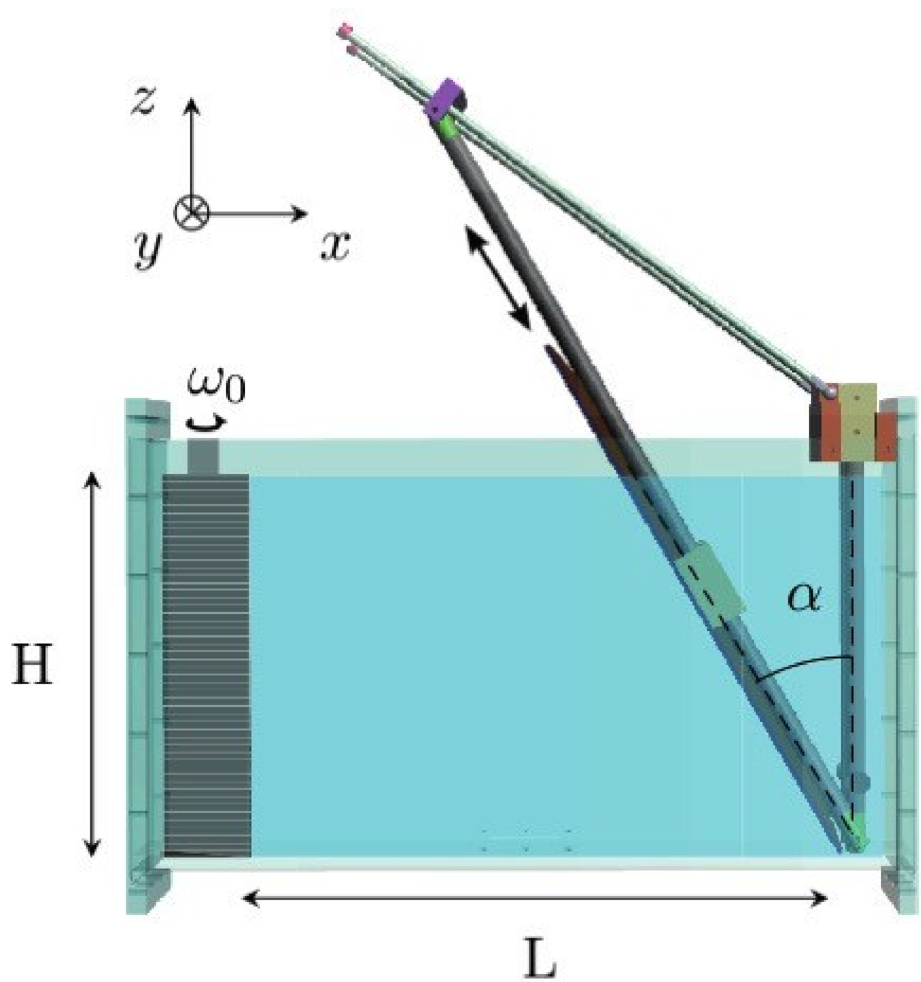
After the two first sections, I will stay in 2D.

Internal waves in physics

Internal waves are waves that oscillate within a fluid, rather than on its surface. To exist, the fluid must be **stratified**: the density must decrease with height due to changes, for example, in temperature and/or salinity. If the density changes over a small vertical distance, the waves propagate horizontally like surface waves. If the density changes continuously, the waves can propagate in several directions through the fluid.

Numerical and lab. experiments performed in the team of Thierry Dauxois, especially by Christophe Brouzet, at ENS Lyon, show that, in 2D, these waves concentrate on attractors for some generic frequencies of the forcing. This wave concentration is the subject of this talk.

