# FEM Exercise 3:
# A first FEM code

$f = x^2$
$\alpha = 1$
$\beta = 1$

Let us consider the problem $-\alpha u'' + \beta u = f$, in $]0,1[$ with $\alpha, \beta > 0$, $f \in L^2(]0,1[)$ and boundary conditions $u(0) = 0$, $u(1) = 0$.

**2.1.** Compute the analytical solution of the considered problem.

**2.2.** State the variational formulation of the proposed problem and prove the existence and the uniqueness of a weak solution.

**2.3.** Let us consider a uniform mesh over the interval $[0,1]$ with $N+1$ nodes and elements of length $h$. Write the space $V_h$ for a piece-wise linear FE approximation of the weak problem and the discrete problem to solve in $V_h$.

**2.4.** Prove that $V_h \subset H_0^1(]0,1[)$.

**2.5.** Prove the existence and uniqueness of the discrete solution $u_h \in V_h$.

**2.6.** Prove that there exists a unique function $\varphi_i \in V_h$ such that $\varphi_i(x_j) = \delta_{ij}$, $i,j = 1, N+1$ and that $\{\varphi_i\}_{i=1,N+1}$ is a basis of $V_h$. Give the expression of $\varphi_i$, for a generic index $i$.

**2.7.** Compute the local stiffness and mass matrices on a mesh element, with $\alpha$ and $\beta$ equal to 1.

**2.8.** Let us now write a code in Scilab to compute explicitly the discrete solution on the given mesh and evaluate the method convergence rate with respect to $h \to 0$ in different norms. Let us consider $f(x) = x^2$. The program can be structured as follows:

1. Define the mesh (number of nodes, number of elements, size $h$ of the mesh elements, coordinates of the mesh nodes, connectivity of the mesh elements).
2. Dimension and initialize to zero of the global arrays $A$, $F$, $U$, $E$.
3. Loop on the element: for each element
   - Use the local stiffness and mass matrices on a mesh element, computed in **2.3.** Are these matrices different from one element to the other ?
   - Compute the local right-hand side by using the (Gauss Legendre) two-point quadrature formula over $[-1,1]$, with nodes $\hat{x}_i = \pm 1/\sqrt{3}$ and weights $\hat{w}_i = 1$.
   - Assemble the local contributions into the global arrays by using the element connectivity.
4. Impose the boundary conditions on the system matrix and right-hand side.
5. Solve the (discrete problem) linear system by the command U = A\F.
6. Visualize the results by the command plot(x,U), where the $x$ vector contains the coordinates of the mesh nodes and $U$ the solution approximated value at them.
7. Compute the error in the $L^\infty$- and $L^2$-norms for $h$, $h/2$, $h/4$, $h/8$, by using the analytical solution computed in **2.1** and visualize the error in a log-log plot.