FEM Exercise 3: A first FEM code

Let us consider the problem $-\alpha u'' + \beta u = f$, in $]0,1[$ with $\alpha, \beta > 0$, $f \in L^2(]0,1[)$ and boundary conditions $u(0) = 0$, $u(1) = 0$.

2.1. Compute the analytical solution of the considered problem.

The characteristic equation associated to the corresponding homogeneous equation is $-\alpha\, r^2 + \beta \;= 0$, whose roots are $r_{1,2} = \pm\sqrt{\dfrac{\beta}{\alpha}}$. The corresponding solution is then $u(x) = c_1\, \exp(r_1\, x) + c_2\, \exp(r_2\, x)$.

Let ϕ_1 and ϕ_2 be the functions $\phi_1(x) = \exp(r_1\, x)$ and $\phi_2(x) = \exp(r_2\, x)$. The solution for the non-homogeneous equation is the following

$$u(x) = c_1(x)\, \phi_1(x) + c_2(x)\, \phi_2(x)\,, \tag{0.1}$$

where the functions c_1' and c_2' are solutions of the linear system

$$\begin{cases} c_1'(x)\, \phi_1(x) + c_2'(x)\, \phi_2(x) = 0\,, \\ c_1'(x)\, \phi_1'(x) + c_2'(x)\, \phi_2'(x) = \dfrac{f(x)}{\alpha}\,. \end{cases}$$

One gets

$$\begin{cases} c_1'(x) = \dfrac{-\dfrac{f(x)}{\alpha}\, \phi_2(x)}{\phi_1(x)\, \phi_2'(x) - \phi_1'(x)\, \phi_2(x)} = \dfrac{-\dfrac{f(x)}{\alpha}\, \exp\left(\sqrt{\dfrac{\beta}{\alpha}} x\right)}{2\sqrt{\dfrac{\beta}{\alpha}}}\,, \\[2ex] c_2'(x) = \dfrac{\dfrac{f(x)}{\alpha}\, \phi_1(x)}{\phi_1(x)\, \phi_2'(x) - \phi_1'(x)\, \phi_2(x)} = \dfrac{\dfrac{f(x)}{\alpha}\, \exp\left(-\sqrt{\dfrac{\beta}{\alpha}} x\right)}{2\sqrt{\dfrac{\beta}{\alpha}}}\,. \end{cases}$$

Then integrating the above equations over $(0, x)$ where $x \in (0,1)$ one obtains

$$\begin{cases} c_1(x) = c_1(0) + \displaystyle\int_0^x \dfrac{-\dfrac{f(s)}{\alpha}\, \exp\left(\sqrt{\dfrac{\beta}{\alpha}} s\right)}{2\sqrt{\dfrac{\beta}{\alpha}}}\, ds\,, \\[2ex] c_2(x) = c_2(0) + \displaystyle\int_0^x \dfrac{\dfrac{f(s)}{\alpha}\, \exp\left(-\sqrt{\dfrac{\beta}{\alpha}} s\right)}{2\sqrt{\dfrac{\beta}{\alpha}}}\, ds\,. \end{cases}$$

The constants $c_1(0)$ and $c_2(0)$ are computed by using the boundary conditions $u(0) = 0$ and $u(1) = 0$ after substuting $c_1(x)$ and $c_2(x)$ in (0.1).

2.2. State the variational formulation of the proposed problem and prove the existence and the uniqueness of a weak solution.

Let $a : (u,v) \mapsto \alpha \int_0^1 u'v'\,dx + \beta \int_0^1 uv\,dx$ and $L : v \mapsto \int_0^1 fv\,dx$. Set $V = H_0^1(0,1) = \{v \in H^1(0,1)\,, v(0) = v(1) = 0\}$.

The variational formulation is the following:

find $u \in V$ such that $a(u,v) = L(v)$, $\forall v \in V$.

• The space $V = H_0^1(0,1)$ is a hilbert since $V = Ker\gamma$, is closed, where γ is the trace function $\gamma : H^1(0,1) \to L^2(\{0\,1\})\,, v \mapsto \{v(0)\,, v(1)\}$, which is continuous.