The experimental setup



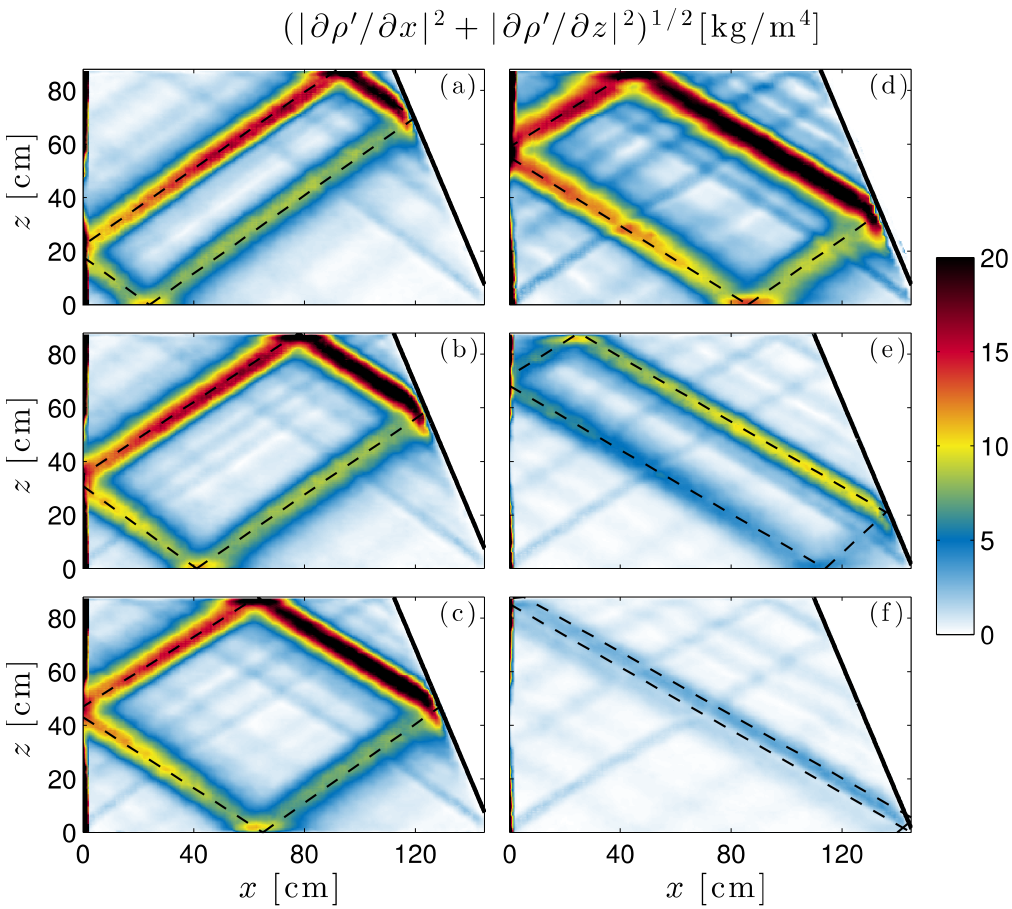
Stratification is affine (salt water).

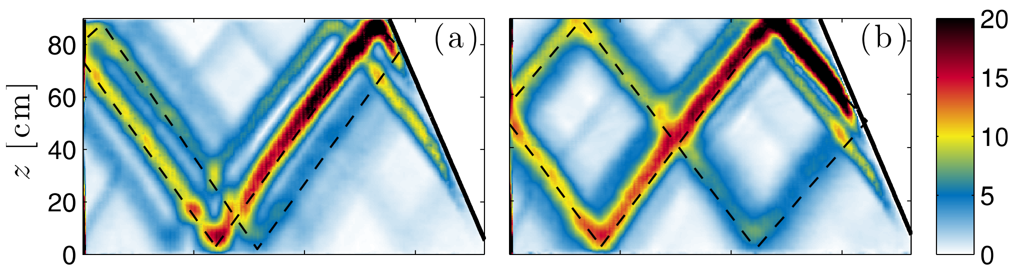
The domain is 2D with an adjustable slope.

The forcing is monochromatic (localized on the boundary)

Small markers allow to visualize the motion of the fluid (PIV).

Attractors





- The energy concentrates on **attractors**.
- The geometry of these attractors (number of branches) depends on the **forcing frequency** and on the **slope**.
- Some branches are more energetic than others (**focusing** after reflection on the slope)

From the PhD thesis of C. Brouzet,

Equations for forced internal waves in the linear regime

The incompressible Navier-Stokes equations in a domain Ω write:

$$\rho\,(\dot{u} + (u.\nabla)u) = \rho\vec{g} - \vec{\text{grad}}\ p + \nu\rho\Delta u,\ \ \text{div}u = 0,$$

$$\dot{\rho} + u.\vec{\text{grad}}\ \rho = k\Delta\rho,$$

+ the boundary condition: u tangent to $\partial\Omega$. The unknowns are the velocity field $u(t)$, the density $\rho(t)$ and the pressure $p(t)$. Here ν and k are ≥ 0 scalar coefficients: ν is the viscosity coefficient, k the diffusion coefficient; $\vec{g}$ the gravity field. We will assume in what follows the $\nu = k = 0$.