• The bilinear form a is continuous :

$$\begin{aligned}
|a(u,v)| &\le \alpha \left|\int_0^1 u'v'\,dx\right| + \beta \left|\int_0^1 uv\,dx\right| \\
&\le \alpha \|u'\|_2 \|v'\|_2 + \beta \|u\|_2 \|v\|_2 \\
&\le \max(\alpha\,,\beta) \left| \begin{pmatrix} \|u\|_2 \\ \|u'\|_2 \end{pmatrix} \cdot \begin{pmatrix} \|v\|_2 \\ \|v'\|_2 \end{pmatrix} \right| \\
&\le \max(\alpha\,,\beta) \left\| \begin{pmatrix} \|u\|_2 \\ \|u'\|_2 \end{pmatrix} \right\|_{2,\mathbb{R}^2} \left\| \begin{pmatrix} \|v\|_2 \\ \|v'\|_2 \end{pmatrix} \right\|_{2,\mathbb{R}^2} \\
&= \max(\alpha\,,\beta) \left(\|u\|_2^2 + \|u'\|_2^2\right)^{1/2} \left(\|v\|_2^2 + \|v'\|_2^2\right)^{1/2} \\
&= \max(\alpha\,,\beta) \|u\|_{H^1(0,1)} \|v\|_{H^1(0,1)}\,.
\end{aligned}$$

• The bilinear form a is coercive :

$$\begin{aligned}
a(u,u) &= \alpha \left|\int_0^1 u'^2\,dx\right| + \beta \left|\int_0^1 u^2\,dx\right| \\
&\ge \min(\alpha\,,\beta) \left(\int_0^1 u'^2\,dx + \int_0^1 u^2\,dx\right) = \min(\alpha\,,\beta) \|u\|_{H^1(0,1)}^2\,.
\end{aligned}$$

• The linear form L is continuous on V:

$$|L(v)| = \left|\int_0^1 fv\,dx\right| \le \|f\|_2 \|v\|_2 \le \|f\|_2 \|v\|_{H^1(0,1)}\,.$$

Thanks to Lax-Milgram theorem, there exists a unique u that solves the variational formulation.

2.3. Let us consider a uniform mesh over the interval $[0,1]$ with $N+1$ nodes and elements of length h. Write the space V_h for a piece-wise linear FE approximation of the weak problem and the discrete problem to solve in V_h.

Let $\mathcal{P}_1$ be the set of polynomials of degree less than or equal to 1 in the variable $x \in \mathbb{R}$, $C^0([0,1])$ the set of continuous functions on the closed interval $[0,1]$. Then the set V is the following

$$V = \{v \in C^0([0,1]),\, v\big|_{[x_i,x_{i+1}]} \in \mathcal{P}_1,\, i = 1\,,\dots,n\,, v(0) = v(1) = 0\}\,.$$

2.4. Prove that $V_h \subset H_0^1(]0,1[)$.

• Let $v \in V_h$. Then $v(0) = v(1) = 0$. Moreover $v \in L^2(0,1)$ since $v \in C^0([0,1])$.

• Let $v \in V_h$. Since $v\big|_{[x_i,x_{i+1}]} \in \mathcal{P}_1$, its derivative in classical sens on each interval $[x_i, x_{i+1}]$ belongs to $\mathcal{P}_1$. Let $w_i = v'\big|_{[x_i,x_{i+1}]}$. For $\phi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned}
\langle v', \phi\rangle &= -\langle v, \phi'\rangle \\
&= -\int_0^1 v\,\phi'\,dx \\
&= -\sum_{i=1}^{n}\int_{x_i}^{x_{i+1}} v\,\phi'\,dx \\
&= -\sum_{i=1}^{n}\big[v\,\phi\big]_{x_i}^{x_{i+1}} + \sum_{i=1}^{n}\int_{x_i}^{x_{i+1}} v'\,\phi\,dx \\
&= v(x_1)\,\phi(x_1) - v(x_{N+1})\,\phi(x_{N+1}) + \sum_{i=1}^{n}\int_0^1 w_i\,\phi'\,dx \\
&= \langle w, \phi\rangle\,,
\end{aligned}$$

where w is defined by $w_i = w\big|_{[x_i,x_{i+1}]}$. Since $w \in L^2(0,1)$, $v \in H^1(0,1)$.

2.5. Prove the existence and uniqueness of the discrete solution $u_h \in V_h$.

The discrete problem is

find $u_h \in V$ such that $a(u_h, v_h) = L(v_h)$, $\forall v_h \in V_h$.

By taking $v_h = \phi_i\,,\, i = 1\,,\dots,N{+}1$ and $u_h = \sum_{j=1}^{N+1} u_{h_j}\,\phi_j$, one gets $a(u_h,\phi_i) = \sum_{j=1}^{N+1} u_{h_j}\,a(\phi_j,\phi_i) = L(\phi_i)\,,\, i = 1\,,\dots,N{+}1$, which is a linear system. Let A be the matrix $A = (a(\phi_i\,,\phi_j))_{i,j=1\,,\dots,N+1}$ and u_h^{N+1} the vector $u_h^{N+1} = \begin{pmatrix} u_{h_1} \\ \vdots \\ u_{h_j} \\ \vdots \\ u_{h_{N+1}} \end{pmatrix}$. Then to prove the existence and uniqueness of u_h, it amounts to prove that A is invertible, which becomes to prove that $Au_h^{N+1} = 0$ leads to $u_h^{N+1} = 0$. Since $Au_h^{N+1} = 0$ implies $a(u_h,u_h) = (Au_h^{N+1}, u_h^{N+1}) = 0$, one gets $0 = a(u_h,u_h) \geq \min(\alpha\,,\beta)\,\|u\|^2_{H^1(0,1)}$ because a is coercive on $V_h \subset V$. This leads to $u_h = 0$ which turns into $u_h^{N+1} = 0$.

2.6. Prove that there exists a unique function $\varphi_i \in V_h$ such that $\varphi_i(x_j) = \delta_{ij}$,

$i,j = 1, N+1$ **and that** $\{\varphi_i\}_{i=1,N+1}$ **is a basis of** V_h**. Give the expression of** φ_i**, for a generic index** i**.**

Let W be the space generated by the family $\{\varphi_i\}_{i=1,N+1}$.

• One has $W \subset V_h$: let $w = \displaystyle\sum_{i=1}^{N+1} \alpha_i \, \phi_i \in W$ where $\alpha_i \in \mathbb{R}$. Then one has $w(0) = w(1) = 0$. Since $\phi_i \in \mathcal{P}_\infty$, one has $\phi_i \in H^1(0,1)$, finally $w \in V_h$.

• One has $V_h \subset W$. Indeed let $v_h \in V_h$. We set $w = \displaystyle\sum_{i=1}^{N+1} v_h(x_i) \, \phi_i$. Clearly one has $w \in W$. Moreover, in each interval $[x_i \, , x_{i+1}]$, $i = 1 \, , \ldots , n$, w and v_h belong to $\mathcal{P}_1$ and coincide in two points, $w(x_i) = v_h(x_i)$, $w(x_{i+1}) = v_h(x_{i+1})$. The functions w and v_h are equal on the all intervalls $[x_i \, , x_{i+1}]$, $i = 1 \, , \ldots , n$, then $w = v_h$ on $(0,1)$.

• The family ϕ_i , $i = 1 \, , \ldots , n$, is linearly independent. Let $0 = w = \sum_{i=1}^{N+1} \alpha_i \, \phi_i$ where α_i are scalars. Since $w(x_j) = \sum_{i=1}^{N+1} \alpha_i \, \phi_i(x_j) = \alpha_j \, \phi_j(x_j) = \alpha_j$, on has $\alpha_i = 0$, $i = 1 \, , \ldots , n$. Hence $\{\varphi_i\}_{i=1,N+1}$ is a basis of V_h.