We will linearize the equations near the equilibrium $u = 0$, $\rho = \bar{\rho}$, $p = \bar{p}$ with $\bar{\rho}\vec{g} - \vec{\text{grad}}\ \bar{p} = 0$.

Assuming that u is small while $\rho = \bar{\rho} + \delta\rho$ with $\delta\rho$ small and $p = \bar{p} + \delta p$ with δp small. We get the linearized system:

$$\bar{\rho}\dot{u} = \delta\rho\vec{g} - \vec{\text{grad}}\ \delta p, \ \text{div} u = 0,$$

$$\dot{\delta\rho} + u.\vec{\text{grad}}\ \bar{\rho} = 0.$$

Differentiating w.r. to time, we get the equation for internal waves:

$$\ddot{u} = -\left(u.\frac{\vec{\text{grad}}\ \bar{\rho}}{\bar{\rho}}\right)\vec{g} - \frac{\vec{\text{grad}}\ \dot{\delta p}}{\bar{\rho}},\ \text{div}u = 0.$$

We now use the Helmholtz-Weyl decomposition relative to the density $\bar{\rho}$ and the associated Leray projector P onto the L^2-closure of divergence free vector fields vanishing near the boundary. We get the following wave equation for the velocity:

$$\ddot{u} + Qu = 0 \text{ with } Qu = P\left(\left(u.\frac{\vec{\text{grad}}\ \bar{\rho}}{\bar{\rho}}\right)\vec{g}\right)$$

This is a pseudo-differential wave equation. **Note that the operator Q is bounded!** This kind of wave equation has not been much studied by mathematicians; it leads to strange wave propagations.

If $\bar{\rho}$ is a function of z only, we get the dispersion relation

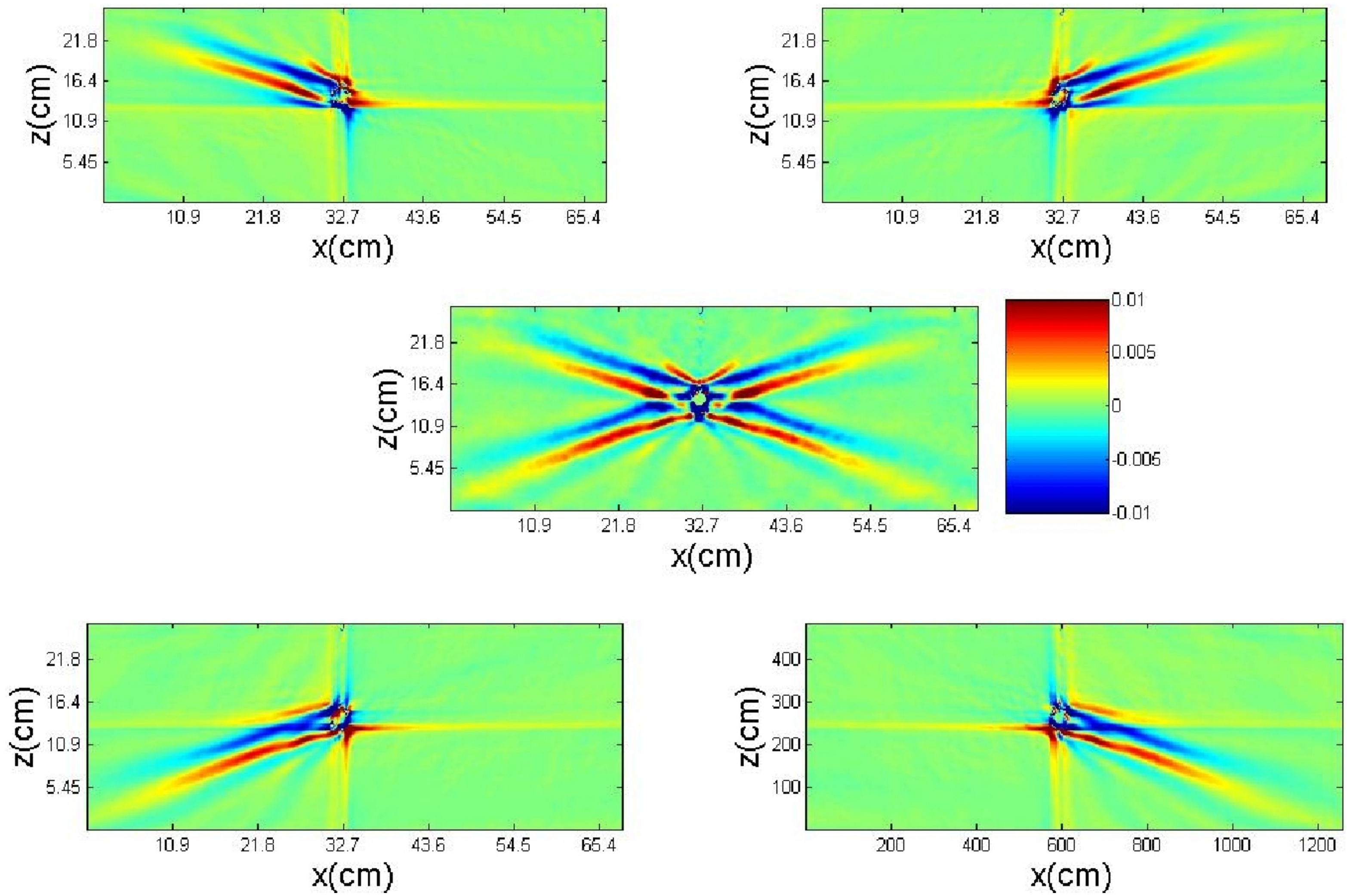
$$\omega^2 - N^2(z)\frac{\xi^2+\eta^2}{\xi^2+\eta^2+\zeta^2} = 0$$