The functions ϕ_i are given by

• $\phi_1(x) = \begin{cases} \dfrac{x_2 - x}{h} & \text{if } x \in [x_1, x_2] \, , \\ 0 & \text{otherwise} \, , \end{cases}$

• $\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x \in [x_{i-1}, x_i] \, , \\ \dfrac{x_{i+1} - x}{h} & \text{if } x \in [x_i, x_{i+1}] \, , \quad \text{for } i = 2 \, , \ldots , N \, , \\ 0 & \text{otherwise} \, , \end{cases}$

• $\phi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{h} & \text{if } x \in [x_N, x_{N+1}] \, , \\ 0 & \text{otherwise} \, . \end{cases}$

2.7. Compute the local stiffness and mass matrices on a mesh element, with α **and** β **equal to 1.**

Let us consider the ith element $[x_i \, , x_{i+1}]$ where $i = 1 \, , \ldots , N$. The local stiff matrix $(a_{lk})_{l,k=1,2}$ entries are given by

$$a_{11} = \alpha \int_{x_i}^{x_{i+1}} \phi_i'(x) \, \phi_i'(x) \, dx = \frac{\alpha}{h} \, , \qquad a_{12} = \alpha \int_{x_i}^{x_{i+1}} \phi_i'(x) \, \phi_{i+1}'(x) \, dx = -\frac{\alpha}{h} \, ,$$

$$a_{21} = \alpha \int_{x_i}^{x_{i+1}} \phi_{i+1}'(x) \, \phi_i'(x) \, dx = -\frac{\alpha}{h} \, , \quad a_{22} = \alpha \int_{x_i}^{x_{i+1}} \phi_{i+1}'(x) \, \phi_{i+1}'(x) \, dx = \frac{\alpha}{h} \, ,$$

which gives

$$(a) = \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} .$$

The local mass matrix $(m_{lk})_{l,k=1,2}$ entries are given by

$$m_{11} = \beta \int_{x_i}^{x_{i+1}} \phi_i(x)\, \phi_i(x)\, dx = \frac{\beta}{3}\, h\,, \qquad m_{12} = \beta \int_{x_i}^{x_{i+1}} \phi_i(x)\, \phi_{i+1}(x)\, dx = \frac{\beta}{6}\, h\,,$$

$$m_{21} = \beta \int_{x_i}^{x_{i+1}} \phi_{i+1}(x)\, \phi_i(x)\, dx = \frac{\beta}{6}\, h\,, \quad m_{22} = \beta \int_{x_i}^{x_{i+1}} \phi_{i+1}(x)\, \phi_{i+1}(x)\, dx = \frac{\beta}{3}\, h\,,$$

which leads to

$$(m) = \frac{\beta}{3}\, h \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$

2.8. Let us now write a code in Scilab to compute explicitly the discrete solution on the given mesh and evaluate the method convergence rate with respect to $h \to 0$ in different norms. Let us consider $f(x) = x^2$. The program can be structured as follows:

1. Define the mesh (number of nodes, number of elements, size h of the mesh elements, coordinates of the mesh nodes, connectivity of the mesh elements).
2. Dimension and initialize to zero of the global arrays A, F, U, E.
3. Loop on the element: for each element
   - Use the local stiffness and mass matrices on a mesh element, computed in **2.3**. Are these matrices different from one element to the other ?
   - Compute the local right-hand side by using the (Gauss Legendre) two-point quadrature formula over $[-1, 1]$, with nodes $\hat{x_i} = \pm 1/\sqrt{3}$ and weights $\hat{w}_i = 1$.
   - Assemble the local contributions into the global arrays by using the element connectivity.
4. Impose the boundary conditions on the system matrix and right-hand side.
5. Solve the (discrete problem) linear system by the command U = A F.
6. Visualize the results by the command plot(x,U), where the x vector contains the coordinates of the mesh nodes and U the solution approximated value at them.
7. Compute the error in the L^∞- and L^2-norms for h, $h/2$, $h/4$, $h/8$, by using the analytical solution computed in **2.1** and visualize the error in a log-log plot.