or

$$N(z)\sin\varphi = \pm\omega$$

with φ the angle of the momentum with the vertical and N is called the "Brunt Vaïsala" or "buoyancy frequency" :

$$N^2 := -g\partial_z \log\bar{\rho}$$

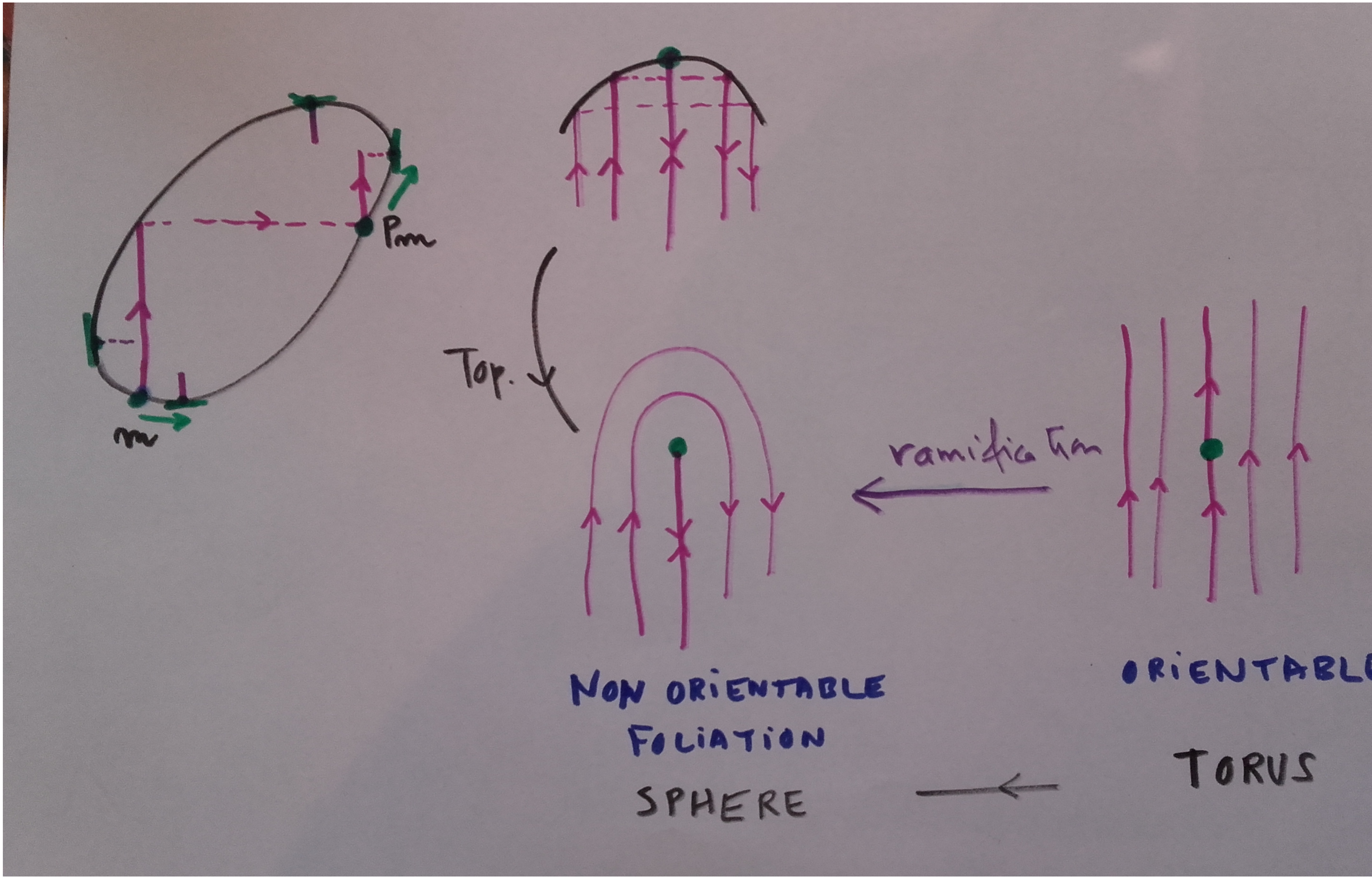


z(cm)
21.8
16.4
10.9
5.45
10.9
21.8
32.7
43.6
54.5
65.4
x(cm)
0.01
0.005
0
-0.005
-0.01
400
300
200
100
200
400
600
800
1000
1200

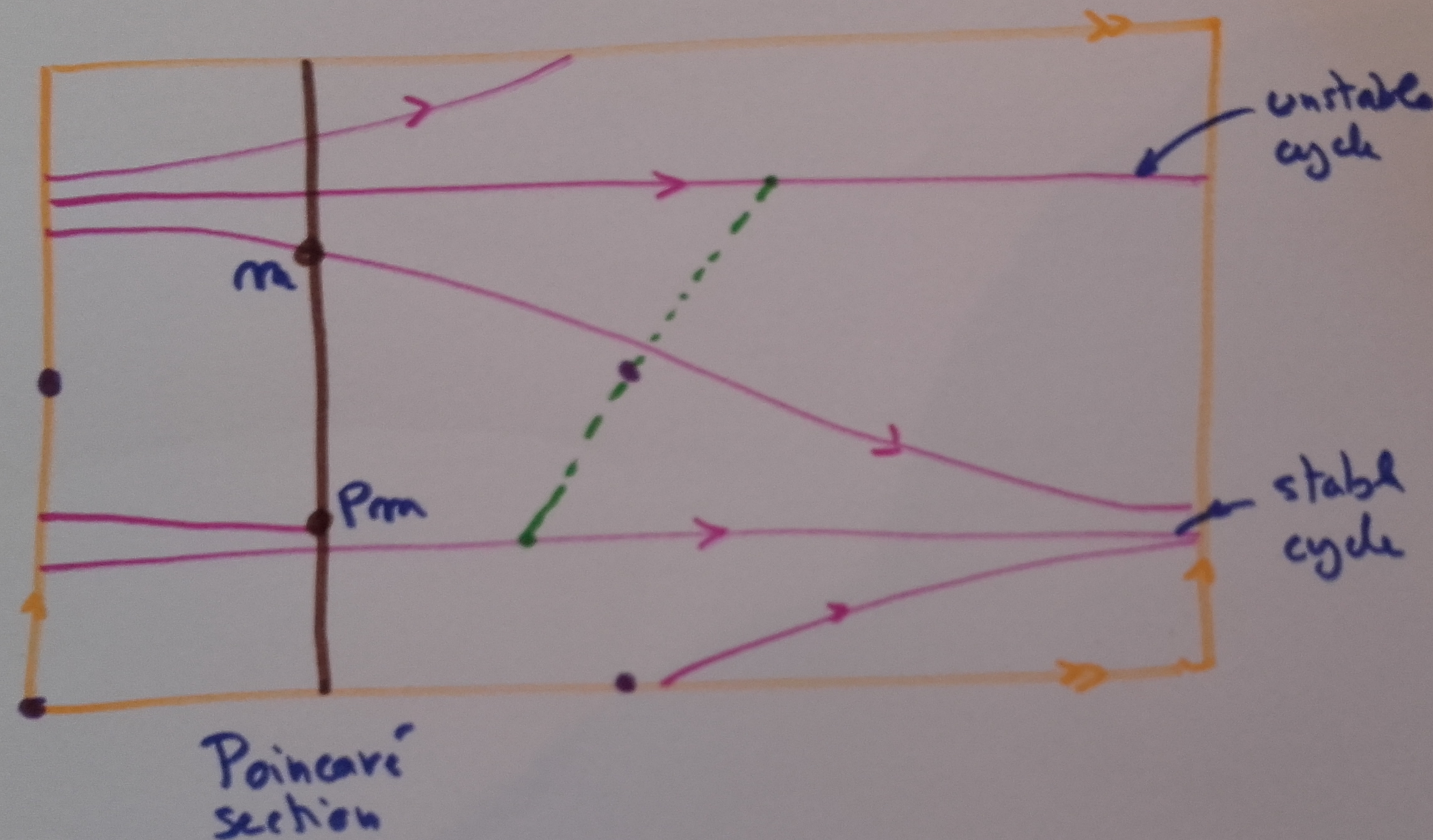
In what follows, we will restrict ourselves to the 2D case and try to study the classical dynamics and the spectral theory associated to such an Hamiltonian system.

Desingularizations

The following pictures will explain how it is possible to use a four-fold ramified covering of the domain Ω which is a torus with an oriented foliation.



Pm
m
Top.
ramification
NON ORIENTABLE
FOLIATION
SPHERE
TORUS



TORUS WITH ORIENTED FOLIATION

A simplified 2D-mathematical model

Let us fix a 2-torus $X = (\mathbb{R}/2\pi\mathbb{Z})^2_{x,y}$ and consider, on X, a self-adjoint $L^2(X)$-bounded pseudo-differential operator $\widehat{H}$ of degree 0 with a principal symbol $H(q,p)$, homogeneous of degree 0, and a vanishing sub-principal symbol. $\widehat{H}$ is the Weyl quantization of H. Changing H to $H-\omega$, we can reduce to study the dynamics on the energy shell $\Sigma := \{(q,p) | p \neq 0,\ H(q,p) = 0\} \subset T^\star X \setminus 0$.

In what follows we will assume that Σ is a finite union of half-lines bundles on X and that dH is non vanishing on Σ. We will reduce our dynamical study to one component of the bundle.

Classical dynamics: foliations of X

We have

$$\dot{x} = \frac{\partial H}{\partial \xi}, \ \dot{y} = \frac{\partial H}{\partial \eta}$$

From Euler relation, we see that the vector $\dot{q} = (\dot{x}, \dot{y})$ lies in the orthogonal of the fiber Σ_q of Σ over q for the duality bracket.

Physicists will say: *the group velocity* $\dot{q}$ *is orthogonal to the phase velocity* p. Note that the foliation is oriented, but there is no canonical parametrization.

Conversely, the oriented foliation determines Σ and, hence, the ideal of functions vanishing on Σ.

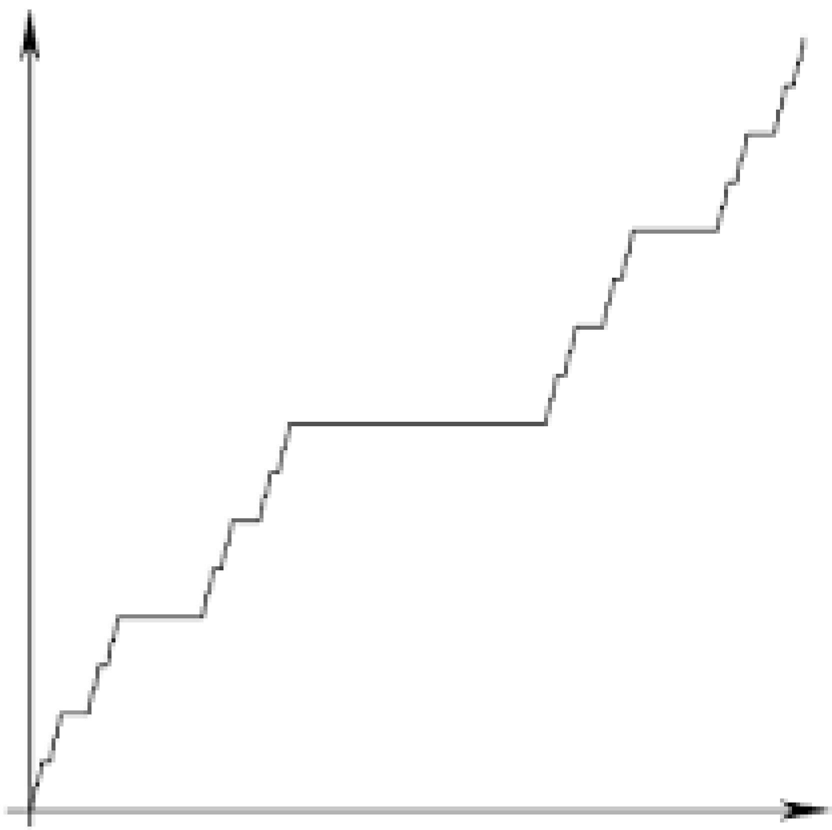
Classical dynamics of the foliation: Poincaré sections

As we have seen in the example with boundaries, we can assume the existence of a global Poincaré section S (no Reeb components). The returm map P is then an orientation preserving diffeomorphism of the circle (depending on the frequency ω). An important invariant is the rotation number $\rho(P)$:

- The rotation number is rational iff P has periodic points and hence the foliation has closed cycles

- The rotation number is irrational iff P is C^0 conjugated (Denjoy) to a rotation (all leaves are dense). Later works by Arnold, Herman, Yoccoz ...

Let us consider a generic map $\omega \to P_\omega$ from $[0,1]$ into $\mathrm{Diff}^+(S)$. The function $\omega \to \rho(P_\omega)$ is continuous and is locally constant near the points where it takes rational values. Those points are dense and the complement is a Cantor set (Devil's staircase).



The Morse-Smale case

We will assume that ω is choosen (generic choice) so that the foliation admits a finite number of closed leaves which are hyperbolic (the rotation number is rational). We can choose a Poincaré section crossing each cycle only at one point. We choose the coordinates on X so that this Poincaré section is given by $x = 0$ and y is constant on each cycle.

Classical dynamics in the phase space: normal form

In the bassin of each hyperbolic cycle, the foliation admits a normal form which follows from the normal form for the Poincaré map: in some coordinates, this map is linear! This implies that there exists a normal form in each bassin, $x \in \mathbb{R}/2\pi\mathbb{Z},\ y \in \mathbb{R}$,

$$H \equiv H_0 := \frac{\xi}{\eta} - \lambda y$$

with $\lambda > 0$ if the cycle is stable and $\lambda < 0$ if it is unstable. The notation $\equiv$ means that the generated ideals are the same. So up to reparametrization with derivative homogenous of degree 0, the Hamiltonians dynamics are the same.

Let us look at the dynamics of this normal form: we have a phase space Poincaré section $x = 0, \xi = \epsilon\lambda y\eta$ with $\eta > 0$. The associated Poincaré map is

$$P(y,\eta) = (e^{-\lambda}y, e^{\lambda}\eta)$$

The return times are longer and longer: $T_n \sim C\text{exp}(\lambda n)$.

Wave dynamics: escape function and Mourre theory

In each bassin with the normal form we take $L := \pm\psi(y)\eta$ with $\psi \in C_o^\infty(]-1,1[)$, $\psi \equiv 1$ near 0, ψ even and $y\psi' \leq 0$. We get

$$\{H_0, L\} = \pm\lambda(\psi(y) - y\psi'(y))$$

which is > 0 inside the support of ψ.

Quantifying the escape functions, gives a positive commutator modulo compact operators. This means that we can apply Mourre result to a small spectral interval around ω. Even more the spectral projector is smooth w.r. to λ outside possibly of a discrete set of eigenvalues.

As a result, we get that the spectrum is absolutely continuous near that frequency ω, except maybe a discrete set of embedded eigenvalues.

These eigenvalues are probably absent in the generic situation.

Wave dynamics: generalized eigenfunctions

Let us quantify our normal form $H = F\left(\frac{\xi}{\eta} - \lambda y\right)$. Let us simplify assuming that

$$\mathrm{Op}\left(F\left(\frac{\xi}{\eta} - \lambda y\right)\right) u = 0$$

is equivalent to

$$\mathrm{Op}\left(\frac{\xi}{\eta} - \lambda y\right) u = 0$$

Any generalized eigenfunction u is a solution of

$$\frac{\partial}{\partial y} \circ \mathrm{Op}\left(\frac{\xi}{\eta} - \lambda y\right) u = 0$$

which is

$$\mathrm{Op}\left(\eta \star \left(\frac{\xi}{\eta} - \lambda y\right)\right) u = O$$

The eigenfunctions will satisfy approximatively:

$$u_x - \lambda(yu_y + 1) = 0$$

and are then given by:

$$u(x,y) = \sum_{\pm} \frac{1}{y_\pm} U_\pm \left(x + \log|y_\pm|/\lambda\right)$$

with $U_\pm \in L^2_{\text{loc}}$. We see that u is not L^2 near the cycle $y = 0$. It remains to see how the forcing at a frequency ω exibhits some generalized eigenmodes at that frequency.

Abstract theory of linear forcing: quasi-resonant states

Let us consider the following forced wave equation:

$$\frac{du}{dt} - iHu = e^{i\omega t} f, \;\; u(0) = 0$$

where H is a self-adjoint operator on some Hilbert space $\mathcal{H}$ and $f \in \mathcal{H}$ is independent of t.

We get

$$u(t) = -i\frac{e^{it\omega} - e^{itH}}{\omega - H}f$$

Note that this expression makes sense because it is a time dependent bounded function of H applied to f.

If ω is not in the spectrum of H, then $u(t)$ stays bounded. It implies that modulo bounded functions, we can assume that the spectral decomposition of f lies in some intervall $I := [\omega - a, \omega + a]$.

It is very well known that if ω is an isolated eigenvalue of H and the component f_ω of f in this eigenspace does not vanish, we get $u(t) = te^{it\omega}f_\omega$ and hence a linear growth of u and a quadratic growth of the energy $\|u(t)\|^2$.

We are interested in the situation where ω belongs to the continuous spectrum of H. In this case, assuming that

$$f = \int_I f_\lambda d\lambda$$

where f_λ belongs to some Hilbert space $\mathcal{F}$ larger than $\mathcal{H}$ and, assuming that f_λ depends C^1 of λ, we get

$$J(t) := u(t)e^{-it\omega} = i\int_{-a}^{a} \frac{1 - e^{its}}{s} f_{s+\omega} ds$$

We get the following result:

For any $\epsilon > 0$, we can write

$$u(t) = v(t) + w_\epsilon(t) + b_\epsilon(t)$$

where

- *As $t \to \infty$, $v(t)e^{-i\omega t}$ converges to πf_ω in $\mathcal{F}$*

- *For any t, $\|w_\epsilon(t)\|_{\mathcal{F}} \leq \epsilon$*

- *$b_\epsilon(t)$ is a bounded function of t with values in $\mathcal{H}$.*

Bandwidth squeezing:

The quasi-resonant process is governed by the following scheme:

(i) the system pumps energy exciting a small band of frequencies around the forcing frequency;

(ii) this band shrinks gradually as time increases;

(iii) as t tends to infinity, the energy grows linearly in time and becomes infinite : what we observe looks more and more like the generalized eigenfunction.

Summary:

We tried to show that the classical dynamics of the 2D-internal waves can have attracting hyperbolic cycles for some open sets of frequencies. This implies that the spectrum of the associated wave Hamiltonian is continuous on this open set (modulo possibly a discrete set of L^2 eigenvalues). Forcing the dynamics at these frequencies gives waves concentrating on these attracting cycles. We gave an indication on the shape of the limit waves in terms of the generalized (non L^2) eigenfunctions.

Mathematics Perspectives

- Say more about the generalized eigendecomposition if the forcing is smooth...

- Extend to more general Morse-Smale singular foliations: foci, saddle points

- Look more carefully at the case with boundary

- Spectrum near ω's with $\rho(\omega)$ irrational

- 3D case

- Non-linearities in Navier-Stokes

Thank you very much